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A new bridge index for links with trivial knot components

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Abstract

Let $L = K_1 \cup K_2$ be a 2-component link in the 3-sphere such that K_1 is a trivial knot. In this paper, we introduce a new bridge index, denoted by $b_{K_1=1}([L])$, for L. Roughly speaking, $b_{K_1=1}([L])$ is the minimum of the bridge numbers of the links ambient isotopic to L under the constraint that all of the bridge numbers of the components corresponding to K_1 are 1. We provide a lower bound estimate of $b_{K_1=1}([L])$ in the case when L is a non-split satellite link. By using this result, we show that for each integer $n(\geqslant 2)$, there exists a link $L_n = K_{1n} \cup K_{2n}$ with K_{1n} a trivial knot such that $b_{K_{1n}=1}([L_n]) - b([L_n]) = n - 1$, where $b([L_n])$ is the bridge index of L_n.

1. *Introduction*

The bridge index $b([L])$ of a link L is a fundamental invariant in knot and link theory. Let K be a satellite knot with a companion L^0 and the pattern $(\widehat{V}, K^0)$ with the index k (for the definitions of these terms, see Section 2). In [**2**], H.Schubert proved the following:

$$b([K]) \geqslant k \cdot b([L^0]).$$

In this paper, let $L = K_1 \cup K_2$ be a non-split 2-component link such that K_1 is a trivial knot. In Section 3, we introduce a new bridge index of L, denoted by $b_{K_1=1}([L])$. Roughly speaking, this is the minimum of the bridge numbers of the links ambient isotopic to L under the constraint that all of the bridge numbers of the components corresponding to K_1 are 1. Then in Theorem 1·1, we show that an inequality which is similar to that of Schubert's holds for the new bridge index. In fact, we have the following.

Suppose the above link $L = K_1 \cup K_2$ is a satellite link with a companion $L^0 = L^0_1 \cup L^0_2$ and a pattern $(\widehat{V}_i, K^0_i)$ $(i = 1, 2)$ (for the definitions of satellite links, companion and pattern, see Section 3). Note that we can define the index of the pattern $(\widehat{V}_i, K^0_i)$ as in the case of knots. Then in Section 3, we define the dual index of L^0_1. With these terms, we have the following theorem:

THEOREM 1·1. *Let $L = K_1 \cup K_2$ be a non-split satellite link with a companion $L^0 = L^0_1 \cup L^0_2$ and a pattern $(\widehat{V}_i, K^0_i)$ $(i = 1, 2)$ such that K_1 is a trivial knot. Let k'_1 be the dual* *index of* L_1^0*, and* k_i *be the index of* $(\widehat{V}_i, K_i^0)$*. Suppose that* K_1^0 *is not ambient isotopic in* $\widehat{V}_1$ *to the core of* $\widehat{V}_1$*. Then the following inequality holds.*

$$b_{K_1=1}([L]) \geqslant 1 + k_1' \cdot k_2 \, .$$

The proof of Theorem 1·1 is carried out by using the arguments of J.Schultens' paper [**3**], where a modern proof of the Schubert's theorem is given.

In general, the inequality $b_{K_1=1}([L]) \geqslant b([L])$ holds, and it is natural to ask whether there exist examples which make the inequalities strict. Then in Section 5, by using Theorem 1·1, we show that for each $n(\geqslant 2)$ there exists a link $L_n = K_{1n} \cup K_{2n}$ satisfying $b_{K_1=1}([L_n]) - b([L_n]) = n - 1$.

2. *Preliminaries*

Throughout this paper, we work in the smooth category. An n-component link is the union of n mutually disjoint 1-spheres in the 3-sphere S^3. In particular, we call a 1-component link a *knot*. A link L is called a *split link* if there exists a 2-sphere S^2 in S^3 such that $S^2 \cap L = \emptyset$, and that S^2 separates components of L. Otherwise, L is a *non-split link*. We take a height function $h : S^3 \to [0, 1]$, that is, h is a Morse function whose critical point set consists of two points, a maximum p_1 of height 1 and a minimum p_0 of height 0.

