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# THE NONHOMOGENEOUS FROG MODEL ON $\mathbb{Z}$

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**Abstract**

We examine a system of interacting random walks with leftward drift on $\mathbb{Z}$, which begins with a single active particle at the origin and some distribution of inactive particles on the positive integers. Inactive particles become activated when landed on by other particles, and all particles beginning at the same point possess equal leftward drift. Once activated, the trajectories of distinct particles are independent. This system belongs to a broader class of problems involving interacting random walks on rooted graphs, referred to collectively as the frog model. Additional conditions that we impose on our model include that the number of frogs (i.e. particles) at positive integer points is a sequence of independent random variables which is increasing in terms of the standard stochastic order, and that the sequence of leftward drifts associated with frogs originating at these points is decreasing. Our results include sharp conditions with respect to the sequence of random variables and the sequence of drifts that determine whether the model is transient (meaning the probability infinitely many frogs return to the origin is 0) or nontransient. We consider several, more specific, versions of the model described, and a cleaner, more simplified set of sharp conditions will be established for each case.

*Keywords:* Frog model; transience; drift; coupling

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## 1. Introduction

The frog model is a system of interacting random walks on a rooted graph. It begins with a single 'active' frog at the root and some distribution of sleeping frogs (either random or deterministic) at the nonroot vertices. The active frog performs a discrete-time nearest-neighbor random walk on the graph (biased or unbiased) and any time an active frog lands on a vertex containing sleeping frogs, all of these frogs become active and begin performing their own discrete-time nearest-neighbor random walks, activating sleeping frogs along the way. Previous work on the frog model has included investigating the model on infinite $n$-ary trees [4] as well as on Euclidean lattices; see [2] and [6]. In particular, a number of authors have studied a variety of different versions of the frog model on $\mathbb{Z}$, often focusing on establishing conditions that determine whether the model is recurrent or transient with respect to the number of distinct frogs that visit the root. In this paper we will focus on exploring several of these models while building on, expanding, and synthesizing some of the existing results pertaining to them.

There are three existing results, each addressing a different version of the frog model on $\mathbb{Z}$, that serve as a jumping off point for the present work. The first concerns a model in which all non-zero vertices contain an independent and identically distributed (i.i.d.) number of sleeping frogs, and activated frogs perform mutually independent random walks that go left with probability $p$

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(where $\frac{1}{2} < p < 1$) and right with probability $1 - p$. Gantert and Schmidt [3] proved that if $\eta$ represents a random variable with the same distribution as the number of sleeping frogs at each nonzero vertex, then

$$\mathbb{P}_\eta(\text{the origin is visited i.o.}) = \begin{cases} 0 & \text{if } \mathbb{E}[\log^+ \eta] < \infty, \\ 1 & \text{if } \mathbb{E}[\log^+ \eta] = \infty, \end{cases} \tag{1.1}$$

where we abbreviate infinitely often to i.o. (note that this condition does not depend on the particular value of $p$).

The second result that served to motivate the present investigations involved a model in which negative integer vertices contain no sleeping frogs and positive integer vertices each contain a single sleeping frog. For each $n > 0$, the frog at $x = n$ (if activated) performs a random walk (independently of the other active frogs) that goes left with probability $p_n$ (with $\frac{1}{2} < p_n < 1$), and right with probability $1 - p_n$ (i.e. the particular drift value depends on where the frog originated). Bertacchi *et al.* [1] established (in addition to a number of other results) that if there exists some increasing sequence of positive integers $\{n_k\}_{k \in \mathbb{N}}$ such that

$$\sum_{k=0}^{\infty} \prod_{i=0}^{n_k} \bigg(1 - \bigg(\frac{1 - p_i}{p_i}\bigg)^{n_{k+1} - i}\bigg) < \infty, \tag{1.2}$$

then the model is nontransient (i.e. infinitely many frogs hit the origin with positive probability).

The third and final result we build on again looks at a frog model on $\mathbb{Z}$ for which no sleeping frogs reside to the left of the origin, and activated frogs perform random walks with leftward drift. This time the number of sleeping frogs $X_j$ at $x = j$ (for $j \geq 1$) has Poisson distribution $\text{Poi}(\eta_j)$, where the $X_j$ are mutually independent and $\{\eta_j\}$ is an increasing sequence. At each step activated frogs go left with probability $p$ (for $\frac{1}{2} < p < 1$) and right with probability $1 - p$. This model was introduced in [5], where it was established that the model is nontransient if and only if

$$\sum_{j=1}^{\infty} \exp\bigg(-\frac{1 - p}{2p - 1} \eta_j\bigg) < \infty. \tag{1.3}$$

*Statement and discussion of results.* In our first result we establish a sharp condition distinguishing between transience and nontransience for a more general frog model on $\mathbb{Z}$ that subsumes all three of the models described above. In this model points to the left of the origin contain no sleeping frogs and, for $j \geq 1$, the number of sleeping frogs at $x = j$ is a random variable $X_j$, where the $X_j$ are independent, nonzero with positive probability, and where $X_{j+1} \succeq X_j$ (here '$\succeq$' represents stochastic dominance). In addition, for each $j \geq 1$, frogs originating at $x = j$ (if activated) go left with probability $p_j$ (where $\frac{1}{2} < p_j < 1$) and right with probability $1 - p_j$, where the $p_j$ are decreasing and the random walks are all mutually independent (the frog beginning at the origin goes left with probability $p_0$, where $p_0$ also satisfies $\frac{1}{2} < p_0 < 1$). This model will be referred to as the nonhomogeneous frog model on $\mathbb{Z}$, and the sharp condition we eluded to will come in the form of the following theorem.

**Theorem 1.1.** *Let $f_j$ be the probability generating function of $X_j$ for the nonhomogeneous frog model on $\mathbb{Z}$. The model is transient if and only if*

$$\sum_{n=2}^{\infty} \prod_{j=1}^{n-1} f_j \bigg(1 - \bigg(\frac{1 - p_j}{p_j}\bigg)^{n-j}\bigg) = \infty. \tag{1.4}$$

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After establishing Theorem 1.1, the focus will shift towards showing how it can be applied in a number of more specific cases. The first application of the theorem will involve the Gantert and Schmidt model from [3], and will entail showing how (1.1) can be achieved quite easily using (1.4). Following this, we use Theorem 1.1 to obtain a formula (see Theorem 3.1) that provides a sharp condition distinguishing between transience and nontransience in the case where the $X_j$ are i.i.d. and which, for the particular case where $X_j = 1$, builds on the result from [1] by giving a sharp result that supersedes the soft condition in (1.2) and, for the case where $p_j = \frac{1}{2} + C/\log j$ (for all but finitely many $j$), implies the existence of a phase transition at $C = \frac{1}{24}\pi^2$. Finally, we also employ Theorem 1.1 to obtain a formula that builds on the result from [5] by generalizing (1.3) to cases where the $p_j$ are not constant. For these last two results, the proofs will require some light assumptions relating to the concavity of the sequences $\{p_j^{-1}\}$ and $\{\lambda_j\}$, where $\lambda_j$ represents the Poisson mean of the distribution of $X_j$ in the final model discussed.

## 2. Transience versus nontransience for the general case

### 2.1. Constructing $M_j$ and $N_j$

In order to move towards a proof of Theorem 1.1, we begin by defining the process $\{M_j\}$, where, for each $j \geq 1$, $M_j$ represents the number of frogs originating in $\{0, 1, \dots, j-1\}$ that ever hit the point $x = j$. We now identify $\{M_j\}$ with a triple $(\Omega, \mathcal{F}, \boldsymbol{P})$ defined as follows: $\Omega$ represents the set of all functions $\omega\colon \mathbb{Z}^+ \to \mathbb{N}$ (i.e. the set of all possible trajectories of $\{M_j\}$), $\mathcal{F}$ represents the $\sigma$-field on $\Omega$ generated by the finite-dimensional sets, and $\boldsymbol{P}$ refers to the probability measure induced on $(\Omega, \mathcal{F})$ by the process $\{M_j\}$. Since $\mathbb{P}(X_n \geq 1) \geq \mathbb{P}(X_1 \geq 1) > 0$ for all $n \geq 1$ (recall that $X_{j+1} \succeq X_j$ for all $j \geq 1$) and the $X_j$ are independent, it follows from the second Borel–Cantelli lemma (BC2) that $\{X_j \geq 1 \text{ i.o.}\}$ almost surely (a.s.). Additionally, since each activated frog performs a random walk with nonzero leftward drift, this means that each activated frog will eventually hit the origin with probability 1. Coupling this with the fact that $\{X_j \geq 1 \text{ i.o.}\}$ a.s. implies that $\sum_{j=1}^{\infty} X_j = \infty$ a.s., along with the implication $M_l = 0 \implies M_j = 0$ for all $j > l$, we find that

$$\{\text{infinitely many frogs hit the origin}\} \quad \iff \quad \min M_j > 0. \tag{2.1}$$

Now, on account of (2.1), it follows that in order to establish Theorem 1.1, it suffices to show that

$$\min M_j = 0, \qquad \boldsymbol{P}\text{-a.s.} \quad \iff \quad \sum_{n=2}^{\infty} \prod_{j=1}^{n-1} f_j\Bigl(1 - \Bigl(\frac{1-p_j}{p_j}\Bigr)^{n-j}\Bigr) = \infty. \tag{2.2}$$

With this in mind, we define a new model which we call the $\mathrm{F}^+$ model. This model resembles the nonhomogeneous frog model on $\mathbb{Z}$ in that the distribution of the number of frogs beginning at every vertex is the same in the two cases, as are the drifts of the active frogs. The only difference is that in the $\mathrm{F}^+$ model all frogs begin as active frogs (i.e. they do not need to be landed on to be activated). The next step is to use the $\mathrm{F}^+$ model to define the process $\{N_j\}$, where, for each $j \geq 1$, $N_j$ is equal to the number of frogs originating in $\{0, 1, \dots, j-1\}$ that ever hit the point $x = j$ in the $\mathrm{F}^+$ model (i.e. $\{N_j\}$ is identical to $\{M_j\}$ except that the $\mathrm{F}^+$ model replaces the nonhomogeneous frog model on $\mathbb{Z}$ in the definition). We identify $\{N_j\}$ with the triple $(\Omega, \mathcal{F}, \boldsymbol{Q})$, where $\boldsymbol{Q}$ refers to the probability measure induced on $(\Omega, \mathcal{F})$ by the process $\{N_j\}$. Having defined this construction, we establish the following proposition which will serve as the key step in proving Theorem 1.1.

