

# On intrinsic and extrinsic rational approximation to Cantor sets

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*Abstract.* We establish various new results on a problem proposed by Mahler [Some suggestions for further research. *Bull. Aust. Math. Soc.* **29** (1984), 101–108] concerning rational approximation to fractal sets by rational numbers inside and outside the set in question. Some of them provide a natural continuation and improvement of recent results of Broderick, Fishman and Reich, and Fishman and Simmons. A key feature is that many of our new results apply to more general, multi-dimensional fractal sets and require only mild assumptions on the iterated function system. Moreover, we provide a non-trivial lower bound for the distance of a rational number  $p/q$  outside the Cantor middle-third set  $C$  to the set  $C$ , in terms of the denominator  $q$ . We further discuss patterns of rational numbers in fractal sets. We highlight two of them: firstly, an upper bound for the number of rational (algebraic) numbers in a fractal set up to a given height (and degree) for a wide class of fractal sets; and secondly, we find properties of the denominator structure of rational points in ‘missing-digit’ Cantor sets, generalizing claims of Nagy and Bloshchitsyn.

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## 1. A question of Mahler and generalizations

1.1. *Introduction and notation.* In 1883, Cantor introduced what is today referred to as the Cantor middle-third set. This consists of the real numbers in  $[0, 1]$  whose infinite base 3 representation avoids the digit 1, that is, numbers of the form

$$\xi = \sum_{j \geq 1} \frac{w_j}{3^j}, \quad w_j \in \{0, 2\}.$$

In 1984, Mahler [21] proposed the problem of studying how well elements in this set can be approximated by rational numbers within the set, and rational numbers outside of it. Problems of this type are usually referred to as intrinsic and extrinsic approximation,

respectively. He noticed that convergents of the continued fraction expansion to numbers in the Cantor middle-third set may or may not lie in the Cantor middle-third set. His problem can be naturally generalized to wider families of fractal sets. The easiest twist is to take  $b \geq 3$  an integer and a digit set  $W \subseteq \{0, 1, \dots, b - 1\}$  of cardinality at least 2 and at most  $b - 1$ , and to consider all numbers in  $[0, 1]$  whose base  $b$  digits belong to  $W$ . We call such sets *missing-digit Cantor sets* and write  $C = C_{b,W}$ . We will consider wider classes of  $d$ -dimensional fractal sets, comprised in the following Definition 1. For convenience we always consider  $\mathbb{R}^d$  equipped with the supremum norm  $\|\underline{z}\| = \max_{1 \leq j \leq d} |z_j|$ ; however, all proofs below can be easily modified if we work with the usual Euclidean norm instead.

*Definition 1.* (IFS, Cantor set) We call a function  $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$  a *contraction* if, for some fixed  $0 < \tau < 1$ , we have

$$\|f(\underline{x}) - f(\underline{y})\| \leq \tau \|\underline{x} - \underline{y}\|, \quad \underline{x}, \underline{y} \in \mathbb{R}^d.$$

In this paper, an *iterated function system* (IFS)  $F = (f_1, \dots, f_J)$  is a finite set of contractions. We call the IFS a *similarity IFS* if the contractions are similarities, that is, for  $1 \leq j \leq J$  there exist  $0 < \tau_j < 1$  with the property

$$\|f_j(\underline{x}) - f_j(\underline{y})\| = \tau_j \|\underline{x} - \underline{y}\|, \quad \underline{x}, \underline{y} \in \mathbb{R}^d.$$

We call the IFS *affine* if the contractions are affine functions on  $\mathbb{R}^d$ , that is, functions

$$f_j(\underline{y}) = A_j \underline{y} + \underline{b}_j, \quad 1 \leq j \leq J, \tag{1}$$

where  $A_j \in \mathbb{R}^{d \times d}$  and  $\underline{b}_j \in \mathbb{R}^d$ . We call an IFS *rational-preserving* if  $f_j(\mathbb{Q}^d) \subseteq \mathbb{Q}^d$  for  $1 \leq j \leq J$ . Any IFS induces a compact set  $C \subseteq \mathbb{R}^d$ , called the attractor of the IFS, given as the unique solution of  $\bigcup_{1 \leq j \leq J} f_j(C) = C$ . We call any such  $C$  a *Cantor set* and carry over the definitions above in the obvious way to Cantor sets (for example,  $C$  is called an affine and rational-preserving Cantor set if the corresponding IFS is affine and rational-preserving).

We discuss relations between the notions of the above definition. Any similarity IFS is an affine IFS with matrices  $A_j = S_j \cdot O_j$  for orthogonal matrices  $O_j$  and for  $S_j = \tau_j I_d$  scalar multiples of the identity matrix; see Hutchinson [16]. Hence any similarity Cantor set is an affine Cantor set, and for  $d = 1$  the concepts coincide. An affine IFS is rational-preserving if and only if  $A_j \in \mathbb{Q}^{d \times d}$  and  $\underline{b}_j \in \mathbb{Q}^d$  for  $1 \leq j \leq J$ . By the above characterization of  $A_j$ , to obtain a rational-preserving similarity Cantor set we require orthogonal matrices with rational entries (then we can choose  $\tau_j \in \mathbb{Q}$ ). A comprehensive description of all such matrices was given in [20]. Clearly, a special choice is  $\tau_j \in \mathbb{Q} \cap (-1, 1)$  and  $O_j = I_d$  for  $1 \leq j \leq J$ . There are also various examples of non-affine, rational-preserving IFS. For  $d = 1$ , one may consider  $f(y) = f_j(y)$  any collection of rational functions  $cP(y)/Q(y) + y/2$  with  $P, Q \in \mathbb{Q}[Y]$ , with  $Q$  not having real roots, the degree of  $P$  not exceeding the degree of  $Q$ , and rational  $c > 0$  sufficiently small to guarantee  $|f'(y)| \in [\epsilon, 1 - \epsilon]$  uniformly, for example  $f(y) = y/(3 + 3y^2) + y/2$ . It is further possible to construct real analytic, transcendental functions with this property (here transcendental means that  $f$  does not satisfy a polynomial identity  $P(z, f(z)) = 0$

with  $P \in \mathbb{C}[X, Y]$ ); it suffices to multiply the functions obtained by Marques and Moreira in [22, Theorem 1.2] by any non-zero rational factor of absolute value less than  $2/3$ .

The existence of the unique fixed point  $C$  in Definition 1 follows from Banach's fixed point theorem [16] applied with respect to the Hausdorff metric on compact subsets of  $\mathbb{R}^d$ . Any  $\underline{\xi} \in C$  has an *address*  $(\omega_1, \omega_2, \dots) \in \{1, 2, \dots, J\}^{\mathbb{N}}$ , that is, it can be written in the form

$$\underline{\xi} = \lim_{i \rightarrow \infty} \pi_1 \circ \dots \circ \pi_i(\underline{0}), \quad \pi_j = f_{\omega_j} \in F.$$

Addresses may not be unique: to guarantee uniqueness one typically has to assume the so-called strong separation condition (SSC); see Definition 2 below. We do not dig deeper into this topic here. We will further need the open set condition (OSC) for some of our results.

*Definition 2.* (OSC, SSC) An IFS satisfies the *open set condition* if there exists a bounded open set  $O \subseteq \mathbb{R}^d$  so that  $f_j(O) \subseteq O$  for  $1 \leq j \leq J$  and for  $i \neq j$  we have  $f_i(O) \cap f_j(O) = \emptyset$ . For simplicity we will say that a Cantor set  $C$  as in Definition 1 satisfies the OSC if its associated IFS does. A Cantor set  $C$  satisfies the *strong separation condition* if the sets  $f_j(C)$  are disjoint.

We briefly discuss measures and dimension of fractals. Any Cantor set  $C$  in Definition 1 supports a natural probability measure by Frostmann's lemma (see [11]). For similarity Cantor sets that satisfy the OSC, it is just a multiple of some  $\Delta$ -dimensional Hausdorff measure, and there is a well-known general formula to determine  $\Delta$  in terms of the contraction factors [16]. See also [23] for a generalization. Clearly this value  $\Delta \in [0, d]$  equals the Hausdorff dimension of  $C$ . For  $C_{b,W}$  it just becomes  $\Delta = \log |W| / \log b$ . We refer to Falconer [11] for an introduction to metric theory on fractals. Rational approximation to Cantor type sets has been intensely studied, especially metrical questions with respect to the measure mentioned. However, until recently, approximation to fractals in the sense of Mahler's question with restrictions on the rationals had not been studied in detail. Only in 2007 was a first attempt made by Levesley, Salp and Velani [19]. In 2011 a paper by Broderick, Fishman and Reich [5] dealt with intrinsic approximation. Two recent papers by Fishman and Simmons [13, 14] shed more light on the topic as well. We rephrase important results from [5, 13, 14] in §1.2 below. The purpose of this paper is to establish further results on Mahler's question, where possible in the very general settings of Definition 1. In §2 we generalize results on intrinsic approximation from [5, 13]. The focus of the paper lies on extrinsic approximation in §3. As a byproduct we obtain related results, especially on the cardinality of rational/algebraic vectors of bounded degree and height and their period lengths. Moreover, we gain some new insight into the structure of rational numbers in missing-digit Cantor sets  $C_{b,W}$ . We gather all these findings in §4.

**1.2. Recent results on Mahler's problem.** The main result of [5] shows a Dirichlet type result for intrinsic rational approximation to missing-digit Cantor sets  $C_{b,W}$ . We present a slightly simplified version to avoid dealing with technical details.

**THEOREM 1.1.** (Broderick, Fishman and Reich [5]) *Let  $C = C_{b,W}$  be a missing-digit Cantor set and  $\xi \in C$ . If  $\Delta = \log |W| / \log b$  denotes the Hausdorff dimension of  $C$ , the*

inequalities

$$1 \leq q \leq b^{Q^\Delta}, \quad \left| \xi - \frac{p}{q} \right| \leq \frac{b}{Qq}, \tag{2}$$

have a solution  $p/q \in \mathbb{Q} \cap C$  for every parameter  $Q \geq 1$ . In particular, if  $\xi \notin \mathbb{Q}$ , there exist infinitely many  $p/q \in \mathbb{Q} \cap C$  with the property

$$\left| \xi - \frac{p}{q} \right| \leq \frac{1}{q(\log_b q)^{1/\Delta}}. \tag{3}$$

As observed in [5], the bounds (2) and (3) become weaker when for fixed  $b$  we extend the digit set and thereby increase the Hausdorff dimension  $\Delta$ . This seems counter-intuitive as there are more rational numbers in  $C$ ; however, in larger Cantor sets there exist more irrational  $\xi \in C$  to be approximated as well. A generalization of Theorem 1.1 was given by Fishman and Simmons [13].

**THEOREM 1.2.** (Fishman and Simmons [13]) *Let  $C \subseteq \mathbb{R}$  be a one-dimensional, affine, rational-preserving Cantor set, derived from an IFS as in (4), that satisfies the OSC. Let  $\Delta$  be its Hausdorff dimension and let  $\gamma$  be as in (5). Let  $\xi \in C$ . Then there is a constant  $K$  such that for any  $Q \geq \max_{1 \leq j \leq J} q_j$  the estimate*

$$\left| \xi - \frac{p}{q} \right| \leq Kq^{\gamma-1}(\log Q)^{-1/\Delta}$$

admits a solution  $p/q \in \mathbb{Q} \cap C$  with  $1 \leq q \leq Q$ .

For one-dimensional, affine, rational-preserving Cantor sets as in the theorem,  $C$  is the attractor of an IFS

$$f_j(y) = \frac{p_j}{q_j}y + \frac{r_j}{q_j}, \quad 1 \leq j \leq J, \tag{4}$$

for  $p_j/q_j$  of absolute value at most 1 and  $r_j/q_j$  a rational number. We can assume that  $q_j > 0$ . Then let

$$\gamma = \max_{1 \leq j \leq J} \frac{\log |p_j|}{\log q_j}. \tag{5}$$

*Definition 3.* If  $d = 1$ , we call an affine, rational-preserving IFS *monic* if  $p_j \in \{1, -1\}$  in (4) for all  $1 \leq j \leq J$ , or equivalently if  $\gamma = 0$ . We call the derived Cantor set a *one-dimensional, affine, monic, rational-preserving Cantor set*.

Note that the missing Cantor sets  $C_{b,W}$  defined above are monic. Indeed, the similarities can be written  $f_j(y) = y/b + w_j/b$  for  $w_j \in W$ , with every contraction factor equal to  $1/b$ . For monic Cantor sets as in the theorem where  $\gamma = 0$ , the estimate becomes

$$\left| \xi - \frac{p}{q} \right| \leq Kq^{-1}(\log Q)^{-1/\Delta}.$$

We thus identify Theorem 1.1 (up to the value of the constant) as a special case. We turn to extrinsic approximation. A result of Fishman and Simmons reads as follows [14, Corollary 1.2].

**THEOREM 1.3.** (Fishman and Simmons [14]) *Let  $C \subseteq \mathbb{R}^d$  be a Cantor set that satisfies the OSC and does not contain a line segment. Assume the additional property that for any compact set  $K \subseteq \mathbb{R}^d$  there is a constant  $\kappa > 0$  with*

$$\|f_j(\underline{x}) - f_j(\underline{y})\| \geq \kappa \|\underline{x} - \underline{y}\|, \quad \underline{x}, \underline{y} \in K, \quad 1 \leq j \leq J. \quad (6)$$

*Let  $\underline{\xi} \in C \setminus \mathbb{Q}^d$ . Then, for some constant  $c = c(C) > 0$ , there exist infinitely many  $\underline{p}/q \in \mathbb{Q}^d \setminus C$  such that*

$$\|\underline{\xi} - \underline{p}/q\| \leq c \cdot q^{-1-1/d}.$$

Originally this result was only formulated for the narrower class of similarity Cantor sets in [14]; however, the proof given extends to our more general situation, once small modifications of the proof of [14, Lemma 2.12] have been made. Indeed, our local hypothesis (6) on the contractions suffices to derive  $\gamma$  as in its proof, and the equality in the last displayed formula must be altered to ‘greater than or equal to’, not affecting the implication of the lemma. Up to the constant  $c$ , the bound is of best possible order we can expect for generic  $\underline{\xi} \in C$ . As observed in [14], if  $d = 1$ , by Dirichlet’s theorem the claim follows directly with  $c = 1$  whenever infinitely many convergents to  $\xi$  lie outside  $C$ . However, it was recently shown [25] that any missing Cantor set  $C = C_{b,W}$  contains irrational numbers  $\xi$  with almost all convergents in  $C_{b,W}$ . The actual proof of Theorem 1.3 employed a variant of Lemma 5.3 below.

## 2. Intrinsic approximation

In this section we provide a variant of Theorem 1.2 for Cantor sets in higher dimension.