Let L be a link. Then $[L]$ denotes the ambient isotopy class of L. Suppose that $h|_L : L \to [0, 1]$ is a Morse function. Then the *bridge number* of L, denoted by $b(L)$, is the number of maxima (= the number of minima) for $h|_L$. The *bridge index* of L, denoted by $b([L])$, is defined as follows;

$$b([L]) = \min\{b(L') \mid L' \in [L],\ h|_{L'} \text{ is a Morse function}\}.$$

We say that a knot K is *trivial* if $b([K]) = 1$. We say that L is in a *minimal bridge position* if L satisfies $b(L) = b([L])$.

Let L^0 be a non-trivial knot, $\widetilde{V}$ be a small regular neighbourhood of L^0. Let $\widehat{V}$ be an unknotted solid torus embedded in S^3, and K^0 be a knot in $\widehat{V}$, which is not ambient isotopic in $\widehat{V}$ to the core of $\widehat{V}$, and is not contained in a 3-ball in $\widehat{V}$. We fix a homeomorphism $\Psi : \widehat{V} \to \widetilde{V}$. Then $\Psi(K^0)$, which is denoted by K, is a knot in S^3. We say that K is a *satellite knot*. The image $\Psi(\widehat{V})$ is denoted by V. Now, we call L^0 a *companion* of K, V a *companion torus* of K with respect to L^0, and the pair $(\widehat{V}, K^0)$ the *pattern* of K with respect to L^0. Then, $\min\{\sharp(D \cap K^0) \mid D : \text{a meridian disk of } \widehat{V}\}$ is called the *index* of the pattern.

Schubert [**2**] proved the following:

THEOREM 2·1 (Schubert). *Let* K *be a satellite knot with* L^0 *and* $(\widehat{V}, K^0)$ *as above. Let* k *be the index of* $(\widehat{V}, K^0)$*. Suppose* L^0 *is a non-trivial knot. Then the following inequality holds*;

$$b([K]) \geqslant k \cdot b([L^0]).$$

In [**3**], Schultens gave a modern proof of this theorem. We will use some ideas from [**3**]. Particularly Lemma 2·2 below is essential. For the statement of the lemma, we introduce some terms. Let K, V be as above. Then T denotes ∂V. We suppose that $h|_T : T \to [0, 1]$ is a Morse function. Then $\mathcal{F}_T$ denotes the singular foliation on T induced by the levels of $h|_T$. Let σ be a singular leaf corresponding to a saddle singularity in $\mathcal{F}_T$. We call σ a *saddle*

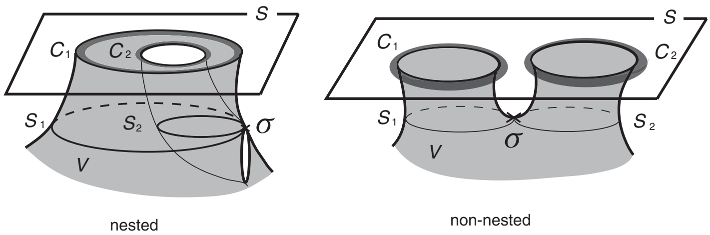

Fig. 1. Nested saddle and non-nested saddle.

of $\mathcal{F}_T$. We note that σ has a representative as a wedge product $\sigma = s_1 \vee s_2$, where s_1 and s_2 are circles in T. If either s_1 or s_2 is inessential in T, we call σ an *inessential saddle*, and we call σ an *essential saddle* if it is not an inessential saddle. Let S_σ be the level sphere which contains σ. Then we can choose circles c_1, c_2 in T, which are parallel to s_1, s_2 respectively, in a certain level sphere S which is either slightly higher or slightly lower to S_σ. Now, $c_1 \cup c_2$ bounds an annulus on the level sphere S. Then σ is called a *nested saddle* if a small regular neighborhood of $c_1 \cup c_2$ in the annulus is contained in V(for example, it is as the left one in Figure 1). Otherwise, σ is a *non-nested saddle* (for example, it is as the right one in Figure 1). We say that V is *taut* with respect to $b([K])$ if the number of critical points of $h|_T$ is minimal in the isotopy class of T under the constraint that the knot which is ambient isotopic to K is in a minimal bridge position. Now, the following holds ([**3**, remark 2]).

