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Convergence results for primal and dual history-dependent quasivariational inequalities

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We consider a class of history-dependent quasivariational inequalities for which we prove the continuous dependence of the solution with respect to the set of constraints. Then, under additional assumptions, we associate with each inequality in the class a new inequality, the so-called dual variational inequality, for which we state and prove existence, uniqueness, equivalence and convergence results. The proofs are based on various estimates, monotonicity and fixed-point arguments for history-dependent operators. Our abstract results are useful in the study of various mathematical models of contact. To provide an example, we consider a boundary value problem which describes the equilibrium of a viscoelastic body in contact with an elastic-rigid foundation. We list the assumptions on the data and derive both the primal and the dual variational formulation of the problem. Then, we state and prove existence, uniqueness and convergence results. We also provide the link between the two formulations, together with their mechanical interpretation.

Keywords: history-dependent quasivariational inequality; primal and dual variational formulation; convergence result; unilateral constraint; frictionless contact; normal compliance; weak solution

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1. Introduction

Variational inequalities play an important role in the study of both the mathematical and numerical analysis of various nonlinear boundary value problems. Reference could be found in the books [**2**, **5**, **6**, **9**, **12**] for instance. The study of contact problems with deformable materials within the framework of variational inequalities was made in various works, including [**4**, **8**, **11**, **14**, **16**, **18**].

History-dependent variational inequalities are inequalities involving a special class of operators, the so-called history-dependent operators. A large number of mathematical models which describe the contact between a deformable body and a foundation lead to such kind of inequalities, in which the unknown is either the displacement or the velocity field. This explains why the history-dependent variational inequalities originate and have many applications in Contact Mechanics. The first abstract study of such inequalities was carried out in [**17**] where existence,

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uniqueness and regularity results have been obtained. This study was continued in [20, 21], where a more general existence and uniqueness result and a convergence result for a penalty method were proved, respectively. Part of these results has been extended to hemivariational inequalities with history-dependent operators, see for instance [13, 19, 25] and the references therein. In most of these papers, the history-dependent inequalities were associated with a set of constraints and the question of the dependence of the solution with respect to the constraints have been left open. Moreover, a large number of contact models lead to variational formulations in which the unknown is the stress field. Such kind of inequalities represents the so-called dual variational formulations of the corresponding contact models and give rise to interesting mathematical problems. For instance, examples and details can be found in [1, 7, 18, 22, 24].

In this paper, we consider a class of abstract inequalities which can be formulated as follows.

Problem $\mathcal{P}$. *Find a function $u : \mathbb{R}_+ \to X$ such that, for all $t \in \mathbb{R}_+$, the inequality below holds:*

$$u(t) \in K, \quad (Au(t), v - u(t))_X + (\mathcal{S}u(t), v - u(t))_X \geqslant (f(t), v - u(t))_X \quad \forall\, v \in K. \tag{1.1}$$

Here and below X is a real Hilbert space with inner product $(\cdot,\cdot)_X$ and associated norm $\|\cdot\|_X$, K is a subset of X, $A : X \to X$ and $\mathcal{S} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; X)$ are given operators and $f : \mathbb{R}_+ \to X$. Moreover, $\mathbb{R}_+ = [0, \infty)$ and $C(\mathbb{R}_+; X)$ represents the space of continuous functions defined on $\mathbb{R}_+$ with values in X. We still use the notation $C(\mathbb{R}_+; K)$ for the set of continuous functions defined on $\mathbb{R}_+$ with values in K and $\mathbb{N}$ will represent the set of positive integers.

In the study of problem $\mathcal{P}$, we consider the following assumptions.

$$K \text{ is a nonempty closed convex subset of } X. \tag{1.2}$$

The operator A is strongly monotone and Lipschitz continuous operator, that is,

$$\begin{cases} \text{(a) There exists } m > 0 \text{ such that} \\ \quad (Au_1 - Au_2, u_1 - u_2)_X \geqslant m\,\|u_1 - u_2\|_X^2 \\ \quad\ \forall\, u_1,\, u_2 \in X. \\ \text{(b) There exists } M > 0 \text{ such that} \\ \quad \|Au_1 - Au_2\|_X \leqslant M\,\|u_1 - u_2\|_X \quad \forall\, u_1,\, u_2 \in X. \end{cases} \tag{1.3}$$

The operator $\mathcal{S}$ is a history-dependent operator, that is,

$$\begin{cases} \text{For every } n \in \mathbb{N} \text{ there exists } s_n > 0 \text{ such that} \\ \quad \|\mathcal{S}u_1(t) - \mathcal{S}u_2(t)\|_X \leqslant s_n \displaystyle\int_0^t \|u_1(s) - u_2(s)\|_X \,\mathrm{d}s \\ \qquad \forall\, u_1,\, u_2 \in C(\mathbb{R}_+; X),\ \forall\, t \in [0, n]. \end{cases} \tag{1.4}$$

The function f has the regularity

$$f \in C(\mathbb{R}_+; X). \tag{1.5}$$

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Under these assumptions, we have the following existence and uniqueness result.

THEOREM 1. *Assume that (1.2)–(1.5) hold. Then, the history-dependent quasivariational inequality (1.1) has a unique solution* $u \in C(\mathbb{R}_+; K)$.

Theorem 1 represents a particular case of more general existence and uniqueness results proved in [**17**, **21**] and, therefore, we skip its proof. We restrict ourselves to mention that it is based on arguments on time-dependent variational inequalities with monotone operators and fixed point.

The aim of the current paper is threefold. The first one is to study the dependence of the solution of problem $\mathcal{P}$ with respect to the set K. The second one is to introduce a new inequality associated with (1.1), so-called dual variational inequality, and to provide its analysis. Finally, our third aim is to illustrate these abstract results in the study of a contact model with viscoelastic materials.

The rest of the paper is structured as follows. In § 2, we state and prove our first convergence result, theorem 2. In § 3, we introduce the dual variational inequality of (1.1) and prove its equivalence with the primal variational inequality, theorem 4. This result implicitly proves the unique solvability of the dual variational inequality. Then, in § 4, we extend the convergence result in theorem 2 to the dual variational inequality. In § 5, we introduce a viscoelastic problem of contact and list the assumption of the data. Finally, in §§ 6 and 7, we provide its analysis by using the primal and the dual variational formulation of the problem, in terms of displacements and stress, respectively. To this end, we use the abstract results obtained in §§ 2–4.

2. A first convergence result

In this section, we investigate the dependence of the solution with respect to the set K. To this end, we consider a perturbation K_ρ of the set K together with the following perturbation of problem $\mathcal{P}$.

PROBLEM $\mathcal{P}_\rho$. *Find a function* $u_\rho : \mathbb{R}_+ \to X$ *such that, for all* $t \in \mathbb{R}_+$, *the inequality below holds:*

$$\begin{aligned} u_\rho(t) \in K_\rho, \quad (Au_\rho(t), v_\rho - u_\rho(t))_X + (\mathcal{S}u_\rho(t), v_\rho - u_\rho(t))_X \\ \geqslant (f(t), v_\rho - u_\rho(t))_X \quad \forall\, v_\rho \in K_\rho. \end{aligned} \tag{2.1}$$

Assume that

$$K_\rho = c(\rho)K + d(\rho)\theta \tag{2.2}$$

where θ is a given element of X and $c : (0, +\infty) \to \mathbb{R}_+$, $d : (0, +\infty) \to \mathbb{R}$ are given functions such that

$$c(\rho) \longrightarrow 1 \quad \text{and} \quad d(\rho) \longrightarrow 0 \quad \text{as} \quad \rho \longrightarrow 0. \tag{2.3}$$

Then, the existence of a unique solution of problem $\mathcal{P}_\rho$ as well as its convergence to the solution of problem $\mathcal{P}$ as $\rho \to 0$ could be obtained by using a result in [**3**], where a more general class of history-dependent variational inequalities has been considered. Nevertheless, in the current paper, we decided to present a different

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version of this convergence result, under different assumptions. The reason is that these assumptions allow us to prove additional results and, moreover, are needed in the study of the dual formulation of problem $\mathcal{P}$. Therefore, we consider in what follows the following hypothesis.

$$\begin{cases}\text{There exists a set } K_0 \text{ and an element } g \in X \text{ such that}\\ \text{(a)}\quad K_0 \text{ is a nonempty closed subset of } X.\\ \text{(b)}\quad u, v \in K_0 \implies u + v \in K_0.\\ \text{(c)}\quad \lambda \geqslant 0,\ \ u \in K_0 \implies \lambda u \in K_0.\\ \text{(d)}\quad K = K_0 + g.\end{cases} \tag{2.4}$$

$$\begin{cases}\text{For each } \rho > 0 \text{ there exists an element } g_\rho \in X \text{ such that}\\ \text{(a)}\quad K_\rho = K_0 + g_\rho \quad \forall\, \rho > 0.\\ \text{(b)}\quad g_\rho \to g \quad \text{in } X, \quad \text{as } \rho \to 0.\end{cases} \tag{2.5}$$

Note that the assumptions (2.4)(a)–(b) show that K_0 is a closed cone in X. Moreover, note that if (2.4) holds, then (1.2) holds, too. Finally, it is easy to see that, if (2.2)–(2.4) hold, then (2.5) holds too, with g_ρ given by $g_\rho = c(\rho)g + d(\rho)\theta$, for all $\rho > 0$.

Our main result in this section is the following.

THEOREM 2. *Assume that (1.3)–(1.5), (2.4) and (2.5) hold. Then:*

(i) *For each $\rho > 0$ inequality (2.1) has a unique solution which satisfies $u_\rho \in C(\mathbb{R}_+; K_\rho)$.*

(ii) *The convergence below holds:*

$$\|u_\rho - u\|_{C(\mathbb{R}_+;X)} \to 0 \quad \text{as} \quad \rho \to 0. \tag{2.6}$$

Related to the statement (ii) in theorem 2, we recall that $u \in C(\mathbb{R}_+; X)$ represents the solution of the inequality (1.1) obtained in theorem 1. Moreover, we recall that $C(\mathbb{R}_+; X)$ can be organized in a canonical way as a Fréchet space, that is, as a complete metric space in which the corresponding topology is induced by a countable family of seminorms. Therefore, the convergence (2.6) is understood with respect to the structure of this space, that is,

$$\begin{cases}\|u_\rho - u\|_{C(\mathbb{R}_+;X)} \to 0 \quad \text{as} \quad \rho \to 0 \quad \text{if and only if}\\ \max\limits_{r\in[0,n]} \|u_\rho(r) - u(r)\|_X \to 0 \ \text{ as } \rho \to 0, \quad \text{for all } \ n \in \mathbb{N}.\end{cases} \tag{2.7}$$

Proof. (i) . Let $\rho > 0$. Using (2.4)(a)–(c) and (2.5)(a) it is easy to show that the set K_ρ is a nonempty closed convex subset of X. Therefore, the unique solvability of problem $\mathcal{P}_\rho$ is a direct consequence of theorem 1, used with $K = K_\rho$.

