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Extensions of valuations

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Continuous valuations have been proposed by several authors as a way of modelling probabilistic non-determinism in programming language semantics. Let $(X, \mathcal{O})$ be a topological space. A quasi-simple valuation on X is the sup of a directed family of simple valuations. We show that quasi-simple valuations are exactly those valuations that extend to continuous valuations to the Alexandroff topology on the specialisation preordering of the topology $\mathcal{O}$. A number of applications are presented. In particular, we recover Jones' result that every continuous valuation is quasi-simple if X is a continuous dcpo – in this case there is a least extension to the Alexandroff topology. We show that this can be refined if X is algebraic, where every continuous valuation is the sup of a directed family of simple valuations based on finite elements. We exhibit another class of spaces in which every continuous valuation is quasi-simple, the so-called finitarily coherent spaces – in this case there is a largest extension to the Alexandroff topology. In general, the extension to the Alexandroff topology is not unique, unless, for example, the original valuation is bicontinuous. We also show that other natural spaces of valuations, namely those of discrete valuations and point-continuous valuations, can be characterised by similar extension theorems.

1. Introduction

Giving a faithful account of probabilistic non-determinism in denotational semantics has attracted quite a lot of research since the pioneering work of Saheb-Djahromi, and of Jones and Plotkin. Mixing domain theory with probabilities seems to require one to use *continuous valuations* rather than measures, as justified and studied in Claire Jones' Ph. D. thesis (Jones 1990). Amongst all the continuous valuations, those that can be built as sups of directed families of simple valuations are the most natural, and can be handled most conveniently. (We will give definitions shortly.) For example, if v is a continuous valuation that can be written as the sup of the directed family

$$\left(\sum_{j=1}^{n_i} a_{ij}\delta_{x_{ij}}\right)_{i \in I},$$

then the integral $\int_{x \in X} f(x) dv$ (as defined in Tix (1995) for example, extending Jones (1990)) can be *computed*, not just characterised, as the sup of all finite sums $\sum_{j=1}^{n_i} a_{ij} f(x_{ij})$. This is one of the notable ingredients in Edalat (1995).

We will call those valuations that are sups of directed families of simple valuations *quasi-simple* for short. The initial purpose of this paper is to refine our understanding of quasi-simple valuations. This will be achieved by proving Theorem 1.1 below, and exploring some of its applications.

We will begin by giving a few definitions. Let $(X, \mathcal{O})$ be a topological space. A *valuation* on X is a map $\nu : \mathcal{O} \to (\mathbb{R}_+ \cup \{+\infty\})$ such that $\nu(\emptyset) = 0$ (*strictness*); $U \subseteq V$ implies $\nu(U) \leqslant \nu(V)$ for all opens $U, V \in \mathcal{O}$ (*monotonicity*); and $\nu(U \cup V) + \nu(U \cap V) = \nu(U) + \nu(V)$ for all opens $U, V \in \mathcal{O}$ (*modularity*). A *continuous* valuation also satisfies $\nu(\bigcup_i U_i) = \sup_i \nu(U_i)$ for every directed family $(U_i)_{i \in I}$ of opens: such a family is *directed* if and only if $I \neq \emptyset$ and for every $i, j \in I$, there is $k \in I$ such that $U_i, U_j \subseteq U_k$. Continuous valuations are a concept close to that of measure; while measures are defined on σ-algebras, valuations are naturally defined on topological spaces.

We can build continuous valuations using a number of different means. First, the *Dirac valuation* δ_x maps every open containing x to 1, and every open not containing x to 0. Second, any finite linear combination $\sum_{i=1}^n a_i \nu_i$ of continuous valuations $(\nu_i)_{1 \leqslant i \leqslant n}$ $(a_i \in \mathbb{R}_+)$ is again a continuous valuation; a linear combination $\sum_{i=1}^n a_i \delta_{x_i}$ of Dirac valuations is called a *simple valuation*. Third, ordering valuations pointwise, any sup of directed families of continuous valuations is again a continuous valuation. We shall call directed sups of simple valuations *quasi-simple valuations*. In general, it is not the case that every bounded continuous valuation is quasi-simple. However, Jones (1990) showed that every bounded continuous valuation is indeed quasi-simple when X is the topological space underlying a continuous dcpo.

The main point of this paper is to give an equivalent characterisation of quasi-simple valuations, as in the following theorem.

Theorem 1.1. Let $(X, \mathcal{O})$ be any topological space. A bounded function $\nu : \mathcal{O} \to \mathbb{R}_+$ is a quasi-simple valuation if and only if it extends to a continuous valuation on the Alexandroff extension of the topology $\mathcal{O}$.

Here ν is *bounded* just means that the range of ν is included in $\mathbb{R}_+$, in particular, $\nu(X) < +\infty$. The major ingredient of the theorem is the Alexandroff extension of a topology. First, recall that on every topological space $(X, \mathcal{O})$ we may define a *specialisation preordering* $\leqslant_s$ by $x \leqslant_s y$ if and only if x is in the closure of y; or, equivalently, if every closed set containing y contains x; or, equivalently, if every open set containing x contains y. This is an ordering exactly when X is T_0. Conversely, given any preordering $\leqslant$, the set of sets that are upper with respect to $\leqslant$ is a topology, called the *Alexandroff topology* over $\leqslant$. We call the Alexandroff topology over the specialisation preordering $\leqslant_s$ the *Alexandroff extension* of the topology $\mathcal{O}$. Notice that every $U \in \mathcal{O}$ is upper in $\leqslant_s$, so all opens are opens of the Alexandroff extension. In the rest of the paper, we shall call any extension of a continuous valuation ν to the Alexandroff extension of the topology an *A-extension* of ν.

Outline of the paper

We prove Theorem 1.1 in Section 2, then explore a number of applications in Section 3. First, we shall show in Section 3.1 that Jones' Theorem that every bounded continuous valuation ν on a continuous dcpo is quasi-simple is an easy corollary of this theorem. In

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fact, v not only has an A-extension, but also a *least* one in this case. We shall then refine this in Section 3.2 to show that every bounded continuous valuation on an algebraic dcpo is the sup of a directed family of simple valuations *based on finite elements*, which fills a gap in Sünderhauf's argument (Sünderhauf 1997). We note that the space of A-extensions does not, in general, have any remarkable structure: in particular, A-extensions need not be unique. However, a case where A-extensions are necessarily unique is that of bicontinuous valuations *à la* Bukatin and Shorina, which we briefly touch upon in Section 3.3. Dually to the continuous dcpo case, we exhibit in Section 3.4 a new class of topological spaces in which every continuous valuation is quasi-simple, this time because there is a *largest* A-extension: the *finitarily coherent spaces*. We derive other, simpler extension results for other classes of continuous valuations in Section 4: the discrete and the point-continuous valuations, respectively. We sprinkle the text liberally with remarks and examples. We conclude by reviewing related work in Section 5.

2. Proof of Theorem 1.1

2.1. *Only if*

The *only if* direction of the Theorem is relatively easy.

First, any simple valuation $v = \sum_{i=1}^{n} a_i \delta_{x_i}$ extends to the Alexandroff valuation of $\mathcal{O}$, namely the valuation $\overline{v}$ that is just $\sum_{i=1}^{n} a_i \delta_{x_i}$ again, only defined on all $\leqslant_s$-upper subsets of X.

Second, we note the well-known fact that any sup v of a directed family of continuous valuations v_i, $i \in I$, on any topology is a continuous valuation on the same topology. We leave this to the reader; see Jones (1990, Theorem 4.1) for a possible proof.

We then need the following auxiliary lemma, which slightly extends Jones' Splitting Lemma.

Lemma 2.1 (Sünderhauf 1997, Theorem 2.4, Corollary 2.6). Let $(X, \mathcal{O})$ be a topological space, and $\leqslant_s$ be its specialisation preordering. Define the following binary relations on the set of simple valuations on X:

1 $v \leqslant_1 v'$ if and only if $v' = v + a\delta_x$ for some $a \in \mathbb{R}_+$, and $x \in X$.
2 $v \leqslant_2 v'$ if and only if v is of the form $v_0 + a\delta_x$, and $v' = v_0 + a\delta_y$ for some simple valuation v_0, some $a \in \mathbb{R}_+$, and some pair x, y of elements of x such that $x \leqslant_s y$.

The preordering $\leqslant$ on simple valuations is exactly the smallest preordering containing $\leqslant_1 \cup \leqslant_2$.

An equivalent, but more synthetic characterisation, is to say that $\sum_{i=1}^{n} a_i \delta_{x_i} \leqslant \sum_{j=1}^{n} b_j \delta_{x_j}$ if and only if there is a matrix $(t_{ij})_{1 \leqslant i,j \leqslant n}$ of non-negative reals such that $a_i = \sum_{j=1}^{n} t_{ij}$ for all i, $b_j = \sum_{i=1}^{n} t_{ij}$ for all j, and whenever $t_{ij} \neq 0$, we have $x_i \leqslant_s x_j$.

Lemma 2.1 implies, in particular, that the property $\sum_{i=1} a_i \delta_{x_i} \leqslant \sum_{j=1}^{n} b_j \delta_{x_j}$ between simple valuations does not depend on the topology $\mathcal{O}$ but only on its specialisation preordering $\leqslant_s$. It follows that, given any directed family $(v_i)_{i \in I}$ of simple valuations on $(X, \mathcal{O})$, the family $(\overline{v}_i)_{i \in I}$ of simple valuations on the Alexandroff extension of $\mathcal{O}$ is also directed. Indeed, $\mathcal{O}$ and its Alexandroff extension have the same specialisation preordering.

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Then the sup of the family $(\overline{\nu}_i)_{i\in I}$ is a continuous valuation on the Alexandroff extension of $\mathcal{O}$, as a sup of a directed family of continuous valuations on the same Alexandroff extension. Furthermore, $\sup_{i\in I}\overline{\nu}_i$ clearly extends $\sup_{i\in I}\nu_i$.

2.2. *If*

The if direction is more complex, but proceeds along lines very similar to Jones (1990, Section 5.1). We mimic the latter to draw attention to the similarity.

Fix a set X and a preordering $\leqslant$ on X. We write $\uparrow A$ for $\{y | \exists x \in A \cdot x \leqslant y\}$, for each $A \subseteq X$, and $\uparrow x$ for $\uparrow\{x\}$. Let X_0 be some fixed subset of X, and assume that the family of sets $\uparrow A$, where $A \subseteq X_0$, forms a topology $\mathcal{O}$. (For example, this is the case if $X_0 = X$, where we recover the Alexandroff topology.) We shall show that every bounded continuous valuation on $(X, \mathcal{O})$ is a sup of a directed family of simple valuations $\sum_{i=1}^n a_i\delta_{x_i}$, where $x_i \in X_0$ for every i.

A *crescent* C is any difference $U \setminus V$ of (Alexandroff) opens U, V. A *field* of sets is any family of sets such that every finite union and every complement of elements of the field is again in the field. This includes $\emptyset$ and the whole space X. The field generated by the open sets is the smallest field containing the opens; this is exactly the set of all finite disjoint unions of crescents. Any bounded valuation ν extends to a unique finitely additive function on the field of all crescents, defined by $\nu(U \setminus V) = \nu(U) - \nu(U \cap V) = \nu(U \cup V) - \nu(V)$. (A *finitely additive function* ν maps any finite disjoint union $\bigcup_i C_i$ to $\sum_i \nu(C_i)$.) This is the Smiley–Horn–Tarski Theorem, which is often attributed to Pettis (Alvarez-Manilla 2000, Section 1.11). We equate continuous valuations on $(X, \mathcal{O})$ with their unique extension to the field of disjoint unions of crescents.

Notice that, for every continuous valuation ν, for every family $(O_i)_{i\in I}$ of opens,

$$\nu\left(\bigcup_{i\in I} O_i\right) = \sup_{J \text{ finite } \subseteq I} \nu\left(\bigcup_{j\in J} O_j\right).$$

This is because the family of finite unions of O_i is directed.

