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Three positive solutions for semilinear elliptic problems involving concave and convex nonlinearities

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We study the existence and multiplicity of positive solutions for the Dirichlet problem

$$-\Delta u = \lambda f(z)|u|^{q-2}u + |u|^{p-2}u \quad \text{in } \Omega,$$

where $\lambda > 0$, $1 < q < 2$, $p = 2^* = 2N/(N-2)$, $0 \in \Omega \subset \mathbb{R}^N$, $N \geqslant 3$, is a bounded domain with smooth boundary $\partial\Omega$ and f is a non-negative continuous function on $\bar{\Omega}$. Assuming that f satisfies some hypothesis, we prove that the equation admits at least three positive solutions for sufficiently small λ.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 3$, be a bounded domain with smooth boundary $\partial\Omega$, and consider the semilinear elliptic problems involving concave–convex nonlinearities

$$-\Delta u = \lambda f(z)|u|^{q-2}u + |u|^{p-2}u \quad \text{in } \Omega, \quad u \in H_0^1(\Omega),$$

where $\lambda > 0$, $1 < q < 2$ and f is a continuous function on $\bar{\Omega}$. Ambrosetti *et al*. [3] ($f \equiv 1$, $2 < p \leqslant 2^* = 2N/(N-2)$) and Wu [11] ($f \in C(\bar{\Omega})$ and changes sign, $2 < p < 2^*$) showed that this equation has at least two positive solutions for λ sufficiently small. Li *et al*. [7] proved that the nonlinear Dirichlet problem

$$-\Delta u = \lambda f(z)|u|^{q-2}u + g(z,u) \quad \text{in } \Omega, \quad u \in H_0^1(\Omega),$$

admits at least two non-negative solutions under suitable assumptions on $g(z,u)$. It is well known that the critical problem

$$\left.\begin{aligned} -\Delta u &= u^{2^*-1} && \text{in } \Omega, \\ u &> 0 && \text{in } \Omega, \\ u &\in H_0^1(\Omega), \end{aligned}\right\} \tag{1.1}$$

has no solution if Ω is a bounded star-shaped domain (Pohozaev identity). Adding a lower perturbation term $f(z,u)$ to (1.1), Brézis and Nirenberg [4] proved the existence of a positive solution by using the mountain-pass theorem.

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In this paper, adding a perturbation term to (1.1), we show the multiplicity of positive solutions for the semilinear elliptic equations

$$-\Delta u = \lambda f(z)|u|^{q-2}u + |u|^{2^*-2}u \quad \text{in } \Omega, \quad u \in H_0^1(\Omega), \qquad (E_{\lambda f})$$

where $\lambda > 0$, $1 < q < 2$, $2^* = 2N/(N-2)$, $0 \in \Omega \subset \mathbb{R}^N$, $N \geqslant 3$, is a bounded domain with smooth boundary $\partial\Omega$ and f is a continuous function on $\bar{\Omega}$. Associated with equation $(E_{\lambda f})$, we define the energy functional J_λ, for $u \in H_0^1(\Omega)$,

$$J_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{q}\int_\Omega f(z)(u^+)^q - \frac{1}{2^*}\int_\Omega (u^+)^{2^*},$$

where $u^+ = \max\{u, 0\} \geqslant 0$. By the result in [8], the functional J_λ is of class C^1. We know that the weak solutions of equation $(E_{\lambda f})$ are equivalent to the critical points of J_λ.

Assume that f satisfies the following conditions:

(f1) $f \in C(\bar{\Omega})$ and $f \gneqq 0$;

(f2) there exist positive numbers d_0 and ρ_0 such that $B^N(0; 3\rho_0) \subset \Omega$ and $f(z) \geqslant d_0 > 0$ for any $z \in B^N(0; 3\rho_0)$.