(ii) In order to prove the convergence (2.6), we consider the following claims that we state here and prove at the end of this section.

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Claim 1. *The following properties hold:*

$$\begin{cases} \text{(a)}\quad g\in K \quad \text{and} \quad g_\rho\in K_\rho, \quad \forall\,\rho>0. \\ \text{(b)}\quad 2\,v-g\in K \quad \forall\,v\in K \quad \text{and} \quad 2\,v_\rho-g_\rho\in K_\rho \quad \forall\,v_\rho\in K_\rho, \quad \forall\,\rho>0. \end{cases} \tag{2.8}$$

Claim 2. *For all $n\in\mathbb{N}$, there exist $w_n>0$ such that*

$$\|u_\rho(t)\|_X\leqslant w_n \quad \text{and} \quad \|u(t)\|_X\leqslant w_n \quad \forall\,t\in[0,n], \quad \forall\rho>0. \tag{2.9}$$

Let $n\in\mathbb{N}$, $t\in[0,n]$ and $\rho>0$. We use (2.5)(a) and (2.4)(d) to see that $u_\rho(t)+g-g_\rho\in K$. Therefore, testing in (1.1) with $v=u_\rho(t)+g-g_\rho$, we obtain

$$\begin{aligned} &(Au(t),(u_\rho(t)-u(t))+(g-g_\rho))_X+(\mathcal{S}u(t),(u_\rho(t)-u(t))+(g-g_\rho))_X \\ &\quad\geqslant (f(t),(u_\rho(t)-u(t))+(g-g_\rho))_X. \end{aligned}$$

Similarly, (2.4)(d) and (2.5)(a) imply that $u(t)+g_\rho-g\in K_\rho$ and, therefore, taking $v_\rho=u(t)+g_\rho-g$ in (2.1) yields

$$\begin{aligned} &(Au_\rho(t),(u(t)-u_\rho(t))+(g_\rho-g))_X+(\mathcal{S}u_\rho(t),(u(t)-u_\rho(t))+(g_\rho-g))_X \\ &\quad\geqslant (f(t),(u(t)-u_\rho(t))+(g_\rho-g))_X. \end{aligned}$$

We now add the previous two inequalities to find that

$$\begin{aligned} (Au_\rho(t)-Au(t),u_\rho(t)-u(t))_X&\leqslant(Au(t)-Au_\rho(t),g-g_\rho)_X \\ &\quad+(\mathcal{S}u_\rho(t)-\mathcal{S}u(t),u(t)-u_\rho(t))_X+(\mathcal{S}u(t)-\mathcal{S}u_\rho(t),g-g_\rho)_X, \end{aligned}$$

then we use condition (1.3) to deduce that

$$\begin{aligned} m\|u_\rho(t)-u(t)\|_X^2&\leqslant\Big(M\|g_\rho-g\|_X+\|\mathcal{S}u_\rho(t)-\mathcal{S}(t)\|_X\Big)\|u_\rho(t)-u(t)\|_X \\ &\quad+\ \|\mathcal{S}u_\rho(t)-\mathcal{S}u(t)\|_X\|g_\rho-g\|_X. \end{aligned} \tag{2.10}$$

Moreover, using (1.4) and (2.9), we obtain

$$\begin{aligned} \|\mathcal{S}u_\rho(t)-\mathcal{S}u(t)\|_X&\leqslant s_n\int_0^t\|u_\rho(s)-u(s)\|_X\,\mathrm{d}s \\ &\leqslant s_n\int_0^t\|u_\rho(s)\|_X\,\mathrm{d}s+s_n\int_0^t\|u(s)\|_X\,\mathrm{d}s\leqslant 2\,nw_ns_n \end{aligned}$$

and, therefore, (2.10) yields

$$\begin{aligned} &m\|u_\rho(t)-u(t)\|_X^2 \\ &\quad\leqslant\Big(M\|g_\rho-g\|_X+\|\mathcal{S}u_\rho(t)-\mathcal{S}u(t)\|_X\Big)\|u_\rho(t)-u(t)\|_X+\,2\,nw_ns_n\|g_\rho-g\|_X. \end{aligned}$$

Next, we use the elementary inequality

$$x,\,a,\,b\geqslant 0 \quad \text{and} \quad x^2\leqslant ax+b \implies x\leqslant a+\sqrt{b} \tag{2.11}$$

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and condition (1.4) to obtain

$$\|u_\rho(t) - u(t)\|_X \leqslant \frac{M}{m}\|g_\rho - g\|_X + \left(\frac{2\,nw_n s_n\|g_\rho - g\|_X}{m}\right)^{1/2} + \frac{s_n}{m}\int_0^t \|u_\rho(s) - u(s)\|_X\,\mathrm{d}s. \tag{2.12}$$

Let

$$T_n(\rho) = \frac{M}{m}\|g_\rho - g\|_X + \left(\frac{2\,nw_n s_n\|g_\rho - g\|_X}{m}\right)^{1/2}.$$

Then, using the Gronwall argument, (2.12) yields

$$\|u_\rho(t) - u(t)\|_X \leqslant T_n(\rho)e^{((s_n)/(m))t}$$

and, therefore,

$$\max_{t\in[0,n]} \|u_\rho(t) - u(t)\|_X \leqslant T_n(\rho)e^{((ns_n)/(m))}. \tag{2.13}$$

Using now (2.5)(b), it is easy to see that

$$T_n(\rho) \longrightarrow 0 \quad \text{as} \quad \rho \longrightarrow 0. \tag{2.14}$$

The convergence (2.6) is now a direct consequence of (2.13), (2.14) and (2.7). □

We turn now to the proof of the claims.

Proof of the Claim 1. First, assumption (2.4)(c) implies that $0_X \in K_0$ and, therefore, equalities (2.4)(d) and (2.5)(a) show that (2.8)(a) holds. Let $v \in K$. Then, equality (2.4)(d) shows that there exists $v_0 \in K_0$ such that $v = v_0 + g$. Therefore, $2v - g = 2(v_0 + g) - g = 2v_0 + g$ and, since (2.4)(c) implies that $2v_0 \in K_0$, we deduce by (2.4)(d) that $2v - g \in K$. Similar arguments show that $2v_\rho - g_\rho \in K_\rho$ for all $v_\rho \in K_\rho$ and, therefore (2.8)(b) holds, which concludes the proof. □

Proof of Claim 2. Let $n \in \mathbb{N}$, $t \in [0, n]$ and $\rho > 0$. We test in (2.1) with $v_\rho = g_\rho \in K_\rho$ to obtain

$$(Au_\rho(t) - Ag_\rho, u_\rho(t) - g_\rho)_X \leqslant (f(t), u_\rho(t) - g_\rho)_X + (\mathcal{S}u_\rho(t), g_\rho - u_\rho(t))_X + (Ag_\rho, g_\rho - u_\rho(t))_X.$$

Then, we use condition (1.3)(a) to see that

$$m\,\|u_\rho(t) - g_\rho\|_X \leqslant \left(\|Ag_\rho\|_X + \max_{t\in[0,n]}(\|f(t)\|_X) + \|\mathcal{S}u_\rho(t)\|_X\right). \tag{2.15}$$

Next, using assumption (1.4), we find that

$$\|\mathcal{S}u_\rho(t)\|_X \leqslant \|\mathcal{S}u_\rho(t) - \mathcal{S}g_\rho\|_X + \|\mathcal{S}g_\rho - \mathcal{S}0_X\|_X + \|\mathcal{S}0_X\|_X \leqslant s_n\int_0^t \|u_\rho(t) - g_\rho\|_X\,\mathrm{d}s + s_n\int_0^t \|g_\rho\|_X\,\mathrm{d}s + \|\mathcal{S}0_X\|_X. \tag{2.16}$$

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We now use the convergence (2.5)(b) to deduce that there exists $c > 0$ which does not depend on ρ such that

$$\|g_\rho\|_X \leqslant c \tag{2.17}$$

and, using this bound in (2.16), yields

$$\|\mathcal{S}u_\rho(t)\|_X \leqslant n s_n c + \|\mathcal{S}0_X\|_X + s_n \int_0^t \|u_\rho(t) - g_\rho\|_X \, \mathrm{d}s. \tag{2.18}$$

Moreover, we use assumption (1.3)(b) and (2.17) to see that

$$\|Ag_\rho\|_X \leqslant c\,M + \|A0_X\|_X. \tag{2.19}$$

We now use the bounds (2.15), (2.18) and (2.19) to deduce that there exists $\lambda_n > 0$ which depends on A, f and $\mathcal{S}$ but does not depends on ρ such that

$$\|u_\rho(t) - g_\rho\|_X \leqslant \lambda_n + \frac{s_n}{m} \int_0^t \|u_\rho(t) - g_\rho\|_X \, \mathrm{d}s$$

and, using a Gronwall's argument, we obtain that

$$\|u_\rho(t) - g_\rho\|_X \leqslant \frac{\lambda_n s_n}{m} e^{n s_n / m}.$$

Therefore,

$$\|u_\rho(t) - g_\rho\|_X \leqslant c_n \tag{2.20}$$

where $c_n = \lambda_n\, s_n / m e^{n\, s_n/m}$. Now, we combine the inequality

$$\|u_\rho(t)\|_X \leqslant \|u_\rho(t) - g_\rho\|_X + \|g_\rho\|_X.$$

with (2.20) and (2.17) to see that

$$\|u_\rho(t)\|_X \leqslant c_n + c. \tag{2.21}$$

Let

$$w_n = \max(c_n + c, \max_{t \in [0,n]} (\|u(t)\|_X)). \tag{2.22}$$

Then, it is easy to see that (2.9) is a direct consequence of (2.21) and (2.22) □

3. Dual variational inequality

In this section, we associate with inequality (1.1) a new inequality, called the dual variational inequality, in which the unknown is the function $\sigma = Au + \mathcal{S}u$. To this end, everywhere in what follows, we assume that $A : X \to X$ is a linear continuous and positively definite operator that is, besides condition (1.3), we assume that A is linear. It is well known that in this case, A is invertible. Moreover, its inverse, denoted by $A^{-1} : X \to X$, satisfies the inequalities

$$\begin{cases} \text{(a)} \quad (A^{-1}u, u)_X \geqslant m/M^2 \, \|u\|_X^2 \quad \forall\, u \in X, \\ \text{(b)} \quad \|A^{-1}u\|_X \leqslant 1/m \, \|u\|_X \quad \forall\, u \in X. \end{cases} \tag{3.1}$$

We start with the following result.