Lemma 2.2. Let ν be a continuous valuation, O and O_i, $i \in I$, be open. Then

$$\nu\left(\bigcup_{i\in I} O_i \setminus O\right) = \sup_{J \text{ finite } \subseteq I} \nu\left(\bigcup_{j\in J} O_j \setminus O\right). \tag{1}$$

Proof. This is exactly Lemma 5.1 of Jones (1990), although on a different topology. For completeness, here is the proof. The left-hand side of (1) is:

$$\begin{aligned}\nu\left(\bigcup_{i\in I} O_i \setminus O\right) &= \nu\left(\bigcup_{i\in I} O_i\right) - \nu\left(\bigcup_{i\in I} O_i \cap O\right)\\ &= \sup_{J \text{ finite } \subseteq I} \nu\left(\bigcup_{j\in J} O_j\right) - \sup_{J \text{ finite } \subseteq I} \nu\left(\bigcup_{j\in J} O_j \cap O\right)\end{aligned}$$

by continuity of ν. Noting that sups of upper bounded directed families of real numbers

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are just their limits, in the sense of Moore–Smith convergence, and since differences of limits are just limits of differences, this is exactly

$$\sup_{J \text{ finite } \subseteq I} \left[v\left(\bigcup_{j\in J} O_j\right) - v\left(\bigcup_{j\in J} O_j \cap O\right)\right] = \sup_{J \text{ finite } \subseteq I} v\left(\bigcup_{j\in J} O_j \setminus O\right). \qquad \square$$

We define a *pre-dissection* D to be any finite set of pairs (x_i, U_i), $1 \leqslant i \leqslant n$, where $x_i \in X_0$ and U_i is open in $\mathcal{O}$: that is, U_i is of the form $\uparrow A_i$ for $A_i \subseteq X_0$. By convention, we write $C_i = \uparrow x_i \setminus U_i$. D is a *dissection* if, additionally, $C_1, \ldots, C_n$ are pairwise disjoint.

Given a dissection D and a continuous valuation v, we define

$$v[D] = \sum_{i=1}^{n} v(C_i)\delta_{x_i}. \tag{2}$$

It is clear that $v[D] \leqslant v$, since for every open U we have

$$v[D](U) = \sum_{x_i\in U} v(C_i) = v\left(\bigcup_{i/x_i\in U} C_i\right)$$

(since v is finitely additive and the C_is are pairwise disjoint), and

$$\bigcup_{i/x_i\in U} C_i \subseteq \bigcup_{i/x_i\in U} \uparrow x_i \subseteq U$$

(because U is upper). In fact we have the following lemma.

Lemma 2.3. Let X_0 be some fixed subset of X, and assume that the family of sets $\uparrow A$, $A \subseteq X_0$, forms a topology $\mathcal{O}$. Every bounded continuous valuation v on $\mathcal{O}$ is the sup of $(v[D])_{D \text{ dissection}}$.

Proof. Let U be any open of the form $\uparrow\{x_1, \ldots, x_n\}$: we call this a *finitary* open. Let U_i be the open $\uparrow\{x_1, \ldots, x_{i-1}\}$ for each i, $1 \leqslant i \leqslant n$. The set D of all pairs (x_i, U_i) is a pre-dissection. We claim that it is a dissection. Indeed, for any $i < j$, we know C_j does not intersect U_j by construction; but U_j contains $\uparrow x_i$ (since $i < j$), which contains C_i, so C_j cannot intersect C_i.

On the other hand, $\bigcup_{i=1}^n C_i = U$. Indeed, by induction on k, $\bigcup_{i=1}^k C_i = \uparrow\{x_1, \ldots, x_k\}$. So $v[D](U) = \sum_{i/x_i\in U} v(C_i) = \sum_{i=1}^n v(C_i) = v(\bigcup_{i=1}^n C_i) = v(U)$.

Fix an arbitrary open U, and an arbitrary $\epsilon > 0$. Since every open is the union of the directed family of all finitary opens contained in it, and since v is continuous, there is a finitary open $U_0 \subseteq U$ such that $v(U) \leqslant v(U_0) + \epsilon$. Now, by the above, there is a dissection D such that $v(U_0) = v[D](U_0)$. Since $U_0 \subseteq U$, we obtain $v[D](U_0) \leqslant v[D](U)$, so $v(U) \leqslant v[D](U) + \epsilon$. As $\epsilon > 0$ is arbitrary, $v(U) \leqslant \sup_D v[D](U)$. Since U is arbitrary, $v \leqslant \sup_D v[D]$. This completes the proof since $v[D] \leqslant v$ for all dissections D. $\square$

However, the family of all $v[D]$, where D ranges over all dissections, is not necessarily directed. Lemma 2.3 clearly entails the following corollary.

Corollary 2.4. Under the assumptions of Lemma 2.3, v is the sup of all valuations $r \cdot v[D]$, where D ranges over all dissections, and $0 < r < 1$.

We shall show that the family $(r \cdot \nu[D])_{D \text{ dissection}, 0<r<1}$ is directed. The key observation is the following lemma, which is roughly similar to Jones' Lemma 5.3.

Lemma 2.5. Let X_0 be some fixed subset of X, and assume that the family of sets $\uparrow A$, $A \subseteq X_0$, forms a topology $\mathcal{O}$.

For any open sets $O_1, \ldots, O_n$ in $\mathcal{O}$, for every $0 \leqslant s < 1$, there is a dissection D such that, for any $B \subseteq \{1, \ldots, n\}$,

$$s \cdot \nu \left(\bigcup_{i \in B} O_i \right) \leqslant \nu[D] \left(\bigcup_{i \in B} O_i \right). \tag{3}$$

Proof. For every $J \subseteq \{1, \ldots, n\}$, let C_J be the crescent $\bigcap_{j \in J} O_j \setminus \bigcup_{j \notin J} O_j$; in the rest of the proof we exclude the case where $J = \emptyset$. Clearly the C_Js are pairwise disjoint, and their union is $\bigcup_{i=1}^n O_i$. In fact, for every $B \subseteq \{1, \ldots, n\}$, $\bigcup_{i \in B} O_i = \bigcup_{J / J \cap B \neq \emptyset} C_J$, where the union on the right is a disjoint union. In particular,

$$\nu \left(\bigcup_{i \in B} O_i \right) = \sum_{J / J \cap B \neq \emptyset} \nu(C_J). \tag{4}$$

We now extract a dissection from each C_J, whose measure approximates that of C_J, using the continuity Lemma 2.2. Observe that since $\bigcap_{i \in J} O_i$ is open, it is a union of basic opens $\uparrow x$ with $x \in X_0$, and hence it is the union of all $\uparrow x$ with $x \in \bigcap_{i \in J} O_i \cap X_0$. We claim that

$$C_J = \left(\bigcup_{x \in C_J \cap X_0} \uparrow x \right) \Big\backslash \bigcup_{j \notin J} O_j \tag{5}$$

for every J. Indeed, let y be an arbitrary element of C_J. Then

$$y \in \bigcap_{i \in J} O_i = \bigcup_{x \in \bigcap_{i \in J} O_i \cap X_0} \uparrow x,$$

so let x be some element of $\bigcap_{i \in J} O_i \cap X_0$ such that $x \leqslant y$. Since $y \in C_J$, we obtain $y \notin \bigcup_{j \notin J} O_j$, and since $\bigcup_{j \notin J} O_j$ is open, and hence upper, x is not in $\bigcup_{j \notin J} O_j$ either. So x is in

$$\left(\bigcap_{i \in J} O_i \cap X_0 \right) \Big\backslash \bigcup_{j \notin J} O_j = C_J \cap X_0.$$

Since $x \leqslant y$, in particular, y is in $\bigcup_{x \in C_J \cap X_0} \uparrow x$. So y is in the right-hand side of (5). Conversely, let y be arbitrary in the right-hand side of (5). In particular, $x \leqslant y$ for some $x \in C_J \cap X_0$. Since $x \in C_J$, in particular, $x \in \bigcap_{j \in J} O_j$. Since the open $\bigcap_{j \in J} O_j$ is upper, $y \in \bigcap_{j \in J} O_j$. Since, on the other hand, $y \notin \bigcup_{j \notin J} O_j$, we obtain $y \in C_J$.

Let $\epsilon_J = (1 - s)\nu(C_J)$. From (5) and Lemma 2.2 we infer that, when $\epsilon_J > 0$, there is a finite set of points $x_{J1}, \ldots, x_{Jm_J}$ in $C_J \cap X_0$ such that

$$\nu(C_J) \leqslant \nu \left(\bigcup_{i=1}^{m_J} \uparrow x_{Ji} \Big\backslash \bigcup_{j \notin J} O_j \right) + \epsilon_J. \tag{6}$$

This also holds when $\epsilon_J = 0$, since in this case $\nu(C_J) = 0$, and therefore (6) is trivial.

Let D_J be the dissection $\{(x_{Ji}, U_{Ji}) | 1 \leqslant i \leqslant m_J\}$, with

$$U_{Ji} = \uparrow\{x_{J1}, \ldots, x_{J(i-1)}\} \cup \bigcup_{j \notin J} O_j$$

(compare with the proof of Lemma 2.3). Note, in particular, that since every x_{Ji} is in X_0, U_{Ji} is indeed open. Let C_{Ji} be the pairwise disjoint sets $\uparrow x_{Ji} \setminus U_{Ji}$, $1 \leqslant i \leqslant m_J$. Then $\bigcup_{i=1}^{m_J} C_{Ji} = \bigcup_{i=1}^{m_J} \uparrow x_{Ji} \setminus \bigcup_{j \notin J} O_j$, so, by (6),

$$v(C_J) \leqslant \sum_{i=1}^{m_J} v(C_{Ji}) + \epsilon_J. \tag{7}$$

Let D be the union of all D_J, $J \subseteq \{1, \ldots, n\}$. This is a pre-dissection. To show that it is a dissection, it remains to show that C_{Ji} does not intersect $C_{J'i'}$ when $J \neq J'$. This is straightforward, since $C_{Ji} \subseteq \bigcup_{i=1}^{m_J} C_{Ji} \subseteq C_J$, and, similarly, $C_{J'i'} \subseteq C_{J'}$, and C_J and $C_{J'}$ do not intersect.