Let $D^{1,2}(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N) \text{ and } \nabla u \in L^2(\mathbb{R}^N)\}$ with the norm

$$\|u\|_D^2 = \int_{\mathbb{R}^N} |\nabla u|^2$$

and let S be the best Sobolev constant defined by

$$S = \inf_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \int_{\mathbb{R}^N} |\nabla u|^2 \biggl(\int_{\mathbb{R}^N} |u|^{2^*}\biggr)^{2^*/2} > 0.$$

Set

$$\Lambda = \left(\frac{2-q}{2^*-q}\right)^{(2-q)/(2^*-2)} \left(\frac{2^*-2}{(2^*-q)|f|_\infty}\right) |\Omega|^{(q-2^*)/2^*} S^{N/2-Nq/4+q/2} > 0. \quad (1.2)$$

This paper is organized as follows. In § 2, we use the argument of Tarantello [10] to divide the Nehari manifold $\boldsymbol{M}_\lambda$ into two parts $\boldsymbol{M}_\lambda^+$ and $\boldsymbol{M}_\lambda^-$ for $\lambda \in (0, \Lambda)$. In § 3, we prove that if f satisfies (f1), then for $\lambda \in (0, \Lambda)$ there is a positive ground-state solution $u_\lambda \in \boldsymbol{M}_\lambda^+$ of equation $(E_{\lambda f})$ in Ω. In § 4, we study the idea of category to show that if f satisfies (f1) and (f2), then for sufficiently small λ there exist at least three positive solution of equation $(E_{\lambda f})$ in Ω (one is the ground-state solution $u_\lambda \in \boldsymbol{M}_\lambda^+$ and the others are in $\boldsymbol{M}_\lambda^-$).

2. Nehari manifold

We define the Palais–Smale (PS) sequences, (PS)-values, and (PS)-conditions in $H_0^1(\Omega)$ for J_λ as follows.

Definition 2.1.

(i) For $\beta \in \mathbb{R}$, a sequence $\{u_n\}$ is a $(\mathrm{PS})_\beta$-sequence in $H_0^1(\Omega)$ for J_λ if $J_\lambda(u_n) = \beta + o(1)$ and $J'_\lambda(u_n) = o(1)$ strongly in $H^{-1}(\Omega)$ as $n \to \infty$.

(ii) $\beta \in \mathbb{R}$ is a (PS)-value in $H_0^1(\Omega)$ for J_λ if there is a $(\mathrm{PS})_\beta$-sequence in $H_0^1(\Omega)$ for J_λ.

(iii) J_λ satisfies the $(\mathrm{PS})_\beta$-condition in $H_0^1(\Omega)$ if every $(\mathrm{PS})_\beta$-sequence in $H_0^1(\Omega)$ for J_λ contains a convergent subsequence.

Since J_λ is not bounded below on $H_0^1(\Omega)$, we consider the Nehari manifold

$$\boldsymbol{M}_\lambda = \{u \in H_0^1(\Omega) \setminus \{0\} \mid \langle J'_\lambda(u), u\rangle = 0\},$$

where

$$\langle J'_\lambda(u), u\rangle = \|u\|_H^2 - \lambda \int_\Omega f(z)(u^+)^q - \int_\Omega (u^+)^{2^*} = 0. \tag{2.1}$$

Note that $\boldsymbol{M}_\lambda$ contains all non-zero solutions of equation $(E_{\lambda f})$. Moreover, we have that J_λ is bounded below on $\boldsymbol{M}_\lambda$.

Lemma 2.2. *The energy functional J_λ is coercive and bounded below on $\boldsymbol{M}_\lambda$.*

Proof. For $u \in \boldsymbol{M}_\lambda$, by (2.1), the Hölder inequality ($p_1 = 2^*/(2^*-q)$ and $p_2 = 2^*/q$) and the Sobolev embedding theorem, we get

$$J_\lambda(u) = \frac{2^*-2}{2^*2}\|u\|_H^2 - \lambda\left(\frac{2^*-q}{2^*q}\right)\int_\Omega f(z)(u^+)^q \tag{2.2 a}$$

$$\geqslant \frac{1}{N}\|u\|_H^2 - \lambda\left(\frac{2^*-q}{2^*q}\right)|\Omega|^{(2^*-q)/2^*}S^{-q/2}\|u\|_H^q|f|_\infty. \tag{2.2 b}$$