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**Proposition 2.1.** *Define the random variable* $K(\omega) = \#\{j \in \mathbb{Z}^+ : \omega(j) = 0\}$. *Then* $\boldsymbol{Q}(K = \infty) = 1$ *if and only if*

$$\sum_{n=2}^{\infty} \prod_{j=1}^{n-1} f_j\left(1 - \left(\frac{1-p_j}{p_j}\right)^{n-j}\right) = \infty. \tag{2.3}$$

*If (2.3) does not hold then* $\boldsymbol{Q}(K = \infty) = 0$.

**Remark 2.1.** It is worth noting that we cannot assume that $\{M_j\}$ and $\{N_j\}$ are Markov processes since $M_j$ (respectively, $N_j$) gives only the number of frogs originating to the left of the point $x = j$ that ever hit $x = j$, rather than also providing the information about where each such frog originated (a significant detail, since frog origin determines the drift). Nevertheless, since the only conditioning we will do with respect to these two processes will involve conditioning on $M_j$ (respectively, $N_j$) equalling 0, they prove to be sufficient for our purposes.

*Proof of Proposition 2.1.* By a simple martingale argument, the probability that a frog starting at $x = j$ ever hits $x = n$ (for $n > j$) is $((1 - p_j)/p_j)^{n-j}$. Hence, the probability that no frogs beginning at $x = j$ ever hit $x = n$ is

$$\sum_{i=0}^{\infty} \mathbb{P}(X_j = i)\left(1 - \left(\frac{1-p_j}{p_j}\right)^{n-j}\right)^i = f_j\left(1 - \left(\frac{1-p_j}{p_j}\right)^{n-j}\right).$$

It then follows that for every $n \ge 1$, we have

$$\boldsymbol{Q}(\omega(n) = 0) = \left(1 - \left(\frac{1-p_0}{p_0}\right)^n\right) \prod_{j=1}^{n-1} f_j\left(1 - \left(\frac{1-p_j}{p_j}\right)^{n-j}\right)$$

$$\implies \quad \boldsymbol{E}[K] = \frac{2p_0 - 1}{p_0} + \sum_{n=2}^{\infty}\left(1 - \left(\frac{1-p_0}{p_0}\right)^n\right) \prod_{j=1}^{n-1} f_j\left(1 - \left(\frac{1-p_j}{p_j}\right)^{n-j}\right),$$

where $\boldsymbol{E}$ refers to expectation with respect to the probability measure $\boldsymbol{Q}$. Since $(1 - ((1 - p_0)/p_0)^n) \to 1$ as $n \to \infty$, this means

$$\boldsymbol{E}[K] < \infty \quad \iff \quad \sum_{n=2}^{\infty} \prod_{j=1}^{n-1} f_j\left(1 - \left(\frac{1-p_j}{p_j}\right)^{n-j}\right) < \infty. \tag{2.4}$$

It now immediately follows that if the right-hand side of (2.4) holds then $\boldsymbol{Q}(K = \infty) = 0$. Hence, to prove the proposition it suffices to establish the implication $\boldsymbol{Q}(K = \infty) < 1 \implies \boldsymbol{E}[K] < \infty$ (note this is just the contrapositive of $\boldsymbol{E}[K] = \infty \implies \boldsymbol{Q}(K = \infty) = 1$).

Now, since the event $\{K = \infty\}$ cannot depend on the behavior of the frogs from any finite collection of vertices (for the process $\{N_j\}$), it follows that $\boldsymbol{Q}(K = \infty \mid \omega(1) = 0) = \boldsymbol{Q}(K = \infty \mid \omega(1) = 1)$, which, in turn, establishes the implication

$$\boldsymbol{Q}(K = \infty) < 1 \quad \implies \quad \boldsymbol{Q}(1 \le K < \infty) > 0. \tag{2.5}$$

Next define $V_n = \{\omega \in \Omega : \omega(j) > 0 \text{ for all } j > n\}$ and assume that $\boldsymbol{Q}(K = \infty) < 1$. Letting $\boldsymbol{Q}^{(n)}$ denote the probability measure obtained by conditioning on the event $\omega(n) = 0$, (2.5) then implies that there must exist $L \ge 1$ such that $\boldsymbol{Q}^{(L)}(V_L) > 0$. Additionally, because $X_{i_1+i_2} \succeq X_{i_1}$ for all $i_1, i_2 \ge 1$ (since $X_{i+1} \succeq X_i$ for all $i \ge 1$ and $\succeq$ is transitive) and because the sequence of drifts $\{p_j\}$ is decreasing with respect to $j$, this implies that for any $L' > L$ there exists a coupling of the models $(\mathrm{F}^+ \mid N_L = 0)$ and $(\mathrm{F}^+ \mid N_{L'} = 0)$ (i.e. the $\mathrm{F}^+$ model with all frogs to the left of the point $x = L'$ removed) with the following properties:

(i) every frog originating at $x = L+j$ in $(\mathrm{F}^+ \mid N_L = 0)$ has a particular frog that corresponds to it originating at $x = L' + j$ in the coupled model $(\mathrm{F}^+ \mid N_{L'} = 0)$ (note that unless $X_{L+j}$ and $X_{L'+j}$ are identically distributed, there can be frogs originating at $x = L'+j$ in $(F^+ \mid N_{L'} = 0)$ that do not correspond to frogs originating at $x = L+j$ in $(F^+ \mid N_L = 0)$), and

(ii) whenever a frog in $(\mathrm{F}^+ \mid N_L = 0)$ takes a step to the right, the corresponding frog in $(\mathrm{F}^+ \mid N_{L'} = 0)$ does as well (and where if a frog with drift $p_{L+j}$ in $(F^+ \mid N_L = 0)$ takes a step to the left, then the corresponding frog in $(F^+ \mid N_{L'} = 0)$ must have its step go to the right with probability $(p_{L+j} - p_{L'+j})/p_{L+j}$). Letting $K_n(\omega) = \#\{j > n \colon \omega(j) = 0\}$, the above coupling then implies that

$$(K_L \mid \omega(L) = 0) \succeq (K_{L'} \mid \omega(L') = 0) \quad \Longrightarrow \quad \boldsymbol{Q}^{(L')}(V_{L'}) \geq \boldsymbol{Q}^{(L)}(V_L). \tag{2.6}$$

Now, if we define the stopping times $T_n$ where $T_1(\omega) = \min\{j \geq 1 \colon \omega(L+j) = 0\}$ and, for $n \geq 2$, $T_n(\omega) = \min\{j > T_{n-1}(\omega) \colon \omega(L+j) = 0\}$, we find that for every $n \geq 2$,

$$\boldsymbol{Q}^{(L)}(K_L \geq n) = \sum_{j=1}^{\infty} \boldsymbol{Q}^{(L)}(T_{n-1} = j)\,\boldsymbol{Q}^{(L+j)}(V_{L+j}^c) \leq \boldsymbol{Q}^{(L)}(V_L^c)\,\boldsymbol{Q}^{(L)}(K_L \geq n-1),$$

where the inequality follows from (2.6). From this, it then follows that for $n \geq 1$,

$$\begin{aligned} &\boldsymbol{Q}^{(L)}(K_L \geq n) \leq (1 - \boldsymbol{Q}^{(L)}(V_L))^n \\ &\qquad \Longrightarrow \quad \boldsymbol{E}[K_L \mid \omega(L) = 0] \leq \sum_{n=1}^{\infty}(1 - \boldsymbol{Q}^{(L)}(V_L))^n = \frac{1 - \boldsymbol{Q}^{(L)}(V_L)}{\boldsymbol{Q}^{(L)}(V_L)} < \infty. \end{aligned}$$

Since $\boldsymbol{E}[K_L] \leq \boldsymbol{E}[K_L \mid \omega(L) = 0]$ and $\boldsymbol{E}[K] \leq L + \boldsymbol{E}[K_L]$, we find that

$$\boldsymbol{E}[K] \leq L + \frac{1 - \boldsymbol{Q}^{(L)}(V_L)}{\boldsymbol{Q}^{(L)}(V_L)} < \infty.$$