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Lemma 3. *Assume that (1.3) and (1.4) hold and, moreover, assume that A is linear. Then, there exists a unique operator $\mathcal{R} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; X)$ such that, for each functions $\sigma, u \in C(\mathbb{R}_+; X)$, the following equivalence hold:*

$$\sigma(t) = Au(t) + \mathcal{S}u(t) \quad \forall\, t \in \mathbb{R}_+ \tag{3.2}$$

if and only if

$$u(t) = A^{-1}\sigma(t) + \mathcal{R}\sigma(t) \quad \forall\, t \in \mathbb{R}_+. \tag{3.3}$$

Moreover, $\mathcal{R}$ is a history-dependent operator, that is, for each $n \in \mathbb{N}$ there exists $r_n > 0$ such that

$$\|\mathcal{R}\sigma_1(t) - \mathcal{R}\sigma_2(t)\|_X \leqslant r_n \int_0^t \|\sigma_1(s) - \sigma_2(s)\|_X \,\mathrm{d}s \tag{3.4}$$
$$\forall\, \sigma_1,\, \sigma_2 \in C(\mathbb{R}_+; X), \quad \forall\, t \in [0, n].$$

Proof. Let $\sigma \in C(\mathbb{R}_+; X)$. We consider the operator $\Lambda : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; X)$ defined by equality

$$\Lambda\eta(t) = A^{-1}\sigma(t) - A^{-1}\mathcal{S}\eta(t) \quad \forall\, \eta \in C(\mathbb{R}_+; X),\ t \in \mathbb{R}_+. \tag{3.5}$$

Note that the operator Λ depends on σ but, for simplicity, we do not indicate explicitly this dependence. Let $\eta_1,\, \eta_2 \in C(\mathbb{R}_+; X)$. We use the definition (3.5) and the properties (3.1)(b) and (1.4) of the operators A^{-1} and $\mathcal{S}$, respectively, to see that

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_X \leqslant \frac{s_n}{m} \int_0^t \|\eta_1(s) - \eta_2(s)\|_X \,\mathrm{d}s \tag{3.6}$$
$$\forall\, n \in \mathbb{N},\ t \in [0, n].$$

Therefore, by using a fixed point result in [**23**] it follows that Λ has a unique fixed point $\eta^* \in C(\mathbb{R}_+; X)$. We note that, again, the fixed point η^* depends on σ but, for simplicity, we do not indicate explicitly this dependence. We combine (3.5) with equality $\Lambda\eta^* = \eta^*$ to see that

$$\eta^*(t) = A^{-1}\sigma(t) - A^{-1}\mathcal{S}\eta^*(t) \quad \forall\, t \in \mathbb{R}_+. \tag{3.7}$$

This equality allows to consider the operator $\mathcal{R} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; X)$ defined as follows:

$$\mathcal{R}\sigma(t) = \eta^*(t) - A^{-1}\sigma(t) = -A^{-1}\mathcal{S}\eta^*(t) \quad \forall\, \sigma \in C(\mathbb{R}_+; X),\ \forall\, t \in \mathbb{R}_+. \tag{3.8}$$

Moreover, (3.8) implies that

$$\eta^*(t) = A^{-1}\sigma(t) + \mathcal{R}\sigma(t) \quad \forall\, t \in \mathbb{R}_+. \tag{3.9}$$

Assume now that $\sigma, u \in C(\mathbb{R}_+; X)$ and, moreover, assume that (3.2) holds. Then it follows that

$$u(t) = A^{-1}\sigma(t) - A^{-1}\mathcal{S}u(t) \quad \forall\, t \in \mathbb{R}_+ \tag{3.10}$$

and, combining this equality with (3.5) it follows that u is a fixed point for the operator Λ. Therefore, by the uniqueness of the fixed point, we deduce that $u = \eta^*$

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and using (3.9) we obtain (3.3). Conversely, assume that (3.3) holds. Then (3.9) shows that $u = \eta^*$ and, therefore, u is a fixed point for the operator Λ. Thus, definition (3.5) implies that

$$u(t) = A^{-1}\sigma(t) - A^{-1}\mathcal{S}u(t) \quad \forall\, t \in \mathbb{R}_+$$

which shows that (3.2) holds, too. This proves the existence of the operator $\mathcal{R}$ and, since its uniqueness is obvious, we conclude the first part of the lemma.

Next, to prove the second part, we consider two functions $\sigma_1, \sigma_2 \in C(\mathbb{R}_+; X)$ and denote by $\eta_1^*, \eta_2^* \in C(\mathbb{R}_+; X)$ the functions η^* obtained as above with $\sigma = \sigma_i$, $i = 1, 2$, respectively. Let $n \in \mathbb{N}$ and let $t \in [0, n]$. Then, using equality (3.8) and the properties of the operators A^{-1} and $\mathcal{S}$, we deduce that

$$\|\mathcal{R}\sigma_1(t) - \mathcal{R}\sigma_2(t)\|_X \leqslant \frac{s_n}{m}\int_0^t \|\eta_1^*(s) - \eta_2^*(s)\|_X \,\mathrm{d}s. \tag{3.11}$$

In addition, equality (3.9) shows that

$$\|\eta_1^*(s) - \eta_2^*(s)\|_X \leqslant \frac{1}{m}\|\sigma_1(s) - \sigma_2(s)\|_X + \|\mathcal{R}\sigma_1(s) - \mathcal{R}\sigma_2(s)\|_X \quad \forall s \in \mathbb{R}_+. \tag{3.12}$$

We now combine inequalities (3.11) and (3.12) to see that

$$\|\mathcal{R}\sigma_1(t) - \mathcal{R}\sigma_2(t)\|_X \leqslant \frac{s_n}{m^2}\int_0^t \|\sigma_1(s) - \sigma_2(s)\|_X \,\mathrm{d}s$$
$$+ \frac{s_n}{m}\int_0^t \|\mathcal{R}\sigma_1(s) - \mathcal{R}\sigma_2(s)\|_X \,\mathrm{d}s.$$

This inequality combined with the Gronwall argument shows that (3.4) holds with $r_n = s_n/m^2\, e^{n s_n/m}$ which concludes the proof. □

Now, for each $t \in \mathbb{R}_+$, we consider the time-dependent set $\Sigma(t) \subset X$ defined by

$$\Sigma(t) = \{\, \tau \in X : (\tau, v - g)_X \geqslant (f(t), v - g)_X \quad \forall\, v \in K \,\}. \tag{3.13}$$

We also define the set Σ_0 by equality

$$\Sigma_0 = \{\, \tau \in X : (\tau, v_0)_X \geqslant 0, \quad \forall\, v_0 \in K_0 \,\}. \tag{3.14}$$

and we note that (2.4)(d) implies that

$$\Sigma(t) = \Sigma_0 + f(t) \quad \forall\, t \in \mathbb{R}_+. \tag{3.15}$$

Moreover, using the operator $\mathcal{R}$ defined in lemma 3, we consider the following variational problem.

Problem $\mathcal{P}^D$. *Find a function $\sigma : \mathbb{R}_+ \to X$ such that, for all $t \in \mathbb{R}_+$, the inequality below holds:*

$$\sigma(t) \in \Sigma(t), (A^{-1}\sigma(t), \tau - \sigma(t))_X + (\mathcal{R}\sigma(t), \tau - \sigma(t))_X \tag{3.16}$$
$$\geqslant (g, \tau - \sigma(t))_X \quad \forall\, \tau \in \Sigma(t).$$

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We refer to inequality (3.16) as the dual variational inequality of the variational inequality (1.1). This terminology could be motivated by the following equivalence result.

THEOREM 4. *Assume that* (1.3)–(1.5), (2.4) *hold and, moreover, assume that* A *is linear. Let* $u \in C(\mathbb{R}_+; X)$ *and* $\sigma \in C(\mathbb{R}_+; X)$ *be given function. The following statements hold.*

(i) *If* u *is a solution to problem* $\mathcal{P}$ *and* $\sigma = Au + \mathcal{S}u$, *then* σ *is a solution to problem* $\mathcal{P}^D$.

(ii) *Conversely, if* σ *is a solution to problem* $\mathcal{P}^D$ *and* $u = A^{-1}\sigma + \mathcal{R}\sigma$, *then* u *is a solution to problem* $\mathcal{P}$.

Proof. (i) Assume that u is a solution to problem $\mathcal{P}$ and $\sigma = Au + \mathcal{S}u$. Let $t \in \mathbb{R}_+$ be given. Then, (1.1) yields

$$(\sigma(t), v - u(t))_X \geqslant (f(t), v - u(t))_X \quad \forall\, v \in K. \tag{3.17}$$

Next, using (2.8), we have $2u(t) - g \in K$ and $g \in K$. Therefore, testing in (3.17) with $v = 2u(t) - g$ and $v = g$, we obtain

$$(\sigma(t), u(t) - g)_X = (f(t), u(t) - g)_X. \tag{3.18}$$

We combine (3.17) and (3.18) to see that

$$(\sigma(t), v - g)_X \geqslant (f(t), v - g)_X \quad \forall\, v \in K$$

which shows that

$$\sigma(t) \in \Sigma(t). \tag{3.19}$$

Next, we use (3.18) and the definition (3.13) of the set $\Sigma(t)$ to obtain that

$$(\tau - \sigma(t), u(t) - g)_X \geqslant 0 \quad \forall\, \tau \in \Sigma(t). \tag{3.20}$$

In addition, lemma 3 implies that $u(t) = A^{-1}\sigma(t) + \mathcal{R}\sigma(t)$ and, therefore, substituting this equality in (3.20), we deduce that

$$(A^{-1}\sigma(t), \tau - \sigma(t))_X + (\mathcal{R}\sigma(t), \tau - \sigma(t))_X \geqslant (g, \tau - \sigma(t))_X \quad \forall\, \tau \in \Sigma(t). \tag{3.21}$$

We now combine (3.19) and (3.21) to see that σ is a solution of problem $\mathcal{P}^D$.