LEMMA 2·2. *Let K, V, T be as above. If V is taut with respect to $b([K])$, then each saddle in $\mathcal{F}_T$ is essential and non-nested.*

3. *A new bridge index*

In this section, we extend the definition of satellite knots to links. Let $L^0 = L_1^0 \cup \cdots \cup L_n^0$ $(n \geqslant 1)$ be an n-component link in S^3, $\widetilde{V}_i$ $(i = 1, \ldots, n)$ be a small regular neighbourhood of L_i^0, and $\widehat{V}_i$ be an unknotted solid torus embedded in S^3. Let $K_i^0(\subset \widehat{V}_i)$ be a knot which is not contained in a 3-ball in $\widehat{V}_i$, where K_j^0 is not ambient isotopic in $\widehat{V}_j$ to the core of $\widehat{V}_j$ for some $j \in \{1, \ldots, n\}$. We fix a homeomorphism $\Psi_i : \widehat{V}_i \to \widetilde{V}_i$ for each i. Then V_i denotes the image of $\widehat{V}_i$. Then T_i denotes ∂V_i, and we put $V = V_1 \cup \cdots \cup V_n$ and $T = T_1 \cup \cdots \cup T_n$. Furthermore, K_i denotes the image of K_i^0. Thus each K_i is a knot in S^3, and then L denotes the link $K_1 \cup \cdots \cup K_n$ in S^3. We call L a *satellite link*, and L^0 a *companion* of L. Moreover, we call the pair $(\widehat{V}_i, K_i^0)$ the *pattern* of K_i with respect to L_i^0. Then we call $\min\{\sharp(D_i \cap K_i^0) \mid D_i$: a meridian disk of $\widehat{V}_i\}$ the *index* of the pattern. We suppose that $h|_T : T \to [0, 1]$ is a Morse function. Then $\mathcal{F}_T$ denotes the singular foliation on T induced by the levels of $h|_T$, and we define a *saddle* σ as in Section 2. Furthermore, we define the terms *taut with respect to* $b([L])$, *nested saddles*, etc. as in Section 2.

Let $L = K_1 \cup K_2$ be a non-split 2-component link such that K_1 is a trivial knot. We define a new bridge index for L, denoted by $b_{K_1=1}([L])$, as follows.

$$b_{K_1=1}([L]) = \min\{b(L') \mid L' = K_1' \cup K_2' \in [L],\ \text{where } h|_{L'} : L' \to [0,1] \text{ is a Morse function with } b(K_1') = 1,\ \text{where } K_1' \text{ is the component corresponding to } K_1\}.$$

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In other words, it is the bridge index under the constraint $b(K_1) = 1$. We say that L is in a *minimal bridge position with respect to trivial* K_1 if L satisfies $b(K_1) = 1$ and $b(L) = b_{K_1=1}([L])$.

Let $L = K_1 \cup K_2$ be a satellite link with a companion $L^0 = L^0_1 \cup L^0_2$. Suppose that K_1 is a trivial knot. By Theorem 2·1, we immediately see that L^0_1 is a trivial knot. Let $N(L^0_1)$ be a small regular neighbourhood of L^0_1. Then $E(L^0_1)$ denotes the closure of the exterior of $N(L^0_1)$. Since L^0_1 is a trivial knot, $E(L^0_1)$ is homeomorphic to a solid torus. We may regard L^0_2 as a knot in $E(L^0_1)$, hence the pair $(E(L^0_1), L^0_2)$ is a pattern. We denote the index of the pattern $(E(L^0_1), L^0_2)$ by k'_1, and call it the *dual index* of L^0_1.