Hence, we have established the implication $\boldsymbol{Q}(K = \infty) < 1 \implies \boldsymbol{E}[K] < \infty$, which then gives the implication $\boldsymbol{E}[K] = \infty \implies \boldsymbol{Q}(K = \infty) = 1$, thus completing the proof of the proposition. □

### 2.2. Proof of Theorem 1.1

Coupling the fact that Theorem 1.1 is equivalent to (2.2) and that $\boldsymbol{P}(\min \omega(j) = 0) = 1$ if and only if $\boldsymbol{Q}(K \geq 1) = 1$, we find the task of proving Theorem 1.1 is reduced to establishing

$$\boldsymbol{Q}(K \geq 1) = 1 \quad \Longleftrightarrow \quad \sum_{n=2}^{\infty} \prod_{j=1}^{n-1} f_j \left(1 - \left(\frac{1-p_j}{p_j}\right)^{n-j}\right) = \infty.$$

Noting that the implication

$$\sum_{n=2}^{\infty} \prod_{j=1}^{n-1} f_j \left(1 - \left(\frac{1-p_j}{p_j}\right)^{n-j}\right) = \infty \quad \Longrightarrow \quad \boldsymbol{Q}(K \geq 1) = 1$$

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follows immediately from Proposition 2.1, as does

$$\sum_{n=2}^{\infty}\prod_{j=1}^{n-1} f_j\left(1-\left(\frac{1-p_j}{p_j}\right)^{n-j}\right) < \infty \quad\Longrightarrow\quad \boldsymbol{Q}(K<\infty)=1,$$

it suffices to establish the implication $\boldsymbol{Q}(K<\infty)=1 \Longrightarrow \boldsymbol{Q}(K=0)>0$. We begin by defining $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\boldsymbol{Q}})$, where $\tilde{\Omega} := \Omega \times \mathbb{N}$, $\tilde{\mathcal{F}}$ is the $\sigma$-field generated by the finite-dimensional sets, and $\tilde{\boldsymbol{Q}}$ is the probability measure induced on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ by $\{N_j\}$ and $R_0$ (where $R_0$ represents the largest $x$-value attained by the frog starting at the origin), and we observe that for any event $A \in \mathcal{F}$, we have $\tilde{\boldsymbol{Q}}(A) = \boldsymbol{Q}(A)$, where $A$, when interpreted as an event in $\tilde{\mathcal{F}}$, represents $A \times \mathbb{N}$. In addition, we note that (2.5) implies that if $\boldsymbol{Q}(K<\infty)=1$ then there exists $L$ such that $\boldsymbol{Q}(V_L \mid \omega(L)=0) > 0$, and we also note that if we define $U_n := \{R_0 \geq n\}$ then $U_L \cap V_L \subseteq \{K=0\}$. Combining these observations with

$$\begin{aligned}
\tilde{\boldsymbol{Q}}(U_L\cap V_L) &= \left(\frac{1-p_0}{p_0}\right)^{L}\tilde{\boldsymbol{Q}}(V_L \mid U_L)\\
&\geq \left(\frac{1-p_0}{p_0}\right)^{L}\tilde{\boldsymbol{Q}}(V_L \mid \omega(L)=0)\\
&= \left(\frac{1-p_0}{p_0}\right)^{L}\boldsymbol{Q}(V_L \mid \omega(L)=0)\\
&> 0
\end{aligned}$$

(where $\tilde{\boldsymbol{Q}}(U_L) = ((1-p_0)/p_0)^L)$, we have

$$\boldsymbol{Q}(K<\infty)=1 \quad\Longrightarrow\quad \boldsymbol{Q}(K=0)=\tilde{\boldsymbol{Q}}(K=0)\geq\tilde{\boldsymbol{Q}}(U_L\cap V_L)>0,$$

thus completing the final step of the proof.

### 2.3. A simple proof of Gantert and Schmidt's result

In order to demonstrate the utility of Theorem 1.1, in this section we show how it can be used to obtain a simple two-step proof of the result from [3] described in the introduction. In part 1 we use a method similar to Gantert and Schmidt's, while in part 2 we employ a more novel approach which simplifies matters considerably.

*Part 1:* $\mathbb{E}[\log^+ \eta] = \infty \Longrightarrow$ *recurrence.* We begin by defining the process $\{A_j\}$, where, for every $j \in \mathbb{Z}/\{0\}$, $A_j$ represents the number of distinct frogs originating at $x = j$ that ever hit the origin in the Gantert–Schmidt model. Next we define the triple $(\Omega^*, \mathcal{F}^*, \boldsymbol{P}^*)$, where $\Omega^*$ represents the set of functions $\omega\colon \mathbb{Z}/\{0\} \to \mathbb{N}$ (i.e. the possible trajectories of $\{A_j\}$), $\mathcal{F}^*$ represents the $\sigma$-field on $\Omega^*$ generated by the finite-dimensional sets, and $\boldsymbol{P}^*$ represents the probability measure induced on $(\Omega^*, \mathcal{F}^*)$ by the process $\{A_j\}$. Additionally, denoting the two-sided sequence $\{\dots, \eta_{-2}, \eta_{-1}, \eta_1, \eta_2, \dots\}$ that gives the number of sleeping frogs beginning at every nonzero vertex as $H$, we define the process $\{A_j^{(H)}\}$ in the same way as $\{A_j\}$, but where the number of sleeping frogs starting at each vertex is given by the terms of $H$. As with $\{A_j\}$, each such process can be identified with a triple $(\Omega^*, \mathcal{F}^*, \boldsymbol{P}_H^*)$, where $\boldsymbol{P}_H^*$ represents the probability measure that $\{A_j^{(H)}\}$ induces on $(\Omega^*, \mathcal{F}^*)$. Now, since the activated frogs in this model all have nonzero leftward drift, this means that all frogs that begin to the left of the origin are activated with probability 1. Hence, for $j \geq 1$ and $H = \{\dots, \eta_{-2}, \eta_{-1}, \eta_1, \eta_2, \dots\}$, we find that

$$\boldsymbol{P}_H^*(\omega(-j)>0) = 1-\left(1-\left(\frac{1-p}{p}\right)^{j}\right)^{\eta_{-j}}.$$

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Now, defining $U(\omega) = \#\{j \in \mathbb{Z}^+ \colon \omega(-j) > 0\}$, noting that the random variables $\omega(-j)$ are independent with respect to $\boldsymbol{P}_H^*$, and noting that if $\eta_{-j} \ge (p/(1-p))^j$ then $\boldsymbol{P}_H^*(\omega(-j) > 0) = 1 - (1 - ((1-p)/p)^j)^{\eta_{-j}} \ge 1 - \mathrm{e}^{-1} > 0$, we see that the implication

$$\Big\{\eta_{-j} \ge \Big(\frac{p}{1-p}\Big)^j \text{ i.o.}\Big\} \quad \Longrightarrow \quad \boldsymbol{P}_H^*(U = \infty) = 1 \tag{2.7}$$

follows from BC2. Furthermore, if we define $\Gamma = \{H \in (\eta_j)_{j\in\mathbb{Z}^*} \colon \eta_{-j} \ge (p/(1-p))^j \text{ i.o.}\}$ and let $\mu$ represent the probability measure associated with $(\eta_j)_{j\in\mathbb{Z}^*}$, then, since

$$\begin{aligned}\sum_{j=1}^{\infty} \mathbb{P}\Big(\eta \ge \Big(\frac{p}{1-p}\Big)^j\Big) &= \sum_{j=1}^{\infty} \mathbb{P}\Big(\log^+ \eta \ge j \log\Big(\frac{p}{1-p}\Big)\Big) \\ &\ge \sum_{j=1}^{\infty} \mathbb{P}\Big(\log^+ \eta \ge j\Big\lceil \log\Big[\frac{p}{1-p}\Big]\Big\rceil\Big) \\ &\ge \frac{1}{\lceil \log[p/(1-p)]\rceil}\Big(\mathbb{E}[\log^+ \eta] - \Big\lceil \log\Big[\frac{p}{1-p}\Big]\Big\rceil\Big),\end{aligned}$$

we find that another application of BC2 yields the implication $\mathbb{E}[\log^+ \eta] = \infty \implies \mu(\Gamma) = 1$. Alongside (2.7), this establishes part 1.

