Ergod. Th. & Dynam. Sys. (2000), **20**, 727–748
Printed in the United Kingdom

Bounded hyperbolic components of quadratic rational maps

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(*Received* 20 *September* 1997 *and accepted in revised form* 4 *June* 1999)

Abstract. Let $\mathcal{H}$ be a hyperbolic component of quadratic rational maps possessing two distinct attracting cycles. We show that $\mathcal{H}$ has compact closure in moduli space if and only if neither attractor is a fixed point.

1. *Introduction*

From the perspective of dynamics, the simplest rational maps are *hyperbolic*: every critical point tends under iteration to some attracting periodic cycle. Such maps constitute an open and conjecturally dense set in parameter space [**8**], whose components are referred to as *hyperbolic components*. Maps in the same component are quasiconformally conjugate near the Julia set, and thus have essentially identical dynamics if critical orbit relations are ignored.

The family $P_c(z) = z^2 + c$ of quadratic polynomials contains one unbounded component, namely $\mathbb{C} - M$ where

$$M = \{c : J(P_c) \text{ is connected}\}$$

is the much studied Mandelbrot set, and infinitely many bounded components; the latter are simply connected regions with smooth real-algebraic boundary, and are naturally parametrized by the eigenvalue $\rho \in \mathbb{D}$ of the unique attracting cycle. Matters become more involved when there are at least two free critical points. The two-parameter families of normalized quadratic rational maps and normalized cubic polynomials are often considered in parallel, as their hyperbolic components admit similar descriptions: there is a single component of maps with a totally disconnected Julia set and all other components are topological 4-cells [**12, 18**]. One essential difference is that cubic polynomials with a connected Julia set form a compact set in parameter space; in particular, every hyperbolic component of maps with two distinct attractors is precompact. By contrast, while many unbounded hyperbolic components of quadratic rational maps have been identified [**6, 16**], bounded components have yet to be exhibited.

Hyperbolic components may also be discussed in the context of Kleinian groups and their quotient 3-manifolds. For finitely generated hyperbolic groups with connected limit set—those whose quotient has incompressible boundary—the corresponding hyperbolic component is precompact if and only the limit set is a Sierpinski carpet: the complement of a countable dense union of Jordan domains with disjoint closures whose diameters tend to zero. Guided by Sullivan's dictionary between these subjects, McMullen conjectured that hyperbolic rational maps with a Sierpinski carpet Julia set lie in bounded hyperbolic components [**9, 10**]. Pilgrim has suggested more precisely that a hyperbolic component is bounded when the Julia set is *almost* a Sierpinski carpet: for example, if every Fatou component is a Jordan domain and no two Fatou components have closures which intersect in more than one periodic point. Here we establish precompactness for hyperbolic components of quadratic rational maps with two attracting cycles, provided that neither attractor is a fixed point. While it is known in this case that every Fatou component is a Jordan domain [**17**], our largely algebraic arguments do not exploit the topology of the Julia set.

We begin in §2 with a review of the theory of the holomorphic index. The index formula

$$\frac{1}{1-\alpha} + \frac{1}{1-\beta} + \frac{1}{1-\gamma} = 1$$

relating the eigenvalues of the three fixed points is fundamental to Milnor's description [**13**] of the moduli space of quadratic rational maps. We survey this work in §3 and show in particular that a sequence of maps is bounded in moduli space if and only if there is an upper bound on the eigenvalues of the fixed points. Moduli space is readily parametrized through the choice of a normal form. For certain purposes it is convenient to work with the family

$$f_{\alpha,\beta}(z) = z\frac{(1-\alpha)z + \alpha(1-\beta)}{\beta(1-\alpha)z + (1-\beta)}$$

of maps fixing zero, ∞ and one with eigenvalues α, β and $\gamma = (2-(\alpha+\beta))/(1-\alpha\beta)$; in other settings it is more useful to work with the family

$$F_{\gamma,\delta}(Z) = \frac{\gamma Z}{Z^2 + \delta Z + 1}$$

of maps with critical points ± 1 and a fixed point at zero with eigenvalue γ.

In §4 we study the limiting dynamics of unbounded sequences in moduli space. Milnor showed that such sequences accumulate at a restricted set of points on a natural infinity locus [**13**], provided that there are cycles with the same period $n > 1$ and uniformly bounded eigenvalues. We sharpen this and related observations in order to show that suitably normalized iterates take limits in the family

$$G_T(Z) = Z + T + \frac{1}{Z}$$

as anticipated by considerations in the thesis of Stimson [**22**]. Cycles with bounded eigenvalues tend in the limit to cycles of G_T or to points in the backward orbit of the parabolic fixed point at ∞; in the latter case this backward orbit contains a critical point.

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In particular, if the maps in the sequence lie in a hyperbolic component where there are two non-fixed attractors then G_T must have either two non-repelling cycles, one non-repelling cycle and one preperiodic critical point, or two preperiodic critical points, in addition to the parabolic fixed point at ∞. As discussed in §5, this violation of the Fatou–Shishikura bound on the number of non-repelling cycles yields the desired contradiction.

§6 gives an intersection-theoretic re-interpretation based on Milnor's observation that $\mathrm{Per}_n(\rho)$, the locus of conjugacy classes of maps with an n-cycle of eigenvalue ρ, is an algebraic curve whose degree depends only on n. The explicit formulae in [**13**] yield a short independent proof of boundedness in the special case of maps with one attracting cycle of period 2 and another of period 3. These considerations suggest a combinatorial expression for the intersection cycle at infinity of a pair of such curves.

2. *Local invariants*

Let g be analytic on $U \subseteq \mathbb{C}$ and $\zeta \in U$ with $g(\zeta) = \zeta$. Assuming that g is not the identity, the *topological multiplicity* is defined as the positive integer

$$\mathrm{mult}_g(\zeta) = \frac{1}{2\pi i}\int_\Gamma \frac{1-g'(z)}{z-g(z)}\,dz$$

where Γ is any sufficiently small, positively-oriented, rectifiable Jordan curve enclosing ζ; the *holomorphic index* is similarly defined as the complex number

$$\mathrm{ind}_g(\zeta) = \frac{1}{2\pi i}\int_\Gamma \frac{1}{z-g(z)}\,dz.$$

One easily checks that these quantities are invariant under a holomorphic change of coordinates and can thereby be sensibly defined for $\zeta = \infty$; moreover, $\mathrm{mult}_g(\zeta) = 1$ if and only if the *eigenvalue* $\rho = g'(\zeta)$ differs from one, and then

$$\mathrm{ind}_g(\zeta) = \frac{1}{1-\rho}. \tag{1}$$

Furthermore, if $|\rho| \neq 1$ or $\rho = 1$ then $\mathrm{mult}_{g^n}(\zeta) = \mathrm{mult}_g(\zeta)$ for every $n \geq 1$.

It follows from Cauchy's integral formula that

$$\sum_{\zeta=g(\zeta)\in V} \mathrm{mult}_g(\zeta) = \frac{1}{2\pi i}\int_{\partial V} \frac{1-g'(z)}{z-g(z)}\,dz$$
$$\sum_{\zeta=g(\zeta)\in V} \mathrm{ind}_g(\zeta) = \frac{1}{2\pi i}\int_{\partial V} \frac{1}{z-g(z)}\,dz$$

for open V with $\overline{V} \subseteq U \subseteq \mathbb{C}$ and with rectifiable boundary containing no fixed points. These sums evidently depend continuously on g. For rational maps $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree D, one sees from the residue theorem that

$$\sum_{\zeta=g(\zeta)\in\widehat{\mathbb{C}}} \mathrm{mult}_g(\zeta) = D+1; \tag{2}$$

the *holomorphic index formula*

$$\sum_{\zeta=g(\zeta)\in\widehat{\mathbb{C}}} \mathrm{ind}_g(\zeta) = 1 \tag{3}$$

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follows similarly. We denote by $\mathrm{Fix}(g)$ the unordered $(D+1)$-tuple of fixed points listed with multiplicity. In general, we denote such collections of possibly identical points as $\{x_1, \ldots, x_n\}$. We similarly write $\mathrm{Crit}(g)$ for the unordered $(2D-2)$-tuple of critical points; note that there are at least two distinct critical points when $D \geq 2$.

A fixed point ζ of an analytic map g is said to be *attracting*, *indifferent* or *repelling* accordingly as the eigenvalue ρ is less than, equal to, or greater than one. If $\rho = e^{2\pi i p/q}$ where $(p, q) = 1$ and g^q is not the identity, then ζ is *parabolic*. A simple calculation then shows that $\mathrm{mult}_{g^q}(\zeta) = \nu q + 1$ for some positive integer ν; we refer to ν as the *degeneracy*, and say that ζ is a *degenerate parabolic* fixed point when $\nu \geq 2$. In view of (1), if $\mathrm{mult}_g(\zeta) = 1$ then ζ is attracting, indifferent or repelling according as the real part of $\mathrm{ind}_g(\zeta)$ is greater than, equal to or less than $\frac{1}{2}$. Following [**1**] we say that a parabolic fixed point ζ with eigenvalue $e^{2\pi i p/q}$ is

$$\begin{cases} \textit{parabolic-attracting}, & \text{when } \Re\, \mathrm{ind}_{g^q}(\zeta) > \dfrac{\nu q + 1}{2} \\ \textit{parabolic-indifferent}, & \text{when } \Re\, \mathrm{ind}_{g^q}(\zeta) = \dfrac{\nu q + 1}{2} \\ \textit{parabolic-repelling}, & \text{when } \Re\, \mathrm{ind}_{g^q}(\zeta) < \dfrac{\nu q + 1}{2}. \end{cases}$$

More generally, we say that ζ is *periodic* under g when $g^n(\zeta) = \zeta$ for some $n \geq 1$, the least such n being referred to as the *period*. The multiplicity, index and eigenvalue of the cycle $\langle\zeta\rangle = \{\zeta, \ldots, g^{n-1}(\zeta)\}$ are the corresponding invariants of ζ as a fixed point of g^n. It follows from the definition of multiplicity that a generic perturbation of g splits an n-cycle with eigenvalue $\rho = e^{2\pi i p/q}$ and degeneracy ν into an n-cycle with eigenvalue close to ρ and a ν-tuple of nq-cycles with eigenvalues close to one. Continuity of the local index sum implies the following.

