CHARACTERIZATIONS OF FINITE PROJECTIVE AND AFFINE SPACES

WILLIAM M. KANTOR

1. Introduction. A well-known result of Dembowski and Wagner (**4**) characterizes the designs of points and hyperplanes of finite projective spaces among all symmetric designs. By passing to a dual situation and approaching this idea from a different direction, we shall obtain common characterizations of finite projective and affine spaces. Our principal result is the following.

THEOREM 1. *A finite incidence structure is isomorphic to the design of points and hyperplanes of a finite projective or affine space of dimension greater than or equal to* 4 *if and only if there are positive integers v, k, and μ with $\mu > 1$ and $(\mu - 1)(v - k) \neq (k - \mu)^2$ such that the following assumptions hold.*

(I) *Every block is on k points, and every two intersecting blocks are on μ common points.*

(II) *Given a point and two distinct blocks, there is a block containing both the point and the intersection of the blocks.*

(III) *Given two distinct points p and q, there is a block on p but not on q.*

(IV) *There are v points, and $v - 2 \geqq k > \mu$.*

Here, (II) is the key assumption. From the point of view of the foundations of geometry, Theorem 1 amounts to a "natural" system of axioms whose only models are the finite projective and affine spaces of dimension $\geqq 4$. This result should be compared with the Dembowski-Wagner Theorem, the dual of which may be stated as follows.

DEMBOWSKI-WAGNER THEOREM (**4**). *A finite incidence structure $\mathscr{I}$ is isomorphic to the design of points and hyperplanes of a finite projective space of dimension $\geqq 2$ if and only if $\mathscr{I}$ satisfies* (I)–(IV) *together with the following condition.*

(0) *Every two blocks meet.*

Originally, this result was stated in a weaker form, which required that $\mathscr{I}$ be a symmetric design (cf. **4**). However, the original proof yields the above stronger version without much difficulty.

In Theorem 1, the restriction to dimension $\geqq 4$ is essential. In fact, there is an example due to Witt (**12**) of a design satisfying our axioms (I)–(IV) with $v = 22$, $k = 6$, $\mu = 2$ (cf. §4). Some further generalizations of the Dembowski-Wagner Theorem avoiding this dimensionality restriction will be mentioned

Received February 24, 1967.

as corollaries of Theorem 1. On the other hand, Theorem 3 provides a common characterization of finite projective and affine spaces and of Witt's design. This theorem may also be stated as the following purely group-theoretic result, providing a characterization of the Mathieu group M_{22}.

THEOREM 2. *Let* Γ *be a 2-transitive permutation group on a finite set. Let* $\Delta < \Gamma$ *fix k points and be transitive on the set of points it moves. Let* $k > \mu > 1$. *If each pair of conjugates of* Δ *have in common either* 0 *or* μ *fixed points, then* Γ *is similar to one of the following groups, in its usual permutation representation.*

(i) M_{22} *or* $\mathrm{Aut}(M_{22})$.

(ii) *A subgroup of* $\mathrm{P\Gamma L}(d, q)$, *containing* $\mathrm{PSL}(d, q)$, *for some* $d \geqq 4$ *and some q.*

(iii) *A group of collineations of a finite affine space containing all perspectivities with centres on the hyperplane at infinity.*

A Jordan group is a permutation group satisfying the hypotheses of the first two sentences of Theorem 2 (cf. Wielandt (**11**, pp. 32–34); Hall (**5**, p. 66)). Jordan groups were first studied geometrically by Hall (**6**), whose definition differs slightly from the above as he requires that Γ not be 3-transitive. Other results concerning both these groups and geometric generalizations of them will appear elsewhere.

Theorems 4 and 5 provide other characterizations of the designs of points and hyperplanes of finite projective spaces.

I am indebted to Dr. P. Dembowski for stimulating my interest in results of the Dembowski-Wagner type. The support of the National Aeronautics and Space Administration during the preparation of this paper is gratefully acknowledged.

2. Definitions. An incidence structure is usually described in terms of a set of points, a set of blocks, and an incidence relation I. In the incidence structures considered here, it will be possible to distinguish blocks by means of the sets of points incident with them. Hence, the relation I will frequently be identified with set-theoretic inclusion. Points are generally denoted by p, q, r, and s, and blocks by B, C, and D. The dual of an incidence structure is the incidence structure derived by reversing the roles of points and blocks, while retaining the same incidence relation.