*Part 2:* $\mathbb{E}[\log^+ \eta] < \infty \implies$ *transience.* We choose a constant $C$ such that $0 < C < 1$ and $C(p/(1-p)) > 1$. Noting that

$$\begin{aligned}\sum_{j=1}^{\infty} \mu\Big(\eta_{-j} \ge C^j\Big(\frac{p}{1-p}\Big)^j\Big) &= \sum_{j=1}^{\infty} \mathbb{P}\Big(\log^+ \eta \ge j \log\Big[\frac{Cp}{1-p}\Big]\Big) \\ &\le \frac{1}{\log[Cp/(1-p)]}\mathbb{E}[\log^+ \eta],\end{aligned}$$

it follows from the Borel–Cantelli lemma (BC1) that

$$\mathbb{E}[\log^+ \eta] < \infty \quad \Longrightarrow \quad \mu\Big(\eta_{-j} \ge C^j\Big(\frac{p}{1-p}\Big)^j \text{ i.o.}\Big) = 0. \tag{2.8}$$

In addition, since, for $j \ge 1$, we have $\boldsymbol{P}_H^*(\omega(-j) > 0) = 1 - (1 - ((1-p)/p)^j)^{\eta_{-j}}$ (see just before (2.7)) and

$$1 - \Big(1 - \Big(\frac{1-p}{p}\Big)^j\Big)^{C^j (p/(1-p))^j} = (1 + o(1))C^j \quad \text{as } j \to \infty,$$

we find that if $\eta_{-j} \ge C^j(p/(1-p))^j$ at only finitely many points, then $\sum_{j=1}^{\infty} \boldsymbol{P}_H^*(\omega(-j) > 0) < \infty$. Now coupling this with (2.8) and employing BC1, we obtain (for $j \ge 1$)

$$\mathbb{E}[\log^+ \eta] < \infty \quad \Longrightarrow \quad \boldsymbol{P}^*(\omega(-j) > 0 \text{ i.o.}) = 0. \tag{2.9}$$

Letting $\mathcal{A} = \sum_{j=1}^{\infty} \omega(-j)$, it follows from (2.9) that

$$\mathbb{E}[\log^+ \eta] < \infty \quad \Longrightarrow \quad \boldsymbol{P}^*(\mathcal{A} < \infty) = 1.$$

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If we now let $\mathcal{B} = \sum_{j=1}^{\infty} \omega(j)$, we find that in order to prove that $\mathbb{E}[\log^+ \eta] < \infty$ implies transience, it suffices to establish that, for each $k$ with $0 \le k < \infty$, the following implication holds:

$$\mathbb{E}[\log^+ \eta] < \infty \qquad \Longrightarrow \qquad \boldsymbol{P}^*(\mathcal{B} < \infty \mid \mathcal{A} = k) = 1. \tag{2.10}$$

Now, note that in terms of whether or not $\mathcal{B} = \infty$, the only relevant detail regarding the frogs beginning to the left of the origin is how far the one(s) that travels the furthest to the right of the origin get. Denoting this value as $\mathcal{C}$, if we assume $\boldsymbol{P}^*(\mathcal{B} = \infty) > 0$, then there would have to exist $r \ge 0$ such that $\boldsymbol{P}^*(\mathcal{B} = \infty \mid \mathcal{C} = r) > 0$. Since the frog beginning at the origin reaches the point $x = r$ with positive probability, it would follow that $\boldsymbol{P}^*(\mathcal{B} = \infty \mid \mathcal{A} = 0) > 0$. Hence, in order to establish (2.10), it suffices to establish the implication $\mathbb{E}[\log^+ \eta] < \infty \implies \boldsymbol{P}^*(\mathcal{B} < \infty \mid \mathcal{A} = 0) = 1$.

The next step is to observe that $(\mathcal{B} \mid \mathcal{A} = 0)$ has the same distribution as the number of distinct (initially sleeping) frogs that hit the origin in the nonhomogeneous model on $\mathbb{Z}$ (in the case where $p_j = p$ for each $j \ge 0$ and the $X_j$ are i.i.d. copies of $\eta$). Using Theorem 1.1, it then follows that in order to establish that $\mathbb{E}[\log^+ \eta] < \infty$ implies transience, it is sufficient to establish the implication

$$\mathbb{E}[\log^+ \eta] < \infty \qquad \Longrightarrow \qquad \sum_{n=2}^{\infty} \prod_{j=1}^{n-1} f\biggl(1 - \biggl(\frac{1-p}{p}\biggr)^{j}\biggr) = \infty, \tag{2.11}$$

where $f$ represents the probability generating function of $\eta$. Now, noting that

$$\sum_{n=2}^{\infty} \prod_{j=1}^{n-1} f\biggl(1 - \biggl(\frac{1-p}{p}\biggr)^{j}\biggr) = \mathbb{E}\biggl[\sum_{n=2}^{\infty} \exp\biggl(\sum_{j=1}^{n-1} \log\biggl[1 - \biggl(\frac{1-p}{p}\biggr)^{j}\biggr] X_j\biggr)\biggr], \tag{2.12}$$

we observe that due to

$$\log\biggl[1 - \biggl(\frac{1-p}{p}\biggr)^{j}\biggr] = -(1 + o(1))\biggl(\frac{1-p}{p}\biggr)^{j} \quad \text{as } j \to \infty,$$

it follows that if we have $0 < C < 1$ such that $Cp/(1-p) > 1$ and $X_j \le (Cp/(1-p))^j$ for all but finitely many $j$, then

$$\sum_{n=2}^{\infty} \exp\biggl(\sum_{j=1}^{n-1} \log\biggl[1 - \biggl(\frac{1-p}{p}\biggr)^{j}\biggr] X_j\biggr) = \infty.$$

When coupled with (2.8) (where we replace $\eta_{-j}$ with $\eta_j$ on the right) and (2.12), this establishes (2.11), which, as we saw, indicates that the left-hand side of (2.11) implies transience, thus completing the proof.

## 3. Applications of Theorem 1.1

### 3.1. Sharp conditions for the i.i.d. case

Having shown in Section 2.3 how Theorem 1.1 can be used to obtain a concise proof of Gantert and Schmidt's result from [3], this subsection is devoted to establishing a new result that involves a model similar to the one from [3], but where the drifts of the individual frogs are dependent on where they originated (it will be assumed that no sleeping frogs reside to the left of the origin). Before presenting this result however, we will first need to give a simple definition and lemma regarding concave sequences.

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**Definition 3.1.** A positive real-valued sequence $\{r_n\}$ will be referred to as concave if and only if

(i) $r_1 \geq r_2 - r_1$, and

(ii) $r_{i+1} - r_i \geq r_{i+2} - r_{i+1}$ for every $i \geq 1$.

**Lemma 3.1.** *If* $\{r_n\}$ *is a positive real-valued concave sequence then for every* $j, n$ *such that* $1 \leq j \leq n$*, it satisfies* $r_j/r_n \geq j/n$.

*Proof.* Based on the concavity of our sequence, we know that

$$\begin{aligned} r_n &= r_j + (r_n - r_j) \\ &= r_j + \sum_{i=j+1}^{n} r_i - r_{i-1} \\ &\leq r_j + \sum_{i=j+1}^{n} r_{j+1} - r_j \\ &= r_j + \frac{n-j}{j} \sum_{i=0}^{j-1} r_{j+1} - r_j \\ &\leq r_j + \frac{n-j}{j} r_j. \end{aligned}$$

Hence, it follows that $r_n \leq (n/j)r_j$, which can be written as $r_j/r_n \geq j/n$. □

Having established the above definition and lemma, we now present the main result of this subsection.

**Theorem 3.1.** *For any version of the nonhomogeneous frog model on* $\mathbb{Z}$ *for which the* $X_j$ *are i.i.d. with* $\mathbb{E}[X_1] < \infty$, $p_j = \frac{1}{2} + a_j$ *with* $g(j) = 1/a_j$ *being concave, and* $d = \min\{j : \mathbb{P}(X_1 = j) > 0\}$*, the model is transient if and only if*

$$\sum_{n=1}^{\infty} \frac{\exp(-\mathcal{K}/4a_n)}{(a_n)^{d/2}} = \infty,$$

*where* $f$ *represents the generating function of* $X_j$ *and* $\mathcal{K} = -\int_0^\infty \log[f(1 - \mathrm{e}^{-x})]\,\mathrm{d}x$.

**Remark 3.1.** Since $X_1$ has a finite first moment (as stated in the theorem), we have

$$\begin{aligned} \mathbb{E}[X_1] < \infty \quad &\Longrightarrow \quad f'(1) = \mathbb{E}[X_1] < \infty \\ &\Longrightarrow \quad \log[f(1 - \mathrm{e}^{-x})] = -q\mathrm{e}^{-x} + o(\mathrm{e}^{-x}) \\ &\Longrightarrow \quad \mathcal{K} < \infty, \end{aligned}$$

where $q = f'(1)$.

**Remark 3.2.** One noteworthy (and immediate) consequence of Theorem 3.1 is that for fixed $f$, $a_n = (\mathcal{K}/4)/\log n$ (for all but finitely many $n$) represents a natural critical case in the sense that, for $a_n = C/\log n$, the model is transient if and only if $C \geq \frac{1}{4}\mathcal{K}$. An instance of particular significance is the case where $X_j = 1$ for all $j$ (i.e. each positive integer point begins with exactly one sleeping frog). Since, in this scenario, $f(x) = x$, we find that

$$\mathcal{K} = \int_0^\infty |\log[1 - \mathrm{e}^{-x}]| \,\mathrm{d}x = \int_0^\infty \sum_{n=1}^\infty \frac{\mathrm{e}^{-nx}}{n} \,\mathrm{d}x = \sum_{n=1}^\infty \int_0^\infty \frac{\mathrm{e}^{-nx}}{n} \,\mathrm{d}x = \sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Hence, it follows that if $a_n = C/\log n$ then the model is transient if and only if $C \geq \frac{1}{24}\pi^2$, thus providing a new phase transition for the model from [1] mentioned in the introduction.