LEMMA 1. *Let g be analytic on U with a parabolic n-cycle $\langle\zeta\rangle$ of eigenvalue $e^{2\pi i p/q}$. Further let g_k be analytic with $g_k \to g$ locally uniformly on U, and with n-cycles $\langle\zeta_k^{[0]}\rangle$ and nq-cycles $\langle\zeta_k^{[1]}\rangle, \ldots, \langle\zeta_k^{[\nu]}\rangle$ converging to $\langle\zeta\rangle$. If all $\langle\zeta_k^{[j]}\rangle$ are attracting for k sufficiently large then $\langle\zeta\rangle$ is parabolic-attracting or parabolic-indifferent.*

Assume now that g is rational of degree D. The *basin* of an attracting cycle $\langle\zeta\rangle$ is the open set consisting of all points $z \in \widehat{\mathbb{C}}$ with $g^n(z) \to \langle\zeta\rangle$. We refer to the connected component containing $\xi \in \langle\zeta\rangle$ as the *immediate basin* of ξ. The basin of a parabolic cycle $\langle\zeta\rangle$ is similarly defined as the open set of all $z \in \widehat{\mathbb{C}} - \bigcup_{n=0}^{\infty} g^{-n}(\zeta)$ with $g^n(z) \to \langle\zeta\rangle$, the νq components adjoining $\xi \in \langle\zeta\rangle$ forming the immediate basin of ξ. In both cases, the immediate basin of $\langle\zeta\rangle$ is taken to be the union of the immediate basins of the points in the cycle. Fatou established the fundamental fact that each cycle of components of the immediate basin of an attracting or parabolic cycle always contains at least one critical value with infinite forward orbit [**11**]. In particular, counting degeneracy there are at most $2D-2$ attracting and parabolic cycles. Shishikura extended this bound to the total count of non-repelling cycles [**20**], and the author proved a refined inequality where the contribution of each parabolic-attracting and parabolic-indifferent cycle is augmented by one [**1**]; consideration of the return maps on Ecalle cylinders shows in fact that there are at

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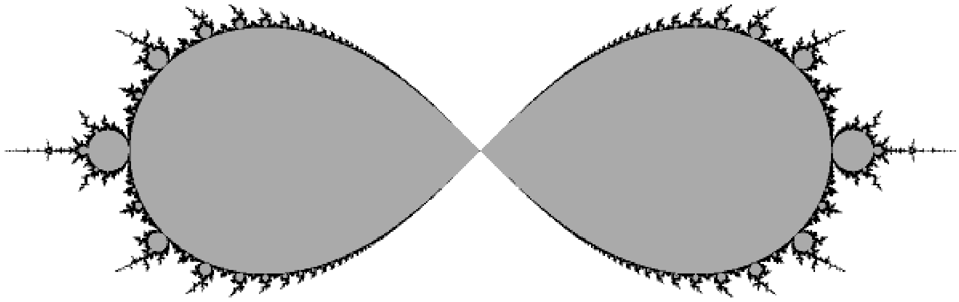

FIGURE 1. Bifurcation locus for the family $G_T(z) = z + T + (1/z)$.

least $\nu + 1$ critical values with infinite forward orbit in the immediate basin of a parabolic-attracting or parabolic-indifferent cycle of degeneracy ν.

Consider the family

$$G_T(Z) = Z + T + \frac{1}{Z}$$

(see Figure 1) of quadratic rational maps with critical points ± 1 and a degenerate fixed point at ∞ with eigenvalue one and holomorphic index $1 - (1/T^2)$; by convention, $G_\infty \equiv \infty$. The Fatou–Shishikura inequality has the following consequences in this special case.

LEMMA 2. *Let $G = G_T$ where $T \in \mathbb{C}$.*

- *If $T = 0$ then ∞ is a degenerate parabolic fixed point. Neither critical point is preperiodic and all other cycles are repelling.*
- *If some $\langle \zeta \rangle \subset \mathbb{C}$ is attracting or indifferent then ∞ is parabolic-repelling, neither critical point is preperiodic and all other cycles are repelling; if parabolic, then $\langle \zeta \rangle$ is non-degenerate parabolic-repelling.*
- *If either critical point is preperiodic then ∞ is parabolic-repelling, the other critical point has infinite forward orbit and all other cycles are repelling.*

3. *Normal forms*

We naturally identify the space of all quadratic rational maps

$$\mathbf{RAT}_2 = \left\{ g(z) = \frac{A_2 z^2 + A_1 z + A_0}{B_2 z^2 + B_1 z + B_0} : \deg g = 2 \right\}$$

with the open subvariety of projective space $\mathbb{P}^5$ where the resultant

$$\det \begin{pmatrix} A_2 & A_1 & A_0 & 0 \\ 0 & A_2 & A_1 & A_0 \\ B_2 & B_1 & B_0 & 0 \\ 0 & B_2 & B_1 & B_0 \end{pmatrix}$$

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is non-vanishing. Various technical purposes require that we work in the spaces

$$\mathbf{RAT}_2^\times = \{(g; \chi^+, \chi^-) \in \mathbf{RAT}_2 \times \widehat{\mathbb{C}}^2 : \mathrm{Crit}(g) = \{\chi^+, \chi^-\}\}$$
$$\mathbf{RAT}_2^\circ = \{(g; a, b, c) \in \mathbf{RAT}_2 \times \widehat{\mathbb{C}}^3 : \mathrm{Fix}(g) = \{a, b, c\}\}$$
$$\mathbf{RAT}_2^\otimes = \left\{ (g; \chi^+, \chi^-; a, b, c) \in \mathbf{RAT}_2 \times \widehat{\mathbb{C}}^5 : \begin{array}{c} \mathrm{Crit}(g) = \{\chi^+, \chi^-\} \\ \text{and} \\ \mathrm{Fix}(g) = \{a, b, c\} \end{array} \right\}$$

where the critical points, fixed points or both have been marked. The quotients under the conjugation action of the Möbius group are the *moduli spaces*

$$\mathbf{rat}_2 = \mathbf{RAT}_2/\mathrm{PSL}_2\mathbb{C}$$
$$\mathbf{rat}_2^\times = \mathbf{RAT}_2^\times/\mathrm{PSL}_2\mathbb{C}$$
$$\mathbf{rat}_2^\circ = \mathbf{RAT}_2^\circ/\mathrm{PSL}_2\mathbb{C}$$
$$\mathbf{rat}_2^\otimes = \mathbf{RAT}_2^\otimes/\mathrm{PSL}_2\mathbb{C}$$

all varieties of complex dimension 2.

Writing α, β, γ for the eigenvalues of the fixed points a, b, c we see from (3) that

$$\frac{1}{1-\alpha} + \frac{1}{1-\beta} + \frac{1}{1-\gamma} = 1$$

so long as $\alpha, \beta, \gamma \neq 1$, and

$$\alpha\beta\gamma - (\alpha + \beta + \gamma) + 2 = 0 \tag{4}$$

always; in particular,

$$\gamma = \frac{2 - (\alpha + \beta)}{1 - \alpha\beta}.$$

Let $[(g; \chi^+, \chi^-; a, b, c)]$ be a class in $\mathbf{rat}_2^\otimes$. Provided that $\chi^+ \neq c \neq \chi^-$, there is a unique representative of the form $(F_{\gamma,\delta}; +1, -1; a, b, 0)$ where

$$F_{\gamma,\delta}(Z) = \frac{\gamma Z}{Z^2 + \delta Z + 1} \tag{5}$$

for some $\gamma, \delta \in \mathbb{C}$ with $\gamma \neq 0$; moreover, every class in $\mathbf{rat}_2^\times$ has a representative of this form. As

$$F'_{\gamma,\delta}(Z) = \frac{\gamma(1 - Z^2)}{Z^2 + \delta Z + 1}$$

it follows that

$$\alpha = \frac{1 - a^2}{\gamma} = \frac{\delta a + 2}{\gamma} - 1$$
$$\beta = \frac{1 - b^2}{\gamma} = \frac{\delta b + 2}{\gamma} - 1$$

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with

$$\{a, b\} = \left\{ \frac{-\delta \pm \sqrt{\delta^2 - 4(1-\gamma)}}{2} \right\}.$$

Alternatively, provided that $a \neq b \neq c \neq a$ there is a unique representative of the form $(f_{\alpha,\beta}; \chi^+, \chi^-; 0, \infty, 1)$ where

$$f(z) = z\frac{(1-\alpha)z + \alpha(1-\beta)}{\beta(1-\alpha)z + (1-\beta)} \tag{6}$$

for some $\alpha, \beta \in \mathbb{C}$ with $\alpha, \beta, \alpha\beta \neq 1$. Writing

$$f_{\alpha,\beta}(z) = z\frac{(1-\alpha)(z-1) + \epsilon}{\beta(1-\alpha)(z-1) + \epsilon} \tag{7}$$

$$= \frac{z}{\beta}\left[\frac{z-\nu}{z-\mu}\right] = \frac{z}{\beta}\left[1 + \frac{\mu\epsilon}{z-\mu}\right], \tag{8}$$

where

$$\epsilon = 1 - \alpha\beta = \frac{(1-\alpha)(1-\beta)}{(\gamma-1)} \tag{9}$$

and

$$\mu = \frac{\beta - 1}{\beta - \alpha\beta} = 1 - \frac{\epsilon}{\beta(1-\alpha)} \tag{10}$$

$$\nu = \frac{\alpha\beta - \alpha}{1-\alpha} = 1 - \frac{\epsilon}{1-\alpha}, \tag{11}$$

we see that $f_{\alpha,\beta}(\mu) = \infty$ and $f_{\alpha,\beta}(\nu) = 0$. Calculating the derivative

$$f'_{\alpha,\beta}(z) = \frac{\beta(1-\alpha)^2(z-1)^2 + (1+\beta)(1-\alpha)(z-1)\epsilon + (2-\alpha-\beta)\epsilon}{[\beta(1-\alpha)(z-1)+\epsilon]^2} \tag{12}$$

$$= \frac{1}{\beta}\left[1 + \frac{\mu\nu - \mu^2}{(z-\mu)^2}\right] = \frac{1}{\beta}\left[1 - \frac{\mu^2\epsilon}{(z-\mu)^2}\right], \tag{13}$$

we find that

$$\mu = \frac{\chi^+ + \chi^-}{2}, \quad \epsilon = \left(\frac{\chi^+ - \chi^-}{\chi^+ + \chi^-}\right)^2$$

whence

$$\chi^{\pm} = \mu(1 \pm \sqrt{\epsilon})$$

for the appropriate choice of $\sqrt{\epsilon}$.