Let $\mathscr{I}$ be any incidence structure. If $p \neq q$ are points such that there is a block I p and q, the intersection of all such blocks is called the line pq. If r is a point not belonging to pq such that there is a block I p, q, and r, the intersection of all such blocks is called the plane pqr. (These definitions differ slightly from those of (**2**, **3**, and **4**).) Note that neither lines nor planes need be independent of the choice of points used to define them. The most one can say in general is that $p' \in pq - \{p\}$ implies $pq \subseteq p'q$, and an analogous inclusion for planes. If B and C are two intersecting blocks, then $B \cap C$ is called a *flat*; thus, a flat is a non-empty set of points. Flats will be denoted by F or G. Assumptions (I) and (II) of Theorem 1 thus deal with flats.

A (balanced incomplete block) design is a finite incidence structure $\mathscr{I}$ such that each point (block) is on r (k) blocks (points), and every two distinct points are on λ blocks. Here, v (the number of points), b (the number of blocks), k, r, and λ are positive integers, the "parameters" of the design. They are assumed to satisfy $v > k + 1$ and $b > r > \lambda$, in addition to $bk = vr$ and $\lambda(v - 1) = r(k - 1)$. $\mathscr{I}$ is called a symmetric design if $r = k$; in this case, every two distinct blocks are on λ points (cf. Ryser (**9**, p. 103)). There is a unique line joining any two distinct points of a design, and every flat of a symmetric design has λ points.

An isomorphism between two incidence structures is a 1-1 incidence-preserving correspondence mapping points to points and blocks to blocks. An automorphism of an incidence structure $\mathscr{I}$ is an isomorphism of $\mathscr{I}$ with itself.

3. Proof of Theorem 1. It is easy to prove that finite affine and projective spaces satisfy (I)–(IV). The proof of the principal assertion of Theorem 1 will require several preliminary results. Throughout our argument, $\mathscr{I}$ will denote an arbitrary finite incidence structure satisfying (I)–(IV). The proof will eventually depend upon induction on μ. Only at the end of the proof will it be assumed that $\mu > 1$ and $(\mu - 1)(v - k) \neq (k - \mu)^2$. The following obvious fact implies that the hypothesis $\mu > 1$ in Theorem 1 is essential.

LEMMA 1. *$\mathscr{I}$ satisfies* (I)–(IV) *with $\mu = 1$ if and only if $\mathscr{I}$ is a design with $\lambda = 1$.*

(3.1) *If two distinct blocks contain a flat, then that flat is their intersection. There is a unique block $\langle p, F\rangle$ containing a given flat F and a given point $p \notin F$.*

Proof. This is clear from (I) and (II).

(3.2) *Every flat is contained in $\rho = (v - \mu)/(k - \mu)$ blocks.*

Proof. If F is a flat and x is the number of blocks containing F, then (3.1) implies that the total number of points is $v = |F| + x(k - |F|)$. (Here, $|F|$ denotes the cardinality of F.)

(3.3) *If $\mu > 1$ and F is a flat, there is a block $B \not\supseteq F$ such that $F \cap B \neq \emptyset$. For any such B, $|F \cap B| = \mu - (k - \mu)^2/(v - k)$.*

Proof. Let p and q be distinct points of F. By (III), there is a block $B \mathrel{I} p$ such that $B \not\mathrel{I} q$, and then $p \in F \cap B$ but $q \notin F \cap B$. For any such B, (I), (3.1), and (3.2) show that $k = |B| = |F \cap B| + \rho(\mu - |F \cap B|)$. Since $\rho > 1$, this may be solved for $|F \cap B|$.

Definition. If $\mu > 1$ and B is a block of $\mathscr{I}$, then $\mathscr{I}(B)$ will denote the incidence structure consisting of the points of B and the distinct (as sets)

https://doi.org/10.4153/CJM-1969-007-9 Published online by Cambridge University Press

flats contained in B, with incidence induced by set-theoretic inclusion. According to (3.1), each such flat may be written in the form $B \cap C$ for a suitable block C. A "block" of $\mathscr{I}(B)$ is thus a flat of $\mathscr{I}$. To avoid confusion, we shall never use the term "flat" in $\mathscr{I}(B)$ in the sense of §2; instead, we shall consider an intersection of two flats contained in B. The expressions "a block of $\mathscr{I}(B)$" and "a flat in B" will be considered as having the same meaning.