**THEOREM 2.1.** *Let  $C \subseteq \mathbb{R}^d$  be a rational-preserving affine Cantor set, that is, the attractor of an IFS  $F$  consisting of contraction maps*

$$f_j(\underline{y}) = \frac{A_j \underline{y}}{q_j} + \frac{\underline{b}_j}{s_j}, \quad 1 \leq j \leq J,$$

*where  $A_j \in \mathbb{Z}^{d \times d}$ ,  $\underline{b}_j \in \mathbb{Z}^d$  and  $q_j, s_j \in \mathbb{N}$ . Let  $\tau_j \in (0, 1)$  be the contraction factor of  $f_j$  and  $\tau = \max_{1 \leq j \leq J} \tau_j$ . Further, let  $S := \prod_{1 \leq j \leq J} s_j$  and*

$$\mu_j = \frac{\log \tau_j}{\log q_j}, \quad \mu = \max_{1 \leq j \leq J} \mu_j < 0.$$

*Then, for any  $\underline{\xi} \in C$  and any parameter  $Q \geq S \cdot 2^d (\max_{1 \leq j \leq J} q_j)^d$ , there exists  $\underline{p}/q \in C \cap \mathbb{Q}^d$  with the properties*

$$1 \leq q \leq Q, \quad \|\underline{\xi} - \underline{p}/q\| \leq q^{\mu/d} (\log Q)^{\log \tau / \log J}.$$

If  $d = 1$  and we identify  $\gamma - 1 = \mu$  with  $\gamma$  in (5), we almost obtain Theorem 1.2. One difference is a slightly altered exponent for  $\log Q$ . We believe the optimal exponent for  $\log Q$  is again  $-1/\Delta$  for  $\Delta$  the Hausdorff dimension of  $C$ . For similarities  $f_j$ , our exponent is slightly worse, unless in the special case that all contraction factors  $\tau_j$  coincide where the identity  $\log \tau / \log J = -1/\Delta$  can be readily verified [16]. In particular, we identify Theorem 1.1 as a special case of Theorem 2.1. On the other hand, even for  $d = 1$  our Theorem 2.1 is slightly more general than Theorem 1.2 in the sense that we

do not require  $s_j = q_j$  as in (4), so  $\mu/d = \mu$  is in general better than the inferred constant  $\gamma - 1$  obtained from transitioning to representation (4). Observe that  $\mu/d \in [-1/d, 0)$ , with  $\mu/d = -1/d$  for instance if all  $A_j$  are permutation matrices (with optional sign conversions 1 to  $-1$ ). Note also that in contrast to [13] we do not require the OSC for the conclusion of Theorem 2.1.

### 3. Extrinsic approximation

3.1. *Uniform extrinsic approximation.* A topic that seems to be untouched so far is uniform extrinsic approximation to Cantor sets. Theorem 3.3 below shows that under mild assumptions on the underlying IFS, any element of the derived Cantor set is uniformly approximable by rationals outside  $C$  with denominator at most  $Q$  of order  $Q^{-1}$ , and this bound is essentially optimal. It is evident that an improvement as for intrinsic approximation in Theorems 1.1 and 1.2 cannot be achieved for extrinsic approximation. Another interpretation is that the exponent  $-1 - 1/d$  from Theorem 1.3 is not valid when it comes to uniform approximation. The result follows from a combination of Theorems 3.1 and 3.2, which are formulated in very general settings.

**THEOREM 3.1.** *Let  $C \subseteq \mathbb{R}^d$  be any Cantor set arising from an IFS with contraction ratios  $\tau_j$ ,  $1 \leq j \leq J$ , and let  $\tau = \max_{1 \leq j \leq J} \tau_j$  and  $D := -\log \tau / \log J > 0$ . Assume that either*

- *$C$  satisfies the OSC and (6), and there is a vector  $\underline{v} \in \mathbb{Z}^d$  such that  $C$  contains no line segment parallel to  $\underline{v}$ ; or*
- *we have  $D < 1/2$ .*

*Then there exists a constant  $K = K(C)$  so that, for every  $\underline{\xi} \in C$  and every  $Q \geq 1$ , the inequality*

$$\|\underline{\xi} - \underline{p}/q\| \leq \frac{K}{Q} \tag{7}$$

*has a solution  $\underline{p}/q \in \mathbb{Q}^d \setminus C$  with  $1 \leq q \leq Q$ .*

We believe that the weak assumption on line segments just means that  $C$  has empty interior; however, for  $d > 1$  a proof of this would be desirable. The implication from this first assumption is in fact a straightforward consequence of results in [14]. The alternative latter assumption essentially says that  $C$  has small Hausdorff dimension, and the implication is a consequence of a new counting result in this case. We now turn to the more challenging reverse estimates.

*Definition 4.* We call a vector  $\underline{v} \in \mathbb{R}^d$  *irrational* if it has at least one irrational coordinate, so that  $\underline{v} \notin \mathbb{Q}^d$ .

**THEOREM 3.2.** *Let  $C \subseteq \mathbb{R}^d$  be a rational-preserving Cantor set. Assume that there is  $f_j \in F$  whose unique fixed point  $\underline{\alpha}_j$  lies in  $\mathbb{Q}^d$ . Assume further that either*

- *all contraction maps  $f_j \in F$  are one-to-one; or*
- *any element in  $C$  has at most countably many addresses (which is true if  $C$  satisfies the SSC).*

*Let  $\Phi : \mathbb{N} \rightarrow \mathbb{R}_{>0}$  be any function that tends to 0 (arbitrarily slowly). Then the set of*

irrational  $\underline{\xi} \in C$  for which

$$\|\underline{\xi} - \underline{p}/q\| > \frac{\Phi(Q)}{Q} \quad (8)$$

holds for infinitely many  $Q \in \mathbb{N}_{>0}$  and every  $\underline{p}/q \in \mathbb{Q}^d \setminus C$  with  $1 \leq q \leq Q$ , is uncountable and dense in  $C$ .

*Remark 1.* Clearly the claim holds if  $\underline{\xi} \in C \cap \mathbb{Q}^d$ , and such vectors are dense in rational-preserving Cantor sets  $C$  (see also the last claim of Theorem 3.3 and its proof below). Hence the irrationality in the claim is an important issue. Ideally we would like to sharpen the claim by requiring the coordinates of  $\underline{\xi} \in C$  to be  $\mathbb{Q}$ -linearly independent together with  $\{1\}$ . We may infer this strengthened version if any hyperplane in  $\mathbb{R}^d$  intersects  $C$  in at most countably many points; however, this assumption seems unnatural.

*Remark 2.* It is possible to relax the assumption that the contraction maps  $f_i$  are one-to-one in various ways. However, we are unable to provide a very natural assumption and do not further elaborate on it here. The condition appears to be unrelated to the OSC, and we believe the OSC is not sufficient to imply the alternative assumption either; however, we are not aware of any concrete example. See Sidorov [29] for similarity Cantor sets in which all but finitely many elements have uncountably many addresses, which however do not satisfy the OSC.

While not satisfied in general, the condition of the rational fixed point in Theorem 3.2 holds if  $C$  is a rational-preserving *affine* Cantor set, derived from an IFS as in (1), as then clearly  $\underline{\alpha}_j = -(A_j - I_d)^{-1} \underline{b}_j \in \mathbb{Q}^d$  (note that 1 is not an eigenvalue of  $A_j$  since it induces a contraction). We may choose  $A_j$  rational scalar multiples of the identity matrix to obtain an IFS that consists of similarity contractions

$$f_j(\underline{y}) = \tau_j \underline{y} + \underline{b}_j, \quad \tau_j \in \mathbb{Q} \cap (-1, 1) \setminus \{0\}, \quad \underline{b}_j \in \mathbb{Q}^d. \quad (9)$$

Recall that a Liouville number is an irrational real number with the property that  $|\xi - p/q| < q^{-N}$  has a rational solution  $p/q$  for arbitrarily large  $N$ . We call  $\underline{\xi} \in \mathbb{R}^d$  a Liouville vector if accordingly  $|\underline{\xi} - \underline{p}/q| < q^{-N}$  has infinitely many solutions for every  $N$ . Theorem 3.2 is essentially equivalent to requiring that  $C$  contains points that are arbitrarily well approximable by rational vectors inside  $C$  (in particular,  $C$  must contain ‘intrinsic Liouville vectors’). Indeed, the proof of Theorem 3.2 relies on the construction of such intrinsic Liouville vectors. Combining these observations, from Theorems 3.1 and 3.2 we immediately deduce the following result.

**THEOREM 3.3.** *Let  $C \subseteq \mathbb{R}^d$  be a rational-preserving similarity Cantor set, for instance derived from an IFS of contractions as in (9). Assume either that the OSC is satisfied and there is a vector  $\underline{v} \in \mathbb{Z}^d$  such that  $C$  contains no line segment parallel to  $\underline{v}$ , or  $D < 1/2$  with  $D$  as in Theorem 3.1. Then there exists  $K > 0$  such that for any  $\underline{\xi} \in C$  and any  $Q > 1$  the inequality*

$$\|\underline{\xi} - \underline{p}/q\| \leq \frac{K}{Q}$$

has a solution  $\underline{p}/q \in \mathbb{Q}^d \setminus C$  with  $1 \leq q \leq Q$ . On the other hand, for any function  $\Phi : \mathbb{N} \rightarrow (0, \infty)$  that tends to 0, there exists  $\underline{\xi} \in C \setminus \mathbb{Q}$  for which

$$\|\underline{\xi} - \underline{p}/q\| > \frac{\Phi(Q)}{Q}$$

holds for certain arbitrarily large  $Q$  and any  $\underline{p}/q \in \mathbb{Q}^d \setminus C$  with  $1 \leq q \leq Q$ . Finally, for  $\Psi : \mathbb{N} \rightarrow (0, \infty)$  any function, there are intrinsically  $\Psi$ -approximable vectors in  $C$ , defined as the vectors  $\underline{\xi} \in C$  for which the inequality

$$\|\underline{\xi} - \underline{p}/q\| < \Psi(q)$$

admits infinitely many solutions  $\underline{p}/q \in \mathbb{Q}^d \cap C$ . In particular,  $C$  contains Liouville vectors.

As observed in §1.1 a wider class of suitable IFSs can be readily found with the aid of [20]. Any reasonable one-dimensional rational-preserving Cantor set with the OSC (in particular, any missing-digit Cantor set  $C_{b,W}$  (with  $W \subsetneq \{0, 1, \dots, b-1\}$ )) satisfies the assumptions of Theorem 3.3. This first claim does not require the rationality of the IFS; however, for the other claims it is probably needed. Intuitively, (similarity) Cantor sets containing no Liouville vector should exist when we drop the rationality condition. For  $d = 1$  and the class of topological Cantor sets, the existence was shown in [1]. See also [4] for fat Cantor sets without rational elements. We remark further that for rational-preserving similarity Cantor sets induced by diagonal matrices  $A_j$  that satisfy the OSC, the existence of very well-approximable vectors that are not Liouville vectors was shown by Baker [2, Theorem 5.6]. Notice also that the claim concerning Liouville vectors cannot be derived from standard metric results as in [13] even for  $C_{b,W}$ , as the set of Liouville numbers has Hausdorff dimension 0 due to Jarník [17].

3.2. *Ordinary extrinsic approximation.* We use Theorems 1.1 and 1.2 to obtain a lower bound on the distance of certain one-dimensional Cantor sets to a rational number not contained in it. This can be viewed as a reverse of Theorem 1.3. For  $A, B \subseteq \mathbb{R}$  we write  $d(A, B) = \inf\{|a - b| : a \in A, b \in B\}$  and denote by  $e = 2.7182 \dots$  Euler’s number.

THEOREM 3.4. *Let  $C$  be a one-dimensional, affine, rational-preserving monic Cantor set, derived from an IFS as in (4) with  $p_j \in \{-1, 1\}$ , that satisfies the OSC. Denote its Hausdorff dimension by  $\Delta$ . Then, for sufficiently large  $\rho = \rho(C)$ , we have the estimate*

$$d\left(C, \frac{p}{q}\right) > e^{-\rho q^\Delta} \tag{10}$$

for every  $p/q \in \mathbb{Q} \setminus C$ .

In the special case of missing-digit Cantor sets  $C = C_{b,W}$ , we have  $\Delta = \log |W|/\log b$  and the inequality

$$d\left(C, \frac{p}{q}\right) > \frac{b^{-(2b)^\Delta} q^\Delta}{2q} \tag{11}$$

holds for every  $p/q \in \mathbb{Q} \setminus C$ . In particular, for every  $\xi \in C_{b,W}$  and any  $\delta > (2b)^\Delta$ , the inequality

$$\left| \xi - \frac{p}{q} \right| \leq b^{-\delta q^\Delta} \tag{12}$$

has only finitely many solutions  $p/q \in \mathbb{Q} \setminus C$ .



We remark that in view of Theorem 2.1, it is possible to provide a variant of Theorem 3.4 that does not require the OSC, at the cost of (possibly) increasing  $\Delta$  slightly. We do not believe that the bounds are optimal. It might be true that there is an absolute upper bound for the order of extrinsic approximation, equivalently  $\lambda_{\text{ext}}(\xi) \ll_C 1$  in the notation of §4.3 below. However, a metric result of Fishman and Simmons [14, Theorem 3.9] demonstrates that we cannot hope for an improvement of Theorems 1.1 and 1.2 underlying our proof. Hence the bounds in Theorem 3.4 are the limit of our method. Our proof does not extend to non-monic Cantor sets, as we require that the right-hand side in Theorem 1.2 decays faster than  $q^{-1}$ . This can only be guaranteed if  $\gamma$  in (5) vanishes. Also for  $d > 1$  we see that Cantor sets as in Theorem 2.1 do not fulfil the requirement. In these cases, no bound seems to be known. We formulate resulting open problems.

*Problem 1.* Improve the bounds in Theorem 3.4. Do there exist extrinsic Liouville numbers, that is, numbers with  $\lambda_{\text{ext}}(\xi) = \infty$  in the notation of §4.3 below?

*Problem 2.* Let  $C$  be a one-dimensional, affine, rational-preserving Cantor set which is not necessarily monic. For  $p/q \notin C$ , find a lower bound for  $d(C, p/q)$  dependent on  $q$ . What about Cantor sets in higher-dimensional Euclidean space?

We believe the latter problem is related to Problem 3 in §4.2 below. For  $C = C_{b,W}$ , an elementary approach, using the fact that the base  $b$  expansion of a rational number  $p/q$  in lowest terms has period length  $\ll q$ , indicates the bound

$$d\left(C, \frac{p}{q}\right) > b^{-cq} \quad \text{for any } \frac{p}{q} \in \mathbb{Q} \setminus C, \tag{13}$$

with a suitable constant  $c = c(b, W) > 0$ . See §4.4 below, in particular Proposition 4.10, for more details. However, the conclusion (11) is stronger than (13) since  $\Delta < 1$ .

We remark that similar patterns to that in Theorem 3.4 are known concerning extrinsic rational approximation to algebraic sets in  $\mathbb{R}^n$ . See [6, Lemma 1], [9, Lemma 1], [10, Lemma 4.1.1], [27, Theorem 2.1]. However, the lower bounds typically decay like a negative power of  $q$  in that case. In [27] it is shown that, for an algebraic variety  $S$  defined by an implicit integral polynomial equation of total degree  $k$ , there are no rational numbers outside  $S$  that approximate  $S$  of order greater than  $k$ . We refer to [12] for approximation to manifolds by rationals within the manifold.

#### 4. Related topics

4.1. *Rational/algebraic vectors in Cantor sets.* As a byproduct of the proof of Theorem 2.1, we show the following upper bound for the number of rational elements with denominator at most  $N$  in a Cantor set.