*Proof of Theorem 3.1.* Given our result in Theorem 1.1, it follows that in order to establish this new result, it will suffice to show that

$$\sum_{n=1}^\infty \frac{\exp(-\mathcal{K}/4a_n)}{(a_n)^{d/2}} = \infty \quad \Longleftrightarrow \quad \sum_{n=2}^\infty \prod_{j=1}^{n-1} f\bigg(1 - \bigg(1 - \frac{4a_{n-j}}{1 + 2a_{n-j}}\bigg)^j\bigg) = \infty, \tag{3.1}$$

where the expression on the right in (3.1) is obtained by substituting $\frac{1}{2} + a_j$ for $p_j$ and switching $j$ and $n - j$ in (1.4). Furthermore, if we define $w_n = 4a_n/(1 + 2a_n)$ and note that

$$\bigg(\frac{\exp(-\mathcal{K}/w_n)}{(w_n)^{d/2}}\bigg)\bigg(\frac{\exp(-\mathcal{K}/4a_n)}{(a_n)^{d/2}}\bigg)^{-1} \to A\mathrm{e}^{-\mathcal{K}/2} \quad \text{as } n \to \infty \tag{3.2}$$

(where $A = (\lim_{n\to\infty} \frac{1}{4}(1 + 2a_n)^{d/2})$, we find that (3.1) is equivalent to

$$\sum_{n=1}^\infty \frac{\exp(-\mathcal{K}/w_n)}{(w_n)^{d/2}} = \infty \quad \Longleftrightarrow \quad \sum_{n=2}^\infty \prod_{j=1}^{n-1} f(1 - (1 - w_{n-j})^j) = \infty. \tag{3.3}$$

We first establish (3.1) (via (3.3)) under the condition that $a_n^{-1}$ is $O(\sqrt{n})$ (see steps 1–4), following which we address the general case.

*Step 1:* $\sum_{n=2}^\infty \prod_{j=1}^{n-1} f(1 - (1 - w_n)^j) = \infty \iff \sum_{n=2}^\infty \prod_{j=1}^{n-1} f(1 - (1 - w_{n-j})^j) = \infty$. Since $a_n$ is decreasing this means that $w_n$ is also decreasing, from which it follows that

$$\prod_{j=1}^{n-1} f(1 - (1 - w_{n-j})^j) \geq \prod_{j=1}^{n-1} f(1 - (1 - w_n)^j) \quad \text{for all } n.$$

Hence, in order to establish this step, it suffices to show that

$$\limsup \sum_{j=1}^{n-1} \log[f(1 - (1 - w_{n-j})^j)] - \log[f(1 - (1 - w_n)^j)] < \infty. \tag{3.4}$$

We can now express (3.4) (see below) as

$$\begin{aligned}
&\limsup \sum_{j=1}^{n-1} \log[f(1 - (1 - w_n)^j + ((1 - w_n)^j - (1 - w_{n-j})^j))] - \log[f(1 - (1 - w_n)^j)] \\
&\quad\leq \limsup \sum_{j=1}^{n-1} \log[f(1 - (1 - w_n)^j + ((j(w_{n-j} - w_n)) \wedge (1 - w_n))(1 - w_n)^{j-1})] \\
&\qquad - \log[f(1 - (1 - w_n)^j)].
\end{aligned} \tag{3.5}$$

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Since $a_n^{-1}$ is $O(\sqrt{n})$ implies that $w_n^{-1}$ is as well, this mean that the first expression in the sum on the left-hand side of (3.5) is equal to

$$\begin{aligned}
&\limsup \sum_{j=1}^{n-1} \log[f(1-(1-w_n)^j + ((j(w_{n-j}-w_n)) \wedge (1-w_n))(1-w_n)^{j-1})] \\
&\qquad - \log[f(1-(1-w_n)^j)] \\
&\quad \le \limsup \sum_{j \le 1/w_n} \log[f(1-(1-w_n)^j + ((j(w_{n-j}-w_n)) \wedge (1-w_n))(1-w_n)^{j-1})] \\
&\qquad\qquad - \log[f(1-(1-w_n)^j)] \\
&\qquad\qquad + \limsup \sum_{1/w_n < j \le n-1} \frac{q(w_{n-j}-w_n)j(1-w_n)^{j-1}}{f(1-(1-w_n)^{1/w_n})} \qquad (3.6)
\end{aligned}$$

(recall that $q = f'(1)$). The second expression on the right-hand side of (3.6) can now be bounded above by

$$\begin{aligned}
&\frac{q}{f(1-\mathrm{e}^{-1})} \limsup \sum_{1/w_n < j \le n-1} (w_{n-j}-w_n)j(1-w_n)^{j-1} \\
&\quad \le \frac{q}{f(1-\mathrm{e}^{-1})} \limsup \sum_{1/w_n < j \le n-1} \bigg(1 - \frac{w_n}{w_{n-j}}\bigg) \frac{w_{n-j}}{w_n} w_n j \mathrm{e}^{-w_n(j-1)} \\
&\quad \le \frac{q\mathrm{e}}{f(1-\mathrm{e}^{-1})} \limsup \sum_{1/w_n < j \le n-1} \frac{1}{w_n(n-j)} (w_n j)^2 \mathrm{e}^{-w_n j}, \qquad (3.7)
\end{aligned}$$

where the final inequality in (3.7) follows from $w_{n-j}/w_n \le n/(n-j)$, which follows from the concavity of $1/w_n$, which, in turn, follows from the concavity of $1/a_n$. Next we bound the final term in (3.7) by

$$\begin{aligned}
&\frac{q\mathrm{e}}{f(1-\mathrm{e}^{-1}) \liminf n w_n^2} \limsup \sum_{1/w_n < j \le n-1} \frac{1}{1-j/n} w_n (w_n j)^2 \mathrm{e}^{-w_n j} \\
&\quad \le \frac{q\mathrm{e}}{f(1-\mathrm{e}^{-1}) \liminf n w_n^2} \limsup \sum_{1/w_n < j \le n-1} w_n (w_n j)^2 \mathrm{e}^{-3 w_n j/4} \qquad (3.8)
\end{aligned}$$

where the above inequality follows from the fact that on account of $w_n^{-1}$ being $O(\sqrt{n})$, we know that for sufficiently large $n$, we have $1/(1-j/n) \le \exp(n^{-2/3} j) \le \exp(\frac{1}{4} w_n j)$ for $1 \le j \le n-1$. Finally, comparing the last sum to the integral of $x^2 \exp(-\frac{3}{4}x)$, we see there must exist $K < \infty$ (independent of $n$) such that the sum is bounded above by $\int_0^\infty x^2 \exp(-\frac{3}{4}x)\,\mathrm{d}x + K$. Combining this with $w_n^{-1}$ being $O(\sqrt{n})$ then implies that the expression on the right-hand side of (3.8) is finite, which when coupled with (3.7) and (3.8) establishes that the second term on the right of the inequality in (3.6) is also finite.

To complete the proof of this step, we just need to show that the first term on the right-hand side of (3.6) is also finite. Since for any probability generating function $f$ of a nonnegative integer-valued random variable with finite mean, the function $f'(x)/f(x)$ is $O(1/x)$, this means

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there must exist a constant $C < \infty$ such that $f'(x)/f(x) \le C/x$ for all $x \in (0, 1]$, from which it follows that the term in question is bounded above by

$$\limsup \sum_{j \le 1/w_n} \frac{C(w_{n-j} - w_n) j (1 - w_n)^{j-1}}{1 - (1 - w_n)^j}. \tag{3.9}$$

Next noting that, for $x \in (0, 1]$ and $m \in \mathbb{Z}^+$, we have

$$\frac{1 - (1 - x)^m}{mx} = \frac{1}{m}(1 + (1 - x) + \cdots + (1 - x)^{m-1}) \ge (1 - x)^{m-1},$$

it follows that (3.9) can be bounded above by

$$C \limsup \sum_{j \le 1/w_n} \frac{(w_{n-j} - w_n)}{w_n} = C \limsup \sum_{j \le 1/w_n} \frac{w_{n-j}}{w_n} - 1.$$

On account of the concavity of $1/w_n$, this last expression can itself be bounded above by

$$C \limsup \sum_{j \le 1/w_n} \frac{j}{n - j} = \frac{C}{2} \limsup \frac{1}{n w_n^2} < \infty,$$

where the equality along with the finiteness of the term on the right-hand side both follow from the fact that $w_n^{-1}$ is $O(\sqrt{n})$. Hence, this establishes that (3.9), as well as the first term on the right-hand side of (3.6), is finite. Now, if we combine this with the finiteness of the second expression on the right-hand side of (3.6), along with the inequality in (3.5), we see that (3.4) follows, thus completing the proof of step 1.