Assuming both restrictions on the marked points, there is a unique Möbius transformation

$$\phi(Z) = \frac{bZ - ab}{aZ - ab} = \frac{(\mu^2\epsilon - \mu^2 + \mu)Z + \mu\sqrt{\epsilon}}{(1-\mu)Z + \mu\sqrt{\epsilon}} \tag{14}$$

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sending $+1, -1, a, b, 0$ to $\chi^+, \chi^-, 0, \infty, 1$. Clearly,

$$F_{\gamma,\delta} = \phi^{-1} \circ f_{\alpha,\beta} \circ \phi$$

where

$$(\alpha, \beta) = \left(\frac{4\chi^+\chi^- - 2\chi^+\chi^-(\chi^+ + \chi^-)}{(\chi^+ + \chi^-)^2 - 2\chi^+\chi^-(\chi^+ + \chi^-)}, \frac{2(\chi^+ + \chi^-) - 4\chi^+\chi^-}{2(\chi^+ + \chi^-) - (\chi^+ + \chi^-)^2} \right)$$

and

$$(\gamma, \delta) = (1 - ab, \ -a - b).$$

Recall that the elementary symmetric functions

$$X(\alpha, \beta, \gamma) = \alpha + \beta + \gamma$$
$$Y(\alpha, \beta, \gamma) = \alpha\beta + \alpha\gamma + \beta\gamma$$
$$Z(\alpha, \beta, \gamma) = \alpha\beta\gamma$$

together determine $\{a, b, c\}$. It follows from (4) that

$$\mathbf{rat}_2^\circ \ni [f; a, b, c] \rightsquigarrow (X(\alpha, \beta, \gamma), Y(\alpha, \beta, \gamma), Z(\alpha, \beta, \gamma)) \in \mathbb{C}^3$$

descends to a map $\mathbf{rat}_2 \to \mathbb{C}^3$ with image in the hyperplane

$$\{(X, Y, Z) \in \mathbb{C}^3 : Z = X - 2\},$$

and we obtain

$$j : \mathbf{rat}_2 \to \mathbb{C}^2$$

on composition with the projection $\mathbb{C}^3 \ni (X, Y, Z) \rightsquigarrow (X, Y) \in \mathbb{C}^2$. Consideration of the normal forms (5) and (6) shows that an unordered triple $\{a, b, c\}$ satisfying (4) determines a unique class in $\mathbf{rat}_2$, and thus j is an isomorphism. As $\{\alpha_k, \beta_k, \gamma_k\}$ and $\{X(\alpha_k, \beta_k, \gamma_k), Y(\alpha_k, \beta_k, \gamma_k)\}$ are simultaneously bounded or unbounded, we recover Milnor's observation [**13**] as follows.

LEMMA 3. *Let g_k be quadratic rational maps with eigenvalues $\alpha_k, \beta_k, \gamma_k$ at the fixed points a, b, c. Then $[g_k]$ is bounded in $\mathbf{rat}_2$ if and only if $\{\alpha_k, \beta_k, \gamma_k\}$ is bounded in $\mathbb{C}$.*

4. *Limit dynamics*

Our first goal follows.

PROPOSITION 1. *Let g_k be quadratic rational maps with eigenvalues $\alpha_k, \beta_k, \gamma_k$ at the fixed points a, b, c, where α_k and β_k converge in $\widehat{\mathbb{C}}$ and $\gamma_k \to \infty$. Assume that there are cycles $\langle z_k \rangle$ with the same period $n > 1$ and uniformly bounded eigenvalues. Then*

$$\alpha_k = \omega + O(\sqrt{\epsilon_k}), \quad \beta_k = \bar{\omega} + O(\sqrt{\epsilon_k})$$

as $k \to \infty$, where ω is an ℓth root of unity for some $1 < \ell \leq n$ and

$$\epsilon_k = 1 - \alpha_k\beta_k = O\left(\frac{1}{\gamma_k}\right).$$

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The proof requires several preliminary lemmas and the elimination of various special cases. Let $\alpha_k, \beta_k, \gamma_k \in \mathbb{C}$ satisfying (4), and suppose that $\gamma_k \to \infty$. Inspection of (9), (10) and (11) shows that:

- $\alpha_k \to \infty$ if and only if $\beta_k \to 0$, and *vice versa*;
- $\alpha_k \to 1$ if and only if $\beta_k \to 1$, and *vice versa*;
- $\epsilon_k = o(\alpha_k - 1)$ if β_k is bounded, and $\epsilon_k = o(\beta_k - 1)$ if α_k is bounded;
- $\epsilon_k = O(\gamma_k^{-1})$ if both α_k and β_k are bounded;
- $\mu_k \to 1$ and $\nu_k \to 1$ if both $\alpha_k, \beta_k \neq 1$ are bounded;
- both μ_k and ν_k are $1 + O(\epsilon_k)$ if α_k, hence also β_k, is bounded away from $\{0, 1, \infty\}$.

Recall from (14) that the choice of $\sqrt{\epsilon_k}$, corresponding to a marking of the critical points, specifies a Möbius transformation ϕ_k which conjugates f_{α_k,β_k} to some F_{γ_k,δ_k}. It follows from these observations that

$$\phi_k(Z) = 1 + Z\sqrt{\epsilon_k} + o(\sqrt{\epsilon_k}) \tag{15}$$

on compact sets in $\mathbb{C}$, provided that α_k and β_k are bounded away from $\{0, 1, \infty\}$.

Let $f_k = f_{\alpha_k,\beta_k}$ where $\gamma_k \to \infty$ and $\alpha_k \to \alpha_\infty \in \mathbb{C}^*$. It follows from (7) and (8) that $f_k(z_k)/z_k = 1 + o(1)$ for any sequence of points $z_k \in \mathbb{C}$ with $z_k - 1 = o(\epsilon_k)$, and that $f_k(z) \to \alpha_\infty z$ locally uniformly on $\mathbb{C} - \{0, 1, \infty\}$. Moreover, if $\alpha_\infty \neq 1$ then

$$\frac{f_k(z_k)}{z_k} = \alpha_k\left(1 + \frac{\epsilon_k}{z_k - 1} + o\left(\frac{\epsilon_k}{z_k - 1}\right) + O(\epsilon_k)\right) \tag{16}$$

when $\epsilon_k = o(z_k - 1)$, so that

$$\frac{f_k(z_k)}{z_k} = \begin{cases} \alpha_k + o(1), & \text{if } \epsilon_k = o(z_k - 1) \\ \alpha_k\left(1 + \dfrac{1}{\tau}\sqrt{\epsilon_k}\right) + o(\sqrt{\epsilon_k}), & \text{if } z_k = 1 + \tau\sqrt{\epsilon_k} + o(\sqrt{\epsilon_k}) \text{ for } \tau \in \mathbb{C}^* \\ \alpha_k + o(\sqrt{\epsilon_k}), & \text{if } \sqrt{\epsilon_k} = o(z_k - 1) \\ \alpha_k + O(\epsilon_k), & \text{if } z_k \text{ is bounded away from one;} \end{cases} \tag{17}$$

furthermore,

$$\frac{f_k(z_k) - 1}{z_k - 1} = \frac{(z_k - \beta_k)(1 - \alpha_k) + \epsilon_k}{\beta_k(1 - \alpha_k)(z_k - 1) + \epsilon_k} \to \infty \tag{18}$$

whenever $z_k \to 1$.

LEMMA 4. *Let $F_k = F_{\gamma_k,\delta_k}$ where $\gamma_k \to \infty$, with $\alpha_k \to \alpha_\infty$ and $\beta_k \to \beta_\infty$ for some $\alpha_\infty, \beta_\infty \notin \{0, 1, \infty\}$; let $Z_k \in \widehat{\mathbb{C}}$ with $F_k^j(Z_k) \to \zeta^{(j)} \in \widehat{\mathbb{C}}$ for $0 \leq j \leq \ell$, where $\ell > 0$, and suppose that $\zeta^{(0)} \notin \{0, \infty\}$ and $\zeta^{(\ell)} \neq \infty$. Then $\ell > 1$. Moreover, if $\zeta^{(j)} \neq 0$ for $0 < j < \ell$ then $\alpha_k = \omega + O(\sqrt{\epsilon_k})$ and $\beta_k = \bar{\omega} + O(\sqrt{\epsilon_k})$, for some ℓth root of unity ω; if $\zeta^{(j)} = \infty$ for every $0 < j < \ell$ then ω is a primitive ℓth root of unity.*

Proof. Consider the points $z_k = \phi_k(Z_k)$. As $\zeta^{(j)} \neq 0$ for $0 \leq j < \ell$ it follows from (15) that $\sqrt{\epsilon_k} = O(f_k^j(z_k) - 1)$, hence $f_k^{j+1}(z_k)/f_k^j(z_k) = \alpha_k + O(\sqrt{\epsilon_k})$ by (17); consequently,

$$\frac{f_k^j(z_k)}{z_k} = \prod_{i=0}^{j-1} \frac{f_k^{i+1}(z_k)}{f_k^i(z_k)} = \alpha_k^j + O(\sqrt{\epsilon_k})$$

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for $0 \leq j \leq \ell$. On the other hand, $f_k^\ell(z_k)/z_k = (1 + O(\sqrt{\epsilon_k}))/(1 + O(\sqrt{\epsilon_k})) = 1 + O(\sqrt{\epsilon_k})$ because $\zeta^{(0)} \neq \infty \neq \zeta^{(\ell)}$, so $\alpha_k^\ell = 1 + O(\sqrt{\epsilon_k})$. It follows that $\alpha_\infty^\ell = 1 = \beta_\infty^\ell$, hence $\alpha_\infty = \omega$ and $\beta_k = \bar{\omega}$ for some ℓth root of unity $\omega \neq 1$, and that $\alpha_k = \omega + O(\sqrt{\epsilon_k})$, hence $\beta_k = \alpha_k^{-1} + O(\epsilon_k) = \bar{\omega} + O(\sqrt{\epsilon_k})$. Furthermore, $f_k^j(z_k) = (\alpha_k^j + O(\sqrt{\epsilon_k}))z_k = (\omega^j + O(\sqrt{\epsilon_k}))(1 + O(\sqrt{\epsilon_k})) = \omega^j + O(\sqrt{\epsilon_k})$ for $0 \leq j \leq \ell$, so that $\zeta^{(j)} \neq \infty$ if $\omega^j \neq 1$. $\square$