(ii) Conversely, assume that σ is a solution to problem $\mathcal{P}^D$ and $u = A^{-1}\sigma + \mathcal{R}\sigma$. Let $t \in \mathbb{R}_+$. Then (3.16) implies that

$$(u(t), \tau - \sigma(t))_X \geqslant (g, \tau - \sigma(t))_X \quad \forall\, \tau \in \Sigma(t). \tag{3.22}$$

We shall prove that u satisfies (1.1) and, to this end, we start by proving that $u(t) \in K$, i.e., $u_0(t) = u(t) - g \in K_0$. Arguing by contradiction, we assume in what follows that $u_0(t) \notin K_0$ and we denote by $Pu_0(t)$ the projection of $u_0(t)$

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on the closed cone K_0. We have $u_0(t) \neq Pu_0(t)$ and, using the characterization of the projection, it follows that

$$(Pu_0(t) - u_0(t), v_0)_X \geqslant (Pu_0(t) - u_0(t), Pu_0(t))_X \\ > (Pu_0(t) - u_0(t), u_0(t))_X \quad \forall\, v_0 \in K_0.$$

This inequality implies that there exists $\alpha \in \mathbb{R}$ such that

$$(Pu_0(t) - u_0(t), v_0)_X > \alpha > (Pu_0(t) - u_0(t), u_0(t))_X \quad \forall\, v_0 \in K_0, \tag{3.23}$$

and, taking $v_0 = 0_X \in K_0$ in (3.23), yields

$$\alpha < 0. \tag{3.24}$$

Next, assume that there exists $\widetilde{v}_0 \in K_0$ such that

$$(Pu_0(t) - u_0(t), \widetilde{v}_0)_X < 0. \tag{3.25}$$

We take $v_0 = \lambda\widetilde{v}_0$ in (3.23) where $\lambda \geqslant 0$ and obtain

$$\lambda(Pu_0(t) - u_0(t), \widetilde{v}_0)_X > \alpha \quad \forall\, \lambda \geqslant 0.$$

Therefore, passing to the limit as $\lambda \to \infty$ and using (3.25) we deduce that $\alpha \leqslant -\infty$ which contradicts $\alpha \in \mathbb{R}$. We conclude from above that

$$(Pu_0(t) - u_0(t), v_0)_X \geqslant 0 \quad \forall\, v_0 \in K_0. \tag{3.26}$$

Moreover, since $\sigma(t) \in \Sigma(t)$, (3.15) shows that $\sigma(t) - f(t) \in \Sigma_0$ and, therefore,

$$(\sigma(t) - f(t), v_0)_X \geqslant 0 \quad \forall\, v_0 \in K_0. \tag{3.27}$$

Combining (3.26) and (3.27) it follows that $Pu_0(t) - u_0(t) + \sigma(t) - f(t) \in \Sigma_0$ and, using (3.15), we deduce that $Pu_0(t) - u_0(t) + \sigma(t) \in \Sigma(t)$. This allows to take $\tau = Pu_0(t) - u_0(t) + \sigma(t)$ in (3.22) and, using equality $u(t) = u_0(t) + g$, we find that

$$(u_0(t), Pu_0(t) - u_0(t))_X \geqslant 0. \tag{3.28}$$

Now, combining (3.23) and (3.24) it follows that

$$(u_0(t), Pu_0(t) - u_0(t))_X < 0. \tag{3.29}$$

The inequalities (3.28) and (3.29) lead to a contradiction. Therefore, we conclude that $u_0(t) \in K_0$, which implies that

$$u(t) \in K. \tag{3.30}$$

We are now in position to verify inequality (1.1). First, we note that $f(t) \in \Sigma(t)$. Therefore, taking $\tau = f(t)$ in (3.22), it follows that

$$(f(t), u(t) - g)_X \geqslant (\sigma(t), u(t) - g)_X. \tag{3.31}$$

In addition, since $\sigma(t) \in \Sigma(t)$ and $u(t) \in K$, the definition (3.13) shows that

$$(\sigma(t), u(t) - g)_X \geqslant (f(t), u(t) - g)_X. \tag{3.32}$$

We combine inequalities (3.31) and (3.32), to see that

$$(\sigma(t), u(t) - g)_X = (f(t), u(t) - g)_X. \tag{3.33}$$

Hence, since $\sigma(t) \in \Sigma(t)$, we deduce that

$$(\sigma(t), v - u(t))_X \geqslant (f(t), v - u(t))_X \quad \forall\, v \in K. \tag{3.34}$$

In addition, since $u(t) = A^{-1}\sigma(t) + \mathcal{R}\sigma(t)$, by lemma 3 we have $\sigma(t) = Au(t) + \mathcal{S}u(t)$ and, therefore, (3.34) yields

$$\begin{aligned}&(Au(t), v - u(t))_X + (\mathcal{S}u(t), v - u(t))_X \\ &\qquad \geqslant (f(t), v - u(t))_X \quad \forall\, v \in K.\end{aligned} \tag{3.35}$$

We combine (3.30) and (3.35) to see that u is a solution to problem $\mathcal{P}$, which concludes the proof.

□

We now turn to the unique solvability of problem $\mathcal{P}^D$ which results from the following result.

THEOREM 5. *Assume that* (1.3)–(1.5), (2.4) *hold and, moreover, assume that A is linear. Then, the history-dependent quasivariational inequality* (3.16) *has a unique solution $\sigma \in C(\mathbb{R}_+; X)$.*

Proof. We provide two different proofs. The first one is based on theorems 2 and 4 and is as follows. We use theorem 2 to see that there exists a unique solution $u \in C(\mathbb{R}_+; X)$ to problem $\mathcal{P}$. Then, theorem 4 (i) implies that $\sigma = Au + \mathcal{S}u$ is a solution to problem $\mathcal{P}^D$. This proves the existence part of the solution. The uniqueness part follows directly from (3.16), by using a Gronwall argument.

The second proof is based on theorem 1. Let $\mathcal{R}_0 : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; X)$ be the operator defined by

$$\mathcal{R}_0\tau(t) = \mathcal{R}(\tau(t) + f(t)) \quad \forall\, \tau \in C(\mathbb{R}_+; X), \quad t \in \mathbb{R}_+ \tag{3.36}$$

and consider the intermediate problem of finding a function $\sigma_0 : \mathbb{R}_+ \to X$ such that, for all $t \in \mathbb{R}_+$, the inequality below holds:

$$\begin{aligned}&\sigma_0(t) \in \Sigma_0, (A^{-1}\sigma_0(t), \tau_0 - \sigma_0(t))_X + (\mathcal{R}_0\sigma_0(t), \tau_0 - \sigma_0(t))_X \\ &\quad \geqslant (g - A^{-1}f(t), \tau_0 - \sigma_0(t))_X \quad \forall\, \tau_0 \in \Sigma_0.\end{aligned} \tag{3.37}$$

We apply theorem 1 in the case when the set K, the operators A, $\mathcal{S}$ and the function f are given by Σ_0, A^{-1}, $\mathcal{R}_0$ and $g - A^{-1}f$, respectively. Therefore, we deduce the existence of a unique solution $\sigma_0 \in C(\mathbb{R}_+, \Sigma_0)$ to inequality (3.37). We now take $\sigma = \sigma_0 + f$ and use equalities (3.15), (3.36) to see that σ is the unique solution of problem $\mathcal{P}^D$ with regularity $\sigma \in C(\mathbb{R}_+, \Sigma)$. □

4. A second convergence result

In this section, we investigate the dependence of the solution of problem $\mathcal{P}^D$ with respect to the set K. To this end, we consider a perturbation K_ρ of the set K and, for each $t \in \mathbb{R}_+$ we denote by $\Sigma_\rho(t)$ the set

$$\Sigma_\rho(t) = \{\tau_\rho \in X : (\tau_\rho, v_\rho - g_\rho)_X \geqslant (f(t), v_\rho - g_\rho)_X \quad \forall\, v_\rho \in K_\rho\}. \tag{4.1}$$

Then, using (2.5)(a) it is easy to see that

$$\Sigma_\rho(t) = \Sigma_0 + f(t), \tag{4.2}$$

where Σ_0 is the set defined by (3.14). We also consider the following perturbation of problem $\mathcal{P}^D$.

Problem $\mathcal{P}^D_\rho$. *Find a function $\sigma_\rho : \mathbb{R}_+ \to X$ such that, for all $t \in \mathbb{R}_+$, the inequality below holds:*

$$\begin{aligned} \sigma_\rho(t) \in \Sigma_\rho(t), \quad & (A^{-1}\sigma_\rho(t), \tau_\rho - \sigma_\rho(t))_X + (\mathcal{R}\sigma_\rho(t), \tau_\rho - \sigma_\rho(t))_X \\ & \geqslant (g_\rho, \tau_\rho - \sigma_\rho(t))_X \quad \forall\, \tau_\rho \in \Sigma_\rho(t). \end{aligned} \tag{4.3}$$

Our main result in this section is the following existence, uniqueness and convergence result.

Theorem 6. *Assume that* (1.3)–(1.5), (2.4), (2.5) (a) *hold and, moreover, assume that A is linear. Then:*

(i) *For each $\rho > 0$ inequality* (4.3) *has a unique solution which satisfies $\sigma_\rho \in C(\mathbb{R}_+; X)$.*

(ii) *In addition, if* (2.5)(b) *holds, then we have the following convergence*

$$\|\sigma_\rho - \sigma\|_{C(\mathbb{R}_+;X)} \to 0 \quad \text{as} \quad \rho \to 0. \tag{4.4}$$

Proof. (i) Let $t \in \mathbb{R}_+$ and $\rho > 0$. We use (4.2) and (3.15) to see that $\Sigma(t) = \Sigma_\rho(t)$. The unique solvability of inequality (4.3) is now a direct consequence of theorem 5, applied with g_ρ instead of g.