**THEOREM 4.1.** *Let  $C \subseteq \mathbb{R}^d$  be any Cantor set as in Definition 1. Let  $\tau = \max_{1 \leq j \leq J} \tau_j \in (0, 1)$  be the maximum contraction rate of the IFS and denote by  $\text{diam}$  the diameter  $\max\{\|\underline{x} - \underline{y}\| : \underline{x}, \underline{y} \in C\}$  of the compact Cantor set. Let  $D = -\log J / \log \tau > 0$ . Then the set of rational vectors in  $C$  up to height  $N$ ,*

$$S(C, N) = \{r/s \in C : (r_1, \dots, r_d, s) = 1, 1 \leq s \leq N\},$$

*has cardinality at most  $J^2 \text{diam}^D N^{2D}$ .*

We require  $D < 1$  or equivalently  $J\tau < 1$  for the result to be non-trivial. When  $C = C_{3,\{0,2\}}$  is the Cantor middle-third set, the bound becomes  $4N^{2 \log 2 / \log 3}$ . The exponent has twice the expected magnitude; indeed, in view of numeric evidence it was conjectured in [5] that  $\mathcal{S}(C, N) \ll_{\epsilon} N^{\log 2 / \log 3 + \epsilon}$  for any  $\epsilon > 0$ . We can extend our claim to algebraic vectors.

**THEOREM 4.2.** *With the assumptions and notation of Theorem 4.1, let  $\mathcal{S}(C, N, n)$  be the set of vectors in  $C$  whose entries are real algebraic numbers of degree at most  $n$  and height at most  $N$ . Then*

$$|\mathcal{S}(C, N, n)| \ll_{C,n} N^{2nD}.$$

Clearly the trivial bound on  $\mathcal{S}(C, N, n)$  is of order  $N^{d(n+1)}$ , so if  $D \leq d/2$  or  $J\tau^{d/2} < 1$  we obtain an improvement simultaneously for every  $n \geq 1$ . For the Cantor middle-third set we get an improvement for  $n \in \{1, 2, 3\}$ . In the case of affine, rational-preserving Cantor sets one might expect  $\mathcal{S}(C, N, n) = \mathcal{S}(C, N)$  for any  $n \geq 1$ , that is, that there are no irrational algebraic vectors in  $C$ . For the Cantor middle-third set  $C_{3,\{0,2\}}$  this is a famous open conjecture of Mahler [21]. It seems that even in this case no non-trivial bound on  $\mathcal{S}(C, N, n)$  had previously been established.

**4.2. Addresses of rational vectors in certain Cantor sets.** We wish to discuss addresses of rational vectors in  $C$ , as the above sections indicate that they are directly linked to the order of rational approximation to Cantor sets. This topic has already been addressed in [13], and Theorem 4.3 below generalizes [13, Lemma 4.2] to certain multi-dimensional settings. We also provide a quantitative version for period lengths. While our proof strategy resembles that in [13] to some extent, we proceed slightly differently.

*Definition 5.* We call an affine, rational-preserving IFS *unimodular* if the contraction maps are of the form  $f_j(\underline{y}) = A_j \underline{y} / q_j + \underline{b}_j / s_j$ , where  $A_j \in \mathbb{Z}^{d \times d}$  with  $\det(A_j) \in \{1, -1\}$ , and  $\underline{b}_j \in \mathbb{Z}^d$  and  $q_j, s_j \in \mathbb{N}$ . We call a Cantor set  $C$  unimodular if it is induced by a unimodular IFS.

Unimodularity generalizes the concept of a monic, affine IFS for  $d = 1$  from §1.2. Note that our set-up is more general than assuming  $f_j(\underline{y}) = (A_j \underline{y} + \underline{b}_j) / q_j$ , that is,  $s_j = q_j$ , with all parameters as in the definition, as the determinant condition is relaxed. Therefore our next theorem in fact even generalizes [13, Lemma 4.2] when  $d = 1$ .

**THEOREM 4.3.** *Let  $C \subseteq \mathbb{R}^d$  be an affine, rational-preserving Cantor set. Then any vector in  $C$  that admits an ultimately periodic address is in  $\mathbb{Q}^d$ . If  $C$  is unimodular, the rational vectors in  $C$  are precisely those vectors in  $C$  that have an ultimately periodic address. Moreover, in this case the period length (including the preperiod) of  $\underline{p}/q \in \mathbb{Q} \cap C$  can be chosen  $\ll_C \min\{q^D, q^d\}$  with  $D = -\log J / \log \tau$  as in Theorem 4.1.*

We remark that the bound supposedly can be reasonably improved, potentially up to  $\ll \log q$ . See the remark after the proof and also [13, Theorem 5.3, Conjecture 5.6]. For  $d = 1$ , the question whether the assumption of  $C$  being unimodular (=monic) is necessary for the conclusion was already raised in [13, §5]. We have no new contribution to this interesting question, but wish to formulate the corresponding generalization.

*Problem 3.* In a Cantor set  $C$  derived from an affine rational-preserving IFS, does every rational vector in  $C$  have an ultimately periodic address? If yes, can it be arranged that the period length of  $p/q \in \mathbb{Q} \cap C$  is of order  $\ll_C \min\{q^D, q^d\}$ ?

From Theorem 4.3 we get some information on the structure of rational numbers in missing-digit Cantor sets. We only highlight special consequences; more information can be extracted from its proof below.

**COROLLARY 4.4.** *Let  $b \geq 3$  and  $W \subsetneq \{0, 1, \dots, b-1\}$ . If  $S = \{q_1, \dots, q_v\}$  is any finite set of prime numbers not dividing  $b$ , then there are only finitely many rational numbers  $r/s$  in  $C_{b,W}$  with  $s$  consisting only of prime factors in  $S$ . In particular, only finitely many integer powers of a rational number  $p/q$  with  $(b, q) = 1$  can lie in  $C_{b,W}$ . Moreover,  $C_{b,W}$  contains at most finitely many rational numbers  $p/q$  where  $q$  is a safe prime, that is,  $q$  and  $(q-1)/2$  are both prime numbers.*

Results of this type (restrictions on rationals in  $C_{b,W}$ ) appear to be rare. The author is only aware of a succession of papers by Wall [30], Nagy [24] and Bloshchitsyn [3], treating rationals with very smooth denominators. The first claim of Corollary 4.4 extends [3, Theorem 2] (and thus also [24]) where the finiteness implication was proved for  $S = \{q\}$  a single prime greater than  $b^2$ . For the second claim about powers of a rational in fact we only require that the denominator  $q$  has a prime divisor that does not divide  $b$ , or equivalently the radical of  $q$  does not divide  $b$ . This condition is easily seen to be necessary in general, as for example any positive integer power of  $1/3$  belongs to  $C_{3,\{0,1\}}$ . For irrational numbers we state the analogous question as an open problem.

*Problem 4.* Does there exist a real number  $\xi$  which is not a root of a rational number and with infinitely many integral powers belonging to a missing-digit Cantor set  $C_{b,W}$ ?

**4.3. Exponents.** We define several exponents to measure the quality of intrinsic and extrinsic rational approximation in a Cantor set. We restrict to the one-dimensional setting in this paper. In what follows, for convenience we agree on  $\sup(\emptyset) = 0$  and  $1/0 = +\infty$ .

*Definition 6.* Let  $S \subseteq \mathbb{R}$  and  $\xi \in \mathbb{R} \setminus \mathbb{Q}$ . Define the ordinary exponent of rational approximation  $\lambda(\xi)$  as the supremum of  $\lambda$  such that

$$\left| \xi - \frac{p}{q} \right| \leq q^{-1} Q^{-\lambda} \quad (14)$$

has a solution  $p/q \in \mathbb{Q}$  with  $1 \leq q \leq Q$  for arbitrarily large values of  $Q$ . Define the uniform exponent  $\widehat{\lambda}(\xi)$  as the supremum of  $\lambda$  for which (14) holds for every large  $Q$ . Define similarly the ordinary and uniform exponents of intrinsic and extrinsic approximation denoted by  $\lambda_{\text{int}}(\xi)$ ,  $\lambda_{\text{ext}}(\xi)$  and  $\widehat{\lambda}_{\text{int}}(\xi)$ ,  $\widehat{\lambda}_{\text{ext}}(\xi)$  respectively, by replacing  $p/q \in \mathbb{Q}$  by  $p/q \in \mathbb{Q} \cap S$  and  $p/q \in \mathbb{Q} \setminus S$  in the definitions accordingly.

The exponents  $\lambda(\xi)$ ,  $\widehat{\lambda}(\xi)$  are classical exponents of Diophantine approximation and are usually denoted by  $\lambda_1(\xi)$  and  $\widehat{\lambda}_1(\xi)$ , respectively. The following properties can be readily checked. We leave the verification to the reader.

PROPOSITION 4.5. *Let  $S \subseteq \mathbb{R}$ . For any irrational  $\xi \in S$  we have*

$$\lambda(\xi) \geq \widehat{\lambda}(\xi) = 1, \quad \lambda_{\text{int}}(\xi) \geq \widehat{\lambda}_{\text{int}}(\xi) \geq 0, \quad \lambda_{\text{ext}}(\xi) \geq \widehat{\lambda}_{\text{ext}}(\xi) \geq 0. \tag{15}$$

Further, we have

$$\lambda(\xi) = \max\{\lambda_{\text{int}}(\xi), \lambda_{\text{ext}}(\xi)\}$$

and

$$\max\{\widehat{\lambda}_{\text{int}}(\xi), \widehat{\lambda}_{\text{ext}}(\xi)\} \leq \widehat{\lambda}(\xi) = 1. \tag{16}$$

Remark 3. The ordinary exponent of approximation  $\lambda(\xi)$  can equivalently be defined as the supremum of  $\lambda$  such that

$$\left| \xi - \frac{p}{q} \right| \leq q^{-1-\lambda}$$

has infinitely many solutions  $p/q \in \mathbb{Q}$ . However, for the intrinsic and extrinsic exponents, for certain sets  $S$  the definition leads to different exponents than those defined above. Several inequalities in (15) may turn out to be false with this altered definition.

The identity  $\widehat{\lambda}(\xi) = 1$  is due to Khintchine [18]. In general there is no equality in the last inequality. Some inequalities concerning special types of Cantor sets can be inferred from [13, 14], quoted above. For example, by Theorem 1.3, in any one-dimensional Cantor set  $C$  that satisfies the OSC we have  $\lambda_{\text{ext}}(\xi) \geq 1$  for any  $\xi \in C \setminus \mathbb{Q}$ . Our main result is inspired by Theorem 3.2 and admits a similar proof.

THEOREM 4.6. *Let  $S \subseteq \mathbb{R}$  be any set and  $\xi \in S$ . We have*

$$\widehat{\lambda}_{\text{ext}}(\xi) \leq \frac{1}{\lambda_{\text{int}}(\xi)}, \quad \widehat{\lambda}_{\text{int}}(\xi) \leq \frac{1}{\lambda_{\text{ext}}(\xi)}. \tag{17}$$

If  $S = C \subseteq \mathbb{R}$  is any one-dimensional Cantor set that satisfies the OSC and  $\xi \in C \setminus \mathbb{Q}$ , then

$$\widehat{\lambda}_{\text{ext}}(\xi) = \min \left\{ \frac{1}{\lambda_{\text{int}}(\xi)}, 1 \right\}. \tag{18}$$

Define the spectrum of an exponent as the set of values taken when inserting any irrational real  $\xi$ . Bugeaud [8] provided a construction of  $\xi$  with  $\lambda(\xi) = \lambda_{\text{int}}(\xi) = \tau$  for any given  $\tau \geq 1$  in missing-digit Cantor sets  $C_{b,W}$ . See also [19, 26]. Therefore the spectrum of  $\lambda$  in  $C_{b,W}$  equals  $[1, \infty]$ , and the spectrum of  $\lambda_{\text{int}}$  contains this interval. Further, in the case where  $b$  is prime and  $W$  contains 0 and  $b - 1$  but does not contain two successive digits, the metrical result [13, Theorem 3.10] implies that for  $r \geq 0$  the set of  $\xi \in C_{b,W}$  with  $\lambda_{\text{int}}(\xi) = r$  (or  $\lambda_{\text{int}}(\xi) \geq r$ ) has Hausdorff dimension  $\Delta / (r + 1)$ , where  $\Delta = \log |W| / \log b$  is the Hausdorff dimension of  $C_{b,W}$ . Thus from Theorem 4.6 we deduce at once the following results.

COROLLARY 4.7. *Consider the missing-digit Cantor set  $C_{b,W}$  of Hausdorff dimension  $\Delta = \log |W| / \log b$ . The spectrum of  $\widehat{\lambda}_{\text{ext}}$  with respect to  $C_{b,W}$  equals  $[0, 1]$ . Assume  $b$  is prime and  $W$  contains  $\{0, b - 1\}$  but does not contain two successive digits. Then for  $\tau \in [0, 1)$  the sets  $\{\xi \in C_{b,W} : \widehat{\lambda}_{\text{ext}}(\xi) = \tau\}$  and  $\{\xi \in C_{b,W} : \widehat{\lambda}_{\text{ext}}(\xi) \leq \tau\}$  have Hausdorff dimension  $\Delta \cdot \tau / (\tau + 1)$ , whereas for  $\tau = 1$  the Hausdorff dimension equals  $\Delta$ . In particular, the set  $\{\xi \in C_{b,W} : \widehat{\lambda}_{\text{ext}}(\xi) < 1\}$  has Hausdorff dimension  $\Delta / 2$ .*

We pose the problem of deciding whether the claims of Corollary 4.7 generalize to one-dimensional Cantor sets that satisfy the OSC. Finally, we point out that numbers with the property  $\widehat{\lambda}_{\text{int}}(\xi) = 1$  do exist in any missing-digit Cantor set  $C_{b,w}$ . Indeed, it is easy to verify this identity for the numbers constructed in [25] which have almost all convergents in  $C_{b,w}$ . The identity can further be checked for Liouville type numbers  $\xi = \sum_{n \geq 0} wb^{-n!}$ ,  $0 \neq w \in W$ ; compare with [26, Lemma 3.10]. Thus, in view of the metric result rephrased above, we believe that the spectrum of  $\widehat{\lambda}_{\text{int}}$  equals  $[0, 1]$  as well. However, we are unable to show this at present.

4.4. *Base expansions.* For missing-digit Cantor sets  $C = C_{b,w}$ , Theorems 4.3 and 3.4 have implications for the base  $b$  digit patterns of rational numbers that can be stated without the framework of Cantor sets. Related considerations concerning the period lengths of base  $b$  representations of rational numbers in  $C_{b,w}$  have been addressed in [13, §5]. However, the results there are almost all conditional in nature. We start with a consequence of Theorem 4.3.

**THEOREM 4.8.** *Let  $W \subseteq \{0, 1, \dots, b - 1\}$  and  $\Delta = \log |W|/\log b$ . Let  $c_0, \dots, c_{N-1} \in W$  and assume that the infinite word  $(c_0c_1 \dots c_{N-1})^\infty$  has period length  $N$ . Then we have*

$$\gcd(c_0b^{N-1} + c_1b^{N-2} + \dots + c_{N-1}, b^N - 1) \ll \frac{b^N}{N^{1/\Delta}}.$$

*Proof.* Write  $p/q$  for the fraction  $(c_0b^{N-1} + c_1b^{N-2} + \dots + c_{N-1})/(b^N - 1)$  in lowest terms. Then the rational number  $p/q$  has base  $b$  expansion  $(0.\overline{c_0c_1 \dots c_{N-1}})_b$ , and by assumption it has period length  $N$ , which by Theorem 4.3 is  $\ll q^\Delta$ . Thus  $q \gg N^{1/\Delta}$ , which implies that the common factor is of size  $\ll b^N/N^{1/\Delta}$ .  $\square$

The natural assumption that the period length is not shorter than expected is necessary, since if we let  $c_0 = c_1 = \dots = c_{N-1} \in W$  then the greatest common divisor in question is a multiple of  $(b^N - 1)/(b - 1)$  and thus  $\gg b^N$ . We now turn to implications of Theorem 3.4.