4. *Proof of Theorem* 1·1

In this section, we introduce some lemmas and prove Theorem 1·1.

Let $L = K_1 \cup \cdots \cup K_n$ $(n \geqslant 1)$ be a satellite link with a companion $L^0 = L^0_1 \cup \cdots \cup L^0_n$. Let V_i, V, T_i, T and $\mathcal{F}_T$ be as in Section 3. Let σ be a saddle of $\mathcal{F}_T$. Recall from Section 2 that σ is a wedge product of two circles s_1, s_2 in T. Then as in Section 2, S_σ denotes the level sphere containing σ.

LEMMA 4·1. *If $\mathcal{F}_T$ contains an inessential saddle, then there exists an ambient isotopy ϕ_t $(0 \leqslant t \leqslant 1)$ in S^3 that satisfies the following conditions*:

(i) *The height function $h|_{\phi_1(L)}$ is a Morse function on $\phi_1(L)$, thus $b(\phi_1(L))$ is defined, and $h|_{\phi_1(T)}$ is a Morse function on $\phi_1(T)$, thus $\mathcal{F}_{\phi_1(T)}$ is defined*;

(ii) *We have $b(\phi_1(K_i)) = b(K_i)$ $(i = 1, \ldots, n)$, and the number of critical points of $h|_{\phi_1(T_i)}$ equals that of $h|_{T_i}$; and*

(iii) *There exists an inessential saddle $\sigma = s_1 \vee s_2$ of $\mathcal{F}_{\phi_1(T)}$, where s_1 bounds a disk D_1 in $\phi_1(T)$ satisfying the following conditions*:

  (a) *The restriction of $\mathcal{F}_{\phi_1(T)}$ to D_1 consists of exactly one central singular point and concentric circles*; *and*

  (b) *There exists a disk component $\widetilde{D}_1$ in $S_\sigma \setminus s_1$ such that we can take a* 3-ball *B in S^3 bounded by $\widetilde{D}_1 \cup D_1$ such that B does not contain p_0 or p_1, where p_0 (p_1 resp.) is the minimum (maximum resp.) of h.*

The proof of the above lemma is carried out by applying the arguments in the proof of [**3**, lemma 1], where knots are treated. It is easy to see that the arguments work for links to give the conclusions in Lemma 4·1, and we omit the description here.

In the remainder of this section, we restrict our attention to non-split 2-component satellite links such that one component of each link is a trivial knot. Let $L = K_1 \cup K_2$ be such a link with K_1 a trivial knot, $L^0 = L^0_1 \cup L^0_2$ a companion of L, and $(\widehat{V}_i, K^0_i)$ $(i = 1, 2)$ a pattern of K_i with respect to L^0_i. We use notations V, T, $\mathcal{F}_T$, k_1, k'_1, k_2 etc. in Section 3. We suppose that L is in a minimal bridge position with respect to trivial K_1. We say that V is *taut with respect to trivial* K_1 if the number of critical points of $h|_T$ is minimal in the isotopy class under the constraint that the link which is ambient isotopic to L is in a minimal bridge position with respect to trivial K_1. It is easy to prove the next lemma by using the arguments in the proof of [**3**, lemma 2] together with Lemma 4·1, and we omit giving the proof here.

LEMMA 4·2. *If V is taut with respect to trivial K_1, then there are no inessential saddles in $\mathcal{F}_T$.*

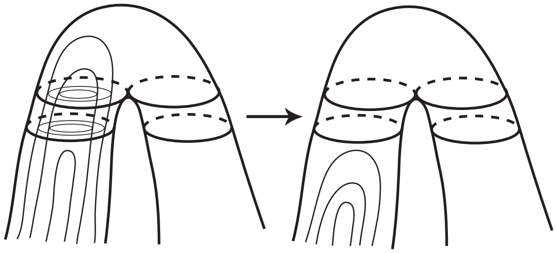

Fig. 2. Pushing down inessential disks.