(ii) Let $t \in \mathbb{R}_+$ and $\rho > 0$. Since $\Sigma(t) = \Sigma_\rho(t)$ we test in (3.16) with $\tau = \sigma_\rho(t) \in \Sigma(t)$, then in (4.3) with $\tau_\rho = \sigma(t) \in \Sigma_\rho(t)$. Next, we add the resulting inequalities to find that

$$\begin{aligned} &(A^{-1}\sigma_\rho(t) - A^{-1}\sigma(t), \sigma_\rho(t) - \sigma(t))_X \\ &\quad \leqslant (\mathcal{R}\sigma_\rho(t) - \mathcal{R}\sigma(t), \sigma(t) - \sigma_\rho(t))_X + (g_\rho - g, \sigma_\rho(t) - \sigma(t))_X. \end{aligned} \tag{4.5}$$

We now use inequalities (3.1) and (3.4) to see that

$$\begin{aligned} &\frac{m}{M^2}\|\sigma_\rho(t) - \sigma(t)\|_X^2 \\ &\quad \leqslant \Big(\|g_\rho - g\|_X + r_n \int_0^t \|\sigma_\rho(s) - \sigma(s)\|_X \mathrm{d}s\Big)\|\sigma_\rho(t) - \sigma(t)\|_X \end{aligned}$$

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and, therefore,

$$\|\sigma_\rho(t) - \sigma(t)\|_X \leqslant \frac{M^2}{m}\|g_\rho - g\|_X + \frac{r_n M^2}{m}\int_0^t \|\sigma_\rho(s) - \sigma(s)\|_X \,\mathrm{d}s.$$

We now use the Gronwall argument to deduce that

$$\|\sigma_\rho(t) - \sigma(t)\| \leqslant \frac{M^2}{m}\|g_\rho - g\|_X \, e^{r_n M^2 t/m}$$

and, therefore,

$$\max_{t\in[0,n]} \|\sigma_\rho(t) - \sigma(t)\|_X \leqslant \frac{M^2}{m}\|g_\rho - g\|_X \, e^{n r_n M^2/m}. \tag{4.6}$$

The convergence (4.4) is now a direct consequence of (4.6) and (2.5)(b). □

5. A viscoelastic contact model

The abstract results presented in §§ 2–4 are useful in the study of various quasistatic models of contact with deformable bodies. To provide an example in this section, we consider a frictionless contact problem for linearly viscoelastic materials with long memory. Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be reference configuration of the viscoelastic body, Γ the boundary of Ω and Γ_1, Γ_2, Γ_3 a partition of Γ such that $meas\,\Gamma_1 > 0$. We denote by $\boldsymbol{\nu}$ the unit outward normal at Γ and by $\mathbb{S}^d$ the space of symmetric tensors of second order on $\mathbb{R}^d$. Then, the classical formulation of the problem is the following.

Problem $\mathcal{Q}$. *Find a displacement field* $\boldsymbol{u} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times \mathbb{R}_+ \to \mathbb{S}^d$ *such that*

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\mathrm{d}s \quad \text{in} \quad \Omega, \tag{5.1}$$

$$\mathrm{Div}\,\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \quad \text{in} \quad \Omega, \tag{5.2}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \quad \text{on} \quad \Gamma_1, \tag{5.3}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2 \quad \text{on} \quad \Gamma_2, \tag{5.4}$$

$$\left.\begin{aligned} &u_\nu(t) \leqslant g_a, \ \ \sigma_\nu(t) + p(u_\nu(t)) \leqslant 0, \\ &(u_\nu(t) - g_a)\big(\sigma_\nu(t) + p(u_\nu(t))\big) = 0 \end{aligned}\right\} \quad \text{on} \quad \Gamma_3, \tag{5.5}$$

$$\boldsymbol{\sigma}_\tau(t) = \boldsymbol{0} \quad \text{on} \quad \Gamma_3, \tag{5.6}$$

for all $t \in \mathbb{R}_+$.

Problem $\mathcal{Q}$ was already introduced in [**18**] and, therefore, to not describe it in detail. We only recall that equation (5.1) represents the constitutive equation in which $\mathcal{A}$ and $\mathcal{B}$ are the elasticity operator and the relaxation tensor, respectively.

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Equation (5.2) is the equilibrium equation in which $\boldsymbol{f}_0$ denotes the density of body forces, and conditions (5.3), (5.4) are the displacement-traction boundary conditions in which $\boldsymbol{f}_2$ represents the density of traction on Γ_2. Condition (5.5) represents the contact condition with normal compliance and unilateral constraint, which models the contact with an elastic-rigid foundation. Here $g_a > 0$ ia a given bound, p is the normal compliance function which will be described below and the index ν denotes the normal components of vectors and tensors. Finally, condition (5.6) represents the frictionless condition, which states that the tangential component of the stress vector, denoted $\boldsymbol{\sigma}_\tau$, vanishes on the contact surface Γ_3.

In the study of problem $\mathcal{Q}$, we use the notation '$\cdot$' and $\|\cdot\|$ for the inner product and the Euclidean norm on the spaces $\mathbb{R}^d$ and $\mathbb{S}^d$ as well as standard notation for the Lebesgue and Sobolev spaces associated with Ω and Γ. Moreover, we consider the spaces

$$V = \{\, \boldsymbol{v} = (v_i) \in H^1(\Omega)^d : v_i = 0 \text{ on } \Gamma_1 \,\},$$
$$Q = \{\ \boldsymbol{\tau} = (\tau_{ij})\ :\ \tau_{ij} = \tau_{ji} \in L^2(\Omega)\ \}.$$

These are real Hilbert spaces endowed with the inner products

$$(\boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, \mathrm{d}x, \quad (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau}\, \mathrm{d}x.$$

and the associated norms denoted by $\|\cdot\|_V$ and $\|\cdot\|_Q$, respectively. Here and below $\boldsymbol{\varepsilon}$ is the deformation operator defined by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \tfrac{1}{2}(u_{i,j} + u_{j,i})$$

Completeness of the space $(V, \|\cdot\|_V)$ follows from the assumption $meas\,(\Gamma_1) > 0$, which allows the use of Korn's inequality.

For an element $\boldsymbol{v} \in V$, we still write $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ on the boundary and we denote by v_ν and $\boldsymbol{v}_\tau$ the normal and tangential components of $\boldsymbol{v}$ on Γ, given by $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$, $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}$. By the Sobolev trace theorem, there exists a positive constant c_0 which depends on Ω, Γ_1 and Γ_3 such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)^d} \leqslant c_0\, \|\boldsymbol{v}\|_V \quad \forall\, \boldsymbol{v} \in V. \tag{5.7}$$

Also, for a regular function $\boldsymbol{\sigma} \in Q$ we use the notation σ_ν and $\boldsymbol{\sigma}_\tau$ for the normal and the tangential trace, that is, $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$. Finally, we denote by $\mathbf{Q}_\infty$ the space of fourth order tensor fields given by

$$\mathbf{Q}_\infty = \{\mathcal{E} = (\mathcal{E}_{ijkl})\ :\ \mathcal{E}_{ijkl} = \mathcal{E}_{jikl} = \mathcal{E}_{klij} \in L^\infty(\Omega), \quad 1 \leqslant i, j, k, l \leqslant d\},$$

and we recall that $\mathbf{Q}_\infty$ is a real Banach space with the norm

$$\|\mathcal{E}\|_{\mathbf{Q}_\infty} = \max_{1 \leqslant i,j,k,l \leqslant d} \|\mathcal{E}_{ijkl}\|_{L^\infty(\Omega)}.$$

Moreover, a simple calculation shows that

$$\|\mathcal{E}\boldsymbol{\tau}\|_Q \leqslant d\, \|\mathcal{E}\|_{\mathbf{Q}_\infty} \|\boldsymbol{\tau}\|_Q \quad \forall\, \mathcal{E} \in \mathbf{Q}_\infty,\ \boldsymbol{\tau} \in Q. \tag{5.8}$$

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The assumptions on the data of problem $\mathcal{Q}$ are the following.

$$\left\{\begin{array}{l}\text{(a) } \mathcal{A}:\Omega\times\mathbb{S}^d\to\mathbb{S}^d.\\ \text{(b) There exists } L_{\mathcal{A}}>0 \text{ such that}\\ \quad \|\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_1)-\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)\|\leqslant L_{\mathcal{A}}\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|\\ \qquad \forall\,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d,\ \text{a.e. } \boldsymbol{x}\in\Omega.\\ \text{(c) There exists } m_{\mathcal{A}}>0 \text{ such that}\\ \quad (\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_1)-\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_2))\cdot(\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2)\geqslant m_{\mathcal{A}}\,\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|^2\\ \qquad \forall\,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d,\ \text{a.e. } \boldsymbol{x}\in\Omega.\\ \text{(d) The mapping } \boldsymbol{x}\mapsto\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}) \text{ is measurable on } \Omega,\\ \quad \text{for any } \boldsymbol{\varepsilon}\in\mathbb{S}^d.\\ \text{(e) The mapping } \boldsymbol{x}\mapsto\mathcal{A}(\boldsymbol{x},\boldsymbol{0}) \text{ belongs to } Q.\end{array}\right. \tag{5.9}$$

$$\mathcal{B}\in C(\mathbb{R}_+;\mathbf{Q}_\infty). \tag{5.10}$$

$$\boldsymbol{f}_0\in C(\mathbb{R}_+;L^2(\Omega)^d),\quad \boldsymbol{f}_2\in C(\mathbb{R}_+;L^2(\Gamma_2)^d). \tag{5.11}$$

$$\left\{\begin{array}{l}\text{(a) } p:\Gamma_3\times\mathbb{R}\to\mathbb{R}_+.\\ \text{(b) There exists } L_p>0 \text{ such that}\\ \quad |p(\boldsymbol{x},r_1)-p(\boldsymbol{x},r_2)|\leqslant L_p\,|r_1-r_2|\\ \quad \forall\,r_1,\,r_2\in\mathbb{R}\quad \text{a.e. } x\in\Gamma_3.\\ \text{(c) } (p(\boldsymbol{x},r_1)-p(\boldsymbol{x},r_2))(r_1-r_2)\geqslant 0\\ \quad \forall\,r_1,\,r_2\in\mathbb{R}\quad \text{a.e. } \boldsymbol{x}\in\Gamma_3.\\ \text{(d) } p(\boldsymbol{x},r)=0 \ \text{ for all } \ r\leqslant 0 \quad \text{a.e. } \boldsymbol{x}\in\Gamma_3.\\ \text{(e) The mapping } \boldsymbol{x}\mapsto p(\boldsymbol{x},r) \text{ is mesurable on } \Gamma_3,\\ \quad \text{for all } r\in\mathbb{R}.\end{array}\right. \tag{5.12}$$

Finally, we assume that there exists an element $\boldsymbol{\zeta}\in V$ such that

$$\boldsymbol{\zeta}=\boldsymbol{\nu}\quad \text{on}\quad \Gamma_3, \tag{5.13}$$

and we send the reader to [**10**, **24**] for examples and details on this condition.