Hence, we have that J_λ is coercive and bounded below on $\boldsymbol{M}_\lambda$. □

Define

$$\psi_\lambda(u) = \langle J'_\lambda(u), u\rangle.$$

Then for $u \in \boldsymbol{M}_\lambda$, we get

$$\langle \psi'_\lambda(u), u\rangle = 2\|u\|_H^2 - \lambda q\int_\Omega f(z)(u^+)^q - 2^*\int_\Omega (u^+)^{2^*}$$

$$= (2-q)\|u\|_H^2 - (2^*-q)\int_\Omega (u^+)^{2^*} \tag{2.3 a}$$

$$= \lambda(2^*-q)\int_\Omega f(z)(u^+)^q - (2^*-2)\|u\|_H^2. \tag{2.3 b}$$

We apply the method in [10]. Let

$$\boldsymbol{M}_\lambda^+ = \{u \in \boldsymbol{M}_\lambda \mid \langle \psi'_\lambda(u), u\rangle > 0\},$$
$$\boldsymbol{M}_\lambda^0 = \{u \in \boldsymbol{M}_\lambda \mid \langle \psi'_\lambda(u), u\rangle = 0\},$$
$$\boldsymbol{M}_\lambda^- = \{u \in \boldsymbol{M}_\lambda \mid \langle \psi'_\lambda(u), u\rangle < 0\}.$$

Then we have the following results.

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LEMMA 2.3. *Let $u \in H_0^1(\Omega)$ be a critical point of J_λ. Then u is a non-negative solution of equation $(E_{\lambda f})$. Moreover, if $u \not\equiv 0$, then u is positive in Ω.*

Proof. Suppose that $u \in H_0^1(\Omega)$ satisfies $\langle J_\lambda'(u), \varphi\rangle = 0$ for any $\varphi \in H_0^1(\Omega)$, that is,

$$\int_\Omega \nabla u \nabla \varphi = \lambda \int_\Omega f(z)(u^+)^{q-1}\varphi + \int_\Omega (u^+)^{2^*-1}\varphi \quad \text{for any } \varphi \in H_0^1(\Omega).$$

Thus, u is a weak solution of $-\Delta u = \lambda f(z)(u^+)^{q-1} + (u^+)^{2^*-1}$ in Ω. Since $f \gneqq 0$ in Ω, by the maximum principle, u is non-negative. If $u \not\equiv 0$, we have that u is positive in Ω. □

LEMMA 2.4. *Let Λ be a constant defined as in (1.2). If $0 < \lambda < \Lambda$, then $\boldsymbol{M}_\lambda^0 = \varnothing$.*

Proof. Assuming the contrary, there exists a number $\lambda_0 \in (0, \Lambda)$ such that $\boldsymbol{M}_{\lambda_0}^0 \neq \varnothing$. Then, for $u \in \boldsymbol{M}_{\lambda_0}^0$, by (2.3), we have

$$\|u\|_H^2 = \frac{2^* - q}{2 - q}\int_\Omega (u^+)^{2^*} = \lambda \frac{2^* - q}{2^* - 2}\int_\Omega f(z)(u^+)^q.$$

Using (f1), the Hölder inequality and the Sobolev embedding theorem, we get

$$\|u\|_H \geqslant \left(\frac{2-q}{2^*-q} S^{2^*/2}\right)^{1/(2^*-2)}$$

and

$$\|u\|_H \leqslant \left(\lambda \frac{2^*-q}{2^*-2}|\Omega|^{(2^*-q)/2^*} S^{-q/2}|f|_\infty\right)^{1/(2-q)}.$$

Thus,

$$\lambda \geqslant \left(\frac{2-q}{2^*-q}\right)^{(2-q)/(2^*-2)}\left(\frac{2^*-2}{(2^*-q)|f|_\infty}\right)|\Omega|^{(q-2^*)/2^*} S^{N/2-Nq/4+q/2} = \Lambda,$$

which is a contradiction. □

Lemma 2.5 can be proved by using the equality (2.3 b).