For every $J, J' \subseteq \{1, \ldots, n\}$, $x_{J'i} \in C_J$ if and only if $J = J'$. Indeed, remember that $x_{J'i}$ is in $C_{J'}$, and the C_Js are pairwise disjoint. So

$$v[D](C_J) = \sum_{J' \subseteq \{1,\ldots,n\}} \sum_{i=1}^{m_{J'}} v(C_{J'i}) \delta_{x_{J'i}}(C_J) = \sum_{1 \leqslant i \leqslant m_J} v(C_{Ji}) \geqslant v(C_J) - \epsilon_J \quad \text{using (7)}$$

Summing over all $J \subseteq \{1, \ldots, n\}$ such that $J \cap B \neq \emptyset$, we obtain

$$\begin{aligned} v[D]\left(\bigcup_{i \in B} O_i\right) &\geqslant v\left(\bigcup_{i \in B} O_i\right) - \sum_{J / J \cap B \neq \emptyset} \epsilon_J \\ &= v\left(\bigcup_{i \in B} O_i\right) - (1-s) \sum_{J / J \cap B \neq \emptyset} v(C_J) \\ &= v\left(\bigcup_{i \in B} O_i\right) - (1-s) v\left(\bigcup_{i \in B} O_i\right) \\ &= s \cdot v\left(\bigcup_{i \in B} O_i\right). \end{aligned}$$

□

We claim that the family $(r \cdot v[D])_{D \text{ dissection}, 0<r<1}$ is directed. Indeed, consider $r_1 \cdot v[D_1]$ and $r_2 \cdot v[D_2]$, where $0 < r_1, r_2 < 1$ and D_1 and D_2 are two dissections. Let D_i be written as the set of all (x_{ij}, U_{ij}), $1 \leqslant j \leqslant n_i$, $i \in \{1, 2\}$. Write C_{ij} for $\uparrow x_{ij} \setminus U_{ij}$. Let

$$O_1 = \uparrow x_{11} \quad \ldots \quad O_{n_1} = \uparrow x_{1n_1} \quad O_{n_1+1} = \uparrow x_{21} \quad \ldots \quad O_{n_2} = \uparrow x_{2n_2}$$

and $n = n_1 + n_2$. Let $0 \leqslant s < 1$. By Lemma 2.5 there is a dissection D such that, for any $B \subseteq \{1, \ldots, n\}$,

$$s \cdot v\left(\bigcup_{i \in B} O_i\right) \leqslant v[D]\left(\bigcup_{i \in B} O_i\right).$$

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Let O be any open. Let B_1 be the set of indices j such that $x_{1j} \in O$. In particular, O contains $\bigcup_{j \in B_1} \uparrow x_j$, so

$$s \cdot v \left(\bigcup_{i \in B_1} O_i \right) \leqslant v[D](O).$$

On the other hand,

$$\begin{aligned} v[D_1](O) &= \sum_{i=1}^{n_1} v(C_{1i}) \delta_{x_{1i}}(O) \\ &= \sum_{i \in B_1} v(C_{1i}) \\ &= v \left(\bigcup_{i \in B_1} C_{1i} \right) \\ &\leqslant v \left(\bigcup_{i \in B_1} \uparrow x_{1i} \right) \\ &= v \left(\bigcup_{i \in B_1} O_i \right). \end{aligned}$$

So $s \cdot v[D_1](O) \leqslant v[D](O)$. Similarly, we prove that $s \cdot v[D_2] \leqslant v[D](O)$. Since O is arbitrary, $s \cdot v[D_1] \leqslant v[D]$ and $s \cdot v[D_2] \leqslant v[D]$. Recall that s is arbitrary. We may choose s and some r such that $0 < s, r < 1$ and $r_1, r_2 \leqslant rs$. It follows that $r_1 \cdot v[D_1] \leqslant r \cdot v[D]$ and $r_2 \cdot v[D_2] \leqslant r \cdot v[D]$. So $(r \cdot v[D])_{D \text{ dissection}, 0<r<1}$ is indeed directed. We then conclude by Corollary 2.4. We have proved the following proposition.

Proposition 2.6. Let X_0 be some fixed subset of X, and assume that the family of sets $\uparrow A$, $A \subseteq X_0$, forms a topology $\mathcal{O}$. Then every bounded continuous valuation on $(X, \mathcal{O})$ is a sup of a directed family of simple valuations $\sum_{i=1}^n a_i \delta_{x_i}$, where $x_i \in X_0$ for every i.

The *if* direction of Theorem 1.1 follows by taking $X_0 = X$.

3. Applications

Given any continuous valuation v on a topological space $(X, \mathcal{O})$, there are two extreme ways of attempting to extend v to the Alexandroff extension of $\mathcal{O}$, that is, to build an A-extension of v. None of these yields a continuous valuation in general, but in particular cases they will.

First, we may approximate v from below: define v_* by

$$v_*(U) = v\left(\overset{\circ}{U}\right)$$

for each upper set U, where $\overset{\circ}{U}$ is the interior of U for the topology $\mathcal{O}$. We may also approximate v from above: define v^* by

$$v^*(U) = \inf_{O \in \mathcal{O}, O \supseteq U} v(O).$$

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Both v^* and v_* are strict and monotonic. Clearly, v_* and v^* bound all A-extensions of v, as in the following lemma.

Lemma 3.1. Let $(X, \mathcal{O})$ be a topological space. For every valuation v on $(X, \mathcal{O})$, for every A-extension $\overline{v}$ of v,

$$v_* \leqslant \overline{v} \leqslant v^*.$$

Proof. For every Alexandroff open U, we have $\overline{v}(U) \leqslant \overline{v}(O) = v(O)$ for every open $O \in \mathcal{O}$ such that $U \subseteq O$. So $\overline{v}(U) \leqslant \inf_{O \in \mathcal{O}, O \supseteq U} v(O) = v^*(U)$. On the other hand, $\mathring{U} \subseteq U$, so $v_*(U) = v(\mathring{U}) = \overline{v}(\mathring{U}) \leqslant \overline{v}(U)$. □

We shall refine the upper bound in Section 3.4, Fact 2.

We shall be particularly interested in topological spaces arising from directed complete partial orders. Recall that a *directed complete partial order*, or *dcpo*, is a partial order $(X, \leqslant)$ such that every directed subset of X has a least upper bound $\sup X$. A subset A of X is *directed* provided A is non-empty and any two elements in A have an upper bound in A. The *Scott topology* on X has as opens all upper subsets O such that, if $\sup D \in O$ for some directed subset D of X, then D contains some element of O. The specialisation ordering of the Scott topology is $\leqslant$, just as for the Alexandroff topology.

Remark 3.1. There is no reason why v^* or v_* should be modular. Let X be $\mathbb{N} \times (\mathbb{N} \cup \{\omega\})$ with the ordering $(j, k) \leqslant (m, n)$ if and only if either $j = m$ and $k \leqslant n$, or $n = \omega$ and $k \leqslant m$. This is a dcpo, which we shall study more closely in Remark 3.6. Its non-empty Scott opens contain all but finitely many points of the form (m, ω). In particular, every finite intersection of non-empty Scott opens is non-empty. This implies that the function v mapping $\emptyset$ to 0 and each non-empty Scott open to 1 is modular, hence a continuous valuation. But v^* maps every non-empty upper set to 1: let $U = \{(2i, \omega) | i \in \mathbb{N}\}$, $V = \{(2i+1, \omega) | i \in \mathbb{N}\}$, then $v^*(U \cup V) + v^*(U \cap V) = 1 + 0 = 1$ while $v^*(U) + v^*(V) = 1 + 1 = 2$. Similarly,

$$v_*(U \cup V) + v_*(U \cap V) = v_*\{(i, \omega) | i \in \mathbb{N}\} + v_*(\emptyset) = v\{(i, \omega) | i \in \mathbb{N}\} = 1,$$

while $v_*(U) + v_*(V) = 0 + 0 = 0$.

There is no reason why v_* should be continuous either. Take v and X again as in Remark 3.1, and let $U_n = \{(i, \omega) | 0 \leqslant i \leqslant n\}$. Then $v_*(U_n) = v(\emptyset) = 0$, so $\sup_{n \in \mathbb{N}} v_*(U_n) = 0$, while $v_*(\sup_{n \in \mathbb{N}} U_n) = v_*\{(i, \omega) | i \in \mathbb{N}\} = v\{(i, \omega) | i \in \mathbb{N}\} = 1$.

Remark 3.2. In general, v^* is not continuous either. Let X be $[0, 1]$ with its usual, metric topology. Its specialisation preordering is just equality, since X is Hausdorff, hence T_1. Let v be the Lebesgue measure on the Borel σ-algebra built on top of this topology. By restriction to the open sets, v yields a strict, monotonic and modular function. It is also continuous, which means that, as a measure, it is τ-smooth: every measure on a fully Lindelöf space, in particular, on a second countable space, is τ-smooth (Alvarez-Manilla 2000, page 35). So v defines a continuous valuation on X.

However, v^* is not continuous. Indeed, let O be any open subset of X. Then O is the directed union of its finite subsets $\mathbb{K} \subseteq O$. But it is clear that $v^*(\mathbb{K}) = 0$. If v^*

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was continuous, therefore, v^* would be identically zero, which is absurd: for example, $v^*(X) = v(X) = 1$.

3.1. *Continuous dcpos, and Erné's C-spaces*

We first prove Jones' result again (Corollary 3.5 below). Let $(X, \leqslant)$ be a partially ordered set. The relation $\ll$ is defined by $x \ll y$ if and only if for every directed subset D of X such that $\sup D$ exists and $y \leqslant \sup D$, then $x \leqslant z$ for some $z \in D$; $\ll$ is called the *way-below* relation, and we write $\twoheaduparrow x = \{y \in X | x \ll y\}$. We say $(X, \leqslant)$ is a *continuous* partially ordered set if and only if, for every $y \in X$, the set $D = \{x \in X | x \ll y\}$ is directed, $\sup D$ exists, and $y = \sup D$. A *continuous dcpo* is a dcpo that is continuous as a partially ordered set. The definition of Scott topology extends to partially ordered sets by: O is Scott open if and only if O is upper and for every directed subset D of X such that $\sup D$ exists and is in O, we have D and O intersect. If X is a continuous partially ordered set, then $\twoheaduparrow x$ is Scott open for every x. In fact, the sets $\twoheaduparrow x$, $x \in X$, form a basis of the Scott topology: that is, every Scott open O is the union of all $\twoheaduparrow x$, $x \in O$.

In any topological space, the interior operator distributes over intersection: the interior of $A \cap B$ is always $\overset{\circ}{A} \cap \overset{\circ}{B}$. In general, interior does not distribute over union: take X to be $\mathbb{R}$ with its usual topology, $A = \mathbb{Q}$, and $B = \mathbb{R} \setminus \mathbb{Q}$, then the interior of $A \cup B$ is $\mathbb{R}$, while both $\overset{\circ}{A}$ and $\overset{\circ}{B}$ are empty. However, interior *does* distribute over unions of upper subsets in continuous dcpos, as we see next.

The correct generalisation of dcpos in this case is given by the following definition.

Definition 3.2 (Erné 1991). A *C-space* is a topological space where every point has a neighbourhood basis of upper sets, that is, for every point y in some open U, there is a point x in U such that y is in the interior of $\uparrow x$.

Every continuous dcpo is a C-space: given y and U, y is the sup of all $x \ll y$; since U is Scott open, some $x \ll y$ must be in U; then y is in $\twoheaduparrow x$, which is an open subset of $\uparrow x$. The same argument shows that, more generally, every continuous partially ordered set is a C-space when equipped with the Scott topology. Slightly specialising Proposition 2.2.C of Erné (1991), we get the following lemma.

Lemma 3.3 (Erné). Let $(X, \mathcal{O})$ be a topological space. $(X, \mathcal{O})$ is a C-space if and only if interior distributes over unions of upper subsets in the specialisation order, that is, if and only if, for any family $(U_i)_{i \in I}$ of upper subsets of X,

$$\overbrace{\bigcup_{i \in I} U_i}^{\circ} = \bigcup_{i \in I} \overset{\circ}{U_i}.$$

Proposition 2.2.C of Erné (1991) actually states a more general result on so-called closure spaces. The same proposition also shows that C-spaces are exactly the locally supercompact spaces, or, equivalently, the spaces whose set of closed subsets is a completely distributive lattice.

Lemma 3.4. Let $(X, \mathcal{O})$ be any C-space. Every bounded continuous valuation on $(X, \mathcal{O})$ is quasi-simple.

Proof. By Theorem 1.1, it suffices to show that every bounded valuation ν extends to the Alexandroff topology over $\leqslant$. Define the extension $\overline{\nu}$ as ν_*; in other words, $\overline{\nu}(U) = \nu(\mathring{U})$ for every upper subset U. Clearly, $\overline{\nu}$ is strict and monotonic. It is also modular:

$$\begin{aligned}
\overline{\nu}(U \cup V) + \overline{\nu}(U \cap V) &= \nu\left(\overbrace{U \cup V}^{\circ}\right) + \nu\left(\overbrace{U \cap V}^{\circ}\right) \\
&= \nu(\mathring{U} \cup \mathring{V}) + \nu(\mathring{U} \cap \mathring{V}) \quad \text{(since interior distributes over } \cup, \cap) \\
&= \nu(\mathring{U}) + \nu(\mathring{V}) \qquad \text{(since } \nu \text{ is modular)} \\
&= \overline{\nu}(U) + \overline{\nu}(V).
\end{aligned}$$

Finally, $\overline{\nu}$ is continuous. Indeed, if $(U_i)_{i \in I}$ is a directed family of upper subsets,

$$\begin{aligned}
\overline{\nu}\left(\bigcup_{i \in I} U_i\right) &= \nu\left(\overbrace{\bigcup_{i \in I} U_i}^{\circ}\right) \\
&= \nu\left(\bigcup_{i \in I} \mathring{U}_i\right) \qquad \text{(interior distributes over directed unions)} \\
&= \sup_{i \in I} \nu(\mathring{U}_i) \qquad (\nu \text{ is continuous}) \\
&= \sup_{i \in I} \overline{\nu}(U_i).
\end{aligned}$$

□

In fact, this proof says something more.