LEMMA 2. *If $\mu > 1$ and B is a block of $\mathscr{I}$, then $\mathscr{I}(B)$ satisfies* (I)–(IV) *with $v(B) = k$, $k(B) = \mu$, and*

$$\mu(B) = \mu - (k - \mu)^2/(v - k) < \mu.$$

Proof. If F and G are distinct flats in B, let $F = B \cap C$ and $G = B \cap D$ for suitable blocks C and D. If F and G meet, then $B \cap (C \cap D) = F \cap G \neq \emptyset$. $C \cap D \not\subseteq B$ (otherwise, $F = B \cap C \cap D = G$ by (3.1)), so that by (3.3),

$$|F \cap G| = |B \cap (C \cap D)| = \mu - (k - \mu)^2/(v - k),$$

proving (I). If p is a point of B, and B' is a block containing both p and $C \cap D$, then $B \cap B'$ is a flat in B containing both p and $F \cap G$; this proves (II). $\mu(B) < \mu = k(B)$ since $\mu < k$. If $v(B) - 2$ were less than $k(B)$, then $k - 1 \leqq \mu$ would imply that $k - 1 = \mu$, and then $\mu(B)$ could not be integral. Hence (IV) holds, and (III) for $\mathscr{I}(B)$ is a consequence of (III) for $\mathscr{I}$.

Lemma 2 is the key to the proof of Theorem 1, as it permits the use of induction on μ.

(3.4) *There is at least one block incident with each of* $\min(\mu + 1, 3)$ *given, distinct points.*

Proof. Let p and q be distinct points. There is a flat containing p; for otherwise, there is at most one block I p, contradicting (III) and $k > 1$. Hence, by (II), there is a block I p and q. This argument, applied to $\mathscr{I}(B)$, provides a flat containing p and q (cf. Lemma 2). Any third point r may then, by (II), be joined to this flat to yield a block I p, q, and r.

(3.5) *There is a unique line containing any two given distinct points. All lines contain the same number $m \geqq 2$ of points.*

Proof. If $\mu = 1$, this is trivial. Assume that $\mu > 1$, and that (3.5) holds for all finite incidence structures satisfying (I)–(IV) but having smaller values of μ than $\mathscr{I}$ has. Let p, q, and r be three distinct points of $\mathscr{I}$. By (3.4), there is a block B I p, q, r, and a flat contained in B and I p, q. Thus, pq consists of the same set of points when regarded as a line of $\mathscr{I}$ or of $\mathscr{I}(B)$. Since (3.5) holds for $\mathscr{I}(B)$, $|pq| = |pr|$. Thus, all lines of $\mathscr{I}$ have the same number of points. Hence, if also $r \in pq$, then $pq = pr$, since, certainly, $pr \supseteq pq$ (cf. the definition of lines).

Definition. If $\mu > 1$ and p is a point of $\mathscr{I}$, then $\mathscr{I}(p)$ will denote the incidence structure whose points are the distinct lines through p and whose blocks are the blocks I p, with incidence induced by set-theoretic inclusion in $\mathscr{I}$.

LEMMA 3. *If $\mu > 1$, then all $\mathscr{I}(p)$, p a point of $\mathscr{I}$, are either isomorphic projective spaces of dimension >2 or projective planes of the same order.*

Proof. According to the Dembowski-Wagner Theorem, in order to prove that $\mathscr{I}(p)$ is a projective space it is only necessary to verify that $\mathscr{I}(p)$ satisfies (0)–(IV). (0) and (II) follow from $\mu > 1$ and (II) for $\mathscr{I}$, respectively. To prove (I), note that (3.5) and (I) for $\mathscr{I}$ together imply that the number of lines containing p is $(v-1)/(m-1)$, and the number of lines containing p and contained in a block or flat containing p is $(k-1)/(m-1)$ or $(\mu-1)/(m-1)$, respectively. This shows, incidentally, that the parameters for $\mathscr{I}(p)$ are independent of p, and hence that Lemma 3 will be proven once we have shown that the $\mathscr{I}(p)$ are projective spaces.

The first part of (3.5) yields (III). $k > \mu$ implies that

$$(k-1)/(m-1) > (\mu-1)/(m-1).$$

Let B be a block of $\mathscr{I}$, not necessarily I p, and let $q, r \not\mathrm{I}\, B$ $(q \neq r)$ and s I B such that $s \notin qr$; this is possible since $v \geqq k+2$ and $|B \cap qr| \leqq 1 < k$ (cf. (3.5)). Considering $\mathscr{I}(s)$ shows that

$$(v-1)/(m-1) \geqq (k-1)/(m-1) + 2.$$

This proves (IV) for $\mathscr{I}(p)$.

(3.6) *For each block B, $\mathscr{I}(B)$ is a design whose parameters are independent of the choice of B. If B contains $b(B)$ flats, and there are r blocks on each point of $\mathscr{I}$, then the number of blocks incident with a point $p \not\mathrm{I}\, B$ and not meeting B is $r - b(B)$; hence, this number is independent of the choices of p and B.*

Proof. Lemmas 1, 2, and 3.