**THEOREM 4.9.** *Let  $b \geq 3$  and  $W \subseteq \{0, 1, \dots, b - 1\}$ . Further, let  $\Delta = \log |W|/\log b$ . Let*

$$\xi = (0.c_0c_1 \dots c_k \overline{c_{k+1}c_{k+2} \dots c_{N-1}})_b, \tag{19}$$

*with  $N > k \geq 0$  and  $c_i \in \{0, 1, \dots, b - 1\}$  be a rational number in  $(0, 1)$  expanded in base  $b$ . Then  $\xi = p/q$  with*

$$p = \sum_{i=0}^{N-1} c_i b^{N-1-i} - \sum_{j=0}^{k-1} c_j b^{k-1-j}, \quad q = b^N - b^k. \tag{20}$$

*Assume there exists an index  $i$  with  $c_i \notin W$  and let  $\phi(\xi) \in \{0, 1, 2, \dots, N - 1\}$  be the smallest index with this property. Further, let  $p_0/q_0$  be the fraction  $p/q$  in lowest terms. Then  $\phi(\xi) \ll q_0^\Delta$ . Equivalently, if for  $\xi$  in (19) the first  $s$  base  $b$  digits  $c_0, c_1, \dots, c_s$  lie in  $W$ , but not all  $c_i$  belong to  $W$ , then its reduced denominator  $q_0$  is  $\gg s^{1/\Delta}$ .*

The relation to Theorem 3.4 arises from the fact that the rational numbers in  $C_{b,w}$  are precisely those where  $c_i \in W$  in (19). For a generic rational  $\xi$  in (19) we expect that  $q_0$  is of exponential order, that is,  $\log q_0 \gg N \geq s$ , stronger than in our claim. However,

elementary methods only yield a bound of order  $q_0 \gg s$  that is valid for all  $\xi$  in (19), as we will see from Proposition 4.10 below. The bound  $s^{1/\Delta}$  of our Theorem 4.9 is stronger since  $\Delta < 1$ . The special case of rationals where  $b$  is a primitive root of the denominator illustrates the improvement well. We denote by  $\varphi$  the Euler totient function. For  $A, B$  coprime integers, as usual  $\text{ord}_A \bmod B$  denotes the smallest positive integer  $m$  with  $A^m \equiv 1 \pmod B$ . The next proposition comprises results on the  $\varphi$ -function and period lengths of rational numbers in a base.

**PROPOSITION 4.10.** *We have  $\varphi(q) \gg q/\log \log q$ . Let  $A, B, C$  be integers with  $(A, C) = 1$ . Write  $C = C_1 C_2$  with  $(C_2, B) = 1$  and  $C_1$  consisting only of factors of  $B$ . Then  $\xi = A/C$  written in base  $B$  has preperiod of length  $P(\xi)$  equal to the smallest integer  $v$  with  $C_1 | B^v$ , followed by a period of length  $L(\xi) = \text{ord}_B \bmod C_2$ . In particular, if  $(A, B) = 1$ , then  $A/C$  has purely periodic base  $B$  expansion of period length  $\text{ord}_B \bmod C$ . In any case, the total period length  $P(\xi) + L(\xi)$  of  $\xi$  is of order  $\ll C$ .*

We omit the proof as the claims are well known. The first claim can be found in the book of Hardy and Wright [15, Theorem 328]. The identities for periods can be checked in an elementary way. The bounds on period lengths follow from  $L(\xi) \leq \text{ord}_B \bmod C_2 \leq \varphi(C_2) \leq C_2 - 1 \leq C - 1$  and  $P(\xi) \leq v \ll \log C_1 \leq \log C$ , or alternatively directly from Theorem 4.3. Recall that  $A$  is called a primitive root modulo  $B$  if  $\text{ord}_A \bmod B = \varphi(B)$ .

**COROLLARY 4.11.** *Let  $b, W, \Delta$  be as in Theorem 4.9. Let  $p_0, q_0$  be coprime integers and assume that  $b$  is a primitive root modulo  $q_0$ . Let (19) be the base  $b$  expansion of  $\xi = p_0/q_0$ . Assume that not all  $c_j$  belong to  $W$  and let  $i$  be the smallest index for which the digit  $c_i$  lies outside  $W$ . Then  $i \ll N^\Delta (\log \log N)^\Delta$ , that is,  $c_i \notin W$  for some  $i \ll N^\Delta (\log \log N)^\Delta$  with an absolute implied constant. In particular, any digit  $v \in \{0, 1, 2, \dots, b - 1\}$  that occurs among the  $c_i$  in (19) already occurs within the first  $\ll N^{\log(b-1)/\log b}$  places.*

*Proof.* If  $b$  is a primitive root modulo  $q_0$  then by Proposition 4.10 the number  $\xi = p_0/q_0$  has no preperiod (i.e.  $k = 0$ ) and the period length  $N$  satisfies  $N = \text{ord}_b \bmod q_0 = \varphi(q_0) \gg q_0/\log \log q_0$ . Thus, using the notation of Theorem 4.9, from its claim we infer  $i = \phi(\xi) \ll q_0^\Delta \ll (N \log \log N)^\Delta$ , as desired. For the particular case, apply the above observation to  $W = \{0, 1, \dots, b - 1\} \setminus \{v\}$  of cardinality  $b - 1$ , which induces  $\Delta = \log(b - 1)/\log b$ . □

An equivalent way to state the claim of Theorem 4.9 is that the numbers  $p, q$  as in (20) have greatest common divisor at most  $\ll b^N/\phi(\xi)^{1/\Delta}$ . If we let  $k = 0$ , another variant on the base  $b$  expansion of large divisors of numbers of the form  $b^N - 1$  is obtained.

**COROLLARY 4.12.** *Let  $b, W$  and  $\Delta < 1$  as in Theorem 4.9. Let  $\psi$  and  $\phi$  be positive integers with the property  $\phi \geq c_1 \psi^\Delta$  for sufficiently large  $c_1 = c_1(b) > 0$ . Assume for some integer  $N \geq 1$  that  $d' \geq b^N/\psi$  divides  $b^N - 1$  and is written in base  $b$  as*

$$d' = (u_0 u_1 \cdots u_{N-1})_b = u_0 b^{N-1} + u_1 b^{N-2} + \cdots + u_{N-2} b + u_{N-1}, \tag{21}$$

with possibly  $c_0 = c_1 = \cdots = c_s = 0$  for some  $s \geq 0$ , and assume that all  $c_i \in W$ . Then for

$$W_1 = \{u_j : 0 \leq j \leq \phi\}, \quad W_2 = \{u_j : 0 \leq j \leq N - 1\},$$

the letters occurring in the first  $\phi$  and all digits of  $d'$ , respectively, we have  $W_1 = W_2$ .

In particular, if  $\phi = \phi(N) = \lfloor rN^\Delta \rfloor$  for some  $r \in (0, 1]$ , and  $\psi = \psi(N) = RN$  for sufficiently small  $R = R(b, r) > 0$ , the following statement holds. Assume that  $b^N - 1$  has a small divisor  $d_N \leq RN$ , and write the complementary divisor  $d'_N = (b^N - 1)/d_N$  as in (21). The sets of digits that occur within  $\{u_0, u_1, \dots, u_{\lfloor N^\Delta r \rfloor}\}$  and  $\{u_0, u_1, \dots, u_{N-1}\}$  coincide.

In particular, if we take  $W$  of cardinality  $b - 1$ , we see that for any large divisor of  $b^N - 1$ , all digits occurring in its base  $b$  expansion can already be found in a relatively short initial sequence. Generically, for large  $N$  we expect  $W_1 = W_2 = \{0, 1, \dots, b - 1\}$ . However,  $d'_N = (b^N - 1)/(b - 1)$  in base  $b$  reads  $(1^N)_b$ , so  $W_1 = W_2 = \{1\}$ . In this case we easily verify the claim. We end this section with an example.

*Example 1.* Let  $b = 3$ ,  $W = \{0, 1\}$  with  $\Delta = \log 2/\log 3$ . Then

$$\gcd(\epsilon_0 3^{N-1} + \epsilon_1 3^{N-2} + \dots + \epsilon_{N-2} 3^1 + 2, 3^N - 1) \ll \frac{3^N}{N^{1/\Delta}},$$

for any choice of  $\epsilon_j \in \{0, 1\}$ ,  $0 \leq j \leq N - 2$  (observe that  $\epsilon_{N-1} = 2 \notin W$ ). Thus,  $3^N - 1$  cannot have a very large divisor given by the sum expression. Consider special cases. First, assume that  $\epsilon_j = 0$  for  $0 \leq j \leq N - m$  and  $\epsilon_j = 1$  for  $N - m + 1 \leq j \leq N - 2$ , for some integer  $2 \leq m \leq N + 1$ . Then the sum expression results in  $(3^m + 1)/2$ , thus

$$\gcd\left(\frac{3^m + 1}{2}, 3^N - 1\right) \ll \frac{3^N}{N^{1/\Delta}}, \quad 0 \leq m \leq N - 2.$$

In particular, the assumption  $(3^m + 1)|(2 \cdot (3^N - 1))$  implies that  $3^m \ll 3^N/N^{1/\Delta}$ , or equivalently  $m < N - \log N/\log 2 + c$  for  $c \in \mathbb{R}$ . In this particular example, in fact the divisibility condition implies the stronger (sharp) estimate  $m \leq N/2$ , since  $(3^m + 1)|(2 \cdot (3^N - 1))$  is equivalent to  $(3^m + 1)|(2 \cdot (3^{N-m} + 1))$ . As a second special case let  $\epsilon_j = 1$  precisely for  $j$  a power of 2 and  $\epsilon_j = 0$  otherwise. We are unable to find an elementary proof that confirms our implication

$$\gcd(3^{2^k} + 3^{2^{k-1}} + \dots + 3^{2^0} + 2, 3^N - 1) \ll \frac{3^N}{N^{1/\Delta}}, \quad k < \frac{\log N}{\log 2}.$$

5. Proofs

5.1. *Proof of Theorem 2.1.* The crucial step for the proof of Theorem 2.1 is to extend [13, Lemma 2.2]. Fortunately, this can be derived quite easily. Compared to [13], it does not involve the measure supported on  $C$ , which might not be nice in our general setting, but stems from the following elementary counting argument.

**PROPOSITION 5.1.** *Let  $C$  be any Cantor set as in Definition 1 and let  $\tau = \max_{1 \leq j \leq J} \tau_j$  be the maximum contraction ratio of the IFS. If  $l \geq 1$  is an integer and  $E$  a subset of  $C$  of cardinality greater than  $J^l$ , then there are two elements  $\underline{x}, \underline{y}$  in  $E$  with distance  $\|\underline{x} - \underline{y}\| \leq \text{diam} \cdot \tau^l$ , where  $\text{diam}$  is the diameter of the compact set  $C$ .*

*Proof.* For any  $\underline{\alpha} \in E$  let  $(\omega_{1,\underline{\alpha}}, \omega_{2,\underline{\alpha}}, \dots)$  be any address of  $\underline{\alpha}$  and let  $\pi_{j,\underline{\alpha}} = f_{\omega_{j,\underline{\alpha}}}$ . Then by the pigeonhole principle, since  $|E| > J^l$ , there are two elements  $\underline{x}, \underline{y}$  whose prefixes  $(\omega_{1,\underline{x}}, \dots, \omega_{l,\underline{x}})$  and  $(\omega_{1,\underline{y}}, \dots, \omega_{l,\underline{y}})$  coincide, so  $\pi_j := \pi_{j,\underline{x}} = \pi_{j,\underline{y}}$  for  $1 \leq j \leq l$ . Let

$$\underline{a} = \lim_{n \rightarrow \infty} \pi_{l+1,\underline{x}} \circ \pi_{l+2,\underline{x}} \cdots \circ \pi_{l+n,\underline{x}}(\underline{0}), \quad \underline{b} = \lim_{n \rightarrow \infty} \pi_{l+1,\underline{y}} \circ \pi_{l+2,\underline{y}} \cdots \circ \pi_{l+n,\underline{y}}(\underline{0}).$$

Clearly  $\underline{a}, \underline{b} \in C$ . Then we have

$$\|\underline{x} - \underline{y}\| = \|\pi_1 \circ \pi_2 \circ \dots \circ \pi_l(\underline{a}) - \pi_1 \circ \pi_2 \circ \dots \circ \pi_l(\underline{b})\| \leq \tau^l \|\underline{a} - \underline{b}\|.$$

Since  $\underline{a}, \underline{b} \in C$  their distance is bounded above by  $\text{diam}$  and the claim follows. □

From the proposition we immediately obtain the required variant of [13, Lemma 2.2].

LEMMA 5.2. *Let  $C, \tau$  be as in Proposition 5.1. Then, if  $N \geq 1$  is an integer and  $\underline{\xi}_1, \dots, \underline{\xi}_N$  belong to  $C$ , there exist a constant  $K_1 > 0$  and two indices  $1 \leq i < j \leq N$  with*

$$\|\underline{\xi}_i - \underline{\xi}_j\| \leq r_N := (N/K_1)^{\log \tau / \log J}.$$

We may choose  $K_1 = (\text{diam}/\tau)^{-\log J / \log \tau}$ .

*Proof.* For given  $N$ , choose the integer  $l \geq 0$  so that  $J^l < N \leq J^{l+1}$ . By Proposition 5.1 there are two elements in the sequence with distance at most  $\text{diam} \cdot \tau^l \leq \text{diam} \cdot \tau^{\log N / \log J - 1} = (\text{diam}/\tau) \cdot N^{\log \tau / \log J}$ , which leads to  $K_1$  in the theorem. □

In what follows, we denote by  $\|A\|_\infty := \max_{\underline{x} \neq \underline{0}} \{\|A_j \underline{x}\| / \|\underline{x}\|\}$  the norm of a matrix  $A \in \mathbb{R}^{d \times d}$ .