LEMMA 5. *Let $F_k = F_{\gamma_k,\delta_k}$ where $\gamma_k \to \infty$, with $\alpha_k = \alpha_\infty + O(\sqrt{\epsilon_k})$ for some $\alpha_\infty \notin \{0, 1, \infty\}$, and let $Z_k \in \widehat{\mathbb{C}}$ with $Z_k \to 0$. If $F_k^j(Z_k)$ is bounded for some $j > 0$ then $Z_k = O(\sqrt{\epsilon_k})$.*

Proof. Set $z_k = \phi_k(Z_k)$; note that $z_k = 1 + o(\sqrt{\epsilon_k})$ by (15), because $Z_k \to 0$. Suppose to the contrary that $\sqrt{\epsilon_k} = o(Z_k)$; then $\epsilon_k/(z_k - 1) = o(1)$, and we claim that

$$f_k^j(z_k) = \alpha_\infty^j \left(1 + \frac{\epsilon_k}{z_k - 1}\right) + o\left(\frac{\epsilon_k}{z_k - 1}\right) \tag{19}$$

for $j > 0$. Indeed, $\alpha_k z_k = \alpha_\infty + O(\sqrt{\epsilon_k}) = \alpha_\infty + o(\epsilon_k/(z_k - 1))$, whence

$$f_k(z_k) = \alpha_\infty \left(1 + \frac{\epsilon_k}{z_k - 1}\right) + o\left(\frac{\epsilon_k}{z_k - 1}\right)$$

by (16); as (19) implies $\sqrt{\epsilon_k} = o(f_k^j(z_k) - 1)$, it follows by (17) and induction that

$$\begin{aligned} f_k^{j+1}(z_k) &= (\alpha_k + o(\sqrt{\epsilon_k}))f_k^j(z_k) = \alpha_\infty \left(1 + o\left(\frac{\epsilon_k}{z_k - 1}\right)\right) f_k^j(z_k) \\ &= \alpha_\infty^{j+1}\left(1 + \frac{\epsilon_k}{z_k - 1}\right) + o\left(\frac{\epsilon_k}{z_k - 1}\right). \end{aligned}$$

On the other hand, if $F_k^j(Z_k)$ is bounded then $f_k^j(z_k) = 1 + O(\sqrt{\epsilon_k})$ by (15); thus, $\alpha_\infty^j = 1$ by (19), but then $\epsilon_k/(z_k - 1) = O(\sqrt{\epsilon_k})$ for a contradiction. $\square$

Suppose now that $f_k^n(z_k) = z_k$ where $z_k \in \widehat{\mathbb{C}} - \{0, 1, \infty\}$ and $n > 1$. As

$$1 = \frac{f_k^n(z_k) - 1}{z_k - 1} = \prod_{j=0}^{n-1} \frac{f_k^{j+1}(z_k) - 1}{f_k^j(z_k) - 1},$$

it follows from (18) that $f_k^j(z_k)$ is bounded away from one for every j in some congruence class modulo n; as

$$1 = \frac{f_k^n(z_k)}{z_k} = \prod_{j=0}^{n-1} \frac{f_k^{j+1}(z_k)}{f_k^j(z_k)}$$

it similarly follows from (17) that

$$\alpha_k^n = \begin{cases} 1 + o(1), & \text{if } \min_j |(f_k^j(z_k) - 1)/\epsilon_k| \to \infty \\ 1 + O(\sqrt{\epsilon_k}), & \text{if } \min_j |(f_k^j(z_k) - 1)/\sqrt{\epsilon_k}| \text{ is bounded away from zero} \\ 1 + o(\sqrt{\epsilon_k}), & \text{if } \min_j |(f_k^j(z_k) - 1)/\sqrt{\epsilon_k}| \to \infty \\ 1 + O(\epsilon_k), & \text{if } \langle z_k \rangle \text{ is bounded away from one.} \end{cases} \tag{20}$$

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LEMMA 6. *Let $F_k = F_{\gamma_k,\delta_k}$ where $\gamma_k \to \infty$, with $\alpha_k \to \alpha_\infty$ and $\beta_k \to \beta_\infty$ for some $\alpha_\infty, \beta_\infty \notin \{0, 1, \infty\}$; let $\langle Z_k \rangle$ be cycles of period $n > 1$, and suppose that $\langle Z_k \rangle \to \Gamma \subset \widehat{\mathbb{C}}$. Then $\infty \in \Gamma$. Moreover:*

- *if $\Gamma \neq \{0, \infty\}$ then $\alpha_k = \omega + O(\sqrt{\epsilon_k})$ and $\beta_k = \bar{\omega} + O(\sqrt{\epsilon_k})$;*
- *if $\Gamma = \{\infty\}$ then $\alpha_k = \omega + o(\sqrt{\epsilon_k})$ and $\beta_k = \bar{\omega} + o(\sqrt{\epsilon_k})$, where ω is an ℓth root of unity for some $1 < \ell \leq n$.*

Proof. We may assume without loss of generality that $F_k^j(Z_k) \to \zeta^{(j)} \in \Gamma$ for each $j \in \mathbb{Z}$; as $f_k^j(z_k)$ is bounded away from one for some j, it follows that $\infty = \zeta^{(j)} \in \Gamma$. If $\Gamma \neq \{0, \infty\}$ then we may further assume that $\zeta^{(n)} = \zeta^{(0)} \notin \{0, \infty\}$. In this case there is a least $\ell > 0$ with $\zeta^{(\ell)} \neq \infty$, so it follows by Lemma 4 that $\alpha_k = \omega + O(\sqrt{\epsilon_k})$ and $\beta_k = \bar{\omega} + O(\sqrt{\epsilon_k})$ where $\omega^\ell = 1$; necessarily, $1 < \ell \leq n$. Finally, if $\Gamma = \{\infty\}$ then $\sqrt{\epsilon_k} = o(f_k^j(z_k) - 1)$ for every j, so that $\alpha_k = \omega + o(\sqrt{\epsilon_k})$, hence $\beta_k = \alpha_k^{-1} + O(\epsilon_k) = \bar{\omega} + o(\sqrt{\epsilon_k})$, by (20). □

Assume now that α_k, hence also β_k, is bounded away from $\{0, 1, \infty\}$, and let z_k be any sequence of points in $\mathbb{C}$. If $z_k - 1 = o(\sqrt{\epsilon_k})$ then

$$f_k'(z_k) = \frac{(2 - \alpha_k - \beta_k)\epsilon_k + o(\epsilon_k)}{o(\epsilon_k)} \to \infty$$

by (12). On the other hand, if $\sqrt{\epsilon_k} = O(z_k - 1)$ then (13) implies

$$f_k'(z_k) = \frac{1}{\beta_k}\left[1 - \frac{\epsilon_k}{(z_k - 1)^2} + o\left(\frac{\epsilon_k}{(z_k - 1)^2}\right)\right]$$

whence

$$f_k'(z_k) = \begin{cases} \alpha_k\left(1 - \dfrac{1}{\tau^2}\right) + o(1), & \text{if } z_k = 1 + \tau\sqrt{\epsilon_k} + o(\sqrt{\epsilon_k}) \text{ for } \tau \in \mathbb{C}^* \\ \alpha_k + o(1), & \text{if } \sqrt{\epsilon_k} = o(z_k - 1). \end{cases}$$

In particular, if the z_k are periodic with period $n > 1$ then the corresponding eigenvalues are

$$\begin{aligned} \rho_k &= \prod_{j=0}^{n-1} f'(f^j(z_k)) \\ &= \begin{cases} O(1), & \text{if } \min_j |(f_k^j(z_k) - 1)/\sqrt{\epsilon_k}| \text{ is bounded away from zero} \\ \alpha_k^n + o(1), & \text{if } \min_j |(f_k^j(z_k) - 1)/\sqrt{\epsilon_k}| \to \infty. \end{cases} \end{aligned}$$

In view of (15) and (20), these observations prove the following.

LEMMA 7. *Let $F_k = F_{\gamma_k,\delta_k}$ where $\gamma_k \to \infty$. Assume that $\alpha_k \to \omega$, hence $\beta_k \to \bar{\omega}$, where $\omega \neq 1$ is a primitive qth root of unity, and let $\langle Z_k \rangle \to \Gamma$ be cycles of period $n > 1$ and eigenvalues ρ_k. If $0 \notin \Gamma$ then $q|n$ and ρ_k is bounded; moreover, if $\Gamma = \{\infty\}$ then $\rho_k \to 1$.*

On the other hand, if $\min_j |(f_k^j(z_k) - 1)/\sqrt{\epsilon_k}| \to 0$ then $\rho_k \to \infty$, unless $f_k^j(z_k) = 1 + \sqrt{\epsilon_k} + o(\sqrt{\epsilon_k})$ for some integer j, so (15) implies the following.

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LEMMA 8. *Let $F_k = F_{\gamma_k,\delta_k}$ where $\gamma_k \to \infty$, with α_k and β_k bounded away from $\{0, 1, \infty\}$, and let $\langle Z_k\rangle \to \Gamma$ be cycles of period $n > 1$ and eigenvalues ρ_k. If $0 \in \Gamma$ and ρ_k is bounded then $+1 \in \Gamma$ or $-1 \in \Gamma$.*

Proof of Proposition 1. Let $\alpha_\infty = \lim_{k\to\infty} \alpha_k$ and $\beta_\infty = \lim_{k\to\infty} \beta_k$. If $\alpha_\infty \notin \{0, 1, \infty\}$ then $\beta_\infty \notin \{0, 1, \infty\}$, so we may represent each class

$$[(g_k; a_k, b_k, c_k)] \in \mathbf{rat}_2^\circ$$

by a map $F_k = F_{\gamma_k,\delta_k}$. Recall that we are given n-cycles $\langle Z_k\rangle$ with uniformly bounded eigenvalues ρ_k. Passing to a subsequence if necessary, we may assume that $\langle Z_k\rangle \to \Gamma \subset \widehat{\mathbb{C}}$. As $\Gamma \neq \{0, \infty\}$ by Lemma 8, it follows by Lemma 6 that $\alpha_k = \omega + O(\sqrt{\epsilon_k})$ and $\beta_k = \bar{\omega} + O(\sqrt{\epsilon_k})$, where ω is an ℓth root of unity for some $1 < \ell \leq n$.