Fact 1. On any C-space, every bounded continuous valuation ν has a *least* A-extension.

Namely $\overline{\nu} = \nu_*$: see Lemma 3.1. In particular, by Lemma 3.3, this is the case for continuous dcpos. Also, the following corollary is an immediate consequence.

Corollary 3.5 (Jones). Every bounded continuous valuation on a continuous dcpo (with the Scott topology) is quasi-simple.

Observe that, without any extra effort, Lemma 3.4 implies that every bounded continuous valuation on a continuous partial ordered set, which is not necessarily a dcpo, is quasi-simple.

Remark 3.3 (Non-uniqueness of extensions). A natural question is whether every bounded continuous valuation extends to a *unique* continuous valuation on the Alexandroff topology, when it extends at all. This is not true, even when X is a continuous dcpo. Indeed, take $X = \mathbb{N} \cup \{\omega\}$, with the natural ordering on $\mathbb{N}$ and $n < \omega$ for all $n \in \mathbb{N}$. This is a continuous dcpo, and the non-empty Scott opens are all sets $[k, \omega] = \{k, k+1, k+2, \ldots, \omega\}$, $k \in \mathbb{N}$. Note that $\{\omega\}$ is upper but not Scott open. Now define $\nu([k, \omega]) = (ak+1)/(k+1)$, where a is some real number between 0 and 1. This is a continuous probability valuation

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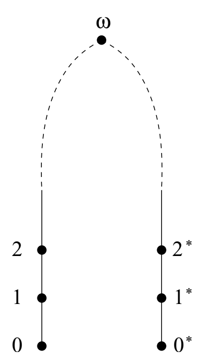
on X, which can be extended by letting $\nu\{\omega\}$ be any real number b between 0 and a. If $a > 0$, we therefore get uncountably many distinct A-extensions of ν. Correspondingly, there are uncountably many distinct ways of writing ν as sups of directed families of simple valuations, for example, as $\sup_{n\in\mathbb{N}} \nu_n$, where ν_n is the increasing (hence directed) sequence of valuations

$$\nu_n = \sum_{k=0}^{n-1} \frac{1-a}{(k+1)(k+2)}\delta_k + \left(\frac{an+1}{n+1} - b\right)\delta_n + b\delta_\omega$$

for all $n \in \mathbb{N}$. Note that ν_n is just $\nu[D]$ where D is the dissection obtained from the finitary open $\uparrow\{\omega, n, n-1, \ldots, 1, 0\}$ as we did in the proof of Lemma 2.3.

Remark 3.4 (On extensions to measures). Recall that the Borel σ-field $\sigma(\mathcal{O})$ generated by a given topology $\mathcal{O}$ on X is the least collection of sets containing $\mathcal{O}$ and such that every complement and every countable union of sets in $\sigma(\mathcal{O})$ is again in $\sigma(\mathcal{O})$. The sets in $\sigma(\mathcal{O})$ are called the *Borel subsets* of X when $\mathcal{O}$ is understood. Theorem 2.16 of Alvarez-Manilla (2000) states that on monotone convergence spaces every bounded quasi-simple valuation extends to a unique measure on the Borel subsets. (A *monotone convergence space* is a T_0 topological space $(X, \mathcal{O})$ whose specialisation ordering $\leqslant_s$ induces a dcpo structure whose Scott topology is finer than $\mathcal{O}$. Every dcpo is a monotone convergence space.) In the example of Remark 3.3, every subset of $\mathbb{N}\cup\{\omega\}$ is a Borel subset, and the unique measure μ that extends ν maps each subset A of $\mathbb{N}\cup\{\omega\}$ to $\sum_{k\in A}\frac{1-a}{(k+1)(k+2)}$, to which we must add a if $\omega \in A$. So, among the A-extensions of ν, only the one that is obtained by letting b be exactly a extends to a measure. In particular the assumption that X be a monotone convergence space in Theorem 2.16 of *op.cit.* is essential: as the example above illustrates, continuous valuations do not, in general, extend to measures, when X is equipped with an Alexandroff topology; a space with the Alexandroff topology is a monotone convergence space only when its topology coincides with the Scott topology.

Remark 3.5 (Extensions on non-continuous dcpos). Another natural question is whether a form of converse to Corollary 3.5 holds: assume $(X, \leqslant)$ is a dcpo on which every bounded continuous valuation is quasi-simple; is X a *continuous* dcpo? Again, the answer is no. Consider the dcpo X given by the Hasse diagram:



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We invite the reader to check that every bounded continuous valuation on X is quasi-simple. The simplest way to show this is to note that the dcpo on the right has a finitarily coherent Scott topology, and to apply the results of Section 3.4.

This can also be shown by direct calculation. While this is instructive, it is also tedious; the Scott open subsets are $\emptyset$ and the sets $A_{m,n}$ of all points at or above m or n^* $(m, n \in \mathbb{N})$, and it can be shown that any continuous valuation v satisfies $v(A_{m,n}) = v(X) - \alpha_m - \alpha_n^*$ for some fixed non-decreasing sequences $(\alpha_m)_{m\in\mathbb{N}}$ and $(\alpha_n^*)_{n\in\mathbb{N}}$ of real numbers; A-extensions $\overline{v}$ of v can be found by letting $\overline{v}\{\omega\} = 0$, and mapping the set of all points at or above m to $a - \alpha_m$, and the set of all points at or above n^* to $a^* - \alpha_n^*$, for some arbitrary constants a and a' such that $a + a^* = 1$, we have $a \geqslant \sup_{m\in\mathbb{N}} \alpha_m$, and $a^* \geqslant \sup_{n\in\mathbb{N}} \alpha_n^*$. In particular, v has *uncountably* many A-extensions.

We might then ask whether there are any cases at all of a continuous valuation on a topological space that is not quasi-simple. Indeed there are; it is well-known (and a simple exercise, to prove it) that the continuous valuation obtained from the Lebesgue measure (see Remark 3.2) cannot be the sup of directed families of simple valuations.

Remark 3.6 (Non-existence of extensions on dcpos). We may refine the observation that there are continuous, non-quasi-simple valuations, even when X is a dcpo. The simplest known example uses Johnstone's famous example of a non-sober dcpo (Alvarez-Manilla 2000, Section 2.7, page 73). At this point, we should warn the reader that non-sobriety is a red herring here; we shall return to this example in Remark 3.10.

The space X is the example we used in Remark 3.1: let X be $\mathbb{N} \times (\mathbb{N} \cup \{\omega\})$ with the ordering $(j, k) \leqslant (m, n)$ if and only if either $j = m$ and $k \leqslant n$, or $n = \omega$ and $k \leqslant m$. Recall that this is a dcpo, and that the function v mapping $\emptyset$ to 0 and each non-empty Scott open to 1 is modular, and hence a continuous valuation. However, it cannot be extended to all upper sets; Alvarez-Manilla relies on two theorems to show this, one saying that any quasi-simple valuation extends uniquely to a measure on the Borel subsets of the topology (Theorem 2.16 in *op.cit.*), and the other stating that for every measure μ, for every decreasing sequence of measurable sets A_n, $n \in \mathbb{N}$, whose intersection is empty, we have $\lim_{n\to+\infty} \mu(A_n) = 0$ (Proposition 1.14 in *op.cit.*). We rely on Theorem 1.1 instead, avoiding the need to introduce measures.

Assume v has an A-extension $\overline{v}$, and fix $\epsilon \geqslant 0$ such that $\epsilon < 1/4$. For each subset S of $\mathbb{N}$, let Ω_S be the upper, but not, in general, Scott open, set $\{(i, \omega) | i \in S\}$. Let O_n be the Scott open $\{(i, j) | i, j \geqslant n\} \cup \Omega_{[n,+\infty)}$. For every $m \geqslant n$, let U_{nm} be the upper set $\{(i, j) | i, j \geqslant n, i \leqslant m\} \cup \Omega_{[n,+\infty)}$; this is not Scott open. (See Figure 1 to see what these subsets look like in the plane.) However, O_n is the directed union of all U_{nm}, $m \geqslant n$. Since $\overline{v}$ is continuous and $\overline{v}(O_n) = v(O_n) = 1$, there is an $m \geqslant n$ such that $\overline{v}(U_{nm}) \geqslant 1 - \epsilon$. By modularity,

$$\overline{v}(U_{nm}) + \overline{v}(O_{m+1}) = \overline{v}(U_{nm} \cup O_{m+1}) + \overline{v}(U_{nm} \cap O_{m+1}).$$

The left-hand side is greater than or equal to $2 - \epsilon$, while $\overline{v}(U_{nm} \cup O_{m+1}) \leqslant \overline{v}(O_n) = 1$ by monotonicity. Since $U_{nm} \cap O_{m+1} = \Omega_{[m+1,+\infty)}$, it follows that $\overline{v}(\Omega_{[m+1,+\infty)}) \geqslant 1 - \epsilon$. Since $\Omega_{[m+1,+\infty)}$ is the directed union of all $\Omega_{[m+1,p]}$, $p \geqslant m + 1$, by continuity again, there is

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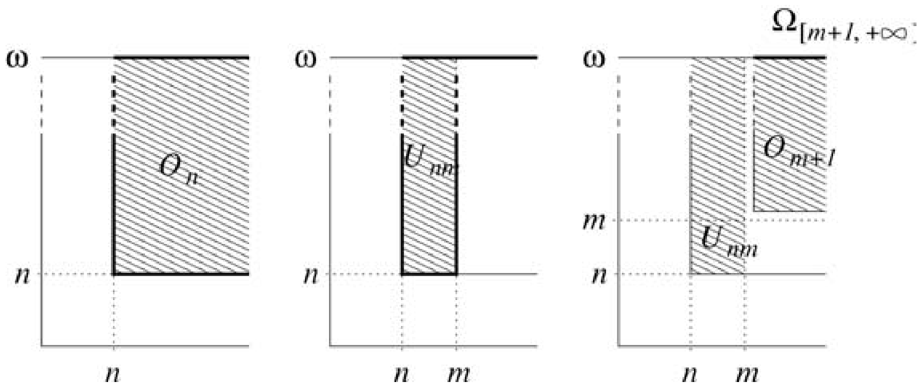

Fig. 1. The various sets used in Remark 3.6.

$p \geqslant m+1$ such that $\overline{v}(\Omega_{[m+1,p]}) \geqslant 1-2\epsilon$. Remember that n was arbitrary, so we have proved:

$$\text{For every } n \in \mathbb{N}, \text{ there are } p > m \geqslant n \text{ such that } \overline{v}(\Omega_{[m+1,p]}) \geqslant 1-2\epsilon. \qquad (*)$$

Now use (*) twice. First take $n = 0$, giving $p > m$ such that $\overline{v}(\Omega_{[m+1,p]}) \geqslant 1-2\epsilon$. Then take $n = p+1$, yielding $p' > m' \geqslant p+1$ such that $\overline{v}(\Omega_{[m'+1,p']}) \geqslant 1-2\epsilon$. By modularity, it then follows that $\overline{v}(\Omega_{[m,p]\cup[m'+1,p']}) \geqslant 2-4\epsilon > 1$ since $\epsilon < 1/4$. But this is impossible, since $\overline{v}(\Omega_{[m,p]\cup[m'+1,p']}) \leqslant \overline{v}(X) = v(X) = 1$.