*Proof of Theorem 2.1.* We proceed essentially as in the proof of [13, Theorem 2.1]. Let  $\underline{\xi} \in C$  be arbitrary. Fix an address  $\omega = (\omega_1, \dots) \in \{1, 2, \dots, J\}^{\mathbb{N}}$  of it, so that with  $\pi_j = f_{\omega_j}$  we have  $\underline{\xi} = \lim_{k \rightarrow \infty} \pi_1 \circ \pi_2 \circ \dots \circ \pi_k(\underline{0})$ . If  $\sigma$  is the left shift on the space of infinite formal words, for every large  $N$  we consider  $\{\sigma^n(\omega)(\underline{0}) : 0 \leq n \leq N\}$ , a finite sequence in  $C$ . By Lemma 5.2 there are two integers  $0 \leq n < m + n \leq N$  so that if

$$\begin{aligned} \underline{y} &:= \sigma^n(\omega)(\underline{0}) = \lim_{k \rightarrow \infty} \pi_{n+1} \circ \dots \circ \pi_{n+k}(\underline{0}), \\ \underline{z} &:= \sigma^{m+n}(\omega)(\underline{0}) = \lim_{k \rightarrow \infty} \pi_{m+n+1} \circ \dots \circ \pi_{m+n+k}(\underline{0}), \end{aligned}$$

then  $\|\underline{y} - \underline{z}\| \leq r_N$ . Define endomorphisms on  $\mathbb{R}^d$  by  $u_{(1)} = \pi_1 \circ \dots \circ \pi_n$  and  $u_{(2)} = \pi_{n+1} \circ \dots \circ \pi_{m+n}$ , so that  $\underline{\xi} = u_{(1)}(\underline{y})$  and  $\underline{y} = u_{(2)}(\underline{z})$ . Consider  $A_{\omega_j} \in \mathbb{Z}^{d \times d}$  and  $q_{\omega_j} \in \mathbb{Z}$  as in the theorem. Then let

$$\begin{aligned} P_{(1)} &= A_{\omega_1} \cdots A_{\omega_n}, & P_{(2)} &= A_{\omega_{n+1}} \cdots A_{\omega_{m+n}}, \\ q_{(1)} &= q_{\omega_1} \cdots q_{\omega_n}, & q_{(2)} &= q_{\omega_{n+1}} \cdots q_{\omega_{m+n}}. \end{aligned}$$

Further, with  $S := \prod_{1 \leq j \leq J} s_j$ , define the integer vectors

$$\underline{r}_{(1)} = \sum_{i=1}^n A_{\omega_1} \cdots A_{\omega_{i-1}} \underline{r}_{\omega_i} q_{\omega_{i+1}} \cdots q_{\omega_n},$$

and

$$\underline{r}_{(2)} = \sum_{i=1}^m A_{\omega_{n+1}} \cdots A_{\omega_{n+i-1}} \underline{r}_{\omega_{n+i}} \frac{S}{s_{\omega_{n+i}}} q_{\omega_{n+i}} \cdots q_{\omega_{m+n}}.$$



In  $r_{(2)}$  the additional factor  $S/(q_{\omega_{n+1}}s_{\omega_{n+1}})$  compared to [13] has entered according to the notational difference  $s_j \neq q_j$  for the shift vector. By the recursive process, the maps  $u_{(i)}$  are accordingly given as

$$u_{(i)}(\underline{t}) = \frac{P_{(i)}\underline{t}}{q_{(i)}} + \frac{\underline{r}_{(i)}}{Sq_{(i)}}, \quad i = 1, 2.$$

The unique fixed point of  $u_{(2)}$  denoted by  $\underline{F}_2$  is given by the solution of

$$\underline{F}_2 = \frac{P_{(2)}}{q_{(2)}}\underline{F}_2 + \frac{\underline{r}_{(2)}}{Sq_{(2)}},$$

which yields

$$\underline{F}_2 = \frac{1}{S}(q_{(2)}I_d - P_{(2)})^{-1}\underline{r}_{(2)}.$$

We show that the inverse matrix is well defined. All  $A_j$  have norm  $\|A_j\|_\infty < q_j$  since  $A_j/q_j$  are contractions for  $1 \leq j \leq J$ , thus their product which is  $P_{(2)}$  has norm less than  $q_{(2)}$ , that is,

$$\|P_{(2)}\|_\infty < q_{(2)}. \tag{22}$$

Consequently, all eigenvalues of  $P_{(2)}$  are of absolute value smaller than  $q_{(2)}$ , proving the regularity of  $q_{(2)}I_d - P_{(2)}$ . We also observe that  $\underline{F}_2 \in \mathbb{Q}^d$  by Cramer’s rule, hence this applies to  $u_{(1)}(\underline{F}_2)$  as well so we may write

$$\underline{p}/q = u_{(1)}(\underline{F}_2) = \frac{P_{(1)}(q_{(2)}I_d - P_{(2)})^{-1}\underline{r}_{(2)} + \underline{r}_{(1)}}{Sq_{(1)}},$$

with  $\underline{p} \in \mathbb{Z}^d$  and  $q \in \mathbb{N}$ . Concretely, by Cramer’s rule the inverse matrix contributes a factor  $\det(q_{(2)}I_d - P_{(2)}) > 0$  in the denominator, so the total denominator will be  $q = Sq_{(1)} \det(q_{(2)}I_d - P_{(2)})$  if we do not simplify the vector  $\underline{p}/q$  to lowest terms. For any matrix  $A \in \mathbb{R}^{d \times d}$  we have  $|\det A| \leq \|A\|_\infty^d$  (the determinant is at most the product of column norms, and columns are images of canonical base vectors), and from (22) we infer  $\|q_{(2)}I_d - P_{(2)}\|_\infty \leq \|q_{(2)}I_d\|_\infty + \|P_{(2)}\|_\infty \leq 2q_{(2)}$ . Applied to  $A = q_{(2)}I_d - P_{(2)}$ , we derive

$$0 < q \leq Sq_{(1)} \cdot (2q_{(2)})^d \leq S(2q_{(1)}q_{(2)})^d. \tag{23}$$

Taking logarithms yields

$$\log q \leq d \sum_{i=1}^{m+n} \log q_{\omega_i} + c, \tag{24}$$

with the constant  $c = d \log 2 + \log S$ . Let  $\tau_{(i)}$  be the contraction rates on  $u_{(i)}$ ,  $i = 1, 2$ . Then, on the other hand,

$$\|\underline{y} - \underline{F}_2\| \leq \tau_{(2)}\|\underline{z} - \underline{F}_2\| \leq \tau_{(2)}\|\underline{y} - \underline{F}_2\| + \tau_{(2)}\|\underline{z} - \underline{y}\|,$$

hence

$$\|\underline{y} - \underline{F}_2\| \leq \frac{\mu_{(2)}}{1 - \mu_{(2)}}\|\underline{z} - \underline{y}\|.$$

Applying  $u_{(1)}$  gives

$$\|\underline{\xi} - \underline{p}/q\| \leq \frac{\tau_{(1)}\tau_{(2)}}{1 - \tau_{(2)}}\|\underline{z} - \underline{y}\| \leq \frac{\tau_{(1)}\tau_{(2)}}{1 - \tau_{(2)}}r_N.$$

Now  $\tau_{(2)} \leq \max_{1 \leq j \leq J} \tau_j = \tau$ , thus with  $K_2 = (1 - \tau)^{-1}$  we infer that

$$\log \|\underline{\xi} - \underline{p}/q\| \leq \log K_2 + \log r_N + \mu \sum_{i=1}^{m+n} \log \tau_{\omega_i} + c.$$

By assumption,  $\log \tau_{\omega_i} = \mu_{\omega_i} \log q_i \leq \mu \log q_i$  for every  $1 \leq i \leq m$ . Thus from (24) and since  $\mu \leq 0$ , further,

$$\log \|\underline{\xi} - \underline{p}/q\| \leq \log K_2 + \log r_N + \frac{\mu}{d} \log q + \frac{c}{d},$$

and after exponentiating again we get

$$\|\underline{\xi} - \underline{p}/q\| \leq K_3 r_N q^{\mu/d}, \tag{25}$$

for some constant  $K_3$ . On the other hand, again from (23) we derive

$$\log q \leq d \sum_{i=1}^{m+n} \log q_{\omega_i} + d \log 2 + \log S.$$

Let  $q_{\max} = \max_{1 \leq j \leq J} q_j$ . Finally since  $q \leq S \cdot 2^d q_{\max}^{d(m+n)} \leq S \cdot 2^{dN} q_{\max}^{dN} = S \cdot (2q_{\max})^{dN}$ , for  $N := \lfloor \log Q / (d(\log(2q_{\max})) + \log S) \rfloor$  and  $Q \geq S 2^d q_{\max}^d$  we conclude that  $q \leq Q$  and

$$r_N \leq \left(\frac{N}{K_1}\right)^{\log \tau / \log J} \leq K_4 \left(\frac{\log Q}{2dK_1 \log q_{\max}}\right)^{\log \tau / \log J} = K_5 \log Q^{\log \tau / \log J}.$$

Putting  $K_1 = K_3 K_5$ , we derive the claim from (25). □

5.2. *Proofs from Section 3.* For the proof of Theorem 3.1 we utilize an auxiliary result established within the proof of [14, Theorem 1.1]. It deals with almost arithmetic sequences in Cantor sets. While originally formulated only for similarity Cantor sets, as pointed out in §1.2 we may extend its claim to Cantor sets as in Definition 1. We do not rephrase the notion of almost arithmetic from [14] and restrict to the arithmetic sequence setting that suffices for our proof.

LEMMA 5.3. (Fishman and Simmons [14]) *Let  $C \subseteq \mathbb{R}^d$  be a Cantor set which satisfies the OSC. There exists a positive integer  $N = N(C)$  with the property that if the set  $C$  contains an arithmetic progression of length  $N$ , then the entire line segment joining these points is contained in  $C$ . In particular, if  $C$  contains no line segment, then no non-constant arithmetic progression of length  $N$  is contained in  $C$ .*

The lemma will apply to the first condition of Theorem 3.1. For the implication from the latter condition, we will employ our counting result Theorem 4.1.

*Proof of Theorem 3.1.* As pointed out, the claim is almost an immediate consequence of Lemma 5.3. In our situation, given  $\underline{\xi} \in C$  and  $Q > 1$ , we consider  $\underline{p}/\lfloor Q \rfloor$  for  $\underline{p} = (p_1, \dots, p_d)$  chosen such that  $p_i/\lfloor Q \rfloor$  is the rational number with denominator  $\lfloor Q \rfloor$  that is closest to the  $i$ th coordinate  $\xi_i$  of  $\underline{\xi} \in C$ . Clearly  $\|\underline{\xi} - \underline{p}/\lfloor Q \rfloor\| \ll Q^{-1}$ . Assume the first condition of the theorem. Let  $N = N(C)$  be large enough that  $C$  contains no arithmetic

sequence of length  $N$  as in Lemma 5.3. Then with  $\underline{v} \in \mathbb{Z}^d$  as in the theorem, the rational vectors

$$\underline{p}/\lfloor Q \rfloor, \underline{p}/\lfloor Q \rfloor + \frac{1}{\lfloor Q \rfloor} \underline{v}, \underline{p}/\lfloor Q \rfloor + \frac{2}{\lfloor Q \rfloor} \underline{v}, \dots, \underline{p}/\lfloor Q \rfloor + \frac{N}{\lfloor Q \rfloor} \underline{v}$$

have common denominator at most  $\lfloor Q \rfloor \leq Q$  and form an arithmetic progression. Thus not all can belong to  $C$  as otherwise by Lemma 5.3 the line segment joining  $\underline{p}/q$  and  $\underline{p}/\lfloor Q \rfloor + N/\lfloor Q \rfloor \cdot \underline{v}$  would be contained in  $C$ , contradicting our hypothesis. On the other hand, we easily see that any element of the progression has distance  $\ll_{N, \underline{v}} Q^{-1} \ll_{C, \underline{v}} Q^{-1}$  from  $\underline{\xi}$ .

Finally, assume the second condition  $D < 1/2$ . Then for every prime number  $N$  with  $Q/2 \leq N < Q$  again let  $\underline{p}_N/N$  be the rational vector with numerator coordinates  $p_{N,i}$  chosen so that  $p_{N,i}/N$  is closest to  $\xi_i$ . We see that  $\|\underline{p}_N/N - \underline{\xi}\| \ll Q^{-1}$ . Moreover, for  $Q > \text{diam}$  these vectors are clearly pairwise distinct and by the prime number theorem there are  $\gg Q/\log Q$  such vectors. On the other hand, by Theorem 4.1 there are only  $\ll Q^{2D}$  rational vectors in  $C$  with denominator at most  $Q$ . Thus if  $D < 1/2$ , for large  $Q$  some must lie outside  $C$  and satisfy the desired property.  $\square$

The proof of Theorem 3.2 requires more preparation. The next lemma comprises some estimates on continued fractions and is partly well known.

LEMMA 5.4. *Let  $\xi$  be a real number with sequence of convergents  $(u_t/v_t)_{t \geq 1}$ . Then*

$$\frac{1}{2v_t v_{t+1}} \leq \left| \xi - \frac{u_t}{v_t} \right| \leq \frac{1}{v_t v_{t+1}}, \quad t \geq 1. \tag{26}$$

Moreover, if  $t \geq 2$  is given and  $r/s \neq u_t/v_t$  is any rational number with  $0 < s < v_{t+1}/2$ , then

$$|s\xi - r| \geq \frac{1}{2v_t}. \tag{27}$$

For the proof, recall that for an origin-symmetric convex set  $K \subseteq \mathbb{R}^h$  and a lattice  $\Lambda \subseteq \mathbb{R}^h$ , the  $i$ th successive minimum  $\lambda_i(K, \Lambda)$  of  $\Lambda$  with respect to  $K$  is defined as the infimum of real numbers  $\lambda$  such that  $\lambda\Lambda \cap K$  contains  $i$  linearly independent vectors (for  $1 \leq i \leq h$ ). Minkowski’s second convex body theorem then bounds the product of all successive minima in terms of the volume of  $K$ , the fundamental area of the lattice  $\Lambda$  and  $h$ . In particular, the case  $h = 2$  we will require reads

$$\frac{2 \det \Lambda}{V(K)} \leq \lambda_1(K, \Lambda) \lambda_2(K, \Lambda) \leq \frac{4 \det \Lambda}{V(K)}. \tag{28}$$

*Proof of Lemma 5.4.* The inequalities of (26) are in fact well known. Recall that two consecutive convergents have distance  $u_{t+1}/v_{t+1} - u_t/v_t = (-1)^t/(v_t v_{t+1})$ . In particular, convergents lie alternately on the left and right of the limit, and the right inequality follows. The left estimate follows similarly, incorporating also the well-known fact that  $u_{t+1}/v_{t+1}$  lies closer to  $\xi$  than  $u_t/v_t$ . We have to show the second claim involving (27). Define the convex body

$$K = [-v_t, v_t] \times [-v_{t+1}^{-1}, v_{t+1}^{-1}],$$

and the lattice

$$\Lambda_\xi = \{(m, m\xi - n) \in \mathbb{R}^2 : m, n \in \mathbb{Z}\}.$$

The lattice has determinant  $\det(\Lambda_\xi) = 1$  and the rectangular convex body has volume  $\text{vol}(K) = 4v_l/v_{l+1}$ . We may assume that  $\xi \in (0, 1)$  and thus  $0 < u_l < v_l$  for all large  $l$ . Since we know from the theory of continued fractions and (26) that

$$|u_{l+1}\xi - v_{l+1}| < |u_l\xi - v_l| \leq \frac{1}{v_{l+1}},$$

we see that the point  $(u_l, u_l\xi - v_l)$  lies in  $\Lambda_\xi \cap K$ . Hence the first successive minimum  $\lambda_1(K, \Lambda_\xi)$  of  $\Lambda$  with respect to  $K$  satisfies  $\lambda_1(K, \Lambda_\xi) \leq 1$ . By Minkowski's second convex body theorem (28) we conclude that the second successive minimum satisfies

$$\lambda_2(K, \Lambda_\xi) \geq \frac{2 \det(\Lambda_\xi)}{V(K)\lambda_1(\Lambda_\xi, K)} \geq \frac{v_{l+1}}{2v_l}.$$

Thus there is no lattice point linearly independent of  $(u_l, u_l\xi - v_l)$  in the region

$$\frac{v_{l+1}}{2v_l} \cdot K = \left[-\frac{v_{l+1}}{2}, \frac{v_{l+1}}{2}\right] \times \left[-\frac{1}{2v_l}, \frac{1}{2v_l}\right].$$

In other words, for any  $(r, s)$  linearly independent of  $(u_l, v_l)$  and with  $|s| \leq v_{l+1}/2$  we have  $|s\xi - r| \geq 1/(2v_l)$ . The proof of the second claim is finished. □

For the density result in Theorem 3.2 we utilize the following lemma.