LEMMA 2.5. *If $u \in \boldsymbol{M}_\lambda^+$, then*

$$\int_\Omega f(z)(u^+)^q > 0.$$

For $u \in H^+ = \{u \in H_0^1(\Omega) \mid u^+ \not\equiv 0\}$, let

$$t_{\max} = t_{\max}(u) = \left[\frac{(2-q)\|u\|_H^2}{(2^*-q)\int_\Omega (u^+)^{2^*}}\right]^{1/(2^*-2)} > 0.$$

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LEMMA 2.6. *For each $u \in H^+$, we have that*

(i) *if*

$$\int_\Omega f(z)(u^+)^q = 0,$$

then there exists a unique positive number $t^- = t^-(u) > t_{\max}$ such that $t^- u \in \boldsymbol{M}_\lambda^-$ and $J_\lambda(t^- u) = \sup_{t \geqslant 0} J_\lambda(tu)$,

(ii) *if $0 < \lambda < \Lambda$ and*

$$\int_\Omega f(z)(u^+)^q > 0,$$

then there exist unique positive numbers $t^+ = t^+(u) < t_{\max} < t^- = t^-(u)$ such that $t^+ u \in \boldsymbol{M}_\lambda^+$, $t^- u \in \boldsymbol{M}_\lambda^-$ and

$$J_\lambda(t^+ u) = \inf_{0 \leqslant t \leqslant t_{\max}} J_\lambda(tu), \qquad J_\lambda(t^- u) = \sup_{t \geqslant t_{\max}} J_\lambda(tu).$$

Proof. For each $u \in H^+$, define

$$k(t) = k_u(t) = t^{2-q}\|u\|_H^2 - t^{2^*-q}\int_\Omega (u^+)^{2^*}.$$

Clearly, we get that $k(0) = 0$ and $k(t) \to -\infty$ as $t \to \infty$. Since

$$k'(t) = (2-q)t^{1-q}\|u\|_H^2 - (2^*-q)t^{2^*-q-1}\int_\Omega (u^+)^{2^*},$$

we have $k'(t_{\max}) = 0$, $k'(t) > 0$ for $0 < t < t_{\max}$, and $k'(t) < 0$ for $t > t_{\max}$. Thus, $k(t)$ achieves its maximum at $t_{\max}$. Moreover, we have

$$\begin{aligned} k(t_{\max}) &= \left[\frac{(2-q)\|u\|_H^2}{(2^*-q)\int_\Omega (u^+)^{2^*}}\right]^{(2-q)/(2^*-2)} \|u\|_H^2 \\ &\quad - \left[\frac{(2-q)\|u\|_H^2}{(2^*-q)\int_\Omega (u^+)^{2^*}}\right]^{(2^*-q)/(2^*-2)} \int_\Omega (u^+)^{2^*} \\ &= \|u\|_H^q\left[\left(\frac{2-q}{2^*-q}\right)^{(2-q)/2^*-2} - \left(\frac{2-q}{2^*-q}\right)^{(2^*-q)/(2^*-2)}\right] \\ &\quad \times \left(\frac{\|u\|_H^{2^*}}{\int_\Omega (u^+)^{2^*}}\right)^{(2-q)/(2^*-2)} \\ &\geqslant \|u\|_H^q\left(\frac{2^*-2}{2^*-q}\right)\left(\frac{2-q}{2^*-q}S^{2^*/2}\right)^{(2-q)/(2^*-2)}. \end{aligned} \tag{2.4}$$