It is now possible to complete the proof of Theorem 1. Assume that μ and $\mu(B) > 1$. Fix a point p of $\mathscr{I}$. As $\mu(B) > 1$, the intersection of two blocks containing p contains a line not through p, so that by Lemma 3 it may be assumed that $\mathscr{I}(p)$ is the design of points and hyperplanes of $PG(d-1, q)$ for some $d \geqq 4$ and some q. Let $B \not\mathrm{I}\, p$. $\mathscr{I}(B)$ may be isomorphically embedded into the projective space $\mathscr{I}(p)$ by mapping $s \rightarrow ps$ and $F \rightarrow \langle p, F\rangle$ for each point s I B and each flat $F \subset B$. Lines of $\mathscr{I}(B)$ are mapped into lines of $\mathscr{I}(p)$. By (3.6) and the Dembowski-Wagner Theorem it may be assumed that $r - b(B) > 0$, otherwise, $\mathscr{I}$ is isomorphic to $PG(d, q)$. Then the set of points of $\mathscr{I}(B)$ is embedded in $AG(d-1, q)$ as a set S of $k > 1$ points such that each hyperplane (line) meets S in 0 or μ (0 or m) points. k/μ hyperplanes from each parallel class of hyperplanes meet S. However,

$$k - 1 = (q^{d-1} - 1)(q-1)^{-1}(m-1)$$

and $\mu - 1 = (q^{d-2}-1)(q-1)^{-1}(m-1)$, so that $\mu | (q\mu - k) = q - m$. Since $m \geqq 2$ and $d \geqq 4$,

$$\mu \geqq (q^{d-2}-1)(q-1)^{-1} > q > q - m,$$

https://doi.org/10.4153/CJM-1969-007-9 Published online by Cambridge University Press

whence $q = m$, $k = q^{d-1}$, and S consists of all the points of AG$(d - 1, q)$.

Hence, each $\mathscr{I}(B)$ is isomorphic to the design of points and hyperplanes of AG$(d - 1, q)$. If two lines of $\mathscr{I}$ are called "parallel" if and only if they lie in a common block B and are parallel lines of $\mathscr{I}(B)$, then the parallel postulate holds for lines (by (3.4)). Also, parallelism is an equivalence relation among lines. For, by $d \geqq 4$ and (3.4), two parallel lines lie in a unique plane, which lies in a block containing a given third line parallel to one of the first two. In the usual manner, adjoin ideal points to $\mathscr{I}$ (an ideal point is an equivalence class of parallel lines) each of which is incident with precisely those blocks of $\mathscr{I}$ containing a member of the equivalence class; and adjoin an ideal block incident with every ideal point but with no point of $\mathscr{I}$. The result is another incidence structure $\mathscr{I}^*$. $\mathscr{I}^*(B^*)$ is isomorphic to the design of points and hyperplanes of PG$(d - 1, q)$ for each block B^* of $\mathscr{I}^*$. Thus, since $d - 1 > 2$, the Veblen and Young axioms (**10**) hold for $\mathscr{I}^*$, whence $\mathscr{I}$ is an affine space.

The following two corollaries are consequences of the preceding proof (cf. (3.6)).

COROLLARY 1. *A finite incidence structure $\mathscr{I}$ is isomorphic to the design of points and hyperplanes of a finite projective space if and only if $\mathscr{I}$ satisfies* (I)–(IV) *and some block meets every other block.*

COROLLARY 2. *A finite incidence structure $\mathscr{I}$ is isomorphic to the design of points and hyperplanes of a finite affine space if and only if $\mathscr{I}$ satisfies* (I)–(IV) *and for some non-incident point-block pair there is precisely one block not meeting the given block and incident with the given point.*

COROLLARY 3. *A finite incidence structure $\mathscr{I}$ is isomorphic to the design of points and hyperplanes of a finite projective or affine space of dimension $\geqq 2$ if and only if $\mathscr{I}$ satisfies* (I)–(IV) *and some line contains at least* $(v - k)/(k - \mu)$ *points.*

Proof. Since $k - 1 = (q^{d-1} - 1)(q - 1)^{-1}(m - 1)$,

$$\mu - 1 = (q^{d-2} - 1)(q - 1)^{-1}(m - 1),$$

$r = (q^d - 1)(q - 1)^{-1}$, and $\lambda = (q^{d-1} - 1)(q - 1)^{-1}$, it follows that

$$m \geqq (v - k)/(k - \mu) = q.$$

Hence, $m = q$ or $q + 1$. As in the proof of the last part of Theorem 1, this implies that $\mathscr{I}$ is an affine or projective space, respectively.