Suppose next that α_∞, hence also β_∞, is in $\{0, \infty\}$. Permuting the fixed points if necessary, we may assume on passage to a subsequence that $\alpha_k \to \infty$ and $\alpha_k = O(\gamma_k)$. Following Milnor [**13**] we work with the representatives $(\hat{f}_k; 0, \infty, c_k)$ where

$$\hat{f}_k(z) = z\frac{z + \alpha_k}{\beta_k z + 1}$$

and

$$c_k = \frac{1 - \alpha_k}{1 - \beta_k} = -\alpha_k + o(\alpha_k).$$

Calculating the derivative

$$\hat{f}_k'(z) = \frac{\beta_k z^2 + 2z + \alpha_k}{(\beta_k z + 1)^2}$$

we see that $\hat{f}_k'(z) = \alpha_k + O(1)$ on the disc $|z| < 4$. In particular, $\hat{f}_k$ is univalent on $|z| < 4$ with the image containing the disc $|z| < 3|c_k|$, and both critical values lie outside the latter region. Consequently, there are univalent inverse branches A_k and C_k, fixing zero and c_k, defined on the disc $|z| < 3|c_k|$. As $D_k = \{z : |2z - c_k| < 2|c_k|\}$ lies in the image of the disc $|z| < 4$, it follows that $A_k'(D_k) = O(\alpha_k^{-1})$ on D_k and $A_k(D_k) \subset D_k$. On the other hand,

$$C_k'(z) = O(\gamma_k^{-1}) = O(\alpha_k^{-1})$$

for $|z| < \frac{5}{2}|c_k|$ by the compactness of normalized univalent functions; consequently, $|C_k(z) - c_k| = O(c_k\gamma_k^{-1}) = O(1)$ for $|z - c_k| < \frac{3}{2}|c_k|$, and in particular $C_k(D_k) \subset D_k$. We deduce that $J(\hat{f}_k) \subset \hat{f}_k^{-1}(D_k)$ is a Cantor set containing all periodic points other than the fixed point at ∞. Thus, $\langle z_k\rangle \subset J(\hat{f}_k)$ and $\rho_k^{-1} = O(\alpha_k^{-n})$ whence $\rho_k \to \infty$.

It remains to treat the case $\alpha_\infty = 1 = \beta_\infty$. Now it is advantageous to choose representatives $(\hat{g}_k; \infty, b_k, c_k)$ where

$$\hat{g}_k(z) = \frac{(\alpha_k\gamma_k - 1)z^2 + (\alpha_k^2\gamma_k - \alpha_k^2)z + \alpha_k^2}{(\alpha_k^2\gamma_k - \alpha_k)z}$$

and

$$(b_k, c_k) = \left(\frac{\alpha_k}{\alpha_k - 1}, \frac{\alpha_k}{1 - \alpha_k\gamma_k}\right) \to (\infty, 0).$$

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Notice that $\hat{g}_k(z) \to z+1$ locally uniformly on $\widehat{\mathbb{C}} - \{0\}$, and thus $\hat{g}_k^n(z) \to z+n$ locally uniformly on $\widehat{\mathbb{C}} - \{-(n-1), \ldots, 0\}$. As the translation $z \rightsquigarrow z+1$ has a fixed point of multiplicity 2 at ∞ and no other periodic points, b_k and ∞ are the only fixed points of $\hat{g}_k^n$ outside the circle $|z| = n$. We may therefore assume without loss of generality that

$$\hat{g}_k^j(z_k) \to \zeta^{(j)} \in \{-(n-1), \ldots, 0\}$$

with $\zeta^{(j+1)} = \zeta^{(j)} + 1$ whenever $\zeta^{(j)} \neq 0$. It follows that some $\zeta^{(j)} = 0$, whence $\hat{g}_k^j(z_k) = O(\gamma_k^{-1})$ as $\hat{g}_k^{j+1}(z_k)$ is bounded away from one. Calculating the derivative

$$\hat{g}_k'(z) = \frac{1}{\alpha_k} - \frac{\alpha_k}{(\alpha_k\gamma_k - 1)z^2}$$

we see that $\hat{g}_k'(\hat{g}_k^j(z_k)) \to \infty$ when $\zeta^{(j)} = 0$, while $\hat{g}_k'(\hat{g}_k^j(z_k)) \to 1$ otherwise, and we conclude that $\rho_k \to \infty$. □

Now let $\alpha_k = \omega(1 + \tau\sqrt{\epsilon_k}) + o(\sqrt{\epsilon_k})$, so that $\beta_k = \bar{\omega}(1 - \tau\sqrt{\epsilon_k}) + o(\sqrt{\epsilon_k})$, where $\omega \neq 1$ is a primitive qth root of unity and $\tau \in \mathbb{C}$. For $Z \in \mathbb{C}^*$ and $0 \leq j < q$, it follows from (17) that

$$f_k^q(\bar{\omega}^j(1 + Z\sqrt{\epsilon_k})) = \bar{\omega}^j\left[1 + \left(Z + q\tau + \frac{1}{Z + j\tau}\right)\sqrt{\epsilon_k}\right] + o(\sqrt{\epsilon_k})$$

whence

$$\psi_{k,(j)}^{-1} \circ f_k^q \circ \psi_{k,(j)}(Z) \to Z + q\tau + \frac{1}{Z + j\tau} = G_{q\tau}(Z + j\tau) - j\tau \tag{21}$$

locally uniformly on $\mathbb{C}^*$, where

$$\psi_{k,(j)}(Z) = \bar{\omega}^j(1 + Z\sqrt{\epsilon_k}).$$

Similarly, if $\alpha_k \to \omega$ but $(\alpha_k - \omega)/\sqrt{\epsilon_k} \to \infty$ then

$$\frac{f_k^q(\bar{\omega}^j(1 + Z\sqrt{\epsilon_k}))}{\alpha_k^q} = \begin{cases} 1 + \left(Z + \dfrac{1}{Z}\right)\sqrt{\epsilon_k} + o(\sqrt{\epsilon_k}), & \text{for } j = 0 \\ \bar{\omega}^j(1 + Z\sqrt{\epsilon_k}) + o(\sqrt{\epsilon_k}), & \text{for } j \neq 0 \end{cases}$$

and thus

$$\psi_{k,(j)}^{-1} \circ f_k^q \circ \psi_{k,(j)}(Z) \to \infty$$

locally uniformly on $\mathbb{C}^*$. Applying (15) to the case $j = 0$, we deduce the following.

PROPOSITION 2. *Let $F_k = F_{\gamma_k,\delta_k}$ where $\gamma_k \to \infty$. Assume that $\alpha_k \to \omega$, hence $\beta_k \to \bar{\omega}$, where $\omega \neq 1$ is a primitive qth root of unity, and assume further that $(\alpha_k - \omega)/\sqrt{\epsilon_k} \to \omega\tau$, hence $(\beta_k - \bar{\omega})/\sqrt{\epsilon_k} \to -\bar{\omega}\tau$, for some $\tau \in \widehat{\mathbb{C}}$. Then $F_k^q \to G_{q\tau}$ locally uniformly on $\mathbb{C}^*$.*

Recalling Lemmas 6 and 8, we observe the following.

PROPOSITION 3. *Let $F_k = F_{\gamma_k,\delta_k}$ where $\gamma_k \to \infty$. Assume that $\alpha_k \to \omega$, hence $\beta_k \to \bar{\omega}$, where $\omega \neq 1$ is a primitive qth root of unity. Assume further that $F_k^q \to G_T$ for some $T \in \mathbb{C}$, and let $\langle Z_k \rangle \to \Gamma$ be cycles of period $n > 1$ and eigenvalues $\rho_k \to \rho_\infty \in \mathbb{C}$. Then $G_T(\Gamma) \subseteq \Gamma$. Moreover:*

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- *if* $\Gamma = \{\infty\}$ *then* $T = 0$*;*
- *if* $0 \in \Gamma$ *then* $G_T^m(\chi) = 0$*, whence* $G_T^{m+1}(\chi) = \infty = G_T^{m+2}(\chi)$*, for some* $\chi \in \{+1, -1\}$ *and* $1 \leq m < n/q$*;*
- *otherwise,* $\Gamma = \langle\zeta\rangle \cup \{\infty\}$ *where* $\langle\zeta\rangle \subset \mathbb{C}$ *is a cycle of period* $m = n/q$ *and eigenvalue* ρ_∞*, or possibly a parabolic cycle of lower period if* $\rho_\infty = 1$*.*

Conversely, given an m-cycle $\langle\zeta\rangle$ of G_T there exist mq-cycles of F_k converging to $\langle\zeta\rangle \cup \{\infty\}$. In particular, for $T \neq 0$ there is a unique finite fixed point $\zeta = -1/T$ with eigenvalue $1 - T^2$, hence $\langle Z_k\rangle \to \{\zeta, \infty\}$ for some q-cycles $\langle Z_k\rangle$. As $\mathrm{mult}_{G_T}(\zeta) = 1$, it follows from Lemma 6 that $\langle\hat{Z}_k\rangle \to \{0, \infty\}$ for every convergent sequence of q-cycles $\langle\hat{Z}_k\rangle \neq \langle Z_k\rangle$. In view of Lemma 8, the eigenvalues of $\langle\hat{Z}_k\rangle$ tend to ∞, as do those of all ℓ-cycles where $\ell \notin \{1, q\}$ divides q, and thus

$$\frac{1}{1-\alpha_k^q} + \frac{1}{1-\beta_k^q} \to 1 - \frac{1}{T^2} \tag{22}$$

by (3). On the other hand, for $T = 0$ there is only the fixed point at ∞, so every convergent sequence of q-cycles of F_k tends to $\{\infty\}$ or $\{0, \infty\}$. The validity of (22) in this case is a particular consequence of the following.