LEMMA 5.5. *Let  $C \subseteq \mathbb{R}^d$  be a Cantor set. Assume for every function  $\Psi : \mathbb{N} \rightarrow \mathbb{R}_{>0}$  that there exists  $\underline{\xi} \in C$  so that*

$$\|\underline{\xi} - p/q\| \leq \Psi(q)$$

*has infinitely many solutions  $p/q \in \mathbb{Q} \cap C$ . Then the set of  $\underline{\xi}$  with the same property is dense in  $C$ .*

*Proof of Lemma 5.5.* Let  $\underline{\xi} \in C$  be arbitrary with address  $(\omega_1, \omega_2, \dots)$  and write  $\pi_i = f_{\omega_i}$  so that  $\underline{\xi} = \lim_{k \rightarrow \infty} \pi_1 \circ \dots \circ \pi_k(\underline{0})$ . For given  $\Psi$  and  $\epsilon > 0$ , we construct a  $\Psi$ -approximable point  $\underline{\xi}_\epsilon$  with  $\|\underline{\xi}_\epsilon - \underline{\xi}\| < \epsilon$ . Again for  $\tau \in (0, 1)$  the largest absolute value of the contraction factors and  $\text{diam}$  the diameter of  $C$ , take an integer  $u$  large enough so that  $\tau^u \text{diam} < \epsilon$ . Then consider the function  $\Phi(t) := \Psi(Nt) \cdot N^{-1}$  for a large integer  $N$  dependent only on  $u$  (and thus  $\epsilon$ ) to be chosen later. By assumption there exists  $\underline{\zeta} \in C$  that is  $\Phi$ -approximable. Define  $\xi_\epsilon = \pi_1 \circ \pi_2 \circ \dots \circ \pi_u(\underline{\zeta})$ . Then we have

$$\|\underline{\xi}_\epsilon - \underline{\xi}\| \leq \tau^u \text{diam} < \epsilon,$$

as the addresses of  $\underline{\xi}$  and  $\underline{\xi}_\epsilon$  coincide up to the  $u$ th place. On the other hand, we claim that  $\underline{\xi}_\epsilon$  is  $\Psi$ -approximable if  $N$  was chosen large enough. To see this, first notice that as  $\pi_j$  are linear maps with rational coefficients, also  $T := \pi_1 \circ \pi_2 \circ \dots \circ \pi_u$  induces a linear, rational transformation  $T(\underline{y}) = (A\underline{y} + \underline{b})/s$ , with  $A \in \mathbb{Z}^{d \times d}$ ,  $\underline{b} \in \mathbb{Z}^d$  and  $s \in \mathbb{N}$ . It is not hard to see that if some  $\underline{y} \in \mathbb{R}$  is  $h$ -approximable for some function  $h(t)$ , then  $T(\underline{y})$  (with  $T$  as above) is  $Nh(t/N)$ -approximable, for sufficiently large  $N$  depending only on  $s$ . This is readily derived from the facts that  $\|T(\underline{y}) - T(p/q)\| \leq \|\underline{y} - p/q\|$  (since  $T$  is a contraction) and  $\text{den}(T(p/q)) \leq sq$ , where  $\text{den}$  denotes the common denominator of a rational vector. Application to  $h(t) = \Phi(t)$  and  $\underline{y} = \underline{\zeta}$ , starting with some suitable value  $N = N(\epsilon)$ , yields that  $\underline{\xi}_\epsilon = T(\underline{\zeta})$  is  $N\Phi(t/N) = \Psi(t)$ -approximable. □

We can now prove Theorem 3.2.

*Proof of Theorem 3.2.* We recursively define a fast-growing lacunary sequence  $(a_n)_{n \geq 1}$  of positive integers that will induce  $\underline{\xi} \in C$  satisfying (8), via a procedure described below. Assume for the moment that this sequence is fixed. Write  $b_1 = a_1 - 1$  and  $b_l = a_l - a_{l-1} - 1$  for integers  $l \geq 2$ . Let  $f = f_1, g = f_2$  be two different contractions in our IFS and without loss of generality assume that  $f$  is the map whose fixed point  $\underline{\alpha} = \underline{\alpha}_j$  is rational. Let  $\tau_i$ , for  $i = 1, 2$ , be the contraction factors of  $f$  and  $g$  respectively, and let  $\tau = \max_{i=1,2} |\tau_i| < 1$ . We define  $\underline{\xi}$  by

$$\underline{\xi} := \lim_{i \rightarrow \infty} f^{b_1} \circ g \circ f^{b_2} \circ g \circ \dots \circ f^{b_i}(\underline{0}). \tag{29}$$

This means that the contraction is  $g$  at the  $a_i$ th positions and  $f$  otherwise. Assume that we have chosen  $a_1 < a_2 < \dots < a_k$  and wish to construct the remaining sequence with the property stated in the theorem. Take

$$\underline{\theta}_k := f^{b_1} \circ g \circ f^{b_2} \circ g \circ \dots \circ f^{b_k} \circ g \circ f^\infty(\underline{0}). \tag{30}$$

This vector clearly lies in  $C$ . Then clearly the suffix vector  $f^\infty(\underline{0})$  of  $\underline{\theta}_k$  is the fixed point of  $f$ , so that  $f^\infty(\underline{0}) = f(\underline{\alpha}) = \underline{\alpha}$ , and recall  $\underline{\alpha}$  is assumed to be rational. Since both  $f, g$  are rational-preserving, we see that  $\underline{\theta}_k = f^{b_1} \circ g \circ f^{b_2} \circ g \circ \dots \circ f^{b_k} \circ g(\underline{\alpha})$  is also a rational vector (if the IFS is affine then with denominator  $\ll N^{da_k}$  if  $N$  is the largest denominator among the contraction factors, but we will not need this). Write  $\underline{\theta}_k = \underline{p}_k/q_k = (p_{k,1}, \dots, p_{k,d})/q_k$  in lowest terms, with  $q_k \ll N^{da_k}$ . Next, we claim that

$$\|\underline{p}_k/q_k - \underline{\xi}\| \leq \text{diam} \cdot \tau^{a_{k+1}}, \tag{31}$$

where  $\text{diam}$  denotes the diameter of our compact Cantor set  $C$ . Observe that

$$\begin{aligned} \|\underline{p}_k/q_k - \underline{\xi}\| &= \|f^{b_1} \circ g \circ \dots \circ g \circ f^{b_{k+1}}(\underline{\alpha}) - f^{b_1} \circ g \circ \dots \circ g \circ f^{b_{k+1}}(\underline{\beta})\| \\ &\leq \tau^{a_{k+1}} \|\underline{\alpha} - \underline{\beta}\|, \end{aligned}$$

for

$$\underline{\beta} = \lim_{L \rightarrow \infty} f^{b_{k+1}} \circ g \circ \dots \circ f^{b_L}(\underline{0})$$

some well-defined limit in  $C$ , since  $f, g$  are contractions with factor at most  $\tau$  and there are a total of  $a_{k+1}$  contractions applied. To derive (31) it then suffices to notice that  $\|\underline{\xi} - \underline{\beta}\| \leq \text{diam}$  as they both belong to  $C$ . From (31) we see that  $\|\underline{\xi} - \underline{p}_k/q_k\|$  can be made arbitrarily small when taking  $a_{k+1}$  large enough, which is all we need for what follows.

Since  $\tau < 1$  by definition, we may assume  $a_{k+1}$  is large enough that  $p_{k,i}/q_k$  is a convergent (not necessarily in lowest terms!) to  $\xi_i$  for any  $1 \leq i \leq d$ . Let  $s_{k,i}/t_{k,i}$  be the preceding convergent and  $y_{k,i}/z_{k,i}$  be the convergent following  $p_{k,i}/q_k$ , for  $1 \leq i \leq d$ . We have to justify that this is well defined, in the sense that  $s_{k,i}/t_{k,i}$  and  $y_{k,i}/z_{k,i}$  only depend on  $a_{k+1}$  but not on the exact choices of  $a_{k+2}, a_{k+3}, \dots$ , for any given sufficiently large  $a_{k+1}$  and much larger  $a_{k+2}$ . This is clear for the preceding convergent  $s_{k,i}/t_{k,i}$ . For  $y_{k,i}/z_{k,i}$ , notice that when we choose  $a_{k+2}$  much larger than  $a_{k+1}$  as well then we can guarantee that  $p_{k+1,i}/q_{k+1}$  is a convergent to  $\xi_i$  as well, by the same argument as above. We can thus reconstruct the convergents of  $\xi_i$  up to  $\theta_{k+1,i} = p_{k+1,i}/q_{k+1}$ . Now  $z_{k,i} \leq q_{k+1}$  by definition of  $y_{k,i}/z_{k,i}$  as the subsequent convergent of  $p_{k,i}/q_k$ . Hence, indeed, once we

have chosen large  $a_{k+1}$  and assume that  $a_{k+2}$  exceeds  $a_{k+1}$  by a large amount, everything is well defined. Since, by Lemma 5.4,

$$z_{k,i} \geq \frac{1}{2q_k |\xi_i - p_{k,i}/q_k|},$$

and  $q_k$  is fixed, we infer from (31) that we can make  $z_{k,i}$  arbitrarily large by choosing  $a_{k+1}$  accordingly large (note that for this argument we do not require  $p_{k,i}/q_k$  to be in lowest terms; the same applies to any argument below).

By Lemma 5.4 we know that

$$\left| \xi_i - \frac{s_{k,i}}{t_{k,i}} \right| \geq \frac{1}{2t_{k,i}q_k}, \quad \left| \xi_i - \frac{p_{k,i}}{q_k} \right| \geq \frac{1}{2z_{k,i}q_k},$$

and for any rational number  $r_i/s \neq p_{k,i}/q_k$  with  $0 < s < z_{k,i}/2$  we have

$$|s\xi_i - r_i| \geq \frac{1}{2q_k}.$$

So let  $Q_k = z_{k,i}/3$ . Thus, for any  $r_i/s$  as above,

$$\left| \xi_i - \frac{r_i}{s} \right| \geq \frac{1}{2sq_k} \geq \frac{1}{2q_k} \cdot \frac{1}{z_{k,i}}. \tag{32}$$

Recall that  $q_k$  is fixed so  $1/(2q_k)$  is a constant as well. On the other hand,

$$\frac{\Phi(Q_k)}{Q_k} = 3 \cdot \frac{\Phi(z_k/3)}{z_{k,i}}. \tag{33}$$

Now by assumption  $\Phi$  tends to 0, and we have observed above that we may choose  $a_{k+1}$  and consequently  $Q_k$  arbitrarily large. Thus we may choose  $a_{k+1}$  large enough so that  $\Phi(Q_k) < 1/(6q_k)$ , and (32) and (33) imply that

$$\left| \xi_i - \frac{r_i}{s} \right| > \frac{\Phi(Q_k)}{Q_k}$$

for any  $r_i/s \neq p_{k,i}/q_k = \theta_{k,i}$  with  $s < Q_k$ . Since this holds for any  $1 \leq i \leq d$  and  $\underline{\theta}_k \in C$ , this means that any  $r/s \notin C$  with  $s < Q_k$  satisfies

$$\|\underline{\xi} - \frac{r}{s}\| \geq |\xi_i - \frac{r_i}{s}| > \frac{\Phi(Q_k)}{Q_k}.$$

The desired property holds for  $Q = Q_k$ . Repeating this process, we get a sequence of values  $(Q_n)_{n \geq 1} \rightarrow \infty$  with property (8) for  $\underline{\xi}$  defined in (29).

To finish the proof, we show that all vectors  $\underline{\xi} = (\xi_1, \dots, \xi_d)$  derived from our construction are irrational, and that we can construct uncountably many distinct ones among them. The density is then also implied by Lemma 5.5. Clearly the method is flexible enough to provide uncountably many formal elements  $f^{b_1} \circ g \circ f^{b_2} \circ g \circ \dots$  with the given property. If we assume any element has at most countably many addresses, we are done. If not, by hypothesis we may assume that  $f, g$  are one-to-one. This case requires more work. We first show that any  $\underline{\xi}$  constructed above is irrational. Observe that by the fast growth of the  $a_i$ , we may assume that the approximating rational vectors  $\underline{\theta}_k = \underline{p}_k/q_k$  to  $\underline{\xi}$  satisfy

$$\|\underline{\xi} - \frac{\underline{p}_k}{q_k}\| < \frac{1}{2q_k^3}. \tag{34}$$

In particular, any  $\theta_{k,i} = p_{k,i}/q_k$  is a convergent to  $\xi_i$  (not necessarily in lowest terms). Note that the denominators  $q_k$  tend to infinity, since we divide by some integer at each contraction step. If some coordinate  $\xi_i = r_i/s_i$  were rational, then clearly any rational number  $p_{k,i}/q_k$  has distance  $\geq (s_i q_k)^{-1} \gg q_k^{-1}$  unless  $\xi_i = p_{k,i}/q_k$ . Hence, in view of (34), if  $\xi_i$  were rational, then we must have  $\xi_i = p_{k,i}/q_k$  for all large  $k$ . In particular,  $\theta_{k,i} = \theta_{k+1,i}$  for large  $k$ . Hence, if all coordinates  $\xi_i$  are rational, by increasing  $k$  if necessary we have the identity for all coordinates  $i = 1, 2, \dots, d$ , that is,  $\underline{\theta}_k = \underline{\theta}_{k+1} = \underline{\xi}$  for any large  $k$ . Since the addresses of  $\underline{\theta}_k, \underline{\theta}_{k+1}$  share the same prefix up to  $f^{b_k} \circ g$  and  $f, g$  are one-to-one by assumption, applying inverse functions repeatedly, we infer an identity

$$f^{b_{k+1}} \circ g(\underline{\alpha}) = \underline{\alpha}, \quad \underline{\alpha} = f^\infty(0). \tag{35}$$