4. The Witt space $\mathscr{W}_{22}$. There is an incidence structure $\mathscr{W}_{22}$ satisfying (I)–(IV) with $v = 22$, $k = 6$, and $\mu = 2$. This was discovered by Witt (**12**), and will therefore be called the Witt space. $\mathscr{W}_{22}$ has the Mathieu group M_{22} as a 3-transitive automorphism group. The Witt space may be described in terms of $\mathfrak{p} = \mathrm{PG}(2, 4)$ and M_{22} as follows. The points of $\mathscr{W}_{22}$ are the points of $\mathfrak{p}$ together with a new point ∞. The blocks of $\mathscr{W}_{22}$ are (as point-sets) (i) the lines of $\mathfrak{p}$ with ∞ adjoined to each line, and (ii) a single orbit of complete ovals of $\mathfrak{p}$ under $\mathrm{PSL}(3, 4)$ (a complete oval of $\mathfrak{p}$ is a set of six points, no three of which are collinear). $\mathscr{W}_{22}$ is then a design, with full automorphism group $\mathrm{Aut}(M_{22})$, such that any three distinct points are on a unique block. If two distinct blocks contain ∞, they certainly have another common point. The transitivity of M_{22} shows that two intersecting blocks will always have precisely two common points. Hence, (I)–(IV) hold in the Witt space.

It seems unlikely that there are any finite incidence structures, other than the projective, affine, or Witt spaces, satisfying (I)–(IV) with $\mu > 1 = \mu(B)$. Suppose that $\mathscr{I}$ satisfies these conditions, but is neither a projective nor an affine space. If p and p' are distinct points of a flat F, then $pp' \subseteq F$; conversely, if $B \mathrel{\mathrm{I}} p, p'$, then $|B \cap F| \geqq 2 > \mu(B)$ shows, by (3.3), that $B \supseteq F$, yielding the reverse inclusion $pp' \supseteq F$. Thus, flats are lines, and hence blocks are planes by (II). There is a unique plane containing three given non-collinear points. Distinct planes either do not meet or meet in a line. Every line contains $m = \mu$ points. The lines and planes through a point p form a projective plane $\mathscr{I}(p)$ whose order q is independent of the choice of p. $\mathscr{I}$ is a design with $\lambda = q + 1$. By Theorem 1, Corollary 1, there are non-intersecting planes. If B is a plane and l a line not meeting B, then the number of planes containing l and meeting B is k/m (by (I) and (3.1)). Hence, the parameters of $\mathscr{I}$ satisfy the following conditions.

(4.1) $v - 1 = (q^2 + q + 1)(m - 1)$, $k - 1 = (q + 1)(m - 1)$, $m|k$, and $m|q$.

(4.2) *If q is a prime power, then $q = m^2$.*

Proof. The number of planes is $v(q^2 + q + 1)/k$. Since $(q + 1)k - v = qm$, $k/m = q + 1 - q/m$ divides $q(q^2 + q + 1)$. $q > m$ by Theorem 1, Corollary 3; thus $(k/m, q) = 1$. Therefore k/m divides $q^2 + q + 1$. As

$$(q^2 + q + 1) - (q + 1)(k/m) = (q/m)(q + 1 - m),$$

it follows similarly that $(q + 1 - q/m)|(q + 1 - m)$. If $m \neq q/m$, then $2(q + 1 - q/m) \leqq q + 1 - m$, whence $q + 1 \leqq 2(q/m) - m < q$, which is impossible.

(4.3) *If $m = 2$, or q is even and $m = q/2$, then $\mathscr{I}$ is isomorphic to $\mathscr{W}_{22}$.*

Proof. A simple computation following the lines of the proof of (4.2) shows that $q = 4$. Then $v = 22$ and $k = 6$. That $\mathscr{I}$ and $\mathscr{W}_{22}$ are isomorphic now follows from (**13**, *Satz* 4).

(4.4) *If p and B are a non-incident point and plane, projection from p of $\mathscr{I}(B)$ into $\mathscr{I}(p)$ embeds $\mathscr{I}(B)$ into the projective plane $\mathscr{I}(p)$ as a non-empty* *set S of points such that each line of $\mathscr{I}(p)$ either does not meet S or meets S in precisely $m \geqq 2$ points.*

This is clear from the properties of $\mathscr{I}$ (cf. §3). It seems likely that the only possible sets of points of a projective plane of order q satisfying the condition stated in (4.4) are (i) the whole plane, (ii) the set of points not on a line, (iii) the set of points of a complete oval in a plane of even order, or (iv) the dual of the set of lines not meeting such a complete oval. In case (iii), $m = 2$, while in case (iv), $m = q/2$. That cases (i) and (ii) correspond to projective (affine) spaces is essentially the content of Theorem 1, Corollary 3.