PROPOSITION 4. *Let* $F_k = F_{\gamma_k,\delta_k}$ *where* $\gamma_k \to \infty$*. Assume that* $\alpha_k \to \omega$*, hence* $\beta_k \to \bar{\omega}$*, where* $\omega \neq 1$ *is a primitive* q*th root of unity, and let* $\langle Z_k\rangle$ *be cycles of period* $n > 1$*. If* $\langle Z_k\rangle \to \{\infty\}$ *then* $n = q$*, and every convergent sequence of* q*-cycles* $\langle\hat{Z}_k\rangle \neq \langle Z_k\rangle$ *tends to* $\{0, \infty\}$*.*

Proof. In view of Lemma 7, we may assume without loss of generality that $n = mq$ for some positive integer m. By (15), it is enough to show that for r and k sufficiently large at most one mq-cycle of f_k lies completely inside

$$V_k^r = \widehat{\mathbb{C}} - \bigcup_{j=1}^{q-1} \overline{D}_{k,(j)}^r$$

where $D_{k,(j)}^r = \{z \in \mathbb{C} : |z - \bar{\omega}^j| < r\sqrt{|\epsilon_k|}\}$, and that $m = 1$ if there is such a cycle. It follows from (17) that $f_k^{-mq}(\infty) \cap \overline{V}_k^r = \{\infty\}$ for large r and k, and thus all of the $2^{mq} - 1$ finite poles of f_k^{mq} lie in $\bigcup_{j=0}^{q-1} D_{k,(j)}^r$. Consequently,

$$\sum_{z=f_k^{mq}(z)\in V_k^r} \mathrm{mult}_{f_k^{mq}}(z) = 2^{mq} + 1 - \sum_{z=f_k^{mq}(z)\in\widehat{\mathbb{C}}-\overline{V}_k^r} \mathrm{mult}_{f_k^{mq}}(z)$$

provided that f_k^{mq} has no fixed points on ∂V_k^r, whence

$$\sum_{z=f_k^{mq}(z)\in V_k^r} \mathrm{mult}_{f_k^{mq}}(z) = 2 - \sum_{j=0}^{q-1} \frac{1}{2\pi i} \int_{\partial D_{k,(j)}^r} \frac{1 - (f_k^{mq})'(z)}{z - f_k^{mq}(z)}\, dz$$

by the argument principle.

Observe that $G_0(Z) = Z + (1/Z)$ has a fixed point of multiplicity 3 at ∞, and thus G_0^m has $2^{mq} - 2$ finite fixed points and $2^{mq} - 1$ finite poles. It follows as above that

$$\frac{1}{2\pi i} \int_{|Z|=r} \frac{1 - (G_0^m)'(Z)}{Z - G_0^m(Z)}\, dZ = -1$$

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so long as $r > \max\{|Z| : Z \in \mathbb{C} \text{ and } Z - G_0^m(Z) \in \{0, \infty\}\}$. In view of (21),

$$\frac{1}{2\pi i}\int_{\partial D^r_{k,(j)}} \frac{1-(f_k^{mq})'(z)}{z - f_k^{mq}(z)}\,dz = \frac{1}{2\pi i}\int_{|Z|=r} \frac{1-(\psi_{k,(j)}^{-1}\circ f_k^{mq}\circ\psi_{k,(j)})'(Z)}{Z-(\psi_{k,(j)}^{-1}\circ f_k^{mq}\circ\psi_{k,(j)})(Z)}\,dZ$$
$$= \frac{1}{2\pi i}\int_{|Z|=r} \frac{1-(G_0^m)'(Z)}{Z - G_0^m(Z)}\,dZ$$

when k is sufficiently large, and thus

$$\sum_{z=f_k^{mq}(z)\in V_k^r} \operatorname{mult}_{f_k^{mq}}(z) = q+2.$$

We deduce that $f_k^{mq}(z) = z \in V_k^r$ implies $f_k^q(z) = z$ for large r and k depending only on m. If $\alpha_k^q = 1$ then $\operatorname{mult}_{f_k^{mq}}(0) = q+1$ and $\operatorname{mult}_{f_k^{mq}}(\infty) = 1$, while if $\beta_k^q = 1$ then $\operatorname{mult}_{f_k^{mq}}(0) = 1$ and $\operatorname{mult}_{f_k^{mq}}(\infty) = q+1$; in these cases f_k^{mq} has no other fixed points in V_k^r. Otherwise,

$$\operatorname{mult}_{f_k^{mq}}(0) = 1 = \operatorname{mult}_{f_k^{mq}}(\infty)$$

and it follows from (17) that the remaining q fixed points of f_k^q in V_k^r constitute a q-cycle of f_k. □

In view of Fatou's theorem, the second assertion in Proposition 3 is sharpened by Proposition 5.

PROPOSITION 5. *Let $F_k = F_{\gamma_k,\delta_k}$ where $\gamma_k \to \infty$, and let Z_k be attracting points of period $n > 1$ with immediate basins $\mathcal{B}_k$. If $Z_k \to 0$ then $\mathcal{B}_k \to 0$.*

Proof. In view of Proposition 1 we may assume without loss of generality that $\alpha_k = \omega + O(\sqrt{\epsilon_k})$ where $\omega \neq 1$ is a root of unity. If k is large then $Z_k \in \mathbb{D}$, so for $j \geq 0$ there are unique components $W_k^j \ni Z_k$ of $F_k^{-nj}(\mathbb{D})$. We claim first that $W_k^1 \to 0$; otherwise, as W_k^1 is connected there exist $k_\ell \to \infty$ and $\tilde{Z}_{k_\ell} \in W_{k_\ell}^1$ with $\tilde{Z}_{k_\ell} \to 0$, but $\sqrt{\epsilon_{k_\ell}} = o(\tilde{Z}_{k_\ell})$, contradicting Lemma 5. It follows that $W_k^1 \subset \mathbb{D}$, hence $W_k^{j+1} \subset W_k^j$ for $j \geq 0$, if k is sufficiently large. Let $\mathcal{X}_k$ be the component of $\bigcap_{j=0}^{\infty} W_k^j$ containing Z_k, and denote its interior by $\mathcal{W}_k$; we contend that $\mathcal{W}_k = \mathcal{B}_k$, and consequently $\mathcal{B}_k \subset W_k^1 \to 0$. By definition, if $\zeta \in \mathcal{B}_k$ there exists open $U \ni \zeta$ such that $F_k^{nj}(U) \subset \mathbb{D}$ when j is large, while if $\zeta \in \partial\mathcal{X}_k$ there exist $\zeta_j \to \zeta$ with $F_k^{nj}(\zeta_j) \in \partial\mathbb{D}$. Thus, $\mathcal{B}_k \cap \partial\mathcal{X}_k = \emptyset$ so $\mathcal{B}_k \subseteq \mathcal{W}_k$, as $\mathcal{B}_k \ni Z_k$ is a connected open set; conversely, $\mathcal{W}_k \subseteq \mathcal{B}_k$ as F_k^{nj} is bounded, hence normal, on the connected open set $\mathcal{W}_k \ni Z_k$. □

5. *Precompactness*

Recall that a rational map is *hyperbolic* if and only if the orbit of every critical point tends to some attracting cycle. As discussed in [**12, 18**], there are four configurations for quadratics.

B Both critical points lie in the immediate basin of the same attracting cycle, but in different components.

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C Both critical points lie in the basin of the same attracting cycle, but only one lies in the immediate basin.

D The critical points lie in the immediate basins of distinct attracting cycles.

E Both critical points lie in the same component of the immediate basin of an attracting fixed point.

There is in fact a unique hyperbolic component of type E consisting of maps with a totally disconnected Julia set. This component is unbounded; see [**12**] for details. Our main result is that components of type D are bounded, so long as neither attractor is a fixed point.

THEOREM 1. *Let g_k be quadratic rational maps, each having distinct non-repelling cycles of periods $n^\pm > 1$. Then the sequence $[g_k]$ is bounded in* $\mathbf{rat}_2$.

Proof. It follows from the proof of the Fatou–Shishikura inequality that we lose no generality in assuming that these cycles are attracting [**20**]. Suppose to the contrary that $[g_k]$ is unbounded in $\mathbf{rat}_2$. By Lemma 3 we may, passing to a subsequence if necessary, choose representatives $F_k = F_{\gamma_k,\delta_k} \in [g_k]$ with $\gamma_k \to \infty$. Let $\langle Z_k^\pm \rangle$ be the corresponding $n^\pm$-cycles of F_k: we may assume without loss of generality that $\langle Z_k^\pm \rangle \to \Gamma^\pm \subset \widehat{\mathbb{C}}$. In view of Propositions 1 and 2, we may further assume that $\alpha_k \to e^{2\pi i p/q}$ and $F_k^q \to G_T$ for some $q \geq 2$ and $T \in \mathbb{C}$; moreover, it follows from Fatou's theorem that we may label the critical points so that ± 1 lies in the immediate basin of $\langle Z_k^\pm \rangle$.

We derive a contradiction by examining the possibilities listed in Proposition 3. If $\Gamma^+ = \{\infty\}$ then $T = 0$; it follows from Lemma 2 that every finite cycle of G_T is repelling and that neither critical point is preperiodic, so $\Gamma^- = \{\infty\}$ in contradiction to Proposition 4. On the other hand, if $0 \in \Gamma^\pm$ then Proposition 5 implies that the critical point ± 1 is preperiodic; it follows from Lemma 2 that every finite cycle of G_T is repelling and that the other critical point ∓ 1 has infinite forward orbit, contradicting Proposition 3. Consequently, neither of $\Gamma^\pm$ contains zero, so $\Gamma^\pm = \langle \zeta^\pm \rangle \cup \{\infty\}$ for some non-repelling cycles $\langle \zeta^\pm \rangle \subset \mathbb{C}$, and in fact $\langle \zeta^+ \rangle = \langle \zeta^- \rangle$ by Lemma 2; it follows from Lemma 1 that this cycle is parabolic-attracting or parabolic-indifferent, once again contradicting Lemma 2. □

The same considerations apply when there is one non-repelling cycle along with a preperiodic critical point.

THEOREM 2. *Let g_k be quadratic rational maps with non-repelling n-cycles $\langle z_k \rangle$ where $n > 1$. Assume further that $g_k^\ell(\chi_k) \in \langle \hat{z}_k \rangle$ for some $\ell > 0$, critical points χ_k, and $\hat{n}$-cycles $\langle \hat{z}_k \rangle$. Then the sequence $[g_k]$ is bounded in* $\mathbf{rat}_2$.