*Step 2:* $\sum_{n=2}^{\infty} \prod_{j=1}^{n-1} f(1 - \mathrm{e}^{-w_n j}) = \infty \iff \sum_{n=2}^{\infty} \prod_{j=1}^{n-1} f(1 - (1 - w_n)^j) = \infty$. Since we know that

$$1 - w_n \le \mathrm{e}^{-w_n} \quad \Longrightarrow \quad \sum_{n=2}^{\infty} \prod_{j=1}^{n-1} f(1 - \mathrm{e}^{-w_n j}) \le \sum_{n=2}^{\infty} \prod_{j=1}^{n-1} f(1 - (1 - w_n)^j),$$

it follows that in order to establish this step, it suffices to show (much like in the case of step 1) that

$$\limsup \sum_{j=1}^{n-1} \log[f(1 - (1 - w_n)^j)] - \log[f(1 - \mathrm{e}^{-w_n j})] < \infty. \tag{3.10}$$

Defining $C_n = (\mathrm{e}^{-w_n} - (1 - w_n))/w_n^2$, we have the following string of inequalities (where the expression on the first line is equal to the expression in (3.10), and with $S(n, j)$ representing the summand on the second line):

$$\begin{aligned}
&\limsup \sum_{j=1}^{n-1} \log[f(1 - \mathrm{e}^{-w_n j} + ((1 - w_n + C_n w_n^2)^j - (1 - w_n)^j)] - \log[f(1 - \mathrm{e}^{-w_n j})] \\
&\quad \le \limsup \sum_{j=1}^{n-1} \log[f(1 - \mathrm{e}^{-w_n j} + ((j C_n w_n^2) \wedge (1 - w_n))\mathrm{e}^{-w_n (j-1)})] \\
&\qquad\qquad - \log[f(1 - \mathrm{e}^{-w_n j})] \\
&\quad \le \limsup \sum_{j \le 1/w_n} S(n, j) + \limsup \sum_{1/w_n < j \le n-1} S(n, j).
\end{aligned} \tag{3.11}$$

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If we can show that both of the expressions on the last line of (3.11) are finite then (3.10) will immediately follow. Beginning with the first expression, observe that if we use the fact (referenced in the proof of step 1) that there must exist $C < \infty$ such that $f'(x)/f(x) \le C/x$ for all $x \in (0, 1]$, then we can obtain the string of inequalities

$$\begin{aligned}\limsup \sum_{j\le 1/w_n} S(n,j) &\le \limsup \sum_{j\le 1/w_n} \frac{CC_n j w_n^2 \mathrm{e}^{-w_n(j-1)}}{1-\mathrm{e}^{-w_n j}}\\ &\le \frac{C}{2}\limsup \sum_{j\le 1/w_n} \frac{j w_n^2}{1-\mathrm{e}^{-w_n j}},\end{aligned} \tag{3.12}$$

where the second inequality follows from the fact that $C_n \le \frac{1}{2}$ for all $n$. Now, using the fact that

$$1-\mathrm{e}^{-w_n j} = (1-\mathrm{e}^{-w_n})(1+\mathrm{e}^{-w_n}+\cdots+(\mathrm{e}^{-w_n})^{j-1}) \ge j(1-\mathrm{e}^{-w_n})(\mathrm{e}^{-w_n})^{j-1}$$

and that $(1-\mathrm{e}^{-w_n})/w_n \ge 1-\mathrm{e}^{-1}$ (since $0 < w_n < 1$ for all $n$), we find that the expression on the second line in (3.12) is bounded above by

$$\frac{C}{2(1-\mathrm{e}^{-1})}\limsup \sum_{j\le 1/w_n} \frac{j w_n^2}{j w_n \mathrm{e}^{-1}} = \frac{C\mathrm{e}}{2(1-\mathrm{e}^{-1})}\limsup \sum_{j\le 1/w_n} w_n \le \frac{C\mathrm{e}}{2(1-\mathrm{e}^{-1})} < \infty,$$

thus establishing that the first sum on the last line of (3.11) is finite.

In order to establish (3.10), and, thus, complete the proof of this step, it remains only to show that the second sum on the last line of (3.11) is also finite. We accomplish this via the following string of inequalities:

$$\begin{aligned}\limsup \sum_{1/w_n<j\le n-1} S(n,j) &\le \frac{C}{2(1-\mathrm{e}^{-1})}\limsup \sum_{1/w_n<j<\infty} w_n(w_n j)\mathrm{e}^{-w_n(j-1)}\\ &\le \frac{C\mathrm{e}}{2(1-\mathrm{e}^{-1})}\int_0^\infty x\mathrm{e}^{-x}\,\mathrm{d}x + K,\end{aligned}$$

where the first inequality follows from the same argument used in (3.12). Hence, the proof of step 2 is complete.

*Step 3:* $\sum_{n=2}^\infty \prod_{j=1}^\infty f(1-\mathrm{e}^{-w_n j}) = \infty \iff \sum_{n=2}^\infty \prod_{j=1}^{n-1} f(1-\mathrm{e}^{-w_n j}) = \infty$. Since one direction is immediate, it just remains to show that

$$\limsup \sum_{j=n}^\infty -\log[f(1-\mathrm{e}^{-w_n j})] < \infty. \tag{3.13}$$

Observing that

$$\begin{aligned}\limsup \sum_{j=n}^\infty -\log[f(1-\mathrm{e}^{-w_n j})] &= \limsup \frac{1}{w_n}\sum_{j=n}^\infty -w_n \log[f(1-\mathrm{e}^{-w_n j})]\\ &\le \limsup \frac{1}{w_n}\int_{(n-1)w_n}^\infty -\log[f(1-\mathrm{e}^{-x})]\,\mathrm{d}x,\end{aligned}$$

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we find that, as a consequence of the fact that $f'(1) = q < \infty$ and $w_n^{-1}$ is $O(\sqrt{n})$,

$$\begin{aligned}\limsup \frac{1}{w_n}\int_{(n-1)w_n}^{\infty} -\log[f(1-\mathrm{e}^{-x})]\,\mathrm{d}x &= \limsup \frac{1}{w_n}\int_{(n-1)w_n}^{\infty} q\mathrm{e}^{-x}\,\mathrm{d}x \\ &= \limsup \frac{1}{w_n} q\mathrm{e}^{-(n-1)w_n} \\ &\le \limsup \frac{\sqrt{n}}{\sqrt{l}} q\exp\Big(-\frac{n-1}{n}\sqrt{n}\sqrt{l}\Big) \\ &= 0,\end{aligned}$$

where $l$ denotes the value of $\liminf nw_n^2$. Hence, this establishes (3.13), thus completing the proof of step 3.

*Step 4:* $\sum_{n=1}^{\infty} \mathrm{e}^{-\mathcal{K}/w_n}/(w_n)^{d/2} = \infty \iff \sum_{n=2}^{\infty}\prod_{j=1}^{\infty} f(1-\mathrm{e}^{-w_n j}) = \infty$. Denoting $c_d = \mathbb{P}(X_1 = d)$ (recall $d = \min\{j : \mathbb{P}(X_1 = j) > 0\}$), observe that

$$\begin{aligned}\frac{\mathrm{d}(\log[f(x)])}{\mathrm{d}x} &= \frac{f'(x)}{f(x)} \\ &= \frac{dc_d + (d+1)c_{d+1}x + \cdots}{c_d x + c_{d+1}x^2 + \cdots} \\ &= \frac{d}{x}\,\frac{1 + ((d+1)/d)/((c_{d+1})/c_d)/x + \cdots}{1 + ((c_{d+1})/c_d)x + \cdots} \\ &= \frac{d}{x} + O(1). \end{aligned} \tag{3.14}$$

Now we want to approximate

$$\begin{aligned} &-\frac{\mathcal{K}}{w_n} - \log\Big[\prod_{j=1}^{\infty} f(1-\mathrm{e}^{-w_n j})\Big] \\ &\quad = \frac{1}{w_n}\int_0^{\lfloor 1/w_n\rfloor w_n} \log[f(1-\mathrm{e}^{-x})]\,\mathrm{d}x - \frac{1}{w_n}\sum_{j=1}^{\lfloor 1/w_n\rfloor} w_n\log[f(1-\mathrm{e}^{-w_n j})] \\ &\qquad + \frac{1}{w_n}\int_{\lfloor 1/w_n\rfloor w_n}^{\infty} \log[f(1-\mathrm{e}^{-x})]\,\mathrm{d}x - \frac{1}{w_n}\sum_{\lceil 1/w_n\rceil}^{\infty} w_n\log[f(1-\mathrm{e}^{-w_n j})] \end{aligned} \tag{3.15}$$

to within an order of $O(1)$. First noting that the expression on the second line of (3.15) is $O(1)$ as $n \to \infty$ (this follows from the fact that it is bounded above by 0 and below by $\log[f(1-\mathrm{e}^{-w_n\lfloor 1/w_n\rfloor})]$), we see that our task is reduced to approximating

$$\begin{aligned} &\frac{1}{w_n}\int_0^{\lfloor 1/w_n\rfloor w_n} \log[f(1-\mathrm{e}^{-x})]\,\mathrm{d}x - \frac{1}{w_n}\sum_{j=1}^{\lfloor 1/w_n\rfloor} w_n\log[f(1-\mathrm{e}^{-w_n j})] \\ &\quad = \frac{1}{w_n}\sum_{j=2}^{\lfloor 1/w_n\rfloor}\int_0^{w_n} \log[f(1-\mathrm{e}^{-(w_n(j-t))})] - \log[f(1-\mathrm{e}^{-w_n j})]\,\mathrm{d}t + O(1), \end{aligned} \tag{3.16}$$