Recall that  $f$  fixes  $\underline{\alpha}$ . In fact,  $\underline{y} = \underline{\alpha}$  is the only solution to  $f(\underline{y}) = \underline{\alpha}$  since  $f$  is one-to-one. Now note that if all contractions of the IFS fix the same element  $\underline{\alpha} \in \mathbb{Q}^d$ , it is easy to check that the Cantor set collapses to a single point  $C = \{\underline{\alpha}\}$ , but then  $\underline{\alpha}$  has uncountably many addresses, which we have dealt with before. Thus we may assume that  $g$  does not fix  $\underline{\alpha}$ , that is,  $g(\underline{\alpha}) \neq \underline{\alpha}$ . Then, by the above observation that the preimage  $\underline{\alpha}$  under  $f$  is only  $\underline{\alpha}$ , we see that (35) cannot hold for any choice of  $b_{k+1}$ . Thus we have confirmed that any  $\underline{\xi}$  in our construction is irrational. Lemma 5.5 and its proof show that taking the finite forward orbit  $\pi_1 \circ \dots \circ \pi_m(\underline{\xi})$  of some  $\underline{\xi}$  under the IFS essentially preserves the property, so we obtain a countably infinite set of suitable  $\underline{\xi}$  that is dense in  $C$ . We only sketch the proof of why the set is actually uncountable and omit rigorous calculations. Assume  $\underline{\xi}^1, \underline{\xi}^2$  are two  $\underline{\xi}$  as in (29) arising from sequences  $(a_n^1)_{n \geq 1}$  and  $(a_n^2)_{n \geq 1}$ , respectively. We claim that if they are ordered  $a_1^1 < a_1^2 < a_2^1 < a_2^2 < a_3^1 < \dots$  with very large gaps between two consecutive elements, then  $\underline{\xi}^1 \neq \underline{\xi}^2$ . If true, this clearly implies that we get an uncountable family of suitable  $\underline{\xi}$ . The claim is obvious if there is some coordinate  $i$  for which  $\xi_i^1$  is rational and  $\xi_i^2$  is irrational, or vice versa. Thus, as we have shown above that both vectors  $\underline{\xi}^1, \underline{\xi}^2$  are irrational, we may assume that there is an index  $i$  with both  $\xi_i^1, \xi_i^2$  irrational. To show the claim, observe that the respective rational approximations  $\theta_{k,i}^j = p_{k,i}^j/q_{k,i}^j$  as in (30) are again very good approximating convergents to  $\xi_i^j$ . If we choose the gaps between two consecutive elements of  $a_1^1, a_1^2, a_2^1, a_2^2, \dots$  sufficiently large in each step, then we will have  $q_{k,i}^{1'} < q_{k,i}^{2'} < q_{k+1,i}^{1'}$  for any  $k \geq 1$ , where  $p_{k,i}^{j'}/q_{k,i}^{j'}$  denotes  $p_{k,i}^j/q_{k,i}^j$  written in lowest terms. Here we use the well-known fact from the theory of continued fractions that the denominators of convergents grow fast when the approximation is good; see (26). In fact, with a proper choice the convergent  $y_{k,i}^2/z_{k,i}^2$  of  $\underline{\xi}^2$  following  $p_{k-1,i}^2/q_{k-1}^2$  will still have larger denominator than  $q_{k,i}^1$ . Hence  $p_{k,i}^1/q_{k,i}^1$  is a convergent to  $\xi_i^1$  but not convergent to  $\xi_i^2$ , thus the continued fraction expansions of  $\xi_i^1$  and  $\xi_i^2$  are not the same, consequently  $\xi_i^1 \neq \xi_i^2$  and  $\underline{\xi}^1 \neq \underline{\xi}^2$ . □

We turn to the proof of Theorem 3.4. For convenience, we will use the framework of parametric geometry of numbers introduced in [28], essentially a parametric logarithmic version of Minkowski’s second lattice point theorem. We only need the special case of approximation to a single number, which corresponds to  $n = 2$  in the notation of [28]. This means essentially that we use a parametric, logarithmic version of formula (28) above.

Concretely, for given  $\xi \in \mathbb{R}$ , similarly to the proof of Lemma 5.4 we consider the lattice  $\Lambda_\xi$  and the family of convex bodies  $K(T)$  parametrized by  $T \geq 1$  of the form

$$\Lambda_\xi = \{(m, m\xi - n) \in \mathbb{R}^2 : m, n \in \mathbb{Z}\}, \quad K(T) = [-T, T] \times [-T^{-1}, T^{-1}]. \tag{36}$$

Observe that a point in  $\Lambda_\xi \cap K(T)$  corresponds to the system of inequalities

$$|m| \leq T, \quad |m\xi - n| \leq T^{-1}.$$

Denote by  $\lambda_{1,\xi}(T), \lambda_{2,\xi}(T)$  the successive minima of  $K(T)$  with respect to the lattice  $\Lambda_\xi$ , that is,  $\lambda_{i,\xi}$  is the minimum value for which  $\lambda_{i,\xi}(T)K(T)$  contains  $i$  linearly independent points of  $\Lambda_\xi$ , for  $i \in \{1, 2\}$ . According to Minkowski’s second convex body theorem (28), in view of  $\text{vol}(K(T)) = 4$  and  $\det \Lambda_\xi = 1$ , we infer that

$$\frac{1}{2} = 2 \frac{\det \Lambda_\xi}{\text{vol } K(T)} \leq \lambda_{1,\xi}(T)\lambda_{2,\xi}(T) \leq 4 \frac{\det \Lambda_\xi}{\text{vol } K(T)} = 1.$$

If we let  $t = \log T$  and  $L_{i,\xi}(t) = \log \lambda_{i,\xi}(T)$  for  $i \in \{1, 2\}$ , then we obtain

$$-\log 2 \leq L_{1,\xi}(t) + L_{2,\xi}(t) \leq 0, \quad t \geq 0. \tag{37}$$

The functions  $L_{i,\xi}(t)$  are piecewise linear with slopes in  $\{-1, 1\}$ . More precisely, if  $(m_i, n_i) \in \mathbb{Z}^2$ , with  $m_i > 0$ , are the vectors realizing the  $i$ th minimum at the given position  $t = \log T$  for  $i \in \{1, 2\}$ , then

$$L_{i,\xi}(t) = L_{(m_i, n_i)}(t), \quad i \in \{1, 2\},$$

where for  $(m, n) \in \mathbb{Z}^2$  we denote

$$L_{(m, n)}(t) := \max\{\log m - t, \log |m\xi - n| + t\}. \tag{38}$$

In particular,  $L_{(m, n)}(t)$  has its minimum at the point position  $(t_0, L_{(m, n)}(t_0))$  with  $t = t_0 = (\log m - \log |m\xi - n|)/2$ , where the expressions in the maximum coincide. This set-up will suffice for our purpose to prove Theorem 3.4; we refer to [28] for parametric geometry of numbers with respect to simultaneous rational approximation to  $\xi_1, \dots, \xi_d$ .

*Proof of Theorem 3.4.* First consider the special case  $C = C_{b, w}$ . Fix any  $p/q \in \mathbb{Q} \setminus C$ . Let  $Q = 2bq$ . Further, let  $\xi \in C \setminus \mathbb{Q}$  be given. Note that we are in the situation of Theorem 1.1. Application of Theorem 1.1 to  $Q \geq 2b \geq 1$  yields that the system

$$1 \leq m \leq bQ^\Delta, \quad |m\xi - n| \leq \frac{b}{Q}$$

has a solution in positive integers  $m, n$  such that  $n/m \in \mathbb{Q} \cap C$ . Write  $|m\xi - n| = \sigma/Q$  with  $\sigma \in [0, b]$ . Since  $\xi$  is irrational in fact  $\sigma > 0$ . Transition to logarithmic scale yields that the minimum of the function  $L_{(m, n)}(t)$  is attained for

$$t = t_0 := \frac{\log m + \log Q - \log \sigma}{2},$$

and at this position we have

$$L_{(m, n)}(t_0) = \log m - t_0 = \frac{\log m - \log Q + \log \sigma}{2}.$$



Clearly the  $(p, q) \in \mathbb{Z}^2$  we started with is linearly independent of  $(m, n) \in \mathbb{Z}^2$ , since this just rephrases as  $m/n \neq p/q$ , which is true since  $m/n \in C$  but  $p/q \notin C$ . Hence by (37) we have

$$L_{(p,q)}(t_0) \geq \frac{\log Q - \log m - \log \sigma}{2} - \log 2,$$

as the reverse estimate would imply that  $L_{1,\xi}(t_0) + L_{2,\xi}(t_0) \leq L_{(p,q)}(t_0) + L_{(m,n)}(t_0) < -\log 2$ , contradicting (37). In view of the definition of  $L_{(p,q)}$  in (38), equivalently at least one of the inequalities

$$\log q - t_0 \geq \frac{\log Q - \log m - \log \sigma}{2} - \log 2$$

and

$$\log |q\xi - p| + t_0 \geq \frac{\log Q - \log m - \log \sigma}{2} - \log 2$$

holds. Thus, if the first estimate is violated, which just becomes

$$q \leq \frac{Q}{2\sigma}, \tag{39}$$

then the second one is correct, which yields

$$|q\xi - p| \geq \frac{1}{2m}.$$

However, (39) is satisfied since  $q = Q/2b \leq Q/2\sigma$ , therefore we conclude that

$$|q\xi - p| \geq \frac{1}{2m} \geq \frac{b^{-Q^\Delta}}{2} = \frac{b^{-(2b)^\Delta q^\Delta}}{2}. \tag{40}$$

Notice that the bound is independent of the choice of  $\xi \in C \setminus \mathbb{Q}$ . Assume that the distance  $d(C, p/q)$  for some  $p/q \in \mathbb{Q} \setminus C$  was smaller than the right-hand side of (40) divided by  $q$ . Then by compactness of  $C$  there is  $\xi_0 \in C$  realizing this distance  $d(C, p/q) = d(\xi_0, p/q)$ . Clearly  $\xi_0 \notin C \setminus \mathbb{Q}$ , since for these numbers we have (40). However, since  $C \setminus \mathbb{Q}$  is dense in  $C$ , we get a contradiction to (40) anyway by choosing  $\xi \in C \setminus \mathbb{Q}$  sufficiently close to  $\xi_0$ . Hence the right-hand side in (40) divided by  $q$  is a lower bound for  $d(C, p/q)$ , which means (11) is true. Since  $p/q \in \mathbb{Q} \setminus C$  and  $\xi \in C \setminus \mathbb{Q}$  were chosen arbitrary, we readily infer (12) from (40) as well. This finishes the proof of the special case  $C = C_{b,w}$ .

For the general case of any monic, rationally generated Cantor set  $C$  with the OSC, by Theorem 1.2 for any  $\xi \in C \setminus \mathbb{Q}$  the estimates

$$1 \leq m \leq e^{Q^\Delta}, \quad \left| \xi - \frac{m}{n} \right| \leq \frac{K}{Q}$$

admit a solution  $m/n \in C$  for any large  $Q$  and some constant  $K$ . For given  $p/q \in \mathbb{Q} \setminus C$ , we proceed as above for  $Q = 2Kq$  to obtain

$$|q\xi - p| \geq \frac{1}{2m} \geq \frac{e^{-Q^\Delta}}{2} = \frac{e^{-(2K)^\Delta q^\Delta}}{2},$$

and the claim follows again by choosing  $\rho > (2K)^\Delta$  sufficiently large to guarantee

$$e^{-\rho q^\Delta} < \frac{e^{-(2K)^\Delta q^\Delta}}{q}$$

simultaneously for any  $q \geq 1$ . □

A similar strategy is applied for the proof of Theorem 4.6 below, indeed the right-hand claim of (17) rewritten as  $\lambda_{\text{ext}}(\xi) \leq 1/\widehat{\lambda}_{\text{int}}(\xi)$  resembles the assertion of Theorem 3.4.

5.3. *Proofs from Section 4.* We start this section with the proof of the counting results. For this purpose we employ Proposition 5.1 above.

*Proof of Theorem 4.1.* Let  $l \geq 0$  be an integer. Assume that there are more than  $J^l$  distinct rational vectors in  $C$ , each with common denominator at most  $N$ . Then by Proposition 5.1 there are two vectors  $\underline{\alpha}, \underline{\alpha}'$  among them that differ by at most  $\|\underline{\alpha} - \underline{\alpha}'\| \leq \text{diam} \cdot \tau^l$ . On the other hand, if we write  $\underline{\alpha}' = \underline{p}/q, \underline{\alpha}' = \underline{r}/s$ , then

$$\|\underline{\alpha} - \underline{\alpha}'\| = \|\underline{p}/q - \underline{r}/s\| \geq \frac{1}{qs} \geq N^{-2}. \tag{41}$$

We conclude that  $N \geq \text{diam}^{-1/2} \tau^{-l/2}$ . In other words, if  $N < \text{diam}^{-1/2} \tau^{-l/2}$  then there are at most  $J^l = \tau^{-Dl}$  rational vectors with common denominator at most  $N$ . Now, for given  $N$ , let  $l$  be the unique integer with  $\text{diam}^{-1/2} \tau^{-l/2+1} \leq N < \text{diam}^{-1/2} \tau^{-l/2}$ . Then by the above argument there are at most  $J^l = \tau^{-Dl} < \text{diam}^D (N/\tau)^{2D} = J^2 \text{diam}^D \cdot N^{2D}$  rational vectors with common denominator at most  $N$ .  $\square$

The estimate extends to bounding the cardinality of vectors  $(r_1/s_1, \dots, r_d/s_d)$  with each denominator  $\max_{1 \leq j \leq d} s_j \leq N$ . The proof of Theorem 4.2 employs Liouville’s inequality.

*Proof of Theorem 4.2.* Let  $\underline{\alpha} = (\alpha_1, \dots, \alpha_d), \underline{\alpha}' = (\alpha'_1, \dots, \alpha'_d)$  be distinct algebraic vectors with entries of degree at most  $n$  and heights  $H(\underline{\alpha}) = \max H(\alpha_j), H(\underline{\alpha}') = \max H(\alpha'_j)$  at most  $N$ , respectively. Since the vectors are distinct there is an index  $j$  with  $\alpha_j \neq \alpha'_j$ . Then Liouville’s inequality [7, Theorem A.1, Corollary A.2] yields

$$\|\underline{\alpha} - \underline{\alpha}'\| \geq |\alpha_j - \alpha'_j| \gg_n H(\alpha_j)^{-n} H(\alpha'_j)^{-n} \geq N^{-2n}.$$

Using this estimate in place of (41), the claim follows very similarly to Theorem 4.1.  $\square$

Next we show Theorem 4.3.

*Proof of Theorem 4.3.* Let  $(\omega_1, \omega_2, \dots)$  be an address of  $\underline{\xi} \in C$  and  $\pi_i = f_{\omega_i}$ , so that  $\underline{\xi} = \lim_{n \rightarrow \infty} \pi_1 \circ \pi_2 \cdots \circ \pi_n(\underline{0})$ . Assume that  $(\omega_1, \dots, \omega_l)$  is the (possibly empty) preperiod and  $(\omega_{l+1}, \dots, \omega_u)^\infty$  is the successive period. Let  $\underline{\zeta} := (\pi_{l+1} \circ \dots \circ \pi_u)^\infty(\underline{0})$ , so that  $\underline{\xi} = \pi_1 \circ \dots \circ \pi_l(\underline{\zeta})$ . By construction,  $\underline{\zeta} = \pi_{l+1} \circ \dots \circ \pi_u(\underline{\zeta})$ . For simplicity write  $C_j = A_{\omega_j}/q_{\omega_j} \in \mathbb{Q}^{d \times d}$  and  $\underline{c}_j = \underline{b}_{\omega_j}/s_{\omega_j} \in \mathbb{Q}^d$ , with  $A_j, \underline{b}_j$  as in Definition 5, so that  $\pi_j(\underline{y}) = C_j \underline{y} + \underline{c}_j$ . Therefore from the concatenation we obtain an identity  $C_{l+1} C_{l+2} \cdots C_u \underline{\zeta} + \underline{c}' = \underline{\zeta}$ , for some  $\underline{c}' \in \mathbb{Q}^d$ . Since  $C_j$  induce contractions, so does any product, so 1 is not an eigenvalue and we see that  $\underline{\zeta}$  is given by  $\underline{\zeta} = -(C_u C_{u-1} \cdots C_{l+1} - I_d)^{-1} \underline{c}'$ . Hence  $\underline{\zeta} \in \mathbb{Q}^d$  as  $C_j \in \mathbb{Q}^{d \times d}$  and  $\underline{c}' \in \mathbb{Q}^d$ . Thus also  $\underline{\xi} = \pi_1 \circ \dots \circ \pi_l(\underline{\zeta})$  is rational since the  $\pi_j \in F$  are rational-preserving.