(i) Since

$$\int_\Omega f(z)(u^+)^q = 0,$$

there exists a unique positive number $t^- = t^-(u) > t_{\max}$ such that

$$k(t^-) = \lambda \int_\Omega f(z)(u^+)^q \quad \text{and} \quad k'(t^-) < 0.$$

Then it is easy to check that $t^- u \in \boldsymbol{M}_\lambda^-$. For $t > t_{\max}$, we have that

$$\begin{aligned}
k'(t) &= \frac{1}{t^{q+1}}\left[(2-q)\|tu\|_H^2 - (2^*-q)\int_\Omega (tu^+)^{2^*}\right] < 0,\\
\frac{\mathrm{d}}{\mathrm{d}t}J_\lambda(tu) &= t\|u\|_H^2 - t^{q-1}\lambda\int_\Omega f(z)(u^+)^q - t^{2^*-1}\int_\Omega (u^+)^{2^*}\\
&= \frac{1}{t}\left[t^q\left(k(t) - \lambda\int_\Omega f(z)(u^+)^q\right)\right] = 0 \quad \text{as } t = t^-,\\
\frac{\mathrm{d}^2}{\mathrm{d}t^2}J_\lambda(tu) &= \frac{1}{t^2}\left[\|tu\|_H^2 - (2^*-1)\int_\Omega (tu^+)^{2^*} - (q-1)\lambda\int_\Omega f(z)(tu^+)^q\right]\\
&= \frac{1}{t^2}\left[(2-q)\|tu\|_H^2 - (2^*-q)\int_\Omega (tu^+)^{2^*}\right] \quad \text{as } t = t^-\\
&< 0 \quad \text{for } t = t^-,
\end{aligned}$$

and $J_\lambda(tu) \to -\infty$ as $t \to \infty$. Hence, $J_\lambda(t^- u) = \sup_{t \geqslant 0} J_\lambda(tu)$.

(ii) Since $0 < \lambda < \Lambda$ and

$$\int_\Omega f(z)(u^+)^q > 0,$$

by (2.4), we have

$$\begin{aligned}
k(0) &= 0\\
&< \int_\Omega f(z)(u^+)^q \leqslant \lambda|\Omega|^{(2^*-q)/2^*} S^{-q/2}|f|_\infty\|u\|_H^q\\
&< \|u\|_H^q\left(\frac{2^*-2}{2^*-q}\right)\left(\frac{2-q}{2^*-q}S^{2^*/2}\right)^{(2-q)/(2^*-2)}\\
&\leqslant k(t_{\max}).
\end{aligned}$$

It follows that there exist unique positive numbers $t^+ = t^+(u)$ and $t^- = t^-(u)$ such that $t^+ < t_{\max} < t^-$,

$$k(t^+) = \lambda\int_\Omega f(z)|u|^q = k(t^-)$$

and $k'(t^-) < 0 < k'(t^+)$. Similarly, we have that $t^+ u \in \boldsymbol{M}_\lambda^+$, $t^- u \in \boldsymbol{M}_\lambda^-$, $J_\lambda(t^+ u) \leqslant J_\lambda(tu) \leqslant J_\lambda(t^- u)$ for each $t \in [t^+, t^-]$ and $J_\lambda(t^+ u) \leqslant J_\lambda(tu)$ for each $t \in [0, t_{\max}]$. Hence,

$$J_\lambda(t^+ u) = \inf_{0 \leqslant t \leqslant t_{\max}} J_\lambda(tu), \qquad J_\lambda(t^- u) = \sup_{t \geqslant t_{\max}} J_\lambda(tu).$$

□

Applying lemma 2.4 ($\boldsymbol{M}_\lambda^0 = \varnothing$ for $0 < \lambda < \Lambda$), we write $\boldsymbol{M}_\lambda = \boldsymbol{M}_\lambda^+ \cup \boldsymbol{M}_\lambda^-$ and define

$$\alpha_\lambda = \inf_{u \in \boldsymbol{M}_\lambda} J_\lambda(u); \qquad \alpha_\lambda^+ = \inf_{u \in \boldsymbol{M}_\lambda^+} J_\lambda(u); \qquad \alpha_\lambda^- = \inf u \in \boldsymbol{M}_\lambda^- J_\lambda(u).$$

Then we have the following results.