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where the $O(1)$ term represents $(1/w_n)\int_0^{w_n} \log[f(1-\mathrm{e}^{-x})]\,\mathrm{d}x - \log[f(1-\mathrm{e}^{-w_n})]$. Using (3.14), we then see that the integrand on the right-hand side of (3.16) is equal to

$$\begin{aligned}
-\int_{1-\mathrm{e}^{-(w_n(j-t))}}^{1-\mathrm{e}^{-w_n j}} \frac{d}{x} + O(1)\,\mathrm{d}x &= d\log\biggl[\frac{1-\mathrm{e}^{-(w_n(j-t))}}{1-\mathrm{e}^{-w_n j}}\biggr] + O(\mathrm{e}^{-(w_n(j-t))} - \mathrm{e}^{-w_n j}) \\
&= d\log\biggl[\frac{1-\mathrm{e}^{-(w_n(j-t))}}{1-\mathrm{e}^{-w_n j}}\biggr] + O(t) \\
&= d\log\biggl[1 + \frac{\mathrm{e}^{-w_n j}(1-\mathrm{e}^{t})}{1-\mathrm{e}^{-w_n j}}\biggr] + O(t) \\
&= d\log\biggl[1 - \frac{t}{w_n j} + O(t)\biggr] + O(t) \\
&= d\log\biggl[1 - \frac{t}{w_n j}\biggr] + O(t)
\end{aligned}$$

with the final equality following from the fact that $n \ge 2$ implies that $1 - t/w_n j \ge \frac{1}{2} > 0$ for all $t$. Substituting this back into the expression on the second line of (3.16) now yields

$$\begin{aligned}
&\frac{1}{w_n}\sum_{j=2}^{\lfloor 1/w_n\rfloor}\int_0^{w_n} d\log\biggl[1-\frac{t}{w_n j}\biggr] + O(t)\,\mathrm{d}t \\
&\quad = \frac{d}{w_n}\sum_{j=2}^{\lfloor 1/w_n\rfloor} -(j-1)w_n\log\biggl[1-\frac{1}{j}\biggr] - w_n + O(w_n^2) \\
&\quad = d\sum_{j=2}^{\lfloor 1/w_n\rfloor} -(j-1)\log\biggl[1-\frac{1}{j}\biggr] - 1 + O(w_n) \\
&\quad = d\sum_{j=2}^{\lfloor 1/w_n\rfloor} -\frac{1}{2j} + O\biggl(\frac{1}{j^2}\biggr) + O(w_n) \\
&\quad = -\frac{d}{2}\log\biggl[\frac{1}{w_n}\biggr] + O(1),
\end{aligned}$$

where the $O(t)$ expressions indicate that the absolute value of the term in question is bounded above by $ct$ for some $c < \infty$ that is independent of both $n$ and $t$. Returning to the first line of (3.15), we find that

$$\begin{aligned}
&-\frac{\mathcal{K}}{w_n} - \log\biggl[\prod_{j=1}^{\infty} f(1-\mathrm{e}^{-w_n j})\biggr] = -\frac{d}{2}\log\biggl[\frac{1}{w_n}\biggr] + O(1) \\
&\implies \quad C_1\frac{\mathrm{e}^{-\mathcal{K}/w_n}}{(w_n)^{d/2}} \le \prod_{j=1}^{\infty} f(1-\mathrm{e}^{-w_n j}) \le C_2\frac{\mathrm{e}^{-\mathcal{K}/w_n}}{(w_n)^{d/2}}
\end{aligned}$$

(for some $C_1, C_2$ independent of $n$ with $0 < C_1 < C_2 < \infty$), thus completing the proof of step 4.

Having now established (3.1) via steps 1–4 when $a_n^{-1}$ is $O(\sqrt{n})$, our final task is to address the general case. To do this, we first note that because in the proof of step 4 we did not use the

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fact that $a_n^{-1}$ is $O(\sqrt{n})$, it follows that it continues to hold without this assumption. Coupling this with (3.2), along with the fact that

$$\begin{aligned}\sum_{n=2}^{\infty}\prod_{j=1}^{\infty} f(1-\mathrm{e}^{-w_n j}) &\le \sum_{n=2}^{\infty}\prod_{j=1}^{n-1} f(1-\mathrm{e}^{-w_n j})\\ &\le \sum_{n=2}^{\infty}\prod_{j=1}^{n-1} f(1-(1-w_n)^j)\\ &\le \sum_{n=2}^{\infty}\prod_{j=1}^{n-1} f(1-(1-w_{n-j})^j),\end{aligned}$$

we find that the implication going from left to right in (3.1) holds regardless of whether or not $a_n^{-1}$ is $O(\sqrt{n})$. Hence, to complete the proof of the theorem we simply need to show that when $a_n^{-1}$ is not $O(\sqrt{n})$, finiteness of the expression on the left-hand side of (3.1) still implies finiteness of the expression on the right-hand side.

If we define the sequence $\tilde{a}_n$ so that

$$\frac{1}{\tilde{a}_n} = \begin{cases} \dfrac{1}{a_n} & \text{if } 1/a_n < 3\sqrt{n},\\ 3\sqrt{n} & \text{otherwise,}\end{cases}$$

it then follows that $1/\tilde{a}_n$ is concave and $O(\sqrt{n})$ (also note $\frac{1}{2} < \frac{1}{2} + \tilde{a}_n < 1$ still holds). In addition, since we assume that the expression on the left-hand side of (3.1) is finite, this means

$$\sum_{n=1}^{\infty} \frac{\mathrm{e}^{-\mathcal{K}/4\tilde{a}_n}}{(\tilde{a}_n)^{d/2}} \le \sum_{n=1}^{\infty} \frac{\mathrm{e}^{-\mathcal{K}/4a_n}}{(a_n)^{d/2}} + \sum_{n=1}^{\infty} \mathrm{e}^{-\mathcal{K}3\sqrt{n}/4}3^{d/2}n^{d/4} < \infty.$$

Hence, the proof of (3.1), for the case where $a_n^{-1}$ is $O(\sqrt{n})$, implies that

$$\sum_{n=2}^{\infty}\prod_{j=1}^{n-1} f\bigg(1-\bigg(1-\frac{4\tilde{a}_{n-j}}{1+2\tilde{a}_{n-j}}\bigg)^j\bigg) < \infty.$$

Coupling this with the fact that $a_n \le \tilde{a}_n$, we now conclude that

$$\sum_{n=2}^{\infty}\prod_{j=1}^{n-1} f\bigg(1-\bigg(1-\frac{4a_{n-j}}{1+2a_{n-j}}\bigg)^j\bigg) < \infty,$$

which, along with the argument in the previous paragraph, establishes that (3.1) continues to hold when $a_n^{-1}$ is not $O(\sqrt{n})$. Hence, the proof of the theorem is complete. □

### 3.2. Sharp conditions for the (Poi $\lambda_j$) scenario

In this section we address the final model discussed in the introduction (see [5]), and establish sharp conditions for the case where the drift values of individual frogs are dependent on where they originate. Our result is as follows.

**Theorem 3.2.** *For* $X_j = \mathrm{Poi}(\lambda_j)$ *and* $p_j = \frac{1}{2} + a_j$ *(with the sequences* $1/a_j$ *and* $\lambda_j$ *both being concave), the nonhomogeneous frog model on* $\mathbb{Z}$ *is transient if and only if*

$$\sum_{n=1}^{\infty} \exp\biggl(-\lambda_n\biggl(\frac{1}{4a_n} - \frac{1}{2}\biggr)\biggr) = \infty.$$

*Proof.* Since $\mathrm{Poi}(\lambda_j)$ has the generating function $\mathrm{e}^{\lambda_j(x-1)}$, applying Theorem 1.1 reduces our task to showing that

$$\sum_{n=1}^{\infty} \exp\biggl(-\lambda_n\biggl(\frac{1}{4a_n} - \frac{1}{2}\biggr)\biggr) = \infty \quad \iff \quad \sum_{n=2}^{\infty} \exp\biggl(-\sum_{j=1}^{n-1} \lambda_{n-j}\biggl(1 - \frac{4a_{n-j}}{1+2a_{n-j}}\biggr)^j\biggr) = \infty.$$

Noting also that

$$\begin{aligned}\sum_{n=2}^{\infty} \exp\biggl(-\lambda_n\biggl(\frac{1}{4a_n} - \frac{1}{2}\biggr)\biggr) &= \sum_{n=2}^{\infty} \exp\biggl(-\sum_{j=1}^{\infty} \lambda_n\biggl(1 - \frac{4a_n}{1+2a_n}\biggr)^j\biggr) \\ &\le \sum_{n=2}^{\infty} \exp\biggl(-\sum_{j=1}^{n-1} \lambda_n\biggl(1 - \frac{4a_n}{1+2a_n}\biggr)^j\biggr) \\ &\le \sum_{n=2}^{\infty} \exp\biggl(-\sum_{j=1}^{n-1} \lambda_{n-j}\biggl(1 - \frac{4a_{n-j}}{1+2a_{n-j}}\biggr)^j\biggr),\end{aligned}$$

we see that it is, in fact, sufficient to establish the implication

$$\begin{aligned}&\sum_{n=1}^{\infty} \exp\biggl(-\lambda_n\biggl(\frac{1}{4a_n} - \frac{1}{2}\biggr)\biggr) < \infty \\ &\qquad \Longrightarrow \quad \sum_{n=2}^{\infty} \exp\biggl(-\sum_{j=1}^{n-1} \lambda_{n-j}\biggl(1 - \frac{4a_{n-j}}{1+2a_{n-j}}\biggr)^j\biggr) < \infty. \end{aligned} \tag{3.17}$$