Conversely, assume that the IFS is unimodular and take arbitrary  $\underline{\xi} = \underline{p}/q \in \mathbb{Q}^d \cap C$ . Again assume that  $\pi_1 \circ \pi_2 \cdots \circ \pi_n(\underline{0})$  is any formal representation of  $\underline{\xi}$ . Consider the sequence  $\underline{a}_0 = \underline{\xi}, \underline{a}_1 = \pi_1^{-1}(\underline{\xi}), \underline{a}_2 = \pi_2^{-1} \circ \pi_1^{-1}(\underline{\xi}), \dots$ , which is well defined as clearly  $\pi_j$  are bijective if the IFS is unimodular. Then obviously  $\underline{a}_n \in \mathbb{Q}^d \cap C$  for any  $n \geq 0$ . Moreover, when building the inverses to derive  $\underline{a}_{n+1}$  from  $\underline{a}_n$ , it follows from the rational IFS being

unimodular that the denominators that occur are divisors of  $qS$  with  $S := \prod_{1 \leq j \leq J} s_j$ . Here we use the fact that when building the inverses  $\pi_j^{-1}$  with Cramer’s rule we do not get additional factors in the denominator by unimodularity of the matrices, and the shift vectors  $\underline{b}_j/s_j$  can possibly only cause a factor that divides  $S$ . Since  $C$  is compact with some finite diameter  $\text{diam} > 0$ , there are at most  $(qS \cdot \text{diam} + 1)^d \ll_C q^d$  rational vectors in  $C$  with this property. Moreover, since  $\underline{a}_n \in \mathbb{Q}^d \cap C$  with denominator  $\ll_C q$  we infer from Theorem 4.1 and its proof that there are at most  $\ll_C q^D$  such vectors (we may remove the factor 2 in the exponent since all denominators divide  $q$ ; see the proof of Theorem 4.1). Hence the number is  $\ll_C \min\{q^D, q^d\}$ . By this finiteness, some rational vector must occur twice, that is,  $\underline{a}_i = \underline{a}_j$  for some  $i < j \ll_C \min\{q^D, q^d\}$ . This means that  $\underline{a}_j = \pi_{i+1} \circ \pi_{i+2} \circ \dots \circ \pi_j(\underline{a}_i) = \underline{a}_i$ . Hence

$$\begin{aligned} \underline{\xi} &= \pi_1 \circ \dots \circ \pi_i(\underline{a}_i) = \pi_1 \circ \dots \circ \pi_i \circ (\text{id})^\infty(\underline{a}_i) \\ &= \pi_1 \circ \dots \circ \pi_i \circ (\pi_{i+1} \circ \pi_{i+2} \circ \dots \circ \pi_j)^\infty(\underline{a}_i) \\ &= \pi_1 \circ \dots \circ \pi_i \circ (\pi_{i+1} \circ \pi_{i+2} \circ \dots \circ \pi_j)^\infty(0). \end{aligned}$$

In the last identity we used the fact that  $\pi_i \in F$  are contractions. Thus  $(\omega_1, \dots, \omega_i)$  composed with  $(\omega_{i+1}, \omega_{i+2}, \dots, \omega_j)^\infty$  is a periodic address of  $\underline{\xi}$ . □

The estimate on rationals in  $C$  whose denominator divides  $q$  we used is potentially very crude; we believe in fact  $\log q$  should be the correct order of period lengths.

*Proof of Corollary 4.4.* For the first claim let  $q_i$  be as in the theorem and  $Q = q_1 q_2 \dots q_v$  be their product. We notice that for a given prime  $q \nmid b$  it follows directly from [3, Lemma 3] that  $\text{ord}_b \text{ mod } q^n = m q^{n-O(1)} \gg q^n$  for some  $m$  dividing  $q - 1$  and all  $n$ . Thus, since for  $s = q_1^{\alpha_1} q_2^{\alpha_2} \dots q_v^{\alpha_v}$  we have that  $\text{ord}_b \text{ mod } s$  is the lowest common multiple of the orders  $\text{ord}_b \text{ mod } q_i^{\alpha_i}$  over  $1 \leq i \leq v$  and  $q_i$  are distinct primes not dividing  $b$ , we see that  $\text{ord}_b \text{ mod } s$  is divisible by  $s/Q^{O(1)}$  and thus  $\text{ord}_b \text{ mod } s \gg s$ . On the other hand, according to Proposition 4.10 the period length of  $r/s \in C_{b,W}$  equals  $\text{ord}_b \text{ mod } s$ , which is  $\ll_C s^D = s^\Delta$  by Theorem 4.3. For large  $s$  we get a contradiction as  $\Delta < 1$ . For the second claim, again since the period length of  $p/q$  is  $\text{ord}_b \text{ mod } q$ , it divides  $\varphi(q) = q - 1$ . Since  $(q - 1)/2$  is prime the only possible divisors are  $\{1, 2, (q - 1)/2, q - 1\}$ . On the other hand, by Theorem 4.3 the period length is  $\ll q^D = q^\Delta$ . Since  $\Delta < 1$  for large  $q$  the divisors  $(q - 1)/2, q - 1$  can be ruled out, so  $q|b^2 - 1$ , which only gives finitely many values of  $q$  as well. In both claims, since  $C_{b,W} \subseteq [0, 1]$ , this yields only finitely many potential rational numbers of the stated form. □

As indicated above, the proof of Theorem 4.6 is similar to that of Theorem 3.4.

*Proof of Theorem 4.6.* As in Lemma 5.4, let  $u_l/v_l$  be the convergents of  $\xi \in S$ . We only show the left inequality of (17); the right is proved analogously. We may assume that  $\lambda_{\text{int}}(\xi) > 1$ , as otherwise the claim is trivial. For simplicity write  $a = \lambda_{\text{int}}(\xi)$ . Let  $\epsilon > 0$ . Recall from the theory of continued fractions that any fraction  $p/q$  that is not convergent to  $\xi$  satisfies  $|\xi - p/q| > (2q)^{-1}$ . Thus, our assumption implies that there exist arbitrarily large indices  $l$  such that the convergent  $u_l/v_l$  lies in  $S$  and has the property

$$v_l^{-a-\epsilon} \leq |v_l \xi - u_l| \leq v_l^{-a+\epsilon}. \tag{42}$$

By Lemma 5.4 we have  $v_{l+1} \asymp |v_l \xi - u_l|^{-1}$ , so for large  $l$  we infer that

$$v_l^{a-\epsilon} \ll v_{l+1} \ll v_l^{a+\epsilon}. \tag{43}$$

Moreover, again Lemma 5.4 shows that for any  $r/s \neq u_l/v_l$  with  $s < v_{l+1}/2$  we have  $|s\xi - r| \geq v_l^{-1}/2$ . In particular, this estimate holds for any  $r/s \notin S$  with  $s < v_{l+1}/2$ . Thus for  $Q_l := v_{l+1}/3$  and any such  $r/s \notin S$  with  $s < Q$  we have

$$|s\xi - r| \geq \frac{1}{2} v_l^{-1}.$$

On the other hand, from (43) we know that  $v_l \gg v_{l+1}^{1/(a+\epsilon)} \gg Q_l^{1/(a+\epsilon)}$ . Thus the left-hand side is  $\gg Q_l^{-1/(a+\epsilon)}$  for any  $r/s \notin S$  with  $s < Q$ . The left inequality in (17) follows since clearly  $Q_l \rightarrow \infty$  as  $l \rightarrow \infty$  and  $\epsilon$  can be taken arbitrarily small. As indicated above, an analogous argument starting with  $a = \lambda_{\text{ext}}(\xi)$  yields the right estimate in (17).

In the case where  $S = C$  is a Cantor set as in the last claim, we now provide the reverse of the left estimate of (17) in order to show (18). First assume that  $\lambda_{\text{int}}(\xi) =: a \geq 1$ . We may again assume strict inequality  $a > 1$  as if otherwise  $a = 1$ , the claimed estimate is a consequence of (17) and (16). This again implies that certain convergents satisfy  $u_l/v_l \in C$  as above. Let  $Q$  be an arbitrary, large real number. Let  $l$  be the index such that  $v_l \leq Q < v_{l+1}$ . Then by Lemma 5.4 we have

$$Q^{-1} > v_{l+1}^{-1} \geq |v_l \xi - u_l|. \tag{44}$$

If  $u_l/v_l \notin C$ , then since  $1 > 1/a$  we may simply choose  $r/s = u_l/v_l$ . Now assume that  $u_l/v_l \in C$ . Then by (43), for large  $l$  additionally to (44) we have

$$Q^{-1/(a+\epsilon)} \geq v_{l+1}^{-1/(a+\epsilon)} \gg v_l^{-1} \geq |v_{l-1} \xi - u_{l-1}|.$$

Combining, for any given natural number  $N$  and any  $1 \leq n \leq N$ , we have

$$\begin{aligned} |(v_l + nv_{l-1})\xi - u_l - nu_{l-1}| &\leq |v_l \xi - u_l| + n|v_{l-1} \xi - u_{l-1}| \\ &\ll_N \max\{Q^{-1}, Q^{-1/(a+\epsilon)}\} = Q^{-1/(a+\epsilon)}. \end{aligned}$$

Now, from Lemma 5.3 we see that for large  $N = N(C)$  and some  $1 \leq n \leq N$  the expression  $(u_l + nu_{l-1})/(v_l + nv_{l-1})$  does not belong to  $C$ . By the above estimate, as  $\epsilon$  can be chosen arbitrarily small and  $Q$  was chosen arbitrarily, we verify the reverse inequality

$$\widehat{\lambda}_{\text{ext}}(\xi) \geq \frac{1}{a} = \frac{1}{\lambda_{\text{int}}(\xi)},$$

so by (17) we must have equality. Hence (18) holds in this case.

Finally, assume that  $\lambda_{\text{int}}(\xi) < 1$ . Then we have  $|q\xi - p| \gg q^{-1+\delta}$  for some  $\delta > 0$  and any  $p/q \in \mathbb{Q} \cap C$ . On the other hand, by Dirichlet’s theorem we know that for any parameter  $Q$  there is a rational number  $p/q$  with  $q \leq Q$  and  $|q\xi - p| \leq Q^{-1} \leq q^{-1}$ . Combining these observations, for large  $Q$  these rationals  $p/q$  must lie outside  $C$ . Hence  $\widehat{\lambda}_{\text{ext}}(\xi) = \widehat{\lambda}(\xi) = 1$ , where the last identity is Khintchine’s result already mentioned in (16). Again we conclude (18). □

The question arises whether we can reverse the argument of the proof to infer ‘good’ intrinsic rational approximations to  $\xi$  from ‘bad’ uniform extrinsic approximation of  $\xi \in C$ , corresponding to a hypothetical inequality like  $\lambda_{\text{int}}(\xi) \geq 1/\widehat{\lambda}_{\text{ext}}(\xi)$  if  $\widehat{\lambda}_{\text{ext}}(\xi)$  is small enough (it is in general false by a metrical argument if  $\widehat{\lambda}_{\text{ext}}(\xi) = 1$ ). The problem is

that in the lattice point problem studied above, both successive minima may be throughout realized by vectors  $(m_i, n_i) \in \mathbb{Z}^2$  that lead to rational numbers  $m_i/n_i$  in  $C$ , for  $i = 1, 2$ , in which case can only infer  $\lambda_{\text{int}}(\xi) \geq 1$ . Nevertheless, as this seems to be a rather exceptional situation that may not occur, the estimate  $\lambda_{\text{int}}(\xi) \geq 1/\widehat{\lambda}_{\text{ext}}(\xi)$  may be true if  $\widehat{\lambda}_{\text{ext}}(\xi) < 1$ .

*Problem 5.* Assume that  $\widehat{\lambda}_{\text{ext}}(\xi) < 1$ . Does it follow that  $\lambda_{\text{int}}(\xi) \geq 1/\widehat{\lambda}_{\text{ext}}(\xi)$ ?

We now turn to the verification of Theorem 4.9.

**PROPOSITION 5.6.** *Let  $b \geq 3$  an integer and  $W \subsetneq \{0, 1, \dots, b - 1\}$ . Assume that  $\xi = (0.c_1c_2 \dots c_k \overline{c_{k+1} \dots c_{N-1}})_b$  is the base  $b$  representation of a rational number. If  $c_1, \dots, c_{k+1}$  belong to  $W$  and  $c_m \notin W$  for some  $m > k + 1$ , then  $\xi \notin C_{b,W}$ .*

*Proof.* If  $\xi$  has a unique base  $b$  expansion then the claim follows from  $c_j \notin W$ . If the representation is not unique then there are two base  $b$  representations of  $\xi$ , one ending in  $\overline{0}$  and the other ending in  $\overline{b - 1}$ . However, since  $c_{k+1} \in W$  and  $c_j \notin W$  for some  $j > k + 1$  and both letters appear in the period above, clearly  $\xi$  is not of this form, and we have a contradiction. □

*Proof of Theorem 4.9.* Clearly any  $\xi \in \mathbb{Q}$  has ultimately periodic base  $b$  expansion, and the representation of  $\xi$  in (19) via  $\xi = p/q$  with  $p, q$  in (20) is carried out straightforwardly as in [5, Lemma 2.3]. Let  $m = \phi(\xi)$ . If  $m \leq k + 1$  so that the first digit outside  $W$  belongs to the preperiod or equals the first period digit, then the estimate follows from the elementary estimate  $m \leq k + 1 \ll \log q_0 \ll q_0^\Delta$ , as discussed in §4.4. Thus we may assume that  $m > k + 1$ . Obviously  $\xi$  has distance at most  $b \cdot b^{-m}$  from  $r := c_0b^{-1} + c_1b^{-2} + \dots + c_{m-1}b^{-m} \in \mathbb{Q}$ . By definition of  $m$ , clearly  $r \in C_{b,W}$ . On the other hand, by our assumption  $c_m \notin W$  and by Proposition 5.6, we have  $\xi \notin C_{b,W}$ . Hence, by Theorem 3.4 the rational number  $\xi$  has distance at least  $b^{-\delta_0 q_0^\Delta}$  from any element of  $C = C_{b,W}$  for some uniform  $\delta_0$ , in particular from  $r$ . Comparison yields  $\phi(\xi) = m \ll q_0^\Delta$ . □

*Proof of Corollary 4.12.* Obviously  $W_1 \subseteq W_2$ , and if  $W_1 = \{0, 1, \dots, b - 1\}$  the claim is obvious. Otherwise there is some  $w \notin W_1$ . If  $w \in W_2$ , by taking  $W = \{0, 1, \dots, b - 1\} \setminus \{w\}$ , from Theorem 4.9 we get  $d' \geq c(b) \cdot \phi^{1/\Delta}$ . We get a contradiction by our assumptions on  $\phi, \psi$  as soon as  $c_1 \geq c(b)^\Delta$ . Hence  $w \notin W_2$ , and since  $w \notin W_1$  was arbitrary the sets are equal. □

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