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A NOTE ON DERIVED LENGTH AND CHARACTER DEGREES

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Abstract

Isaacs and Seitz conjectured that the derived length of a finite solvable group G is bounded by the cardinality of the set of all irreducible character degrees of G. We prove that the conjecture holds for G if the degrees of nonlinear monolithic characters of G having the same kernels are distinct. Also, we show that the conjecture is true when G has at most three nonlinear monolithic characters. We give some sufficient conditions for the inequality related to monolithic characters or real-valued irreducible characters of G when the commutator subgroup of G is supersolvable.

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1. Introduction

One of the famous open problems in character theory is the Taketa problem. This problem asks whether the inequality $\mathrm{dl}(G) \leq |\mathrm{cd}(G)|$ holds for every finite solvable group G, where $\mathrm{dl}(G)$ is the derived length of G and $|\mathrm{cd}(G)|$ is the cardinality of the set of all irreducible character degrees of G. The validity of this inequality was first seen by Taketa for M-groups (see [4, Theorem 5.12]). Later, Isaacs and Seitz conjectured that the inequality remains true for all solvable groups. Although this conjecture is still open, there are several results which prove that some classes of solvable groups satisfy this inequality. For example, Berger [1] showed that the Taketa inequality holds for all groups of odd order. Also, Isaacs proved that if $|\mathrm{cd}(G)| \leq 3$, then G is solvable and the inequality $\mathrm{dl}(G) \leq |\mathrm{cd}(G)|$ holds for G (see [4, Corollary 12.6 and Theorem 12.15]).

Let G be a finite group and let χ be an irreducible character of G. We say that χ is a real-valued irreducible character if $\chi(g)$ is a real number for all $g \in G$. We say that χ is a monolithic character when $G/\ker(\chi)$ has a unique minimal normal subgroup. We use the notation $\mathrm{Irr}_m(G)$ for the set of all monolithic characters of G. Real-valued irreducible characters and monolithic characters give some important results about

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the structure of G. For example, by [2, Ch. 30, Proposition 18], if the nonlinear monolithic character degrees are even or odd, then G has a normal 2-complement or an abelian normal Sylow 2-subgroup, respectively. It is well known that the order of G is odd if and only if 1_G is the unique real-valued irreducible character of G. Also, by [6, Theorem 2.1], G has a normal 2-complement if and only if every nonlinear real irreducible character degree is divisible by two. In addition to these, there are some results obtained by using these characters for the Taketa problem. For instance, by [3, Corollary 2.6], G is a solvable group and the inequality $\mathrm{dl}(G) \leq |\mathrm{cd}(G)|$ holds for G when every nonlinear real-valued irreducible character of G has odd or even degree. Similarly, from [3, Corollary 2.7], G satisfies the Taketa inequality if the nonlinear monolithic character degrees are odd or even.

In this paper, we provide further sufficient conditions related to monolithic and real irreducible characters for the Taketa problem.

2. The main results

Assume that G is a counterexample of the smallest possible order for the following results whose hypotheses are inherited by factor groups. Then, by [7, Proposition 2.7], we know that G has a unique minimal normal subgroup M, which is an elementary abelian p-group for some prime p and also $\mathrm{cd}(G) = \mathrm{cd}(G/M)$. It follows that the Fitting subgroup $\mathrm{F}(G)$ is a p-group and G has at least one faithful irreducible character. All faithful irreducible characters of G are monolithic. We observe that an irreducible character χ of G is faithful if and only if $M \nleq \ker(\chi)$. This leads to

$$|G| = |G : G'| + \sum_{\substack{\chi \in \mathrm{Irr}_1(G) \\ M \leq \ker(\chi)}} \chi(1)^2 + \sum_{\substack{\chi \in \mathrm{Irr}_1(G) \\ M \nleq \ker(\chi)}} \chi(1)^2, \tag{2.1}$$

$$|G/M| = |G : G'| + \sum_{\substack{\chi \in \mathrm{Irr}_1(G) \\ M \leq \ker(\chi)}} \chi(1)^2, \tag{2.2}$$

where $\mathrm{Irr}_1(G)$ is the set of all nonlinear irreducible characters of G. Thus,

$$\sum_{\substack{\chi \in \mathrm{Irr}_1(G) \\ M \nleq \ker(\chi)}} \chi(1)^2 = |G| - |G : M|, \tag{2.3}$$

$$\sum_{\substack{\chi \in \mathrm{Irr}_1(G) \\ M \leq \ker(\chi)}} \chi(1)^2 = |G : M| - |G : G'|. \tag{2.4}$$