THEOREM 3. *Let $\mathscr{I}$ be a design satisfying* (I), (III), *and* (IV), *with $\mu > 1$. Let Γ be an automorphism group of $\mathscr{I}$ such that the pointwise stabilizer $\Gamma(B)$ of each block B is transitive on the points off B. Then either $\mathscr{I}$ is isomorphic to the design of points and hyperplanes of a finite projective or affine space of dimension $\geqq 3$ and Γ contains the little projective group or the group of all perspectivities with centres on the hyperplane at infinity, respectively; or $\mathscr{I}$ is isomorphic to $\mathscr{W}_{22}$ and $\Gamma \approx M_{22}$ or* $\mathrm{Aut}(M_{22})$.

Proof. Let B and C be distinct blocks and $p \in B \cap C = F$. Let $q \in C - F$, and $q' \not I B, C$. Then $\Gamma(B)$ contains an element moving q to q', hence moving C to a block containing q' and F. Thus, (II) holds. Hence, by Theorem 1, in the proof that $\mathscr{I}$ is one of the desired designs, it may be assumed that $\mathscr{I}$ satisfies the conditions stated at the beginning of this section.

Let D be a plane, $p \in D$, and B a plane meeting D in a line not containing p. $\Gamma(D)$ induces a collineation group of $\mathscr{I}(p)$ which contains the full translation group Δ with respect to D (André (**1**); Pickert (**8**, pp. 313–314)). Hence, $\mathscr{I}(p)$ is a translation plane with respect to D, and therefore has prime power order q (**5**, Chapter 20; **8**). $\Gamma(D)_B$ is transitive on $B - B \cap D$. Using (4.1) and (4.4), $|\Delta_D| = k - m = q(m - 1)$. As $|\Delta| = q^2$, (4.2) and (4.3) show that $m = 2$, $q = 4$, and $\mathscr{I}$ is isomorphic to $\mathscr{W}_{22}$.

The description of Γ in the case of affine spaces is trivial, while for projective spaces the description given follows from a theorem of Lüneburg (**7**, *Satz* 8). If $\mathscr{I}$ is $\mathscr{W}_{22}$, then $\mathrm{Aut}(M_{22})$ is the automorphism group of $\mathscr{I}$ (Witt (**12**, *Satz* 7)). As B varies over the blocks of $\mathscr{I}$, $\Gamma(B)$ varies over a class of conjugate subgroups of M_{22}. The group generated by these groups is contained in Γ and normal in M_{22}. Since M_{22} is simple, Γ is M_{22} or $\mathrm{Aut}(M_{22})$.

Theorem 2 follows from Theorem 3 by letting B be the set of points fixed by Δ.

5. Another characterization of projective spaces. In (**7**), Lüneburg considered symmetric designs $\mathscr{I}$ with the property that some block is met by all lines. In order to prove that $\mathscr{I}$ is a projective space, he assumed the existence of a large number of automorphisms of a very special type (cf. Theorem 5). This latter assumption is not, however, needed.

THEOREM 4. *Let $\mathscr{I}$ be a design satisfying the dual of* (III). *$\mathscr{I}$ is isomorphic to the design of points and hyperplanes of a finite projective space if and only if some block of $\mathscr{I}$ is met by every line of $\mathscr{I}$.*

Proof. If $\lambda = 1$, this is trivial. Hence, assume that $\lambda > 1$. Since $\mathscr{I}$ is a design, there is a unique line joining any two given, distinct points. Let B be a block met by every line.

(5.1) *Every line not contained in B meets every block.*

Proof (Lüneburg (**7**)). Let $D \neq B$ be a block. Lines not contained in B but contained in $B \cup D$ certainly meet D. Suppose that $p \not I B, D$. Since a line through p meets B and D at most once, projection of the points of D into B from p is 1-1, hence onto. Thus, every line through p meets D, since the line meets B.

(5.2) *Every line not contained in B contains $m = (b - \lambda)/(r - \lambda) > 2$ points. Every plane not contained in B is contained in $\lambda - (r - \lambda)^2/(b - r)$ blocks. There is a unique plane containing any three given non-collinear points not all on B.*

Proof. Since $v - 2 \geqq k$, some such line must, by hypothesis, contain at least three points. The remaining assertions are proved as in (3.2) and (3.3).

(5.3) *Let q_1q_2 and r_1r_2 be distinct lines contained in a plane E, neither of which is contained in B. If $B \cap E$ is a line, then q_1q_2 and r_1r_2 meet.*

Proof (Dembowski and Wagner (**4**)). Let $r_1 \notin q_1q_2$. If D is a block containing q_1q_2 and $\not I\, r_1$, then $D \cap E \cap B = q_1q_2 \cap B$ implies that $D \cap E = q_1q_2$. By (5.1), $q_1q_2 \cap r_1r_2 = D \cap r_1r_2 \neq \emptyset$.