Remark 3.7 (Sups of undirected families). Defining quasi-simple valuations as sups *of directed families* of simple valuations is important. First and foremost, the sup of an undirected family of (simple) valuations need not be a valuation, because it need not be modular: take v as the sup of δ_a and δ_b on the two-element set $\{a, b\}$ with $=$ as ordering, then $v(\{a\} \cup \{b\}) + v(\{a\} \cap \{b\}) = 1$ but $v(\{a\}) + v(\{b\}) = 2$. More subtly, there are continuous valuations that are sups of simple valuations but are not quasi-simple: for example, the valuation v of Remark 3.1 and Remark 3.6 is the sup of all δ_x, $x \in X$, but is not quasi-simple. Clearly, the family $(\delta_x)_{x \in X}$ is not directed.

3.2. *Algebraic dcpos*

An element x of a dcpo $(X, \leqslant)$ is *finite* if and only if $x \ll x$. Let X_0 be the set of finite elements in X. X itself is *algebraic* if and only if, for every $y \in X$, the set $D = \{x \in X_0 | x \leqslant y\}$ is directed and $y = \sup D$. Every algebraic dcpo is continuous. Furthermore, if X is algebraic, $\uparrow x$ is Scott open for every $x \in X_0$, and, in fact, every Scott open O is $\bigcup_{x \in O \cap X_0} \uparrow x$. Proposition 2.6 immediately entails the following corollary.

Corollary 3.6. Let $(X, \leqslant)$ be an algebraic dcpo, and X_0 be its subset of finite elements. Then every bounded continuous valuation on X, equipped with its Scott topology, is the sup of a directed family of *simple valuations based on finite elements*, that is, of simple valuations $\sum_{i=1}^{n} a_i \delta_{x_i}$ such that $x_1, \ldots, x_n \in X_0$.

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Remark 3.8. This was first claimed by Sünderhauf (Sünderhauf 1997), whose Theorem 3.4 states the corollary above, in particular, although the fact that we can choose the family of simple valuations to be directed is stated without justification. Theorem 3.4 of Sünderhauf (1997) also states that each simple valuation based on finite elements is finite, but this is wrong: as already observed in Jones (1990, page 84), $\delta_x \ll \delta_y$ holds for no pair of elements x and y such that $x \ll y$; in particular, δ_x is never finite, even when x is finite. In fact, it is easy to see that the only finite valuation ν (in the sense that $\nu \ll \nu$) is the everywhere zero valuation: if $\nu \neq 0$, the directed family $(r\nu)_{0 \leqslant r < 1}$ has ν as supremum, but $r\nu \geqslant \nu$ is false for all $r < 1$.

Corollaries 3.5 and 3.6 apply to bounded continuous valuations: in particular, they apply to *sub-probability* valuations ν, such that $\nu(X) \leqslant 1$. However, the construction of Section 2.1 requires valuations $r\nu[D]$ such that $r\nu[D](X)$ may be strictly less than 1. These results extend, as in the following corollary, to *probability* valuations ν (such that $\nu(X) = 1$) in the case where X has a least element $\perp$.

Corollary 3.7. Let $(X, \leqslant)$ be a continuous dcpo with bottom $\perp$.

Every bounded continuous probability valuation ν on X, with its Scott topology, is the sup of a directed family of simple probability valuations $\sum_{i=1}^{n} a_i \delta_{x_i}$ on X.

If X is algebraic, we may additionally require that x_i is finite, $1 \leqslant i \leqslant n$.

Proof. We use Edalat's trick (Edalat 1995): apply Corollaries 3.5 and 3.6 to the restriction ν' of ν to the dcpo $X' = X \setminus \{\perp\}$. Then ν' is the sup of a directed family of simple sub-probability valuations $(\nu_i')_{i \in I}$ on X', where $\nu_i' = \sum_j a_{ij} \delta_{x_{ij}}$, and $\sum_j a_{ij} \leqslant 1$. Let ν_i be $\sum_j a_{ij} \delta_{x_{ij}} + (\sum_j a_{ij}) \delta_\perp$, that is, add the missing mass at $\perp$. Then $(\nu_i)_{i \in I}$ is directed and $\sup_{i \in I} \nu_i = \nu$. □

3.3. *Bicontinuous valuations*

Before we turn to other kinds of spaces in Section 3.4, we will take a quick look at bicontinuous valuations, as introduced in Bukatin and Shorina (1998). Let $(X, \mathcal{O})$ be a topological space, $\leqslant_s$ be its specialisation preordering, and $\downarrow x$ be $\{y \in X | y \leqslant_s x\}$. Note that $\downarrow x$ is always closed. In fact, $\downarrow x$ is just the closure of x. Also, it is well known (Jones 1990, page 81) that every upper subset U is the intersection of all opens that contain it.

A valuation ν is said to be *cocontinuous* (Bukatin and Shorina 1998; Alvarez-Manilla 2000, Section 1.10, page 41) if and only if, for every filtered family F of opens,

$$\nu\left(\overbrace{\bigcap_{O \in F} O}^{\circ}\right) = \inf_{O \in F} \nu(O).$$

The family F is *filtered* if and only if F is non-empty, and for every $O_1, O_2 \in F$, there is $O \in F$ such that $O \subseteq O_1, O_2$. The valuation ν is *bicontinuous* (Keimel 1997) if and only if it is both continuous and cocontinuous.

Recall that we have defined two approximations of continuous valuations (Lemma 3.1).

Lemma 3.8. For every cocontinuous valuation ν, $\nu_* = \nu^*$.

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Proof. Assume that v is cocontinuous. Let U be any Alexandroff open. Since U is upper, U is the intersection of all opens O that contain U. The family of such opens O is clearly filtered. Since v is cocontinuous, $v(\overset{\circ}{U}) = \inf_{O \in \mathcal{O}, O \supseteq U} v(O)$. That is, $v_* = v^*$. □

It follows that every bicontinuous valuation has at most one A-extension.

The sup of any directed family of continuous valuations is again a continuous valuation (Jones 1990, Theorem 4.1, page 65). It is not clear whether the sup of a directed family of quasi-simple valuations should be again a quasi-simple valuation (although it is a continuous valuation). However, due to Lemma 3.8, we obtain the following lemma.

Lemma 3.9. On any topological space $(X, \mathcal{O})$, the sup of any directed family of quasi-simple bicontinuous valuations is quasi-simple.

Proof. Let $(v_i)_{i \in I}$ be a directed family of quasi-simple bicontinuous valuations, with sup v. By Theorem 1.1, Lemma 3.1 and Lemma 3.8, v_i has a unique A-extension $\overline{v}_i$. Moreover, since the map $v \mapsto v_*$ (or $v \mapsto v^*$) is monotonic, so is $v_i \mapsto \overline{v}_i$. It follows that the family $(\overline{v}_i)_{i \in I}$ is a directed family of continuous valuations on the Alexandroff extension of $\mathcal{O}$. So it has a sup $\overline{v}$, which is indeed an extension of v to the Alexandroff extension of $\mathcal{O}$. By Theorem 1.1, v is quasi-simple. □

3.4. *Finitarily coherent spaces*

Let $(X, \mathcal{O})$ be a topological space. Recall that $K \subseteq X$ is *compact* if and only if for every family of opens $O_i \in \mathcal{O}$, $i \in I$, such that $K \subseteq \bigcup_{i \in I} O_i$, there is a finite sub-family $I_{fin} \subseteq I$ such that $K \subseteq \bigcup_{i \in I_{fin}} O_i$. A subset A of X is *saturated* if and only if $A = \bigcap_{O \in \mathcal{O} \supseteq A} O$; recall that the saturated sets are exactly those that are upper in the specialisation preordering. We use $\mathcal{Q}$ to denote the set of compact saturated subsets of X.

A topological space is *locally compact* if and only if every neighbourhood of any point contains a compact neighbourhood. In other words, for every open O, for every $x \in O$, there is a compact K such that $x \in \overset{\circ}{K} \subseteq K \subseteq O$.

A topological space $(X, \mathcal{O})$ is *sober* if and only if every irreducible closed subset is the closure of a unique point x. A closed set F is *irreducible* if and only if it is non-empty and, for any two closed subsets F_1 and F_2 such that $F \subseteq F_1 \cup F_2$, either $F \subseteq F_1$ or $F \subseteq F_2$. Given the specialisation preordering $\leqslant_s$, the closure of a point x is $\downarrow x$. That x should be unique means that $\leqslant_s$ should be an ordering, that is, X should be T_0. In a sober space, in addition, every irreducible closed subset should be of the form $\downarrow x$ for some x. In the example of Remark 3.1 and Remark 3.6, X is not sober: X itself is irreducible, but is not of the form $\downarrow x$.

The most important property of sober spaces is the Hofmann–Mislove theorem (Abramsky and Jung 1994, Proposition 7.2.9): the sets $\mathcal{N}_Q = \{O \in \mathcal{O} | O \supseteq Q\}$, where Q is compact saturated, are exactly the Scott open filters of opens in $\mathcal{O}$. It follows (Abramsky and Jung 1994, Propositions 7.2.12, 7.2.13) that if $(X, \mathcal{O})$ is sober, then $(X, \leqslant_s)$ is a dcpo, and $\mathcal{O}$ is *order-consistent*, meaning that it is finer than the *upper topology*, which is generated by all complements of sets of the form $\downarrow x$, $x \in X$, and it is coarser than the Scott topology on $\leqslant_s$. Also, every filtered intersection of compact saturated sets is again compact saturated, and if $(Q_i)_{i\in I}$ is a filtered family of compact saturated sets, and $\bigcap_{i\in I} Q_i \subseteq O$ for some open $O \in \mathcal{O}$, then $Q_i \in O$ for some i. (A family is *filtered* for $\subseteq$ if and only if it is directed for $\supseteq$.)

While the union of compact saturated subsets is again compact saturated, their intersection need not be. A topological space that is sober, locally compact, and such that the intersection of any two compact saturated sets is again compact saturated is called a *coherent space* (Tix 1995, Definition 3.1, page 17; Abramsky and Jung 1994, Definition 7.2.17). Some authors use different definitions. For example, Alvarez-Manilla (Alvarez-Manilla 2000, Definition 1.11, page 31) drops local compactness from the definition of coherent spaces. By Alvarez-Manilla (2000, Proposition 1.12, page 32), coherent spaces are also exactly the so-called *stably locally compact* spaces, as well as the locally compact supersober spaces.

A result by Tix, also discovered by Norberg and Vervaat (Tix 1995, Satz 3.4, page 19; Alvarez-Manilla 2000, Proposition 3.33, page 88) is that, on a coherent space $(X, \mathcal{O})$, given any continuous valuation v on $\mathcal{O}$, the restriction of v^* to compact saturated subsets is strict, monotonic, modular, *inf-continuous* (that is, for every filtered family of compact saturated subsets $(Q_i)_{i\in I}$, $v^*(\bigcap_{i\in I} Q_i) = \inf_{i\in I} v^*(Q_i)$), and for every $O \in \mathcal{O}$, $v(O) = \sup_{Q \text{ compact saturated } \subseteq O} v^*(Q)$.

Earlier, we called subsets of the form $\uparrow\{x_1, \ldots, x_n\}$ the *finitary opens* of the Alexandroff topology of a preorder $(X, \leqslant)$. When $\leqslant$ is the specialisation preordering $\leqslant_s$ of a topological space, such subsets are compact saturated. Indeed, by the remark above, they are saturated. Furthermore, if $\uparrow\{x_1, \ldots, x_n\} \subseteq \bigcup_{i\in I} O_i$, where each O_i is open, choose O_{i_1} containing x_1, $\ldots$, O_{i_n} containing x_n, then $\uparrow\{x_1, \ldots, x_n\} \subseteq \bigcup_{k=1}^{n} O_{i_k}$, so $\uparrow\{x_1, \ldots, x_n\}$ is compact. Note that this is true whatever the topology $\mathcal{O}$, as soon as it has $\leqslant$ as specialisation preordering. We call such subsets $\uparrow\{x_1, \ldots, x_n\}$ *finitary compacts*.