On the other hand, $\mathrm{F}(G)$ is not abelian. Otherwise, we get the contradiction

$$\mathrm{dl}(G) \leq \mathrm{dl}(G/\mathrm{F}(G)) + \mathrm{dl}(\mathrm{F}(G)) \leq |\mathrm{cd}(G/\mathrm{F}(G))| + 1 \leq |\mathrm{cd}(G)|$$

by using [4, Theorem 12.19 and Corollary 12.20]. Thus, $1 < M \leq \mathrm{F}'(G)$. By Clifford's theorem, p divides every faithful character degree in $\mathrm{cd}(G)$. Also, G has nonlinear real-valued irreducible characters of odd and even degree and, similarly, G has nonlinear monolithic characters of odd and even degree by [3, Corollaries 2.6 and 2.7].

By [3, Theorem B], finite groups whose nonlinear monolithic characters have distinct degrees satisfy the Taketa inequality. Although the proof of the following theorem is almost the same as the proof of [3, Theorem B], it is definitely a generalisation of [3, Theorem B]. For example, the semidirect product of C_2^2 and A_4 acting via C_3 is a solvable group satisfying the conditions of Theorem 2.1 but this group does not satisfy the hypothesis of [3, Theorem B] since it has five monolithic characters of degree three.

Theorem 2.1. *Let G be a finite solvable group with the property that the degrees of nonlinear monolithic characters having the same kernels are distinct. Then the inequality* $\mathrm{dl}(G) \leq |\mathrm{cd}(G)|$ *holds for G.*

Proof. Let G be a counterexample of the smallest possible order. Since $\mathrm{Irr}_m(G/N) = \mathrm{Irr}(G/N) \cap \mathrm{Irr}_m(G)$ for all $N \trianglelefteq G$, the hypothesis of the theorem is inherited by factor groups. By the first paragraph of this section, all faithful irreducible characters of G are monolithic. Thus, we know that the degrees of all faithful irreducible characters of G are distinct by the hypothesis of the theorem. This gives the inequality

$$\sum_{\substack{\chi \in \mathrm{Irr}_1(G) \\ M \nleq \ker(\chi)}} \chi(1)^2 \leq \sum_{\substack{\theta \in \mathrm{Irr}_1(G) \\ M \leq \ker(\theta)}} \theta(1)^2 \tag{2.5}$$

since $\mathrm{cd}(G) = \mathrm{cd}(G/M)$. From (2.3), (2.4) and (2.5), $(|M| - 2)|G'| \leq -|M| \leq -2$, which is a contradiction. □

The following corollary is a generalisation of [3, Theorem C]. The semidirect product of C_3 and S_4 acting via C_2 is a real solvable group which does not satisfy the hypothesis of [3, Theorem C] since it has four monolithic characters of degree two. But this group satisfies the hypothesis of the following corollary because the kernels of its monolithic characters of even degree are distinct.

Corollary 2.2. *Let G be a finite solvable group whose nonlinear monolithic characters of odd degrees are real-valued. Suppose that $\chi(1) \neq \theta(1)$ if $\ker(\chi) = \ker(\theta)$, where χ and θ are monolithic characters of even degree of G. Then the inequality* $\mathrm{dl}(G) \leq |\mathrm{cd}(G)|$ *holds for G.*

Proof. Assume that G is a counterexample of minimal order. Thus, G has a faithful irreducible character. By the same arguments as in the proof of [3, Theorem C], we can show that all faithful irreducible characters of G have even degrees because all nonlinear monolithic characters of odd degree of G are real-valued. The degrees of faithful irreducible characters are distinct by the hypothesis of the corollary. We finally get a contradiction by using the same arguments as in the proof of Theorem 2.1. □

Corollary 2.3. *Let G be a finite solvable group such that $\chi(1) = \max(\mathrm{cd}(G))$ for every nonlinear odd degree monolithic character χ of G. Assume that $\chi(1) \neq \theta(1)$ if $\ker(\chi) = \ker(\theta)$, where χ and θ are monolithic characters of even degree of G. Then the inequality* $\mathrm{dl}(G) \leq |\mathrm{cd}(G)|$ *holds for G.*