The exceptional type D components are known to be unbounded; see Lemma 10 below. Many, although not all, type D maps arise as *matings* of pairs of hyperbolic quadratic polynomials. In this construction, the filled-in Julia sets are glued back-to-back along complex-conjugate prime ends; see [**2, 23**] for further details. It is tempting to speculate that our arguments could be refined to establish precompactness for large portions of the mating locus, but our results in this direction are rather limited at present. Examination of Figure 2 suggests that the type C components are all bounded. This would follow immediately from our arguments if it could be shown in this case that $F_k^{-1}(\mathcal{B}_k) - \mathcal{B}_k \to 0$ where $\mathcal{B}_k$ is the immediate basin of the unique attracting cycle. There are evidently many

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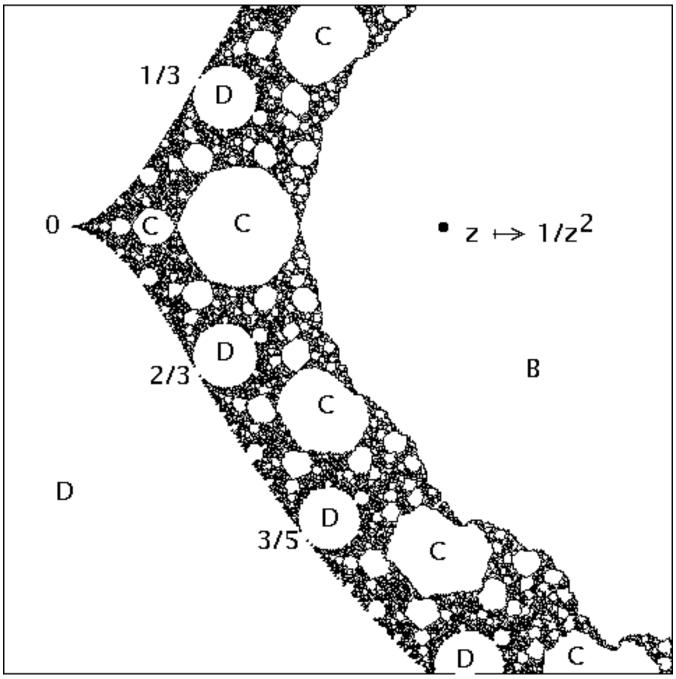

FIGURE 2. Bifurcation locus in $\mathrm{Per}_2(0)$.

unbounded type B components. Makienko [**6**] has obtained a degree-independent sufficient condition for unboundedness, loosely speaking the existence of a family of closed Poincaré geodesics on the basin quotient with lifts linking to separate the Julia set; see also [**16**]. On the other hand, there are type B maps which do not admit such a family: Pilgrim [**16**] cites the example $g(z) = (i\sqrt{3}/2)(z + (1/z))$ and describes its Julia set as an *almost Sierpinski carpet*. Such maps presumably lie in bounded hyperbolic components.

A good deal of what is known about hyperbolic quadratic rational maps—that Fatou components are usually Jordan domains [**17**], that polynomials can be mated if and only if they do not lie in conjugate limbs of the Mandelbrot set [**23**], that mating is discontinuous due to the existence of type D hyperbolic components whose closures are not homeomorphic to $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$ [**2**], that moduli space is isomorphic to $\mathbb{C}^2$—is valid with minor changes for higher degree *bicritical maps* possessing two maximally degenerate critical points. Much of the discussion here extends similarly, and Milnor has recently generalized Lemma 3 to this larger setting: if $[g_k]$ is unbounded then the eigenvalues of all but at most two fixed points tend to infinity [**14**]. However, it is not immediately apparent how best to adapt the brute-force calculations of §4, or better yet, how to replace them with a more conceptual approach applicable to other degenerating families.

6. *Intersection theory*

The results above yield preliminary information about the intersection theory at infinity of dynamically defined curves in moduli space. Milnor's isomorphism $j : \mathbf{rat}_2 \to \mathbb{C}^2$

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induces a natural compactification $\widehat{\mathbf{rat}}_2 \cong \mathbb{P}^2$. Following the discussion in [**13**] we identify the line at infinity $\mathcal{L}$ with the set of unordered triples $\{\alpha, \alpha^{-1}, \infty\}$ where $\alpha \in \widehat{\mathbb{C}}$, so that $\alpha + \alpha^{-1}$ is the limiting ratio of $Y/(X-2)$ in the coordinates of §3; see [**21**] for a treatment in the language of the geometric invariant theory. With this convention, an unbounded sequence $[g_k] \in \mathbf{rat}_2$ converges to the ideal point $\{\alpha, \alpha^{-1}, \infty\}$ if and only if $\{\alpha_k, \beta_k, \gamma_k\} \to \{\alpha, \alpha^{-1}, \infty\}$, where $\alpha_k, \beta_k, \gamma_k$ are the eigenvalues of the fixed points of g_k.

Recall that a *curve* in $\mathbb{P}^2$ may be defined as an equivalence class of non-constant homogeneous polynomials $h \in \mathbb{C}[W, X, Y]$, where $h \sim \tilde{h}$ when $h = \lambda\tilde{h}$ for some $\lambda \in \mathbb{C}^*$, so that a point $P \in \mathbb{P}^2$ with homogeneous coordinates $[w : x : y]$ lies on $C = \lceil h \rceil$ if and only if $h(w, x, y) = 0$; in this situation we write $P \in C$. The *degree* of $C = \lceil h \rceil$ is the natural number $\deg h$; an *algebraic family* of degree d curves parametrized by a variety Λ is a regular map $\Lambda \to \mathcal{C}_d$, where the set $\mathcal{C}_d$ of all degree d curves is naturally regarded as the projective space $\mathbb{P}^{d(d+3)/2}$. If h has no non-trivial factors then $C = \lceil h \rceil$ is said to be *irreducible*. An irreducible curve $\hat{C} = \lceil \hat{h} \rceil$ with $\hat{h}|h$ is a *component* of C, and curves C_1 and C_2 with no common component are said to *intersect properly*. Notice that C intersects $\mathcal{L}$ properly if and only if $\deg h(1, X, Y) = \deg h$. Curves C_1, C_2 which intersect properly have finitely many points in common, and each such point can be assigned an appropriate *intersection multiplicity* $\mathcal{I}_{C_1,C_2}(P) > 0$; by convention, $\mathcal{I}_{C_1,C_2}(P) = 0$ if $P \notin C_1$ or $P \notin C_2$. The *intersection cycle* is the formal sum

$$C_1 \bullet C_2 = \sum_{P \in \mathbb{P}^2} \mathcal{I}_{C_1,C_2}(P) \cdot P;$$

the *intersection cycle at infinity* is

$$C_1 \bullet_\infty C_2 = \sum_{P \in \mathcal{L}} \mathcal{I}_{C_1,C_2}(P) \cdot P.$$

Bezout's theorem asserts that the total intersection multiplicity is the product of the degrees $d_i = \deg C_i$, so that $C_1 \bullet C_2$ may be regarded as an element of the symmetric product

$$\mathcal{S}_{d_1 d_2} = \mathrm{Sym}^{d_1 d_2}(\mathbb{P}^2).$$

Moreover, $(C_1, C_2) \rightsquigarrow C_1 \bullet C_2$ yields a regular map

$$\mathcal{C}_{d_1} \times \mathcal{C}_{d_2} - \mathcal{E}_{d_1,d_2} \to \mathcal{S}_{d_1,d_2}$$

where $\mathcal{E}_{d_1,d_2}$ is the set of pairs of curves with a common component; see [**3**] for further details.

Consider the function $n \rightsquigarrow d(n)$ defined inductively by the relation

$$\sum_{m|n} d(m) = 2^{n-1};$$

equivalently, $d(n)$ is the number of period n hyperbolic components of the Mandelbrot set M. Milnor [**13**] has shown the following.

LEMMA 9. *For each $n \geq 1$ there is a algebraic family of curves*

$$\mathbb{C} \ni \rho \rightsquigarrow \mathrm{Per}_n(\rho) \in \mathcal{C}_{d(n)}$$

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uniquely determined by the condition that $[g] \in \mathrm{Per}_n(\rho)$ *for* $\rho \neq 1$ *if and only if* g *has an* n*-cycle with eigenvalue* ρ*. The curves* $\mathrm{Per}_n(1)$ *are reducible for* $n > 1$*, indeed*

$$\mathrm{Per}_n(1) = \mathrm{Per}_n^{\#}(1) \cup \bigcup_{1<q|n,(p,q)=1} \mathrm{Per}_{n/q}(e^{2\pi i p/q})$$

where the generic $[g] \in \mathrm{Per}_n^{\#}(1)$ *has an* n*-cycle of eigenvalue 1.*

Here are the defining polynomials for $n = 1, 2, 3$:

$$\begin{aligned}
\mathrm{Per}_1(\rho):&\quad \rho^3 W - \rho^2 X + \rho Y - X + 2W\\
\mathrm{Per}_2(\rho):&\quad \rho W - 2X - Y\\
\mathrm{Per}_3(\rho):&\quad \rho^2 W^3 - \rho(WX(2X+Y) + 3W^2X + 2W^3)\\
&\qquad + (X+Y)^2(2X+Y) - WX(X+2Y) + 12W^2X + 28W^3.
\end{aligned}$$

Note that $\mathrm{Per}_1(\rho) \bullet \mathcal{L} = \{\rho, \rho^{-1}, \infty\}$; the degeneration described in Proposition 2 takes place in a parameter space where

$$\infty_{p/q} = \{e^{2\pi i p/q}, e^{-2\pi i p/q}, \infty\} = \infty_{(q-p)/q}$$

has been blown up and replaced by a 2-fold branched cover of the line $\mathrm{Per}_1(1)$. Moreover,

$$\mathrm{Per}_2(\rho) \bullet \mathcal{L} = \infty_{1/2} \tag{23}$$

and

$$\mathrm{Per}_3(\rho) \bullet \mathcal{L} = \infty_{1/2} + 2 \cdot \infty_{1/3}. \tag{24}$$

Recall that $M = \{c : P_c(z) = z^2 + c$ has a connected Julia set$\}$ is the disjoint union of the cardioid $\heartsuit = \{c_\lambda = \frac{1}{2}\lambda - \frac{1}{4}\lambda^2 : \lambda \in \mathbb{D}\}$, the boundary points $c_{e^{2\pi i\theta}}$ for $\theta \in (\mathbb{R} - \mathbb{Q})/\mathbb{Z}$, and the closed *limbs*

$$L_{p/q} = \{c : P_c \text{ has a fixed point of combinatorial rotation number} p/q\}$$

for $p/q \in (\mathbb{Q} - \mathbb{Z})/\mathbb{Z}$; see [**4**] for precise definitions and proofs. It follows from standard deformation considerations [**8**] that each pair $(c, \alpha) \in M \times \mathbb{D}$ determines a unique class $[P_{c,\alpha}] \in \mathrm{Per}_1(\alpha)$ consisting of maps which are quasiconformally conjugate to P_c on a neighbourhood of the filled-in Julia set $K(P_c)$ through conjugacies with vanishing dilatation on $K(P_c)$. Thus, each $c \in M$ is the centre of a disc $\mathcal{D}_c = \{[P_{c,\alpha}] : \alpha \in \mathbb{D}\}$; moreover, $\mathcal{D}_c \cap \mathcal{D}_{\hat{c}} = \emptyset$ for $c \neq \hat{c}$, provided that at least one of $c, \hat{c}$ lies in the complement of $\heartsuit$. Petersen [**13, 15**] proved the following by a modulus estimate similar to the Yoccoz inequality.