While there may be more compact saturated subsets than just the finitary ones in general, the compact saturated subsets of an Alexandroff topology are exactly the finitary compacts (and, therefore, every compact saturated subset is also open). Indeed, given any compact saturated set Q, we have $Q = \bigcup_{x\in Q} \uparrow x$ since Q is saturated, hence upper; since Q is compact, Q is in fact equal to a finite union of sets of the form $\uparrow x$, $x \in Q$, so Q is a finitary compact.

Following Heckmann (1996), we say a topological space is *locally finitary* if and only if, for every open O, for every $x \in O$, there is a *finitary* compact $\mathbb{K}$ such that $x \in \overset{\circ}{\mathbb{K}} \subseteq \mathbb{K} \subseteq O$. In other words, if each point has a fundamental system of finitary compact neighbourhoods. In particular, every locally finitary space is locally compact. This notion appears earlier in the literature, in the case of partially ordered sets, under the name of *quasicontinuous posets* (Gierz *et al.* 1983).

We shall now concentrate on topological spaces that look like coherent spaces. The main role will, however, be played not by compact saturated subsets but by finitary compacts: instead of local compactness, we require locally finitary spaces; instead of requiring that the intersection of two compacts saturated subsets be compact, we require that the intersection of two finitary compacts be finitary compact; finally, we replace sobriety by the weaker property given by the following definition.

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Definition 3.10. A topological space $(X, \mathcal{O})$ is called *finitarily sober* if and only if, for every open O and every filtered family $(\mathbb{K}_i)_{i \in I}$ of finitary compacts such that $\bigcap_{i \in I} \mathbb{K}_i \subseteq O$, we have $\mathbb{K}_i \subseteq O$ for some $i \in I$.

$(X, \mathcal{O})$ is *finitarily coherent* if and only if it is finitarily sober, locally finitary, and the intersection of any two finitary compacts is finitary compact.

Remark 3.9. Every sober space is clearly finitarily sober. The converse does not hold. For instance, a finitarily sober space need not even be T_0. On the other hand, a finitarily sober topology is coarser than the Scott topology with respect to its specialisation preorder $\leqslant_s$: if $(x_i)_{i \in I}$ is a directed family in $(X, \leqslant_s)$, then $(\uparrow x_i)_{i \in I}$ is a filtered family of finitary compacts, so for any open O, if $(x_i)_{i \in I}$ has a sup x in O, then $\bigcap_{i \in I} \uparrow x_i = \uparrow x \subseteq O$, so $\uparrow x_i \in O$, therefore $x_i \in O$ for some $i \in I$. (This discussion requires one to generalise slightly the definition of Scott topology to non-dcpos, see, for example, Alvarez-Manilla (2000, Section 1.6, page 30).)

Conversely, if $(X, \leqslant)$ is a dcpo, the Scott topology on X is always finitarily sober: this is Lemma 3.9.4 of Heckmann's Ph. D. thesis (Heckmann 1990). In particular, if $(X, \mathcal{O})$ is a monotone convergence space (see Remark 3.4 for the definition), it is finitarily sober.

That the intersection of any two finitary compacts is finitary compact is a purely-order theoretic property: it is equivalent to requiring that for any two points x, y of X, there is a finite set $\{x_1, \ldots, x_n\}$ such that $x \leqslant z$ and $y \leqslant z$ is equivalent to $x_i \leqslant z$ for some i. In particular, this is the case when x and y have no common upper bound, or when x and y have a least upper bound.

We do not claim that finitarily coherent spaces are *the* right kind of topological space here. This notion is the one that enabled us to produce Theorem 3.12 below, but it might be that this theorem still holds for a larger class of topological spaces. While we need the intersection of two finitary compacts to be finitary compact again, such a condition is not a standard one in the literature. It was suggested by one referee that Lawson-compact quasicontinuous posets might be the right framework in which to try and establish Theorem 3.12. We leave this as an open problem.

For readability, we write $\mathbb{K}$, possibly subscripted or primed, for finitary compacts, and O for opens. Tix's Satz 3.4 can be adapted to the finitarily coherent case, as follows.

Proposition 3.11. Let $(X, \mathcal{O})$ be a finitarily coherent space. For every continuous valuation ν on $\mathcal{O}$:

1 For every open O, $\nu(O) = \sup_{\mathbb{K} \subseteq O} \nu^*(\mathbb{K})$.
2 ν^* is strict on the finitary compacts, that is, $\nu^*(\emptyset) = 0$.
3 ν^* is monotonic on the finitary compacts, that is, $\nu^*(\mathbb{K}_1) \leqslant \nu^*(\mathbb{K}_2)$ if $\mathbb{K}_1 \subseteq \mathbb{K}_2$.
4 ν^* is modular on the finitary compacts, that is, $\nu^*(\mathbb{K}_1 \cup \mathbb{K}_2) + \nu^*(\mathbb{K}_1 \cap \mathbb{K}_2) = \nu^*(\mathbb{K}_1) + \nu^*(\mathbb{K}_2)$ for every $\mathbb{K}_1, \mathbb{K}_2$.
5 ν^* is inf-continuous, that is, for every filtered family $(\mathbb{K}_i)_{i \in I}$,

$$\nu^*\left(\bigcap_{i \in I} \mathbb{K}_i\right) = \inf_{i \in I} \nu^*(\mathbb{K}_i).$$

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Proof.

1 Clearly $v^*(\mathbb{K}) \leqslant v(O)$ for every $\mathbb{K} \subseteq O$, so $\sup_{\mathbb{K} \subseteq O} v^*(\mathbb{K}) \leqslant v(O)$. Conversely, for every open O, since X is locally finitary, O is the directed union of all $\overset{\circ}{\mathbb{K}}$, for $\mathbb{K} \subseteq O$. Indeed, for each $x \in O$, let $\mathbb{K}_x$ be such that $x \in \overset{\circ}{\mathbb{K}_x} \subseteq \mathbb{K}_x \subseteq O$, then $O \subseteq \bigcup_{x \in O} \overset{\circ}{\mathbb{K}_x} \subseteq \bigcup_{\mathbb{K} \subseteq O} \overset{\circ}{\mathbb{K}}$; the converse inclusion is obvious. So $v(O) = \sup_{\mathbb{K} \subseteq O} v(\overset{\circ}{\mathbb{K}}) \leqslant \sup_{\mathbb{K} \subseteq O} v^*(\mathbb{K})$.

2 This part is obvious, since $\emptyset$ is finitary compact.

3 This part is also straightforward.

4 We begin with a few observations:

(a) For every $\mathbb{K} \subseteq O$, we have $\mathbb{K} \subseteq \overset{\circ}{\mathbb{K}'} \subseteq \mathbb{K}' \subseteq O$ for some finitary compact $\mathbb{K}'$. Indeed, write $\mathbb{K} = \uparrow\{x_1, \dots, x_n\}$. Since X is locally finitary, for every i with $1 \leqslant i \leqslant n$, there is a finitary compact $\mathbb{K}_i$ such that $x_i \in \overset{\circ}{\mathbb{K}_i} \subseteq \mathbb{K}_i \subseteq O$. So we may take $\mathbb{K}' = \bigcup_{i=1}^n \mathbb{K}_i$, noting that $\bigcup_{i=1}^n \overset{\circ}{\mathbb{K}_i} \subseteq \overset{\circ}{\mathbb{K}'}$.

(b) Every finitary compact $\mathbb{K}$ is the filtered intersection of all $\mathbb{K}'$ such that $\mathbb{K} \subseteq \overset{\circ}{\mathbb{K}'}$. Since $\mathbb{K}$ is saturated, $\mathbb{K}$ is the intersection of all opens $O \supseteq \mathbb{K}$: on the one hand, every $\mathbb{K}'$ such that $\overset{\circ}{\mathbb{K}'} \supseteq \mathbb{K}$ contains some open $O \supseteq \mathbb{K}$, trivially; on the other hand, every open $O \supseteq \mathbb{K}$ contains some finitary compact $\mathbb{K}'$ such that $\overset{\circ}{\mathbb{K}'} \supseteq \mathbb{K}$ by (a); so $\mathbb{K}$ is the claimed intersection. This intersection is filtered because the intersection of two finitary compacts is finitary compact, and interior distributes over intersection.

(c) If $\mathbb{K}_1 \cap \mathbb{K}_2 \subseteq O$, there are two opens O_1 and O_2 such that $\mathbb{K}_1 \subseteq O_1$, $\mathbb{K}_2 \subseteq O_2$, and $O_1 \cap O_2 \subseteq O$. Indeed, by (b), $\mathbb{K}_i$ is the filtered intersection of its finitary compact neighbourhoods, $i = 1, 2$. So $\mathbb{K}_1 \cap \mathbb{K}_2$ is the intersection of all $\mathbb{K}'_1 \cap \mathbb{K}'_2$, where $\mathbb{K}'_i$ ranges over the finitary compact neighbourhoods of $\mathbb{K}_i$, $i = 1, 2$. Moreover, this intersection is filtered. Since the intersection of any pair of finitary compacts is finitary compact, each such $\mathbb{K}'_1 \cap \mathbb{K}'_2$ is finitary compact. Since X is finitarily sober, some such $\mathbb{K}'_1 \cap \mathbb{K}'_2$ is included in O, so we let O_i be the interior of $\mathbb{K}'_i$, $i = 1, 2$.

(d) $v^*(\mathbb{K}_1 \cap \mathbb{K}_2) = \inf_{O_1 \supseteq \mathbb{K}_1, O_2 \supseteq \mathbb{K}_2} v(O_1 \cap O_2)$. Indeed, by (c), the two filtered families $\{O | O \supseteq \mathbb{K}_1 \cap \mathbb{K}_2\}$ and $\{O_1 \cap O_2 | O_1 \supseteq \mathbb{K}_1, O_2 \supseteq \mathbb{K}_2\}$ are coinitial, that is, for every element of one there is a smaller element in the other.

(e) $v^*(\mathbb{K}_1 \cup \mathbb{K}_2) = \inf_{O_1 \supseteq \mathbb{K}_1, O_2 \supseteq \mathbb{K}_2} v(O_1 \cup O_2)$. Indeed the two filtered families $\{O | O \supseteq \mathbb{K}_1 \cup \mathbb{K}_2\}$ and $\{O_1 \cup O_2 | O_1 \supseteq \mathbb{K}_1, O_2 \supseteq \mathbb{K}_2\}$ are clearly coinitial.

So:

$$\begin{aligned} v^*(\mathbb{K}_1 \cup \mathbb{K}_2) + v^*(\mathbb{K}_1 \cap \mathbb{K}_2) &= \inf_{O_1 \supseteq \mathbb{K}_1, O_2 \supseteq \mathbb{K}_2} v(O_1 \cup O_2) + \inf_{O_1 \supseteq \mathbb{K}_1, O_2 \supseteq \mathbb{K}_2} v(O_1 \cap O_2) \\ &\qquad \text{(by (d), (e))} \\ &= \inf_{O_1 \supseteq \mathbb{K}_1, O_2 \supseteq \mathbb{K}_2} (v(O_1 \cup O_2) + v(O_1 \cap O_2)) \\ &= \inf_{O_1 \supseteq \mathbb{K}_1, O_2 \supseteq \mathbb{K}_2} (v(O_1) + v(O_2)) \qquad \text{(by modularity)} \end{aligned}$$

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$$= \inf_{O_1 \supseteq \mathbb{K}_1} v(O_1) + \inf_{O_2 \supseteq \mathbb{K}_2} v(O_2)$$
$$= v^*(\mathbb{K}_1) + v^*(\mathbb{K}_2)$$

5 We observe that $v^*\left(\bigcap_{i \in I} \mathbb{K}_i\right) = \inf_{O \supseteq \bigcap_{i \in I} \mathbb{K}_i} v(O) = \inf_{\exists i \in I \cdot O \supseteq \mathbb{K}_i} v(O)$ (since X is finitarily coherent, $O \supseteq \bigcap_{i \in I} \mathbb{K}_i$ is equivalent to $\exists i \in I \cdot O \supseteq \mathbb{K}_i$) $= \inf_{i \in I} v^*(\mathbb{K}_i)$. □

There is a subtle point in the inf-continuity property 5, insofar as $\bigcap_{i \in I} \mathbb{K}_i$ need not be finitarily compact, although each $\mathbb{K}_i$ is. Therefore item 5 is not a property of inf-continuity of v^* *on the finitary compacts.*

As an example of a space where the filtered intersection of finitary compacts is not necessarily finitary compact, let $(X, \leqslant)$ be the set of finite and infinite words over a two-letter alphabet $\mathbb{B} = \{0, 1\}$, ordered by the prefix ordering $\leqslant$, seen as a dcpo. Let $\mathbb{K}_i$, $i \in \mathbb{N}$ be the set of finite words of length at least i: this is finitarily compact, since there are only finitely many words of length exactly i. However, the filtered intersection $\bigcap_{i \in \mathbb{N}} \mathbb{K}_i$ is the infinite set of all infinite words $\mathbb{B}^\omega$, which is not finitary compact. (It is compact, however.)