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Proof. Let G be a counterexample of the smallest possible order. For every nonlinear odd degree monolithic character χ of G, $\chi(1) = \max(\mathrm{cd}(G))$ and so $\ker(\chi) < \mathrm{F}(G)$ by [4, Theorem 12.19 and Corollary 12.20]. Thus, $G/\mathrm{F}(G)$ has a normal 2-complement by [2, Ch. 30, Proposition 18(b)] and $\mathrm{F}(G)$ must be a 2-group by [7, Theorem 2.2]. So, all faithful irreducible characters have even degrees by the first paragraph of this section. The degrees of faithful irreducible characters are distinct by the hypothesis of the corollary. We complete the proof by using the same arguments as in the proof of Theorem 2.1. □

Theorem 2.4. *Let G be a finite solvable group such that $|\mathrm{Irr}_{1,m}(G)| \leq 3$, where $\mathrm{Irr}_{1,m}(G)$ is the set of all nonlinear monolithic characters of G. Then G satisfies the inequality* $\mathrm{dl}(G) \leq |\mathrm{cd}(G)|$.

Proof. If G has only one nonlinear monolithic character, then G satisfies the Taketa inequality by [3, Corollary 2.7]. Next, we assume that $\mathrm{Irr}_{1,m}(G) = \{\chi_1, \chi_2\}$. If $\chi_1(1) = \chi_2(1)$, then the degrees of all nonlinear monolithic characters of G are either even or odd and G satisfies the Taketa inequality by [3, Corollary 2.7]. Therefore, we can suppose that $\chi_1(1) \neq \chi_2(1)$. This yields $\mathrm{dl}(G) \leq |\mathrm{cd}(G)|$ by [3, Theorem B].

It remains to analyse the case $|\mathrm{Irr}_{1,m}(G)| = 3$. Let $\mathrm{Irr}_{1,m}(G) = \{\chi_1, \chi_2, \chi_3\}$ and let G be a minimal counterexample. Since $\mathrm{Irr}_m(G/N) \subseteq \mathrm{Irr}_m(G)$ for all $N \trianglelefteq G$, the hypothesis of the theorem is inherited by factor groups G/N. Since all faithful irreducible characters of G are monolithic, G has at most three faithful irreducible characters. The number of faithful irreducible characters of G must be less than three. Otherwise, G satisfies the inequality since G/M is abelian. Assume that G has exactly one faithful irreducible character, say χ_1. Then

$$\chi_1(1)^2 \leq \sum_{\substack{\theta \in \mathrm{Irr}_1(G) \\ M \leq \ker(\theta)}} \theta(1)^2$$

since $\mathrm{cd}(G) = \mathrm{cd}(G/M)$. Finally, we get the contradiction $(|M| - 2)|G'| \leq -|M| \leq -2$ by using the natural equalities (2.1) and (2.2). It follows that G has exactly two faithful irreducible characters, say χ_1 and χ_2. By Theorem 2.1, we can assume that $\chi_1(1) = \chi_2(1)$. If $\chi_1(1)$ is even, then $\chi_3(1)$ is odd by [3, Corollary 2.7]. It follows that every nonlinear monolithic character of G/M has odd degree and, thus, G/M has an abelian normal Sylow 2-subgroup by [2, Ch. 30, Proposition 18(c)]. By Ito's theorem, we conclude that every irreducible character of G/M has odd degree, which is a contradiction since $\mathrm{cd}(G/M) = \mathrm{cd}(G)$. If $\chi_1(1)$ is odd, then $\chi_3(1)$ is even by [3, Corollary 2.7]. Thus, G/M has a normal 2-complement by [2, Ch. 30, Proposition 18(b)]. Therefore, G has a normal 2-complement since the order of M is odd by the first paragraph of this section. By [7, Theorem 2.2], we have a contradiction. This completes the proof. □

Theorem 2.5. *Let G be a finite solvable group whose nonlinear monolithic characters of odd degree are faithful. Then G satisfies the Taketa inequality.*