LEMMA 2.7.

(i) *If* $\lambda \in (0, \Lambda)$, *then* $\alpha_\lambda \leqslant \alpha_\lambda^+ < 0$.

(ii) *If* $\lambda \in (0, \frac{1}{2}q\Lambda)$, *then* $\alpha_\lambda^- \geqslant d_0 > 0$ *for some constant*

$$d_0 = d_0(N, q, S, |\Omega|, \lambda, |f|_\infty).$$

Proof. (i) Let $u \in \boldsymbol{M}_\lambda^+$, by (2.3 a), we get

$$\frac{2-q}{2^*-q}\|u\|_H^2 > \int_\Omega (u^+)^{2^*}.$$

Then

$$\begin{aligned} J_\lambda(u) &= \left(\frac{1}{2}-\frac{1}{q}\right)\|u\|_H^2 + \left(\frac{1}{q}-\frac{1}{2^*}\right)\int_\Omega (u^+)^{2^*} \\ &< \left[\left(\frac{1}{2}-\frac{1}{q}\right) + \left(\frac{1}{q}-\frac{1}{2^*}\right)\frac{2-q}{2^*-q}\right]\|u\|_H^2 \\ &= -\frac{2-q}{qN}\|u\|_H^2 \\ &< 0. \end{aligned}$$

By the definitions of α_λ and α_λ^+, we deduce that $\alpha_\lambda \leqslant \alpha_\lambda^+ < 0$.

(ii) Let $u \in \boldsymbol{M}_\lambda^-$, by (2.3 a) and the Sobolev embedding theorem, we get

$$\frac{2-q}{2^*-q}\|u\|_H^2 < \int_\Omega (u^+)^{2^*} \leqslant S^{-2^*/2}\|u\|_H^{2^*}.$$

This implies

$$\|u\|_H > \left(\frac{2-q}{2^*-q}\right)^{1/(2^*-2)} S^{N/4} \quad \text{for any } u \in \boldsymbol{M}_\lambda^-. \tag{2.5}$$

Using (2.2 b) and (2.5), we obtain that

$$\begin{aligned} J_\lambda(u) &\geqslant \|u\|_H^q\left[\frac{1}{N}\|u\|_H^{2-q} - \lambda\left(\frac{2^*-q}{2^*q}\right)|\Omega|^{(2^*-q)/2^*}S^{-q/2}|f|_\infty\right] \\ &> \left(\frac{2-q}{2^*-q}\right)^{q/(2^*-2)} S^{qN/4}\left[\frac{1}{N}\left(\frac{2-q}{2^*-q}\right)^{(2-q)/(2^*-2)} S^{(2-q)N/4} \right. \\ &\qquad\qquad \left. - \lambda\left(\frac{2^*-q}{2^*q}\right)|\Omega|^{(2^*-q)/2^*}S^{-q/2}|f|_\infty\right]. \end{aligned}$$

Hence, if $\lambda \in (0, \frac{1}{2}q\Lambda)$, then for any $u \in \boldsymbol{M}_\lambda^-$,

$$J_\lambda(u) \geqslant d_0(N, q, S, |\Omega|, \lambda, |f|_\infty) > 0.$$

□

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For $c > 0$, we define

$$J^c(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{2^*}\int_\Omega c(u^+)^{2^*},$$

$$\boldsymbol{M}^c = \{u \in H_0^1(\Omega) \setminus \{0\} \mid \langle J^{c\prime}(u), u\rangle = 0\}.$$

Note that $J_0 = J^1$ and $\boldsymbol{M}_0 = \boldsymbol{M}^1$. From the results of lemma 2.6, there exist the unique $t^- = t^-(u) > 0$ and a unique $t_0 = t_0(u) > 0$ such that $t^- u \in \boldsymbol{M}_\lambda^-$ and $t_0 u \in \boldsymbol{M}_0 = \boldsymbol{M}^1$. Then we have the following results.