Let p be a point not on B. $\mathscr{I}(p)$, as defined in § 3, is a design by (5.2); for example, $k(p) = (k - 1)/(m - 1)$ and $\lambda(p) = \lambda - (r - \lambda)^2/(b - r)$. By (5.1) and $m > 2$, each plane on p contains a line not on p and not contained in B which meets all blocks I p. The Dembowski-Wagner Theorem then implies that $\mathscr{I}(p)$ is a projective space. As the number of lines on p is both k and $v(p) = b(p) = r$, $\mathscr{I}$ is symmetric. Moreover, the incidence structure $\mathscr{I}_B$ of points $\not I\, B$ and blocks other than B satisfies (I)–(IV).

If $\lambda(p) > 1$, then $\mathscr{I}_B$ is an affine space. If $\lambda(p) = 1$ then a block $C \neq B$ meets B in a line. For otherwise, there are non-collinear points x, y, z in $B \cap C$ and a block D I x, y; $D \not I\, z$. As $B \cap C \neq B \cap D$ and $\mathscr{I}$ is symmetric, there is a point $p' \in C \cap D - B \cap C \cap D$. Then $p'x$ and $p'y$ are points of $\mathscr{I}(p')$ on both C and D, contradicting $\lambda(p') = 1$. By (5.3), planes of $\mathscr{I}_B$ are affine planes. As in § 4, $\mathscr{I}_B$ is again an affine space.

Since $\mathscr{I}$ is symmetric, blocks of $\mathscr{I}$ which are parallel in $\mathscr{I}_B$ meet B in the same flat. As $\mathscr{I}(p)$ is a projective space, (II) thus holds for $\mathscr{I}$. The result now follows from the Dembowski-Wagner Theorem.

6. Transitive groups of special automorphisms. Theorem 2 requires the existence of a large number of automorphisms of a special type. The results of Lüneburg (**7**) mentioned in §5 involve a smaller number of automorphisms of an even more restricted variety. Since Theorem 3 shows that Lüneburg's assumptions are unnecessarily stringent, it is natural to try to find substitutes for his principal result involving much weaker hypotheses. As in §5, we shall work with planes instead of with groups.

THEOREM 5. *Let $\mathscr{I}$ be a finite incidence structure, B a block of $\mathscr{I}$, and Δ a group of automorphisms of $\mathscr{I}$, subject to the following conditions.*

(i) *Δ fixes B pointwise, and is transitive on the points $\not{I}\, B$.*

(ii) *If q and r are distinct points $\not{I}\, B$, there is a point p I B such that the lines pq and qr are defined and equal.*

(iii) *There is a unique line joining any two points of B.*

(iv) *B contains three non-collinear points.*

(v) *Every block has k points. No two blocks are incident with the same set of k points.*

(vi) *If p I $B \not{I}\, q$, there is a block $\not{I}\, p, q$.*

(vii) *There are at least as many blocks as points.*

Then $\mathscr{I}$ is isomorphic to the design of points and hyperplanes of a finite projective space of dimension $\geqq 3$, and Δ contains the group of elations with axis B.

Let us note what is *not* being assumed. The elements of Δ are not assumed to fix all the blocks on some point of $\mathscr{I}$. A non-trivial element of Δ could even conceivably fix several points off B or no blocks other than B. Moreover, $\mathscr{I}$ is assumed to have only a few of the properties of a design.

To prove Theorem 5, it is only necessary to verify the Veblen and Young axioms (**10**) for the points and lines of $\mathscr{I}$. For then, (ii) shows that blocks must be hyperplanes, and (vii) implies that every hyperplane occurs in $\mathscr{I}$, while the result of Lüneburg (**7**, *Satz* 8) referred to in §5 implies that all elations with axis B are in Δ.

(6.1) *There is a unique line containing any two given distinct points p and q.*

Proof. If p, q I B, this is (iii). Let p I B and $q \not{I}\, B$. There is a block D I p, $D \neq B$, and a point r I D but $r \not{I}\, B$ (by (v)). If $r^\delta = q$, $\delta \in \Delta$, then D^δ I p, q. Thus, pq is defined. If $B \not{I}\, q^* \in pq$, then the existence of an element of Δ moving pq to pq^* shows that $|pq| = |pq^*|$, hence $pq = pq^*$. If $p^* \neq p$ were another point of B lying on pq, the transitivity of Δ would imply that $p^* \in px$ for every point $x \not{I}\, B$. Then every block containing p contains p^*. If $p' \notin pp^*$, p' I B, then $p^* \notin pp'$ (cf. (iii) and (iv)), a contradiction. Hence, $pq \cap B = p$.