In spite of this, X is finitarily coherent. It is finitarily sober, since it is a dcpo (Remark 3.9). In fact, since X is algebraic, it is also continuous, and hence sober. Note that every Scott open O of X is such that if some infinite word w is in O, then some prefix of it is in O already. It follows that X is locally finitary: if $x \in O$, then there is a finite prefix y of x in O, so $x \in \overset{\circ}{\mathbb{K}} = \mathbb{K} \subseteq O$, where $\mathbb{K} = \uparrow y$. Finally, the intersection of any two finitary compacts is finitary compact, since any two words have no common upper bound, or they have a sup (when they are comparable).

Theorem 3.12. Let $(X, \mathcal{O})$ be a finitarily coherent space. Every bounded continuous valuation on $(X, \mathcal{O})$ is quasi-simple.

Proof. Define $\overline{v}$ on the Alexandroff extension of $\mathcal{O}$ by $\overline{v}(U) = \sup_{\mathbb{K} \subseteq U} v^*(\mathbb{K})$, for each upper set U. We claim that $\overline{v}$ is a continuous valuation extending v. It is clear that $\overline{v}$ is strict and monotonic. It is continuous, because for any directed family $(U_i)_{i \in I}$ of upper sets, for any finitary compact $\mathbb{K}$, we have $\mathbb{K} \subseteq \bigcup_{i \in I} U_i$ if and only if $\mathbb{K} \subseteq U_i$ for some $i \in I$. Indeed, the *if* direction is clear; for the *only if* direction, since $\mathbb{K}$ is Alexandroff compact, $\mathbb{K} \subseteq \bigcup_{i \in I} U_i$ implies that $\mathbb{K}$ is included in some finite union of U_is, and hence in some U_i, since $(U_i)_{i \in I}$ is directed. It follows that

$$\overline{v}\left(\bigcup_{i \in I} U_i\right) = \sup_{\mathbb{K} \subseteq \bigcup_{i \in I} U_i} v^*(\mathbb{K}) = \sup_{i \in I, \mathbb{K} \subseteq U_i} v^*(\mathbb{K}) = \sup_{i \in I} \sup_{\mathbb{K} \subseteq U_i} v^*(\mathbb{K}) = \sup_{i \in I} \overline{v}(U_i).$$

Next, Proposition 3.11, item 1, implies that $\overline{v}$ extends v.

Let us show finally that $\overline{v}$ is modular. Fix two arbitrary upper subsets U and V. For any two finitary compacts $\mathbb{K}_1 \subseteq U$ and $\mathbb{K}_2 \subseteq V$, there is a finitary compact $\mathbb{K} \subseteq U \cap V$ such that $\mathbb{K}_1 \cap \mathbb{K}_2 \subseteq \mathbb{K}$, namely $\mathbb{K} = \mathbb{K}_1 \cap \mathbb{K}_2$: this is finitary compact by assumption. Conversely, for every finitary compact $\mathbb{K} \subseteq U \cap V$, there are two finitary compacts $\mathbb{K}_1 \subseteq U$

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and $\mathbb{K}_2 \subseteq V$ such that $\mathbb{K} \subseteq \mathbb{K}_1 \cap \mathbb{K}_2$: take $\mathbb{K}_1 = \mathbb{K}_2 = \mathbb{K}$. Therefore

$$\overline{v}(U \cap V) = \sup_{\mathbb{K} \subseteq U \cap V} v^*(\mathbb{K}) = \sup_{\mathbb{K}_1 \subseteq U, \mathbb{K}_2 \subseteq V} v^*(\mathbb{K}_1 \cap \mathbb{K}_2).$$

A similar argument, using the fact that the union of two finitary compacts is always finitary compact, implies

$$\overline{v}(U \cup V) = \sup_{\mathbb{K} \subseteq U \cup V} v^*(\mathbb{K}) = \sup_{\mathbb{K}_1 \subseteq U, \mathbb{K}_2 \subseteq V} v^*(\mathbb{K}_1 \cup \mathbb{K}_2).$$

It follows that

$$\begin{aligned} \overline{v}(U \cup V) + \overline{v}(U \cap V) &= \sup_{\mathbb{K}_1 \subseteq U, \mathbb{K}_2 \subseteq V} v^*(\mathbb{K}_1 \cup \mathbb{K}_2) + v^*(\mathbb{K}_1 \cap \mathbb{K}_2) \\ &= \sup_{\mathbb{K}_1 \subseteq U, \mathbb{K}_2 \subseteq V} v^*(\mathbb{K}_1) + v^*(\mathbb{K}_2) \qquad \text{(by Proposition 3.11, item 4)} \\ &= \sup_{\mathbb{K}_1 \subseteq U} v^*(\mathbb{K}_1) + \sup_{\mathbb{K}_2 \subseteq V} v^*(\mathbb{K}_2) = \overline{v}(U) + \overline{v}(V) \end{aligned}$$

The conclusion follows by Theorem 1.1. □

Conversely to Fact 1, the construction in the proof of Theorem 3.12 entails the following fact.

Fact 2. On a finitarily coherent space, every bounded continuous valuation has a *largest* A-extension.

Proof. By Lemma 3.1, every A-extension $\overline{v}'$ of v is such that $\overline{v}' \leqslant v^*$. In particular, $\overline{v}'(\mathbb{K}) \leqslant v^*(\mathbb{K})$ for every finitary compact $\mathbb{K}$. For every upper set U, we then have $\overline{v}'(U) = \sup_{\mathbb{K} \subseteq U} \overline{v}'(\mathbb{K})$ (since $\overline{v}'$ is continuous) $\leqslant \sup_{\mathbb{K} \subseteq U} v^*(\mathbb{K}) = \overline{v}(U)$, where $\overline{v}$ was constructed in the proof of Theorem 3.12. □

Remark. Every bounded continuous valuation on the dcpo X of Remark 3.5 is quasi-simple, although X is not a continuous dcpo. This can be shown by hand (as we did in Remark 3.5), or using Theorem 3.12.

X is a dcpo, and hence is finitarily sober (Remark 3.9). Closer examination shows that X is even sober. The Scott closed subsets are X and the complement of $A_{i,j}$, $i, j \in \mathbb{N}$. X is irreducible, as the closure of ω. The other irreducible closed subsets are of the form $F = X \setminus A_{i,j}$. So $F \subseteq (X \setminus A_{i,0}) \cup (X \setminus A_{0,j})$, and irreducibility implies that $i = 0$ or $j = 0$ (but not both, since F is irreducible, hence non-empty). However, $X \setminus A_{i,0}$ is the closure of $i - 1$, and $X \setminus A_{0,j}$ is the closure of $(j-1)^*$. Therefore X is sober.

Observe that every upper subset is finitary compact:

$$\emptyset = \uparrow \emptyset \qquad \{\omega\} = \uparrow \{\omega\} \qquad [n, \omega] = \uparrow \{n\} \qquad [n^*, \omega] = \uparrow \{n^*\} \qquad A_{i,j} = \uparrow \{i, j^*\}.$$

In particular, X is locally finitary: for every open $x \in O$, we have $x \in \overset{\circ}{\mathbb{K}} \subseteq \mathbb{K} \subseteq O$ where we take the finitary compact $\mathbb{K}$ to be just O.

Since every upper subset is finitary compact, the compact saturated subsets are exactly the upper subsets, and are exactly the finitary compacts. In particular, the intersection of any two compact saturated subsets are compact saturated (hence X is a coherent space), and the intersection of any two finitary compacts is finitary compact (hence X is finitarily

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coherent too). So Theorem 3.12 applies: every bounded continuous valuation on X is quasi-simple.

Note that Corollary 3.5 does not apply here, since X is not continuous. So Corollary 3.5 does not subsume Theorem 3.12. We may show that the latter does not subsume the former, as follows.

Every continuous dcpo $(X, \leqslant)$ is sober (Abramsky and Jung 1994, Proposition 7.2.27). X is also locally compact, in fact locally finitary: for any open O and $x \in O$, x is the sup of all elements $y \ll x$ since X is continuous; since O is open, some $y \ll x$ must then be in X, which entails that $x \in \twoheaduparrow y \subseteq \uparrow y \subseteq O$. Note that $\uparrow y$ is finitary compact and $\twoheaduparrow y$ is its interior.

However, the interior of two finitary compacts may fail to be finitary compact. Let X be the space consisting of $\mathbb{N}$ union two fresh elements a and b with the smallest ordering such that $a \leqslant n$ and $b \leqslant n$ for every $n \in \mathbb{N}$. This is a continuous dcpo, even an algebraic one: all elements are finite. As such, the Scott topology (which coincides with the Alexandroff topology here) is sober, locally compact and locally finitary, but the intersection of the two finitary compacts $\uparrow a$ and $\uparrow b$ is $\mathbb{N}$, which is not finitary, and not even compact. So X is not finitarily coherent, and not coherent either.

Therefore Theorem 3.12 does not apply, but Corollary 3.5 does: every bounded continuous valuation on X is quasi-simple. So Theorem 3.12 does not subsume Corollary 3.5 either.

Remark 3.10. The space X of Remark 3.6 is often taken as the prime example of a non-sober dcpo. However, the reason the continuous valuation v of Remark 3.6 is not quasi-simple does not lie in non-sobriety. Recall that every non-empty Scott open O contains all but finitely many points of the form (m, ω). Moreover, if O contains (m, ω), it must also contain every (m, j) for large enough j. It follows that no finitary compact can contain any non-empty Scott open, that is, the interior of any finitary compact is empty. So X is not locally finitary. Also, in X the intersection of two finitary compacts need not be finitary compact: if $i \neq i'$, and $j, j' \in \mathbb{N}$, then $\uparrow (i, j) \cap \uparrow (i', j') = \{(n, \omega) | n \geqslant \max(j, j')\}$, which is compact but not finitary compact. These are the two assumptions of Theorem 3.12 that fail. Observe that the third is satisfied: X is finitarily sober, as a dcpo, although it is not sober.

4. Other extension results

Theorem 1.1 is one characterisation of a class of continuous valuations by existence of extensions. We will now present two other similar theorems, which are much easier to prove.

4.1. *Discrete valuations*

We use *discrete valuation* to mean any valuation of the form $\sum_{i=0}^{+\infty} a_i \delta_{x_i}$, where $(a_i)_{i \in \mathbb{N}}$ is a summable family of non-negative reals, that is, $\sum_{i=0}^{+\infty} a_i < +\infty$. Recall that such a family is also *commutatively summable*, that is, for any permutation σ of $\mathbb{N}$, $\sum_{i=0}^{+\infty} a_{\sigma(i)} = \sum_{i=0}^{+\infty} a_i < +\infty$.

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Similarly to Theorem 1.1, we may show the following proposition.

Proposition 4.1. Let $(X, \mathcal{O})$ be a topological space. The bounded function $v : \mathcal{O} \to \mathbb{R}_+$ is a discrete valuation if and only if v extends to a continuous valuation on the discrete topology.

Proof.