Let σ_1, σ_2 be saddles of $\mathcal{F}_T$. We say that the pair σ_1, σ_2 is *adjacent* if there exists a component of $T \setminus (\sigma_1 \cup \sigma_2)$, which is denoted by C, such that C does not include a critical point of $h|_T$. This term will be used in the proof of Lemma 4·3. Recall from Section 3, that T_i $(i = 1, 2)$ is the component of T corresponding to the boundary of the regular neighborhood V_i of L_i^0.

LEMMA 4·3. *Suppose V is taut with respect to trivial K_1. If K_1^0 is not a core of $\widehat{V}_1$, then each saddle of $\mathcal{F}_T$ contained in T_1 is nested, and each saddle of $\mathcal{F}_T$ contained in T_2 is non-nested.*

Proof. We first note that index k_1 is greater than 1, since K_1 is a trivial knot, and K_1^0 is not a core of $\widehat{V}_1$. By Lemma 4.2, each saddle in $\mathcal{F}_T$ is essential. Let $\sigma = s_1 \vee s_2$ be the highest saddle in $\mathcal{F}_T$. Then for σ, the next claim holds:

CLAIM 1. *The saddle σ is non-nested.*

Proof of Claim 1. The following arguments are essentially the same as the proof of the Claim in the proof of [**3**, lemma 3]. Let c_1, c_2 be circles in a level surface S which is slightly lower than S_σ as in Section 2, and let $\widehat{D}_1$, $\widehat{D}_2$ be mutually disjoint disks bounded by c_1, c_2 respectively in S. Let c be a component of $T \cap int(\widehat{D}_i)$ $(i = 1$ or $2)$. Then since σ is the highest saddle, we see that c bounds a disk D_c in T such that:

(i) D_c is included in the region above S; and

(ii) The restriction of $\mathcal{F}_T$ to D_c consists of exactly one central singular point and concentric circles.

We push down the disk D_c slightly below S by an ambient isotopy as in Figure 2. We note that this isotopy can be performed so as not to change $b(K_i)$ $(i = 1, 2)$, and the number of critical points in $\mathcal{F}_T$. By repeating such isotopies, we may suppose that $int(\widehat{D}_i)$ is disjoint from T, i.e. $\widehat{D}_i$ is contained in V or $\mathrm{cl}(S^3 \setminus V)$. Then since s_i is essential in T, we see that c_i is essential in T by the definition of c_i. We note that since L is a non-split link, L^0 is a non-split link. This implies that T is incompressible in $\mathrm{cl}(S^3 \setminus V)$. Hence the disk $\widehat{D}_i$ must be a meridian disk in V. This shows that σ is non-nested.

Then, let $\sigma' = s'_1 \vee s'_2$ be the saddle which is the highest one in the saddles of $\mathcal{F}_T$ contained in T_1. Then we have:

CLAIM 2. *The saddle σ' is nested, in particular the saddle σ in Claim* 1 *is contained in T_2.*

Proof of Claim 2. We take a level sphere S', circles c'_1, c'_2 $(\subset S')$, and disks $\widehat{D}'_1$, $\widehat{D}'_2$ analogous to $S, c_1, c_2, \widehat{D}_1, \widehat{D}_2$ for σ in the proof of Claim 1. Assume that σ' is non-nested.

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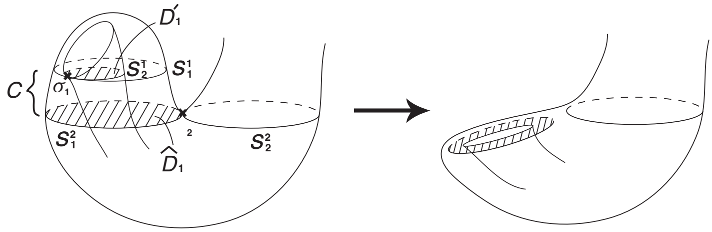

Fig. 3. Removing a nested saddle.