Now suppose that p and q are distinct points off B. If $p^\delta = q$, $\delta \in \Delta$, then by (ii) there is a point q' I B such that $pq' = pq$. As there is a unique line containing p and q', the same must be true for p and q. This proves (6.1). Note that we have also shown that every line not contained in B meets B. The proof of (5.1) applies, thus (5.1) holds.

(6.2) *If q_1, q_2, r_1, r_2, and s are distinct points such that $q_1q_2 \cap r_1r_2 = s$, then q_1q_2 and r_1r_2 meet.*

Proof. Suppose first that not all five points are on B. The proof of (5.3) will apply once it is shown that (a) three non-collinear points, not all on B, are contained in a unique plane, and (b) such a plane meets B in a line.

Let p, q, and r be three non-collinear points with $p, q \text{ I } B$; $r \not{I} B$. (iii) and (iv) imply that every two distinct points of B are contained in some block not equal to B. Hence, we may use the same argument as in the proof of (6.1) to show that $\mathfrak{p} = pqr$ is defined and is independent of the point $r \in \mathfrak{p} - \mathfrak{p} \cap B$ used to define it. If p^* is a point of $pqr \cap B$ not on pq, we again find that every block containing pq also contains p^*, a contradiction.

Next, suppose that p, q, and r are any non-collinear points with, say, $r \not{I} B$. If $x = rp \cap B$ and $y = rq \cap B$, then (6.1) shows that $pqr = xyr$; thus $pqr \cap B = xy$, proving (b). Given any other non-collinear points p', q', and $r' \in pqr$ with $r' \not{I} B$, $p'q'r' \supseteq pqr$ implies that $p'q'r' \cap B$ is a line containing, and hence equal to, xy. Thus, $pqr = xyr = xyr' = p'q'r'$. This proves (a).

(6.2) holds if not all the points q_1, q_2, r_1, r_2, s are on B. Suppose now that they all lie on B. Let $p \not{I} B$. By (vi) and (5.1), there is a point $s' \in ps - \{p, s\}$. Since $ps' \cap q_1q_2 = s$, the part of (6.2) already proved implies that $pq_1 \cap s'q_2$ is a point q_1', and similarly, $pr_1 \cap s'r_2$ is a point r_1'. Then $q_1'q_2 \cap r_1'r_2 = s'$, thus $q_1'r_1' \cap q_2r_2$ is a point; similarly, $q_1'r_1' \cap q_1r_1$ is a point, and these both equal $q_1'r_1' \cap B$.

(6.3) *Every line contains at least three points.*

Proof. This is already known for lines not contained in B (cf. the proof of (6.2)). If $qr \subset B$ and $p \not{I} B$, let $q' \in pq - \{p, q\}$, and $r' \in pr - \{p, r\}$. Then $qr \cap q'r'$ is a third point of qr.

This completes the proof of Theorem 5.

References

1. J. André, *Über Perspektivitäten in endlichen projektiven Ebenen*, Arch. Math. *6* (1954), 29–32.
2. P. Dembowski, *Eine Kennzeichnung der endlichen affinen Räume*, Arch. Math. *15* (1964), 146–154.
3. ——— *Kombinatorik* (Lecture notes, Frankfurt, 1965).
4. P. Dembowski and A. Wagner, *Some characterizations of finite projective spaces*, Arch. Math. *11* (1960), 465–469.
5. Marshall Hall, Jr., *The theory of groups* (MacMillan, New York, 1959).
6. ——— *Automorphisms of Steiner triple systems*, Proc. Sympos. Pure Math. *6* (1962), 47–66.
7. H. Lüneburg, *Zentrale Automorphismen von λ-Räumen*, Arch. Math. *12* (1961), 134–145.
8. G. Pickert, *Projektive Ebenen* (Springer, Berlin, 1955).
9. H. J. Ryser, *Combinatorial mathematics* (Wiley, New York, 1963).
10. O. Veblen and J. W. Young, *Projective geometry*. I (Ginn, Boston, 1910).

11. H. Wielandt, *Finite permutation groups* (Academic Press, New York, 1964).
12. E. Witt, *Die 5-fach transitiven Gruppen von Mathieu*, Abh. Math. Sem. Univ. Hamburg *13* (1938), 256–264.
13. ——— *Über Steinersche Systeme*, Abh. Math. Sem. Univ. Hamburg *13* (1938), 265–275.

University of Wisconsin,
Madison, Wisconsin