Proof. Let G be a minimal counterexample to the assertion. By [3, Corollary 2.7(b)], we can assume that G has at least one nonlinear monolithic character χ of odd degree, which is faithful by the hypothesis of the theorem. Also, by [3, Corollary 2.7], G/N satisfies the Taketa inequality for every $N \trianglelefteq G$. Thus, G has a unique minimal normal subgroup M, which is an elementary abelian p-group. By the first paragraph of this section, p divides every faithful irreducible character degree and so $p \neq 2$. Thus, G has a normal 2-complement since G/M has a normal 2-complement by [2, Ch. 30, Proposition 18(b)]. Finally, we get a contradiction by [7, Theorem 2.2]. □

Theorem 2.6. *Let G be a finite solvable group whose nonlinear monolithic characters of odd degree are real-valued. If all monolithic characters of even degree have the same kernels, then G satisfies the Taketa inequality.*

Proof. Suppose that G is a counterexample of minimal order. We recall that G has at least one faithful irreducible character. Since all nonlinear monolithic characters of odd degree of G are real-valued, all faithful irreducible characters of G have even degree by the same arguments as in the proof of [3, Theorem C]. Thus, every monolithic character of even degree of G is faithful by the hypothesis of the theorem. By [2, Ch. 30, Proposition 18(c)], G/M has an abelian normal Sylow 2-subgroup, where M is the unique minimal normal subgroup of G. By Ito's theorem, every irreducible character degree of G/M is odd, contradicting $\mathrm{cd}(G/M) = \mathrm{cd}(G)$. □

Let $N \trianglelefteq G$. We write

$$\mathrm{Irr}(G|N) = \{\chi \in \mathrm{Irr}(G) \mid N \not\leq \ker(\chi)\}$$

and

$$\mathrm{cd}(G|N) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G|N)\}.$$

We see that $\mathrm{Irr}(G|G')$ is exactly the set of all nonlinear irreducible characters of G and so $|\mathrm{cd}(G|G')| = |\mathrm{cd}(G)| - 1$. If G is a finite group whose commutator subgroup G' is nilpotent, then

$$\mathrm{dl}(G) \leq \mathrm{dl}(G/G') + \mathrm{dl}(G') \leq 1 + |\mathrm{cd}(G|G')| = 1 + |\mathrm{cd}(G)| - 1 = |\mathrm{cd}(G)|$$

since $\mathrm{dl}(G') \leq |\mathrm{cd}(G|G')|$ by [5, Corollary 3.3].

Theorem 2.7. *Let G be a finite group whose commutator subgroup G' is supersolvable. If all nonlinear irreducible characters of odd degree of G are real-valued, then G is a solvable group and the Taketa inequality holds for G.*

Proof. Note that G is solvable since G/G' and G' are solvable. Suppose that G is a minimal counterexample to the assertion. Since every group of odd order satisfies the Taketa inequality by [1], the order of G must be even. As is easily seen, the hypotheses of the theorem are inherited by factor groups. Therefore, G has a unique minimal normal subgroup M, which is an elementary abelian p-group for some prime p and so $\mathrm{F}(G)$, the Fitting subgroup of G, is a p-group.

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Assume first that $p = 2$. Since G' is supersolvable, $Q \trianglelefteq G'$, where $Q \in \mathrm{Syl}_q(G')$ and q is the largest prime divisor of $|G'|$. Thus, $Q \trianglelefteq G$ because Q is a characteristic subgroup of G'. Therefore, $q = 2$ since $\mathrm{F}(G)$ is a 2-group. It follows that $G' \leq \mathrm{F}(G) \leq G$ and so G' is nilpotent. By [5, Corollary 3.3],

$$\mathrm{dl}(G) \leq \mathrm{dl}(G/G') + \mathrm{dl}(G') \leq 1 + |\mathrm{cd}(G|G')| \leq 1 + |\mathrm{cd}(G)| - 1 = |\mathrm{cd}(G)|.$$

Now, we can assume that $p \neq 2$. If all nonlinear irreducible character degrees of G are even, then G has a normal 2-complement by [4, Corollary 12.2] and so $\mathrm{dl}(G) \leq |\mathrm{cd}(G)|$ by [7, Theorem 2.2]. Thus, G has at least one nonlinear irreducible character χ of odd degree. This character χ is a real irreducible character of G by the hypothesis of the theorem. Also, $G'' \leq \mathrm{F}(G)$ since G' is supersolvable. Thus, the order of G'' is odd and so the trivial character $1_{G''}$ is the unique real-valued irreducible character of G''. By Clifford's theorem [4, Theorem 6.2], $\chi_{G''} = \chi(1)1_{G''}$ since χ is a real-valued irreducible character having odd degree. Therefore, $G'' \leq \ker(\chi)$ for every irreducible character χ of odd degree by the hypothesis of the theorem. Furthermore, G'' is a subgroup of the kernel of every linear irreducible character of G. This yields $|\mathrm{cd}(G|G'')| \leq |\mathrm{cd}(G)| - 2$ and, by [5, Corollary 3.3],