Then, the neighbourhood of $\partial \widehat{D}_i'$ in $\widehat{D}_i'(i = 1, 2)$ is contained in V_1. Hence any component of $int(\widehat{D}_i') \cap T$ which is outermost in $int(\widehat{D}_i')$ is contained in T_1. Then the arguments in the proof of Claim 1 apply and we may suppose that each $\widehat{D}_i'$, is contained in V_1. Since $k_1 > 1$, we see that K_1 intersects $\widehat{D}_i'$ $(i = 1, 2)$ in at least 2 points. This shows that $b(K_1) > 1$, a contradiction. Hence σ' is nested. This together with Claim 1 shows that σ is contained in T_2.

Then we have:

CLAIM 3. *Each saddle in T_2 is non-nested.*

Proof of Claim 3. If there exists a nested saddle in T_2, then by Claims 1, and 2, we see that there is a pair of a nested saddle and a non-nested saddle in T_2. In this situation, there exists an adjecent pair of saddles σ_1, σ_2, in $\mathcal{F}_T$ contained in T_2 such that σ_1 is nested and σ_2 is non-nested. Then by the same argument as in the proof of [**3**, lemma 3], we can derive a contradiction to the assumption that V is taut with respect to trivial K_1 (see Figure 3). Thus we have that each saddle in T_2 is non-nested.

Finally, we show:

CLAIM 4. *Each saddle in T_1 is nested.*

Proof of Claim 4. If there exists a non-nested saddle in T_1, then by Claim 2, we see that there is a pair of a nested saddle and a non-nested saddle in T_1. By the arguments in the proof of Claim 3, we can derive a contradiction. Thus we see that any saddle in T_1 is a nested saddle.

Claims 3 and 4 complete the proof of Lemma 4·3.

By using the above arguments, now we prove Theorem 1·1.

Proof of Theorem 1·1. Recall that L_1^0 is a trivial knot, hence the extorior of V_1, say V_1^c, is an unknotted solid torus. By Lemma 4·3, each saddle of T_1 $(= \partial V_1 = \partial V_1^c)$ is essential and nested. Then let $\sigma' = s_1' \vee s_2'$ be the saddle which is the highest one in the saddles of T_1 and $\widehat{D}_i'$ $(i = 1, 2)$ be the disk bounded by c_i' in S' as in the proof of Claim 2 in the proof of Lemma 4·3. We consider about $\widehat{D}_i' \cap T_2$. If there exists a component, say c, of $\widehat{D}_i' \cap T_2$ such that c is inessential in T_2, then by Lemma 4·2, there exists a disk D_c in T_2 such that $\partial D_c = c$ and the restriction of $\mathcal{F}_T$ to D_c consists of one central singularity and concentric circles. We note that D_c might be under S' (as in Figure 4). By using D_c, we can apply an isotopy as in the proof of Claim 1 in the proof of Lemma 4·3 to remove c from $\widehat{D}_i' \cap T_2$. Hence, we may suppose that any component of $\widehat{D}_i' \cap T_2$ is essential in T_2. Thus by the definition of the

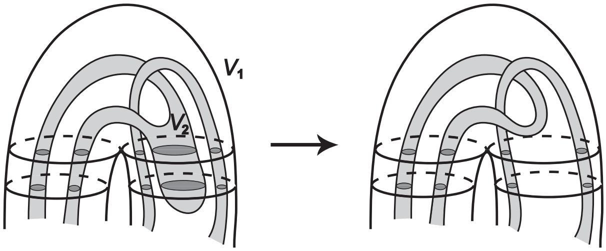

Fig. 4. Removing inessential intersections.

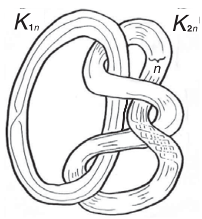

Fig. 5. The link L_n.