We use these assumptions to provide the variational analysis of problem $\mathcal{Q}$ based on the primal variational formulation, in terms of displacements. Then, under additional assumptions that we shall introduce later, we shall provide the variational analysis of the problem by using the dual variational formulation, in terms of stress.

6. Primal variational formulation

Everywhere in this section, we assume that (5.9)–(5.12) hold. Moreover, we define the set $U\subset V$, the operators $\mathcal{S}:C(\mathbb{R}_+;V)\to C(\mathbb{R}_+;V)$, $P:V\to V$ and $A:V\to V$ and the function $\boldsymbol{f}:\mathbb{R}_+\to V$ by equalities

$$U=\{\boldsymbol{v}\in V: v_\nu\leqslant g_a \ \text{ on } \ \Gamma_3\}, \tag{6.1}$$

$$(\mathcal{S}\boldsymbol{u}(t),\boldsymbol{v})_V=\Big(\int_0^t\mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,\mathrm{d}s,\boldsymbol{\varepsilon}(\boldsymbol{v})\Big)_Q\quad \forall\,\boldsymbol{u}\in C(\mathbb{R}_+;V),\ \boldsymbol{v}\in V, \tag{6.2}$$

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$$(P\boldsymbol{u},\boldsymbol{v})_V = \int_{\Gamma_3} p(u_\nu)v_\nu\,\mathrm{d}a \quad \forall\,\boldsymbol{u},\ \boldsymbol{v}\in V, \tag{6.3}$$

$$(A\boldsymbol{u},\boldsymbol{v})_V = (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (P\boldsymbol{u},\boldsymbol{v})_V \quad \forall\,\boldsymbol{u},\,\boldsymbol{v}\in V, \tag{6.4}$$

$$(\boldsymbol{f}(t),\boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t)\cdot\boldsymbol{v}\,\mathrm{d}x + \int_{\Gamma_2} \boldsymbol{f}_2(t)\cdot\boldsymbol{v}\,\mathrm{d}a \quad \forall\,\boldsymbol{v}\in V, \quad \forall\,t\in\mathbb{R}_+. \tag{6.5}$$

Assume in what follows that $(\boldsymbol{u},\boldsymbol{\sigma})$ are sufficiently regular functions which satisfy (5.1)–(5.6). Let $\boldsymbol{v}\in U$ and $t\in\mathbb{R}_+$. Then, using integration by parts combined with standard arguments it is easy to see that

$$\begin{aligned}\int_\Omega \boldsymbol{\sigma}(t)\cdot(\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}(t)))\,\mathrm{d}x &= \int_\Omega \boldsymbol{f}_0(t)\cdot(\boldsymbol{v}-\boldsymbol{u}(t))\,\mathrm{d}x \\ &+ \int_{\Gamma_2} \boldsymbol{f}_2(t)\cdot(\boldsymbol{v}-\boldsymbol{u}(t))\,\mathrm{d}a + \int_{\Gamma_3} \sigma_\nu(t)(v_\nu-u_\nu(t))\,\mathrm{d}a,\end{aligned} \tag{6.6}$$

$$\int_{\Gamma_3} \sigma_\nu(t)(v_\nu-u_\nu(t))\,\mathrm{d}a \geqslant -\int_{\Gamma_3} p(u_\nu(t))(v_\nu-u_\nu(t))\,\mathrm{d}a. \tag{6.7}$$

We now combine (6.6) and (6.7) and use the definitions (6.3), (6.5) to deduce that

$$\begin{aligned}&(\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\boldsymbol{v})-\boldsymbol{\varepsilon}(\boldsymbol{u}(t)))_Q + (P\boldsymbol{u}(t),\boldsymbol{v}-\boldsymbol{u}(t))_V \\ &\quad\geqslant (\boldsymbol{f}(t),\boldsymbol{v}-\boldsymbol{u}(t))_V \quad \forall\,\boldsymbol{v}\in U.\end{aligned} \tag{6.8}$$

Next, we substitute the constitutive law (5.1) in (6.8) and use the definitions (6.2) and (6.4) to deduce the following variational formulation of the contact problem $\mathcal{Q}$.

Problem $\mathcal{Q}_V$. *Find a displacement field* $\boldsymbol{u}:\mathbb{R}_+\to U$ *such that, for all* $t\in\mathbb{R}_+$, *the inequality below holds:*

$$\begin{aligned}&\boldsymbol{u}(t)\in U,\ (A\boldsymbol{u}(t),\boldsymbol{v}-\boldsymbol{u}(t))_V + (\mathcal{S}\boldsymbol{u}(t),\boldsymbol{v}-\boldsymbol{u}(t))_V \\ &\qquad\geqslant (\boldsymbol{f}(t),\boldsymbol{v}-\boldsymbol{u}(t))_V \quad \forall\,\boldsymbol{v}\in U\end{aligned} \tag{6.9}$$

In the study of the problem $\mathcal{Q}_V$ we have the following existence and uniqueness result.

Theorem 7. *Assume that* (5.9)–(5.12) *hold. Then, problem* $\mathcal{Q}_V$ *has a unique solution which satisfies* $\boldsymbol{u}\in C(\mathbb{R}_+;U)$.

Proof. We use theorem 1 with $X=V$, $K=U$. To this end, we note that the set U given by (6.1) is a nonempty closed convex subset of V and, therefore, it satisfies the assumption (1.2). Next, we use assumptions (5.9), (5.12) and inequality (5.7) to see that the operator A defined by (6.4) satisfies the inequalities

$$(A\boldsymbol{u}-A\boldsymbol{v},\boldsymbol{u}-\boldsymbol{v})_V \geqslant m_{\mathcal{A}}\,\|\boldsymbol{u}-\boldsymbol{v}\|_V^2 \quad \forall\,\boldsymbol{u},\boldsymbol{v}\in V, \tag{6.10}$$

$$\|A\boldsymbol{u}-A\boldsymbol{v}\|_V \leqslant (L_{\mathcal{A}}+c_0^2L_p)\|\boldsymbol{u}-\boldsymbol{v}\|_V \quad \forall\,\boldsymbol{u},\,\boldsymbol{v}\in V. \tag{6.11}$$

Therefore, condition (1.3) holds with $m=m_{\mathcal{A}}$ and $M=L_{\mathcal{A}}+c_0^2L_p$. Let $n\in\mathbb{N}$. Then, a simple calculation based on the assumption (5.10) and inequality (5.8)

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shows that

$$\|\mathcal{S}\boldsymbol{u}_1(t) - \mathcal{S}\boldsymbol{u}_2(t)\|_V \leqslant d \max_{r\in[0,n]} \|\mathcal{B}(r)\|_{\mathbf{Q}_\infty} \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|_V \,\mathrm{d}s \qquad (6.12)$$
$$\forall\, \boldsymbol{u}_1,\, \boldsymbol{u}_2 \in C(\mathbb{R}_+; V), \quad \forall\, t \in [0, n].$$

This inequality shows that the operator $\mathcal{S}$ satisfies condition (1.4) with

$$s_n = \max_{r\in[0,n]} \|\mathcal{B}(r)\|_{\mathbf{Q}_\infty}.$$

Finally, note that condition (5.11) on the body forces and tractions imply that $\boldsymbol{f} \in C(\mathbb{R}_+; V)$, which shows that condition (1.5) is satisfied, too. Theorem 7 is now a direct consequence of theorem 1. □

Next, we study the dependence of the solution with respect the bound g_a by using the abstract convergence results in § 2. To this end, we consider the set

$$U_\rho = \{\, \boldsymbol{v} \in V : v_\nu \leqslant g_{a\rho} \text{ on } \Gamma_3 \,\}, \qquad (6.13)$$

where, here and below, $g_{a\rho} > 0$ represents a perturbation of g_a and ρ is a parameter which converges to zero. We also consider the following perturbation of problem $\mathcal{Q}_V$.

Problem $\mathcal{Q}_V^\rho$. *Find a displacement field $\boldsymbol{u}_\rho : \mathbb{R}_+ \to U_\rho$ such that*

$$(A\boldsymbol{u}_\rho(t), \boldsymbol{v}_\rho - \boldsymbol{u}_\rho(t))_V + (\mathcal{S}\boldsymbol{u}_\rho(t), \boldsymbol{v}_\rho - \boldsymbol{u}_\rho(t))_V \qquad (6.14)$$
$$\geqslant (\boldsymbol{f}(t), \boldsymbol{v}_\rho - \boldsymbol{u}_\rho(t))_V \quad \forall\, \boldsymbol{v}_\rho \in U_\rho.$$

We assume that

$$g_{a\rho} \longrightarrow g_a \quad \text{as} \quad \rho \longrightarrow 0. \qquad (6.15)$$

Then, we have the following existence, uniqueness and convergence result.

Theorem 8. *Assume that* (5.9)–(5.12). *Then:*

(i) *For each $\rho > 0$ problems $\mathcal{Q}_V^\rho$ has a unique solution.*

(ii) *In addition, if* (5.13) *and* (6.15) *hold, then the solution $\boldsymbol{u}_\rho$ of inequality* (6.14) *converges to the solution $\boldsymbol{u}$ of inequality* (3.16)*, that is,*

$$\boldsymbol{u}_\rho \longrightarrow \boldsymbol{u} \quad \text{in} \quad C(\mathbb{R}_+; V) \quad \text{as} \quad \rho \longrightarrow 0. \qquad (6.16)$$

Proof. We define the set U_0 by equality

$$U_0 = \{\, \boldsymbol{v} \in V : v_\nu \leqslant 0 \text{ on } \Gamma_3 \,\}. \qquad (6.17)$$

Then it is easy to see that condition (2.4) holds with $X = V$, $K = U$, $K_0 = U_0$ and $\boldsymbol{g} = g_a\boldsymbol{\zeta}$, the element $\boldsymbol{\zeta}$ being defined in condition (5.13). Moreover, the condition (2.5) holds with $X = V$, $K_\rho = U_\rho$, $K_0 = U_0$ and $\boldsymbol{g} = g_a\boldsymbol{\zeta}$. Finally, recall that, as shown in the proof of Theorem 7, conditions (1.3)–(1.5) hold. Theorem 8 is now a direct consequence of theorem 2. □

7. Dual variational formulation

We turn now to the dual variational formulation of problem $\mathcal{Q}$, expressed in terms of stress. To this end, we need additional assumptions which guarantee the linearity of the operator A defined by (6.4). Therefore, in what follows we assume that

$$\begin{cases} \mathcal{A} \in \mathbf{Q}_\infty \text{ and there exists } m_{\mathcal{A}} > 0 \text{ such that} \\ \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \cdot \boldsymbol{\varepsilon} \geqslant m_{\mathcal{A}} \, \|\boldsymbol{\varepsilon}\|^2 \quad \forall \, \boldsymbol{\varepsilon} \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \end{cases} \tag{7.1}$$

$$p(\boldsymbol{x}, r) = 0 \quad \forall \, r \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \tag{7.2}$$

Using these assumptions it follows that the operator (6.4) is given by

$$(A\boldsymbol{u}, \boldsymbol{v})_V = (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q \quad \forall \, \boldsymbol{u}, \, \boldsymbol{v} \in V. \tag{7.3}$$

and, moreover, it is a linear operator.