$$\mathrm{dl}(G) \leq \mathrm{dl}(G/G'') + \mathrm{dl}(G'') \leq 2 + |\mathrm{cd}(G|G'')| \leq 2 + |\mathrm{cd}(G)| - 2 = |\mathrm{cd}(G)|,$$

as desired. □

By [3, Corollary 3.2], if G is a finite real group and the commutator subgroup G' is supersolvable, then the Taketa inequality holds for G. This fact can also be seen as a corollary of Theorem 2.7.

Corollary 2.8. *Let G be a finite group and let the commutator subgroup G' be supersolvable. If $\chi(1) = \max(\mathrm{cd}(G))$ for every nonlinear monolithic character χ of odd degree of G, then G is a solvable group and the inequality* $\mathrm{dl}(G) \leq |\mathrm{cd}(G)|$ *holds for G.*

Proof. Suppose that G is a counterexample of minimum possible order. It is easily seen that the hypotheses of the corollary are inherited by factor groups since $\mathrm{Irr}_m(G/N) \subseteq \mathrm{Irr}_m(G)$ for all $N \trianglelefteq G$. So, G has a unique minimal normal subgroup M which is an elementary abelian p-group for some prime p and also the Fitting subgroup $\mathrm{F}(G)$ is a p-group. Since G' is supersolvable, we see that p is an odd prime by the second paragraph of the proof of Theorem 2.7. By [4, Theorem 12.19 and Corollary 12.20], $\ker(\chi) < \mathrm{F}(G)$ for all nonlinear monolithic characters χ of odd degree because $\chi(1) = \max(\mathrm{cd}(G))$. Therefore, by using [2, Ch. 30, Proposition 18(b)], $G/\mathrm{F}(G)$ has a normal 2-complement. Since p is an odd prime, G has a normal 2-complement. The result now follows from [7, Theorem 2.2]. □

Corollary 2.9. *Let G be a finite group and let the commutator subgroup G' be supersolvable. If $\chi(1) = \max(\mathrm{cd}(G))$ for every nonlinear real-valued irreducible character χ of odd degree of G, then G is a solvable group and the inequality* $\mathrm{dl}(G) \leq |\mathrm{cd}(G)|$ *holds for G.*

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Proof. Let G be a minimal counterexample. Then G has a unique minimal normal subgroup M, which is an elementary abelian p-group for some prime p and hence $F(G)$ is a p-group. Assume that p is odd. Then $F(G) \leq \ker(\chi)$ for every nonlinear real-valued irreducible character χ of odd degree of G, which is a contradiction by [4, Theorem 12.19 and Corollary 12.20]. It follows that $p = 2$. However, this also leads to a contradiction. Since G' is supersolvable, p is odd by the same arguments as in the second paragraph of the proof of Theorem 2.7. □

References

[1] T. R. Berger, 'Characters and derived length in groups of odd order', *J. Algebra* **39** (1976), 199–207.

[2] Y. G. Berkovich and E. M. Zhmud, *Characters of Finite Groups. Part 2*, Translations of Mathematical Monographs, 181 (American Mathematical Society, Providence, RI, 1999).

[3] B. Çınarcı and T. Erkoç, 'Real characters, monolithic characters and the Taketa inequality', *J. Algebra Appl.* **18**(10) (2019), Article ID 19501834. doi:10.1142/S0219498819501834.

[4] I. M. Isaacs, *Character Theory of Finite Groups* (Academic Press, New York, 1976).

[5] I. M. Isaacs and G. Knutson, 'Irreducible character degrees and normal subgroups', *J. Algebra* **199** (1998), 302–326.

[6] G. Navarro, L. Sanus and P. H. Tiep, 'Real characters and degrees', *Israel J. Math.* **171** (2009), 157–173.

[7] U. Yılmaztürk, T. Erkoç and İ. Ş. Güloğlu, 'Some sufficient conditions for the Taketa inequality', *Proc. Japan Acad.* **89** (2013), 103–106.

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