To do this we begin by proving (3.17) for the case where $\lambda_n$ and $a_n^{-1}$ are both $O(n^{1/3})$. Similarly to the proof of Theorem 3.1, we accomplish this by showing that

$$\limsup \lambda_n\biggl(\frac{1}{4a_n} - \frac{1}{2}\biggr) - \sum_{j=1}^{n-1} \lambda_{n-j}\biggl(1 - \frac{4a_{n-j}}{1+2a_{n-j}}\biggr)^j < \infty. \tag{3.18}$$

As a first step towards establishing (3.18), we observe the following string of inequalities (with $\varepsilon_j$ denoting $a_j/(1+2a_j)$):

$$\begin{aligned}&\limsup \sum_{j=1}^{n-1} \lambda_n(1-4\varepsilon_{n-j})^j - \sum_{j=1}^{n-1} \lambda_{n-j}(1-4\varepsilon_{n-j})^j \\ &\quad = \limsup \sum_{j=1}^{n-1} (\lambda_n - \lambda_{n-j})(1-4\varepsilon_{n-j})^j \\ &\quad \le \limsup \sum_{j=1}^{n-1} (\lambda_n - \lambda_{n-j})(1-4\varepsilon_n)^j\end{aligned}$$

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$$
\begin{aligned}
&\le \limsup \sum_{j=1}^{n-1} \frac{j}{n} \lambda_n e^{-4j\varepsilon_n} \\
&\le \limsup \frac{\lambda_n/\varepsilon_n^2}{n} \sum_{j=1}^{\infty} \varepsilon_n (\varepsilon_n j) e^{-4j\varepsilon_n} \\
&< \infty,
\end{aligned} \tag{3.19}
$$

where the second inequality follows from the fact that $\lambda_j$ is concave and $(1-4\varepsilon_n)^j \le e^{-4j\varepsilon_n}$, and where the finiteness of the last expression is derived from the fact that $\lambda_n$ and $\varepsilon_n^{-1}$ are both $O(n^{1/3})$, along with the fact that the sum is bounded above by $\int_0^\infty x e^{-4x}\,dx + K$ for some $K < \infty$. Next, we present another string of inequalities:

$$
\begin{aligned}
\limsup \sum_{j=1}^{n-1} \lambda_n (1-4\varepsilon_n)^j &- \sum_{j=1}^{n-1} \lambda_n (1-4\varepsilon_{n-j})^j \\
&= \limsup \lambda_n \sum_{j=1}^{n-1} (1-4\varepsilon_n)^j - (1-4\varepsilon_{n-j})^j \\
&\le \limsup 4\lambda_n \sum_{j=1}^{n-1} (\varepsilon_{n-j} - \varepsilon_n) j (1-4\varepsilon_n)^{j-1} \\
&= \limsup 4\lambda_n \sum_{j=1}^{n-1} (\varepsilon_n^{-1} - \varepsilon_{n-j}^{-1}) \varepsilon_n \varepsilon_{n-j} j (1-4\varepsilon_n)^{j-1}.
\end{aligned} \tag{3.20}
$$

Since $\varepsilon_n^{-1}$ is concave (since it is equal to $a_n^{-1} + 2$), it follows that the expression on the second line of (3.20) is less than or equal to

$$
\begin{aligned}
\limsup 4\lambda_n \sum_{j=1}^{n-1} \frac{j}{n} \varepsilon_{n-j} j (1-4\varepsilon_n)^{j-1} &\le \limsup 4\lambda_n \sum_{j=1}^{n-1} \varepsilon_{n-j} \frac{j^2}{n} e^{-4(j-1)\varepsilon_n} \\
&\le \limsup 4e\lambda_n \sum_{j=1}^{n-1} \frac{\varepsilon_n}{1-j/n} \frac{j^2}{n} e^{-4j\varepsilon_n} \\
&= \limsup \frac{4e\lambda_n/\varepsilon_n^2}{n} \sum_{j=1}^{n-1} \frac{1}{1-j/n} (j\varepsilon_n)^2 e^{-4j\varepsilon_n} \varepsilon_n \\
&\le \limsup \frac{4e\lambda_n/\varepsilon_n^2}{n} \sum_{j=1}^{\infty} \varepsilon_n (\varepsilon_n j)^2 e^{-3j\varepsilon_n} \\
&< \infty,
\end{aligned}
$$

where the third inequality follows from the fact that, for sufficiently large $n$, we have $1/(1 - j/n) < e^{j\varepsilon_n}$ for all $j$ with $1 \le j < n$, and where the finiteness of the last term follows from $\lambda_n$ and $\varepsilon_n^{-1}$ both being $O(n^{1/3})$, together with the fact that the sum is once again bounded above by

$$\int_0^\infty x^2 e^{-3x}\,dx + K \quad \text{for some } K < \infty.$$

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Combining the above string of inequalities with (3.20), we see that

$$\limsup \sum_{j=1}^{n-1} \lambda_n (1-4\varepsilon_n)^j - \sum_{j=1}^{n-1} \lambda_n (1-4\varepsilon_{n-j})^j < \infty. \tag{3.21}$$

Finally, we observe that

$$\begin{aligned}\limsup \lambda_n\left(\frac{1}{4a_n}-\frac{1}{2}\right) - \sum_{j=1}^{n-1}\lambda_n(1-4\varepsilon_n)^j &= \limsup \sum_{j=1}^{\infty}\lambda_n(1-4\varepsilon_n)^j - \sum_{j=1}^{n-1}\lambda_n(1-4\varepsilon_n)^j \\ &= \limsup \sum_{j=n}^{\infty}\lambda_n(1-4\varepsilon_n)^j \\ &= \limsup \lambda_n \frac{(1-4\varepsilon_n)^n}{4\varepsilon_n} \\ &\le \limsup \frac{\lambda_n}{4\varepsilon_n}\mathrm{e}^{-4n\varepsilon_n} \\ &= 0, \end{aligned} \tag{3.22}$$

where the last equality again follows from $\lambda_n$ and $\varepsilon_n^{-1}$ both being $O(n^{1/3})$. Now combining (3.19), (3.21), and (3.22), we see that (3.18) (and, therefore, (3.17)) does indeed hold if $\lambda_n$ and $a_n^{-1}$ are $O(n^{1/3})$.

To complete the proof of the theorem, we just need to prove (3.17) for the general case (i.e. without the condition that $\lambda_n$ and $a_n^{-1}$ are $O(n^{1/3})$). To do this, we begin by defining $\tilde{\lambda}_n$ and $\tilde{a}_n$ as

$$\tilde{\lambda}_n = \begin{cases} \lambda_n & \text{if } \lambda_n < n^{1/3}, \\ n^{1/3} & \text{otherwise,} \end{cases} \quad \text{and} \quad \frac{1}{\tilde{a}_n} = \begin{cases} \dfrac{1}{a_n} & \text{if } 1/a_n < 3n^{1/3}, \\ 3n^{1/3} & \text{otherwise} \end{cases}$$

(again the coefficient 3 has been chosen so that $\frac{1}{2} < \frac{1}{2} + \tilde{a}_n < 1$ for all $n$). Now, noting that

$$\begin{aligned}&\sum_{n=1}^{\infty} \exp\left(-\tilde{\lambda}_n\left(\frac{1}{4\tilde{a}_n}-\frac{1}{2}\right)\right) \\ &\quad \le \sum_{n=1}^{\infty}\exp\left(-n^{1/3}\left(\frac{3n^{1/3}}{4}-\frac{1}{2}\right)\right) + \sum_{n=1}^{\infty}\exp\left(-n^{1/3}\left(\frac{1}{4a_n}-\frac{1}{2}\right)\right) \\ &\quad\quad + \sum_{n=1}^{\infty}\exp\left(-\lambda_n\left(\frac{3n^{1/3}}{4}-\frac{1}{2}\right)\right) + \sum_{n=1}^{\infty}\exp\left(-\lambda_n\left(\frac{1}{4a_n}-\frac{1}{2}\right)\right) \\ &\quad < \infty \end{aligned}$$

(where the finiteness of the middle two sums on the right of the inequality follows from the fact that $a_n < \frac{1}{2}$ and $\lambda_n > 0$ for all $n \ge 1$), it follows from the proof of (3.17), in the case where $\lambda_n$ and $a_n^{-1}$ are $O(n^{1/3})$, that

$$\sum_{n=2}^{\infty}\exp\left(-\sum_{j=1}^{n-1}\lambda_{n-j}\left(1-\frac{4a_{n-j}}{1+2a_{n-j}}\right)^j\right) \le \sum_{n=2}^{\infty}\exp\left(-\sum_{j=1}^{n-1}\tilde{\lambda}_{n-j}\left(1-\frac{4\tilde{a}_{n-j}}{1+2\tilde{a}_{n-j}}\right)^j\right) < \infty,$$

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where the first inequality follows from the fact that $\tilde{\lambda}_j \le \lambda_j$ and $\tilde{a}_j^{-1} \le a_j^{-1}$. Hence, this establishes (3.17) for the general case, and thus completes the proof of the theorem. □

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