Next, we consider the operator $\mathcal{S}^* : C(\mathbb{R}_+; Q) \to C(\mathbb{R}_+; Q)$ defined by

$$\mathcal{S}^* \boldsymbol{\tau}(t) = \int_0^t \mathcal{B}(t-s) \boldsymbol{\tau}(s) \, \mathrm{d}s \quad \forall \, \boldsymbol{\tau} \in C(\mathbb{R}_+; Q), \ t \in \mathbb{R}_+ \tag{7.4}$$

and note that assumption (5.10) implies that $\mathcal{S}^*$ is a history-dependent operator. Therefore, using lemma 3 in the case $X = Q$, we obtain the following result.

LEMMA 9. *There exists a unique operator $\mathcal{R}^* : C(\mathbb{R}_+; Q) \to C(\mathbb{R}_+; Q)$ such that, for each functions $\boldsymbol{u} \in C(\mathbb{R}_+; V)$, $\boldsymbol{\sigma} \in C(\mathbb{R}_+; Q)$ the following equivalence hold:*

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{B}(t-s) \boldsymbol{\varepsilon}(\boldsymbol{u}(s)) \, \mathrm{d}s \quad \forall \, t \in \mathbb{R}_+, \tag{7.5}$$

if and only if

$$\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) = \mathcal{A}^{-1} \boldsymbol{\sigma}(t) + \mathcal{R}^* \boldsymbol{\sigma}(t) \quad \forall \, t \in \mathbb{R}_+. \tag{7.6}$$

Moreover, $\mathcal{R}^*$ is a history-dependent operator.

In addition of the result in lemma 9, we recall that the operator $\mathcal{R}^*$ is of the form

$$\mathcal{R}^* \boldsymbol{\sigma}(t) = \int_0^t \mathcal{B}^*(t-s) \boldsymbol{\sigma} \, \mathrm{d}s \quad \forall \, \boldsymbol{\sigma} \in C(\mathbb{R}_+; Q), \ \ t \in \mathbb{R}_+,$$

where $\mathcal{B}^* \in C(\mathbb{R}_+; \mathbf{Q}_\infty)$ represents the creep tensor. Details on the inverse of the viscoelastic constitutive law (7.5) by using the creep tensor could be found in [**15**], for instance.

Now, for each $t \in \mathbb{R}_+$ we introduce the set of admissible stress fields defined by

$$\Sigma(t) = \{ \, \boldsymbol{\tau} \in Q : (\boldsymbol{\tau}, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{g}))_Q \geqslant (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{g})_V \quad \forall \, \boldsymbol{v} \in U \, \} \tag{7.7}$$

where, recall, $\boldsymbol{g} = g_a \boldsymbol{\zeta}$ was defined in the proof of Theorem 8. Then, using the operator $\mathcal{R}^*$ defined in lemma 9, we consider the following variational problem.

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Problem $\mathcal{Q}_V^D$. *Find a function* $\boldsymbol{\sigma} : \mathbb{R}_+ \to Q$ *such that, for all* $t \in \mathbb{R}_+$, *the inequality below holds:*

$$\boldsymbol{\sigma}(t) \in \Sigma(t),\ (\mathcal{A}^{-1}\boldsymbol{\sigma}(t), \boldsymbol{\tau} - \boldsymbol{\sigma}(t))_Q + (\mathcal{R}^*\boldsymbol{\sigma}(t), \boldsymbol{\tau} - \boldsymbol{\sigma}(t))_Q \geqslant (\boldsymbol{\varepsilon}(\boldsymbol{g}), \boldsymbol{\tau} - \boldsymbol{\sigma}(t))_Q \quad \forall\, \boldsymbol{\tau} \in \Sigma(t). \tag{7.8}$$

We refer to problem $\mathcal{Q}_V^D$ as the dual variational formulation of the contact problem $\mathcal{Q}$. The link between the variational problems $\mathcal{Q}_V$ and $\mathcal{Q}_V^D$, formulated in terms of displacements and stress, respectively, is provided by the following result.

Theorem 10. *Assume* (5.10), (5.11), (5.13), (7.1) *and* (7.2). *The following statements hold:*

(i) *If* $\boldsymbol{u}$ *is a solution to problem* $\mathcal{Q}_V$ *with regularity* $\boldsymbol{u} \in C(\mathbb{R}_+; V)$, *then the function*

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,\mathrm{d}s \quad \forall\, t \in \mathbb{R}_+ \tag{7.9}$$

is a solution to problem $\mathcal{Q}_V^D$, *with regularity* $\boldsymbol{\sigma} \in C(\mathbb{R}_+; V)$.

(ii) *Conversely, if* $\boldsymbol{\sigma}$ *is a solution to problem* $\mathcal{Q}_V^D$ *with regularity* $\boldsymbol{\sigma} \in C(\mathbb{R}_+; Q)$, *then there exists a unique function* $\boldsymbol{u} \in C(\mathbb{R}_+; V)$ *such that* (7.9) *holds and, moreover,* $\boldsymbol{u}$ *is a solution to problem* $\mathcal{Q}_V$.

To present the proof of Theorem 10 will need some additional notation and preliminaries. First, since $meas\,\Gamma_1 > 0$, the range of the deformation operator $\boldsymbol{\varepsilon}\colon V \to Q$, denoted $\boldsymbol{\varepsilon}(V)$, is a closed subspace of Q. A proof of this result can be found in [**18**, p. 87]. It is a direct consequence of the equality

$$\|\boldsymbol{v}\|_V = \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_Q \quad \forall\, \boldsymbol{v} \in V. \tag{7.10}$$

Denote by $\widetilde{P}\colon Q \to \boldsymbol{\varepsilon}(V)$ the orthogonal projection operator on $\boldsymbol{\varepsilon}(V) \subset Q$ and note that equality (7.10) shows that $\boldsymbol{\varepsilon}\colon V \to \boldsymbol{\varepsilon}(V)$ is an invertible operator. In what follows, we denote by $\boldsymbol{\varepsilon}^{-1}\colon \boldsymbol{\varepsilon}(V) \to V$ the inverse of $\boldsymbol{\varepsilon}$. The ingredients above allow to define the operator $\Theta\colon Q \to V$ by

$$\Theta\boldsymbol{\tau} = \boldsymbol{\varepsilon}^{-1}(\widetilde{P}\boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in Q. \tag{7.11}$$

Then, it is easy to see that

$$(\Theta\boldsymbol{\tau}, \boldsymbol{v})_V = (\boldsymbol{\tau}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q \quad \forall\, \boldsymbol{\tau} \in Q,\ \boldsymbol{v} \in V. \tag{7.12}$$

Next, for each $t \in \mathbb{R}_+$, we define the set $\overline{\Sigma}(t) \subset Q$ by equality

$$\overline{\Sigma}(t) = \{\, \overline{\boldsymbol{\tau}} \in V : (\overline{\boldsymbol{\tau}}, \boldsymbol{v} - \boldsymbol{g})_V \geqslant (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{g})_V \quad \forall\, \boldsymbol{v} \in U \,\} \tag{7.13}$$

where, recall, the set U is defined by (6.1). Then, using (7.7) and (7.12) we deduce that

$$\boldsymbol{\tau} \in \Sigma(t) \iff \Theta\boldsymbol{\tau} \in \overline{\Sigma}. \tag{7.14}$$

We are now in a position to provide the proof of the Theorem 10.

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Proof. (i) Assume that $\boldsymbol{u}$ is a solution to problem $\mathcal{Q}_V$ with regularity $\boldsymbol{u} \in C(\mathbb{R}_+; V)$, and let $\boldsymbol{\sigma}$ be given by (7.9) We also consider the function $\widetilde{\boldsymbol{\sigma}} \in C(\mathbb{R}_+; V)$ defined by

$$\widetilde{\boldsymbol{\sigma}}(t) = A\boldsymbol{u}(t) + \mathcal{S}\boldsymbol{u}(t) \quad \forall\, t \in \mathbb{R}_+ \tag{7.15}$$

where, recall, the operators A and $\mathcal{S}$ are defined by (7.3) and (6.2), respectively. Since $\mathcal{S}$ is a history-dependent operator, it follows from lemma 3 that there exists a unique history-dependent operator $\mathcal{R} : C(\mathbb{R}_+; V) \to C(\mathbb{R}_+; V)$, such that (7.15) holds if and only if

$$\boldsymbol{u}(t) = A^{-1}\widetilde{\boldsymbol{\sigma}}(t) + \mathcal{R}\widetilde{\boldsymbol{\sigma}}(t) \quad \forall\, t \in \mathbb{R}_+. \tag{7.16}$$

Let $t \in \mathbb{R}_+$. Then, theorem 4 (i) shows that

$$\begin{aligned} \widetilde{\boldsymbol{\sigma}}(t) \in \overline{\Sigma}(t), (A^{-1}\widetilde{\boldsymbol{\sigma}}(t), \overline{\boldsymbol{\tau}} - \widetilde{\boldsymbol{\sigma}}(t))_V + (\mathcal{R}\widetilde{\boldsymbol{\sigma}}(t), \overline{\boldsymbol{\tau}} - \widetilde{\boldsymbol{\sigma}}(t))_V \\ \geqslant (\boldsymbol{g}, \overline{\boldsymbol{\tau}} - \widetilde{\boldsymbol{\sigma}}(t))_V \quad \forall\, \overline{\boldsymbol{\tau}} \in \overline{\Sigma}(t). \end{aligned} \tag{7.17}$$

We now substitute (7.16) in (7.17) to deduce that

$$\widetilde{\boldsymbol{\sigma}}(t) \in \overline{\Sigma}(t), \quad (\overline{\boldsymbol{\tau}} - \widetilde{\boldsymbol{\sigma}}(t), \boldsymbol{u}(t))_V \geqslant (\boldsymbol{g}, \overline{\boldsymbol{\tau}} - \widetilde{\boldsymbol{\sigma}}(t))_V \quad \forall\, \overline{\boldsymbol{\tau}} \in \overline{\Sigma}(t). \tag{7.18}$$