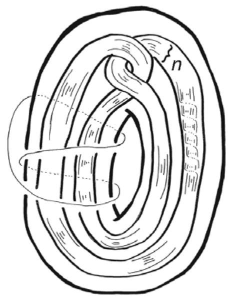

Fig. 6. A minimal bridge position with respect to trivial K_{1n}.

dual index k_1' of L_1^0, $\widehat{D}_i' \cap V_2$ consists of at least k_1' meridian disks of V_2. Furthermore by the definition of k_2, K_2 intersects each meridian disk of V_2 at least k_2 times. This shows that K_2 intersects $\widehat{D}_i'$ at least $k_1' \cdot k_2$ times, and this implies that K_2 has at least $k_1' \cdot k_2$ maxima. This together with the fact $b(K_1) = 1$ gives the conclusion of Theorem 1·1.

5. *Examples*

Let $L = K_1 \cup K_2$ be a non-split 2-component link such that K_1 is a trivial knot. In general, $b([L]) \leqslant b_{K_1=1}([L])$ holds. Thus we would like to ask whether there exists L such that $b([L]) < b_{K_1=1}([L])$ holds. In this section, we show that for each n ($\geqslant 2$), there exists a link $L_n = K_{1n} \cup K_{2n}$ such that $b_{K_{1n}=1}([L_n]) - b([L_n]) = n - 1$. In fact, we prove the following.

PROPOSITION 5·1. *For each n ($\geqslant 2$), let $L_n = K_{1n} \cup K_{2n}$ be the 2-component link such that K_{1n} is a trivial knot as in Figure 5, where K_{2n} is an $(n, n+1)$-torus knot. Then we have:*

(i) $b_{K_{1n}=1}([L_n]) = 1 + 2n$; *and*

(ii) $b([L_n]) = 2 + n$.

Proof. Note that $L_n = K_{1n} \cup K_{2n}$ is a satellite link with the companion $L^0 = L_1^0 \cup L_2^0$ as in Figure 7(a) and the pattern $(\widehat{V}_i, K_i^0)$ $(i = 1, 2)$ as in Figure 7(c). Then let $V = V_1 \cup V_2$, $T = T_1 \cup T_2$ be as in Section 3 (Figure 7(b)). Firstly, we note that the dual index of L_1^0 is 2, and the index of the pattern $(\widehat{V}_2, K_2^0)$ is n. Hence by Theorem 1·1, we have $b_{K_{1n}=1}([L_n]) \geqslant 1 + 2n$.

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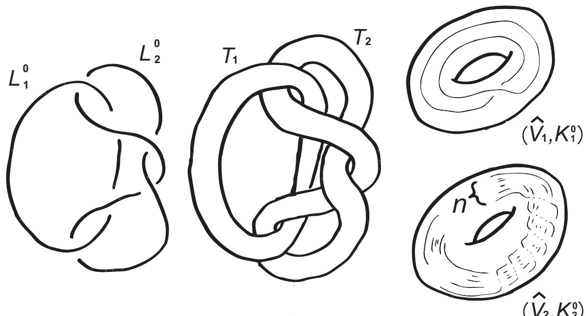

Fig. 7. The construction of L_n.

Note that L_n can be isotoped into a position as in Figure 6, then we see that $b_{K_{1n}=1}([L_n]) \leqslant 1 + 2n$. Thus we obtain $b_{K_{1n}=1}([L_n]) = 1 + 2n$.

Next, by the facts $b([K_{1n}]) = 1$ and $b([K_{2n}]) = n$ ([**1**], Theorem 7.5.3), we have $b([L_n]) \geqslant 1 + n$. Assume that $b([L_n]) = 1 + n$. Let $L'_n = K'_{1n} \cup K'_{2n}$ ($\in [L_n]$) be a position such that $b(L'_n) = 1 + n$. This together with the facts, $b([K'_{1n}]) = 1$, and $b([K'_{2n}]) = n$ shows that $b(K'_{1n}) = 1$ (, and $b(K'_{2n}) = n$). This shows that $b_{K_{1n=1}}([L_n]) \leqslant 1 + n$, but this contradicts the above. Therefore we have $b([L_n]) \geqslant 2 + n$. On the other hand, by Figure 5, we see $b([L_n]) \leqslant 2 + n$. Thus we obtain $b([L_n]) = 2 + n$.

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