— *Only if*: Define $\overline{v}$ as mapping every subset Y of X to $\sum_{i \in \mathbb{N}, x_i \in Y} a_i$. Then $\overline{v}$ extends v to the discrete topology, is strict and monotonic. It is continuous and modular because $(a_i)_{i \in \mathbb{N}}$ is commutatively summable.

— *If*: Assume v extends to a continuous valuation $\overline{v}$ on the discrete topology. For every $x \in X$, let a_x be $\overline{v}(\{x\})$. Then, for every $Y \subseteq X$, we get $\overline{v}(Y) = \sum_{x \in Y} a_x$. We claim that $a_x \neq 0$ for only countably many $x \in X$. Let B be $v(X)$. Since v is bounded, $B \in \mathbb{R}_+$. For every $N \in \mathbb{N} \setminus \{0\}$, there can be at most B/N points x such that $v(\{x\}) \geqslant 1/N$. So $\{x | a_x \geqslant 1/N\}$ is finite. Therefore $\{x | a_x \neq 0\} = \bigcup_{N \geqslant 1} \{x | a_x \geqslant 1/N\}$ is countable. Let $i \mapsto x_i$ be a bijection from $\mathbb{N}$ to $\{x | a_x \neq 0\}$. Then v coincides with $\sum_{i=0}^{+\infty} a_{x_i} \delta_{x_i}$ on $\mathcal{O}$. □

Although every discrete valuation is quasi-simple, the converse does not hold: on $[0, 1]$ with the Scott topology on $\leqslant$, the Scott opens are $[0, 1]$, $\emptyset$, and $(a, 1]$ for each $a \in [0, 1)$. The *Lebesgue valuation* v maps $[0, 1]$ to 1, $\emptyset$ to 0, and $(a, 1]$ to $1 - a$. It extends to all upper sets by $v[a, 1] = 1 - a$. So, by Jones' Theorem (Corollary 3.5), this is a quasi-simple valuation; but it is not discrete, otherwise, by Proposition 4.1, v extends to a continuous valuation $\overline{v}$ on the discrete topology. Then $\overline{v}\{0\} = v[0, 1] - v(0, 1] = 0$. And for every $a \in (0, 1]$, we have $\overline{v}\{a\} \leqslant v(a - \epsilon, 1] - v(a, 1]$ for every sufficiently small $\epsilon > 0$, so $\overline{v}\{a\} \leqslant \epsilon$. Therefore $\overline{v}$ must be the everywhere zero valuation, hence also v, which is impossible. (The purpose of this discussion is not to show that the Lebesgue measure is not discrete, which is well-known, but to illustrate the use of Proposition 4.1.)

Although every simple valuation is discrete, the converse fails. For example, on the dcpo $[0, 1]$ again, $\sum_{i=0}^{+\infty} 1/(n + 1)^2 \delta_{1/n}$ is clearly discrete, but takes infinitely many values, so it cannot be simple.

4.2. *Point-continuous valuations*

A valuation v is *point-continuous* (Heckmann 1996) if and only if, for every open O, for every $r \in \mathbb{R}_+$ such that $v(O) > r$, there is a finitary compact $\mathbb{K} \subseteq O$ such that $v(O') > r$ for every open $O' \supseteq \mathbb{K}$.

Lemma 4.2. Let $(X, \mathcal{O})$ be a topological space. Given a function $v : \mathcal{O} \to \mathbb{R}_+$, the following propositions are equivalent:

1. v is a point-continuous valuation.
2. v is a valuation, and there is a function f mapping each finitary compact $\mathbb{K}$ to a non-negative real such that, for every open O, $v(O) = \sup_{\mathbb{K} \subseteq O} f(\mathbb{K})$.
3. v is a valuation, and for every open O, $v(O) = \sup_{\mathbb{K} \subseteq O} v^*(\mathbb{K})$.
4. v is modular on $\mathcal{O}$, and extends to a strict, monotonic and continuous function $\overline{v}$ on the Alexandroff extension of $\mathcal{O}$.

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Proof. The implication $3 \Rightarrow 2$ is clear.

Let us show $2 \Rightarrow 1$, and assume $\nu(O) = \sup_{\mathbb{K} \subseteq O} f(\mathbb{K})$. If $\nu(O) > r$, there is a finitary compact $\mathbb{K} \subseteq O$ such that $f(\mathbb{K}) > r$. But then, for every open $O' \supseteq \mathbb{K}$, $\nu(O') \geqslant f(\mathbb{K}) > r$. Let us show $1 \Rightarrow 3$. Assume that ν is point-continuous. Recall that $\nu^*(\mathbb{K}) = \inf_{O' \text{ open} \supseteq \mathbb{K}} \nu(O')$. In particular (take $O' = O$), $\sup_{\mathbb{K} \subseteq O} \nu^*(\mathbb{K}) \leqslant \nu(O)$. Conversely, $\sup_{\mathbb{K} \subseteq O} \nu^*(\mathbb{K}) \geqslant \nu(O)$ is equivalent to the property that for every $r < \nu(O)$, there is a finitary compact $\mathbb{K} \subseteq O$ such that $\nu^*(\mathbb{K}) \geqslant r$: this is a direct consequence of 1.

To show the equivalence between 4 and 1–3, we prove $4 \Rightarrow 2$ and $3 \Rightarrow 4$.

— $4 \Rightarrow 2$: Let f be $\overline{\nu}$, then:

$$\begin{aligned} \sup_{\mathbb{K} \subseteq O} f(\mathbb{K}) &= \sup_{\mathbb{K} \subseteq O} \overline{\nu}(\mathbb{K}) \\ &= \overline{\nu}(O) && \text{(since } \overline{\nu} \text{ is continuous)} \\ &= \nu(O) && \text{(since } \overline{\nu} \text{ extends } \nu\text{).} \end{aligned}$$

— $3 \Rightarrow 4$: let $\overline{\nu}$ be defined by $\overline{\nu}(U) = \sup_{\mathbb{K} \subseteq U} \nu^*(\mathbb{K})$. This clearly extends ν to the Alexandroff extension of $\mathcal{O}$, and is strict and monotonic. It is continuous because $\mathbb{K}$ is compact in the Alexandroff topology: see the proof of Theorem 3.12 for a similar argument. □

Item 4 of the previous lemma shows, in particular, that point-continuous valuations can be characterised by the fact that they are exactly the valuations that can be extended to some specific kind of function (strict, monotonic, continuous) on some topology (Alexandroff). In contrast with Theorem 1.1, we allow the extension to be non-modular here.

Every point-continuous valuation is a continuous valuation, but the converse fails: the example of Remark 3.2 yields a continuous valuation ν such that $\nu(O)$ is in general different from $\sup_{\mathbb{K} \subseteq O} \nu^*(\mathbb{K})$, showing that item 3 of the previous lemma fails.

Every quasi-simple valuation is point-continuous, but the converse fails: the example of Remark 3.6 is not quasi-simple but is point-continuous. Indeed, for every open O, if $\nu(O) > r$, then O is non-empty and $r < 1$. Take $x \in O$, then the finitary compact $\uparrow x$ is included in O, and for every open O' containing it, O' is non-empty, hence $\nu(O') = 1 > r$.

Item 4 of Lemma 4.2 shows that quasi-simple and point-continuous valuations are rather similar, although the two notions differ. There are more puzzling similarities. In particular, Heckmann shows that every bounded point-continuous valuation is the sup of a family of simple valuations (Heckmann 1996, Lemma 5.3). This may appear to be a claim that point-continuous valuations are quasi-simple. However, every bounded point-continuous valuation is the sup of a family, but not necessarily a *directed* family of simple valuations. This actually makes a difference: as shown in Remark 3.7, the valuation ν of Remark 3.1 is a sup of simple valuations but is not quasi-simple, although it is point-continuous (see above).

The set of point-continuous valuations is always a dcpo. Indeed, the point of Heckmann (1996) is precisely that this set is the soberification of the set of simple valuations, and its specialisation preordering is just the ordinary ordering of valuations. That this is a dcpo can also be proved directly using Lemma 4.2. We have to show that any sup ν of a directed

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family $(v_i)_{i\in I}$ of point-continuous valuations is again point-continuous. By Lemma 4.2, item 3, $v_i(O) = \sup_{\mathbb{K}\subseteq O} v_i^*(\mathbb{K})$, so $v(O) = \sup_{i\in I} \sup_{\mathbb{K}\subseteq O} v_i^*(\mathbb{K}) = \sup_{\mathbb{K}\subseteq O} \sup_{i\in I} v_i^*(\mathbb{K})$. So v is point-continuous by Lemma 4.2, item 2, taking $f(\mathbb{K}) = \sup_{i\in I} v_i^*(\mathbb{K})$.

On the other hand, it is not known whether the set of quasi-simple valuations is a dcpo, except in special cases (when all continuous valuations are quasi-simple for example, see Corollary 3.5, Theorem 3.12, or when the given quasi-simple valuations are bicontinuous, see Proposition 3.9).

5. Related work and conclusion

Claire Jones' Ph. D. thesis (Jones 1990) was a milestone, and contains a wealth of fundamental results on continuous valuations. She also cites previous work by Graham, Saheb-Djahromi, Plotkin and Frutos Escrig, who were the first to study continuous valuations as models of probabilistic non-determinism. Among other results, Jones shows that every bounded continuous valuation on a continuous dcpo is quasi-simple. This is one of the necessary steps for showing that the dcpo of all continuous subprobability valuations on a continuous dcpo is again a continuous dcpo, one of Jones' major achievements.

A classical theme in continuous valuation theory is to look for conditions under which continuous valuations extend to a (unique) measure on the Borel σ-algebra generated by the topology. This was initiated by Saheb-Djahromi, pursued by Jones, and corrected and extended by several authors (Alvarez-Manilla 2000; Alvarez-Manilla *et al.* 1997; Keimel and Lawson *submitted*). We will not list the many results obtained in this field, as this would side-track us from the theme of this paper, namely extensions of valuations to larger topologies. Perhaps closest to our concerns is Keimel and Lawson's result that any quasi-simple valuation extends to a finitely additive function on the algebra generated by the Alexandroff extension of the original topology (Keimel and Lawson *submitted*, Proposition 4.3). In general, there are certainly many intriguing connections between valuation and measure theory. However, as we noted in Remark 3.3, one difference is that when extensions to measures exist, they seem to be unique (at least when they are bounded): this is not the case with extensions of valuations to Alexandroff extensions of topologies.

Sünderhauf studied spaces of valuations from the quasi-metric point of view (Sünderhauf 1997). Our Corollary 3.6, stating that on algebraic dcpos, every bounded continuous valuation is the sup of a directed family of simple valuations based on finite elements, is part of his Theorem 3.4. We have noted in Remark 3.8 that there remained a gap in his proof, which we filled.

Section 3.4 owes a lot to Tix's Diplomarbeit (Tix 1995). Alvarez-Manilla (Alvarez-Manilla 2000) notes that Tix's results were independently discovered by Norberg and Vervaat.

Finally, we have discussed the subtle relationship between quasi-simple and point-continuous valuations in Section 4.2. The latter were introduced in Heckmann (1996), where it was proved that the space of point-continuous valuations is exactly the soberification of the set of simple valuations. Heckmann also shows (Theorem 4.1 in *op.cit.*) that every continuous valuation on a locally finitary space is point-continuous. We have shown (Theorem 3.12) that every continuous valuation is even quasi-simple (a stronger property) if X is finitarily coherent (a stronger requirement).

To finish, we can sum up the main results of this paper in the following table, where classes of valuations above the double line are equivalent to extension conditions below the double line:

valuation:	simple	$\subsetneq$	discrete	$\subsetneq$	quasi-simple	$\subsetneq$	point-continuous	$\subsetneq$	continuous
extends to:			continuous valuation		continuous valuation		strict, monotonic, continuous function		
on:			discrete topology		Alexandroff extension		Alexandroff extension		

In particular, it is remarkable that several classes of valuations can be characterised by the existence of extensions.

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