In addition, using equality (7.15), the definitions of the operators A, $\mathcal{S}$ and (7.9) it is easy to see that

$$(\widetilde{\boldsymbol{\sigma}}(t), \boldsymbol{v})_V = (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q \quad \forall\, \boldsymbol{v} \in V. \tag{7.19}$$

Therefore, regularity $\widetilde{\boldsymbol{\sigma}}(t) \in \overline{\Sigma}(t)$ implies that

$$\boldsymbol{\sigma}(t) \in \Sigma(t). \tag{7.20}$$

Let $\boldsymbol{\tau} \in \Sigma(t)$. Then (7.14) allows us to test in (7.18) with $\overline{\boldsymbol{\tau}} = \Theta\boldsymbol{\tau}$. Moreover, using (7.12) and (7.19) it follows that

$$(\overline{\boldsymbol{\tau}}, \boldsymbol{u}(t))_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\tau})_Q, \quad (\boldsymbol{g}, \overline{\boldsymbol{\tau}})_V = (\boldsymbol{\varepsilon}(\boldsymbol{g}), \boldsymbol{\tau})_Q,$$

$$(\widetilde{\boldsymbol{\sigma}}(t), \boldsymbol{u}(t))_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\sigma}(t))_Q, \quad (\boldsymbol{g}, \widetilde{\boldsymbol{\sigma}}(t))_V = (\boldsymbol{\varepsilon}(\boldsymbol{g}), \boldsymbol{\sigma}(t))_Q.$$

Substituting these inequalities in (7.18) yields

$$(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\tau} - \boldsymbol{\sigma}(t))_Q \geqslant (\boldsymbol{\varepsilon}(\boldsymbol{g}), \boldsymbol{\tau} - \boldsymbol{\sigma}(t))_Q. \tag{7.21}$$

Recall also that (7.9) and lemma 9 imply that (7.6) holds. Therefore, using (7.6) and (7.21) we deduce that

$$(\mathcal{A}^{-1}\boldsymbol{\sigma}(t), \boldsymbol{\tau} - \boldsymbol{\sigma}(t))_Q + (\mathcal{R}^*\boldsymbol{\sigma}(t), \boldsymbol{\tau} - \boldsymbol{\sigma}(t))_Q \geqslant (\boldsymbol{\varepsilon}(\boldsymbol{g}), \boldsymbol{\tau} - \boldsymbol{\sigma}(t))_Q. \tag{7.22}$$

We now combine (7.20) and (7.22) to see that (7.8) holds, which concludes the first part of the proof.

(ii) Conversely, let $\boldsymbol{\sigma}$ be a solution to problem $\mathcal{Q}_V^D$ with regularity $\boldsymbol{\sigma} \in C(\mathbb{R}_+; Q)$ and define the function $\widetilde{\boldsymbol{\varepsilon}} \in C(\mathbb{R}_+; Q)$ by equality

$$\widetilde{\boldsymbol{\varepsilon}}(t) = A^{-1}\boldsymbol{\sigma}(t) + \mathcal{R}^*\boldsymbol{\sigma}(t) \quad \forall\, t \in \mathbb{R}_+. \tag{7.23}$$

Let $t \in \mathbb{R}_+$. Then, substituting (7.23) in (7.8) yields

$$(\widetilde{\boldsymbol{\varepsilon}}(t), \boldsymbol{\tau} - \boldsymbol{\sigma}(t))_Q \geqslant (\boldsymbol{\varepsilon}(\boldsymbol{g}), \boldsymbol{\tau} - \boldsymbol{\sigma}(t))_Q \quad \forall\, \boldsymbol{\tau} \in \Sigma(t). \tag{7.24}$$

Consider now an element $\boldsymbol{z} \in Q$ such that

$$(z, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q = 0 \quad \forall\, \boldsymbol{v} \in V. \tag{7.25}$$

Since $\boldsymbol{\sigma}(t) \in \Sigma(t)$, using (7.25) and the definition (7.7) of the set $\Sigma(t)$ we deduce that $\boldsymbol{\sigma}(t) \pm \boldsymbol{z} \in \Sigma(t)$ therefore, testing in (7.24) with $\boldsymbol{\tau} = \boldsymbol{\sigma}(t) \pm \boldsymbol{z}$ we obtain that

$$(\widetilde{\boldsymbol{\varepsilon}}(t), \boldsymbol{z})_Q = (\boldsymbol{\varepsilon}(\boldsymbol{g}), \boldsymbol{z})_Q. \tag{7.26}$$

Using (7.26) and (7.25), we have that $\widetilde{\boldsymbol{\varepsilon}}(t) - \boldsymbol{\varepsilon}(\boldsymbol{g}) \in \boldsymbol{\varepsilon}(V)^{\perp\perp}$, where the symbol $\perp$ represents the orthogonal complements in Q. On the contrary, since the space $\boldsymbol{\varepsilon}(V)$ is a closed subspace of Q we deduce that $\boldsymbol{\varepsilon}(V)^{\perp\perp} = \boldsymbol{\varepsilon}(V)$. This implies that there exists an elements $\widetilde{\boldsymbol{u}}(t) \in V$ such that $\widetilde{\boldsymbol{\varepsilon}}(t) - \boldsymbol{\varepsilon}(\boldsymbol{g}) = \boldsymbol{\varepsilon}(\widetilde{\boldsymbol{u}}(t))$ and, denoting $\boldsymbol{u}(t) = \widetilde{\boldsymbol{u}}(t) + \boldsymbol{g}$ we have

$$\widetilde{\boldsymbol{\varepsilon}}(t) = \boldsymbol{\varepsilon}(\boldsymbol{u}(t)). \tag{7.27}$$

We now compare equalities (7.23) and (7.27) to see that

$$\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) = A^{-1}\widetilde{\boldsymbol{\sigma}}(t) + \mathcal{R}^*\widetilde{\boldsymbol{\sigma}}(t)$$

and, using Lemme 9, we deduce that (7.9) holds. The regularity $\boldsymbol{u} \in C(\mathbb{R}_+; V)$ and the uniqueness of this function follows from (7.27) and (7.10).
Next, we prove that $\boldsymbol{u}$ is a solution to problem $\mathcal{Q}_V$. To this end, we consider the function $\widetilde{\boldsymbol{\sigma}} \in C(\mathbb{R}_+; V)$ defined by equality

$$\widetilde{\boldsymbol{\sigma}} = A\boldsymbol{u} + \mathcal{S}\boldsymbol{u}. \tag{7.28}$$

Then, lemma 3 implies that

$$\boldsymbol{u} = A^{-1}\widetilde{\boldsymbol{\sigma}} + \mathcal{R}\widetilde{\boldsymbol{\sigma}}. \tag{7.29}$$

In addition, using (7.9), the definitions of the operators A, $\mathcal{S}$ and (7.28), it is easy to see that

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q = (\widetilde{\boldsymbol{\sigma}}(t), \boldsymbol{v})_V \quad \forall\, \boldsymbol{v} \in V. \tag{7.30}$$

Let $\overline{\boldsymbol{\tau}} \in \overline{\Sigma}(t)$ and denote $\boldsymbol{\tau} = \boldsymbol{\varepsilon}(\overline{\boldsymbol{\tau}})$. Then, it is easy to see that $\boldsymbol{\tau} \in \Sigma(t)$. We now use (7.24) and (7.27) to obtain that

$$(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\tau} - \boldsymbol{\sigma}(t))_Q \geqslant (\boldsymbol{\varepsilon}(\boldsymbol{g}), \boldsymbol{\tau} - \boldsymbol{\sigma}(t))_Q.$$

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Thus, using (7.30) and equality $\boldsymbol{\tau} = \boldsymbol{\varepsilon}(\overline{\boldsymbol{\tau}})$ yields

$$(\boldsymbol{u}(t), \overline{\boldsymbol{\tau}} - \widetilde{\boldsymbol{\sigma}}(t))_V \geqslant (\boldsymbol{g}, \overline{\boldsymbol{\tau}} - \widetilde{\boldsymbol{\sigma}}(t))_V. \tag{7.31}$$

We now combine (7.29) and (7.31) to see that

$$(A^{-1}\widetilde{\boldsymbol{\sigma}}(t) + \mathcal{R}\widetilde{\boldsymbol{\sigma}}(t), \overline{\boldsymbol{\tau}} - \widetilde{\boldsymbol{\sigma}}(t))_V \geqslant (\boldsymbol{g}, \overline{\boldsymbol{\tau}} - \widetilde{\boldsymbol{\sigma}}(t))_V. \tag{7.32}$$

Moreover, since $\boldsymbol{\sigma}$ is a solution to problem $\mathcal{Q}_V^D$ we know that $\boldsymbol{\sigma}(t) \in \Sigma(t)$ and, therefore, equality (7.30) implies that

$$\widetilde{\boldsymbol{\sigma}}(t) \in \overline{\Sigma}(t). \tag{7.33}$$

Relations (7.32) and (7.33) show that the function $\widetilde{\boldsymbol{\sigma}} \in C(\mathbb{R}_+; V)$ is a solution of a history-dependent problem which, with the terminology introduced in § 3, represents the dual variational inequality of the variational inequality (6.9). Therefore, using (7.28) and theorem 4 (ii) we deduce that $\boldsymbol{u}$ satisfies (6.9) at each time moment $t \in \mathbb{R}_+$, which concludes the proof.

□

The unique solvability of problem $\mathcal{Q}_V^D$ can be obtained by using theorem 5 and is as follows.

THEOREM 11. *Assume that* (5.10), (5.11), (5.13), (7.1) *and* (7.2) *hold. Then, problem* $\mathcal{Q}_V^D$ *has a unique solution which satisfies* $\boldsymbol{\sigma} \in C(\mathbb{R}_+; Q)$.

We also note that the continuous dependence of the solution of problem $\mathcal{Q}_V^D$ with respect a perturbation of the bound g_a can be easily obtained by using the abstract result provided by theorem 6. Since the details are obvious, we omit them.

A couple of functions $(\boldsymbol{u}, \boldsymbol{\sigma})$ which satisfies (6.9) and (7.8), respectively, at each $t \in \mathbb{R}_+$, is called a weak solution to the contact problem $\mathcal{Q}$. We conclude by theorems 7 and 11 that problem $\mathcal{Q}$ has a unique weak solution. Moreover, the solution depends continuously on the bound g_a.

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