LEMMA 10. *Let* $P_c(z) = z^2 + c$ *where* $c \in M$*. If* $\alpha_k \in \mathbb{D}$ *converges non-tangentially to* $e^{-2\pi i p/q} \neq 1$ *then* $[P_{c_k,\alpha_k}] \to \infty_{p/q} \in \mathcal{L}$ *uniformly for* $c_k \in L_{p/q}$.

Suppose in particular that c belongs to a hyperbolic component $H \subset L_{p/q}$, say $[P_c] \in \mathrm{Per}_n(\rho)$ for some $n > 1$ and $\rho \in \mathbb{D}$, and let $\omega = e^{-2\pi i p/q}$. As

$$\mathcal{Y} = \{y \in \widehat{\mathbf{rat}}_2 : [P_{c,\alpha_k}] \to y \text{ for some } \alpha_k \to \omega\}$$

is connected, but $\mathcal{Y} \cap \mathbf{rat}_2$ is contained in the finite set $\mathrm{Per}_n(\rho) \cap \mathrm{Per}_1(\omega)$, it follows from Proposition 1 that $[P_{c,\alpha_k}] \to \infty_{p/q}$ for any $\alpha_k \in \mathbb{D}$ with $\alpha_k \to \omega$, and that $\alpha_k - \omega = O(\sqrt{\epsilon_k})$ where $\epsilon_k = 1 - \alpha_k\beta_k = (2(1 - \Re\omega) + o(1))/\gamma_k$. It similarly follows from Propositions 2 and 3 that the connected set

$$\mathcal{T} = \left\{\tau \in \mathbb{C} : \frac{\bar{\omega}\alpha_k - 1}{\sqrt{\epsilon_k}} \to \tau \text{ for some } \alpha_k \to \omega\right\}$$

is discrete, whence $\mathcal{T} = \{\tau_c\}$ for some $\tau_c \in \mathbb{C}^*$; as $\Re(\tau_c^2) > 0$ by Lemma 2, the image of $\mathbb{D} \ni \alpha \rightsquigarrow \gamma = (2(1 - \Re\omega)\tau_c^2 + o(1))/(\bar{\omega}\alpha - 1)^2$ contains some right half-plane. Consequently, $\mathcal{D}_c \cap \mathrm{Per}_1(\gamma) \neq \emptyset$ for any c in a hyperbolic component of M and any sufficiently large $\gamma > 0$, so that $\mathrm{Per}_n(\rho)$ and $\mathrm{Per}_1(\gamma)$ have at least

$$D_{p/q}(n) = \begin{cases} d_{p/q}(n), & \text{for } p/q = \frac{1}{2} \\ 2d_{p/q}(n), & \text{otherwise} \end{cases}$$

intersections near $\infty_{p/q}$, where $d_{p/q}(n)$ is the number of period n hyperbolic components in $L_{p/q}$. As the local intersection multiplicities are stable under perturbation, while the total intersection multiplicity is

$$d(n) = \sum_{\substack{1 \le p < q \le n \\ (p,q)=1}} d_{p/q}(n)$$

by Bezout's theorem, it follows that $\mathcal{I}_{\mathrm{Per}_n(\rho),\mathcal{L}}(\infty_{p/q}) = D_{p/q}(n)$, at least for $\rho \in \mathbb{D}$. In view of Proposition 1 and the continuity of $\rho \rightsquigarrow \mathrm{Per}_n(\rho) \bullet \mathcal{L}$, these considerations prove the following.

PROPOSITION 6. *Let n be an integer greater than one, and let $\rho \in \mathbb{C}$. Then*

$$\mathrm{Per}_n(\rho) \bullet \mathcal{L} = \sum_{\substack{1 \le p < q \le n \\ (p,q)=1}} d_{p/q}(n) \cdot \infty_{p/q}.$$

The number of branches of $\mathrm{Per}_n(\rho)$ near $\mathcal{L}$ is studied in [**22**]. (It is claimed to be $d(n)$. This is not, in fact, correct, but the method used apparently gives a formula for the number of branches: see [**19**].)

It is not hard to show that the intersection of $\mathrm{Per}_{n^+}(\rho^+)$ and $\mathrm{Per}_{n^-}(\rho^-)$ is generically proper [**2**], so it is somewhat surprising that there are non-trivial exceptions: for example,

$$\mathrm{Per}_2(-3) = \mathrm{Per}_3^{\#}(1) \tag{25}$$

as observed in [**13**]. This coincidence yields a short independent proof of Theorem 1 in the special case $(n^+, n^-) = (2, 3)$. Recall that a quadratic rational map has precisely two 3-cycles counting multiplicity, whence

$$\mathrm{Per}_2(-3) \bullet \mathrm{Per}_3(\rho^-) = \mathrm{Per}_3^{\#}(1) \bullet \mathrm{Per}_3(\rho^-) = 3 \cdot \infty_{1/2}$$

for $\rho^- \neq 1$ by (23), (24), (25) and Bezout's theorem; thus, $\mathrm{Per}_2(-3)$ and $\mathrm{Per}_3(\rho^-)$ are tangent at $\infty_{1/2}$. In view of the transversality of distinct lines $\mathrm{Per}_2(\rho^+)$, the curves $\mathrm{Per}_2(\rho^+)$ and $\mathrm{Per}_3(\rho^-)$ are transverse at $\infty_{1/2}$ provided that they intersect properly, so

$$\mathrm{Per}_2(\rho^+) \bullet_\infty \mathrm{Per}_3(\rho^-) = \infty_{1/2} \tag{26}$$

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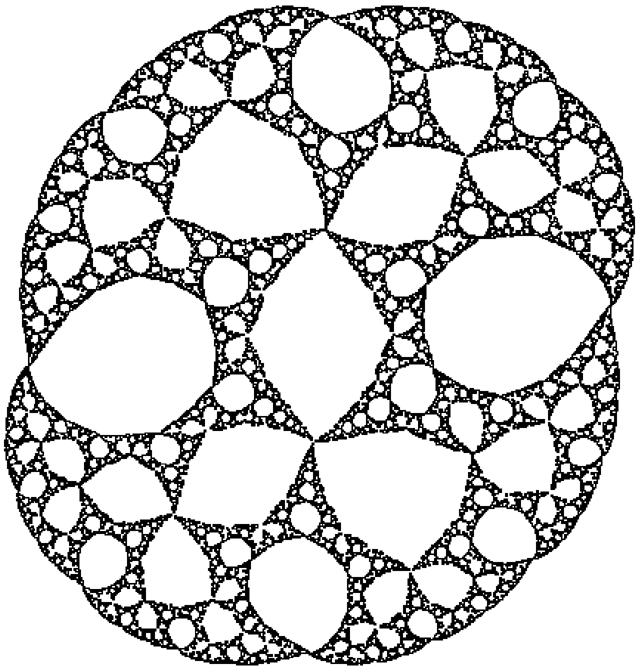

FIGURE 3. $J(f)$ for $f \in \mathbf{RAT}_2$ with critical points of periods 2 and 3.

for $(\rho^+, \rho^-) \neq (-3, 1)$; in particular, for $(\rho^+, \rho^-) \in \overline{\mathbb{D}} \times \overline{\mathbb{D}}$ the points in

$$\mathrm{Per}_2(\rho^+) \bullet \mathrm{Per}_3(\rho^-) - \mathrm{Per}_2(\rho^+) \bullet_\infty \mathrm{Per}_3(\rho^-)$$

are uniformly bounded away from $\mathcal{L}$. These finite intersection points fill out two hyperbolic components, a complex-conjugate pair obtained by mating the unique period 2 component in the Mandelbrot set M with the period 3 components disjoint from the real axis. It is clear from Figure 3 that the corresponding Julia sets are not Sierpinski carpets, but they are almost Sierpinski carpets in the sense of Pilgrim [**16**].

Conversely, we may apply Theorem 1 to deduce transversality principles generalizing (26). Recall that if $n > 1$ then $\mathrm{Per}_n(\rho)$ intersects $\mathcal{L}$ only at points $\infty_{p/q}$ with $1 < q \leq n$. Consequently, if $n^\pm > 1$ then there are natural numbers $I_{p/q}(n^+, n^-)$ such that

$$\mathrm{Per}_{n^+}(\rho^+) \bullet_\infty \mathrm{Per}_{n^-}(\rho^-) = \sum_{\substack{1 \leq p < q \leq \min(n^+, n^-) \\ (p,q)=1}} I_{p/q}(n^+, n^-) \cdot \infty_{p/q}$$

for every pair (ρ^+, ρ^-) in some Zariski open subset $\mathcal{U}(n^+, n^-) \subseteq \mathbb{C}^2$. In view of Theorem 1, if $n^+ \neq n^-$, or if $n^+ = n^-$ but $\rho^+ \neq \rho^-$, then $\mathcal{U}(n^+, n^-) \ni (\rho^+, \rho^-)$ for $\rho^\pm \in \overline{\mathbb{D}}$; if $n^+ = n^-$ and $\rho^+ = \rho^- \in \overline{\mathbb{D}}$ then this relation remains valid provided that we interpret the left-hand side as $\mathrm{Per}_{n^+}(\rho^+) \bullet_\infty \mathrm{Env}_{n^+}$, where Env_n is the *envelope* of the family $\rho \rightsquigarrow \mathrm{Per}_n(\rho)$. Heuristic considerations supported by calculations in [**22**] suggest that

$$I_{p/q}(n^+, n^-) = \sum_{(H^+,H^-)\in L_{p/q}(n^+)\times L_{p/q}(n^-)} \iota(H^+, H^-)$$

where $\iota(H^+, H^-)$ measures the mutual combinatorial depth of $H^\pm$ in M; the language of *internal addresses* [**5**] may be useful in the formulation and proof of this assertion.

Acknowledgements. The author wishes to thank Jack Milnor and Kevin Pilgrim for sharing their insight. Further thanks are due to Xavier Buff and Jack Milnor for graciously providing the graphics, and to Scott Sutherland for tireless technical assistance. We are additionally grateful to the referee for several suggestions which clarified the presentation of various supporting estimates in §4 and alerted us to an expository gap in §6. This research was generously supported by the Institute for Mathematical Sciences at Stony Brook.

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