Lemma 2.8.

(i) *For each*

$$u \in \Sigma = \{u \in H_0^1(\Omega) \mid u^+ \not\equiv 0 \text{ and } \|u\|_H = 1\},$$

there exists a unique number $t^c(u) > 0$ *such that* $t^c(u)u \in \boldsymbol{M}^c$ *and*

$$\max_{t\geqslant 0} J^c(tu) = J^c(t^c(u)u) = \left(\frac{1}{2} - \frac{1}{2^*}\right)\left(c\int_\Omega (u^+)^{2^*}\right)^{-2/(2^*-2)}.$$

(ii) *For each* $u \in H_0^1(\Omega)$ *and* $\lambda \in (0,1)$, *we have*

$$\begin{aligned}(1-\lambda)J^{1/(1-\lambda)}(u) - \frac{2-q}{2q}\lambda\|f\|^{2/(2-q)}_{L^{2/(2-q)}} &\leqslant J_\lambda(u) \\ &\leqslant (1+\lambda)J^{1/(1+\lambda)}(u) + \frac{2-q}{2q}\lambda\|f\|^{2/(2-q)}_{L^{2/(2-q)}}.\end{aligned}$$

(iii) *For each* $u \in \Sigma$ *and* $\lambda \in (0,1)$, *we have*

$$\begin{aligned}(1-\lambda)^{N/2}J_0(t_0 u) - \frac{2-q}{2q}\lambda\|f\|^{2/(2-q)}_{L^{2/(2-q)}} &\leqslant J_\lambda(t^- u) \\ &\leqslant (1+\lambda)^{N/2}J_0(t_0 u) + \frac{2-q}{2q}\lambda\|f\|^{2/(2-q)}_{L^{2/(2-q)}}.\end{aligned}$$

Proof. (i) For each $u \in \Sigma$, let

$$f(t) = J^c(tu) = \tfrac{1}{2}t^2 - \frac{1}{2^*}t^{2^*}\int_\Omega c(u^+)^{2^*}.$$

Then $f(t) \to -\infty$ as $t \to \infty$,

$$f'(t) = t - t^{2^*-1}\int_\Omega c(u^+)^{2^*} \quad \text{and} \quad f''(t) = 1 - (2^*-1)t^{2^*-2}\int_\Omega c(u^+)^{2^*}.$$

Let

$$t^c(u) = \left(\int_\Omega c(u^+)^{2^*}\right)^{-1/(2^*-2)} > 0.$$

Then $f'(t^c(u)) = 0$, $t^c(u)u \in \boldsymbol{M}^c$ and

$$(t^c(u))^2 f''(t^c(u)) = a(t^c(u)u) - (2^* - 1)\int_\Omega c[t^c(u)u^+]^{2^*} = (2 - 2^*)(t^c(u))^2 a(u) < 0.$$

Thus, there exists a unique $t^c(u) > 0$ such that $t^c(u)u \in \boldsymbol{M}^c$ and

$$\max_{t\geqslant 0} J^c(tu) = J^c(t^c(u)u) = \left(\frac{1}{2} - \frac{1}{2^*}\right)\left(c\int_\Omega (u^+)^{2^*}\right)^{-2/(2^*-2)}.$$

(ii) By the Hölder and Young inequalities, we get

$$\left|\int_\Omega f(z)|u|^q\right| \leqslant \|f\|_{L^{2/(2-q)}}\|u\|_H^q \leqslant \frac{2-q}{2}\|f\|_{L^{2/(2-q)}}^{2/(2-q)} + \tfrac{1}{2}q\|u\|_H^2.$$

Then

$$(1-\lambda)J^{1/(1-\lambda)}(u) - \frac{2-q}{2q}\lambda\|f\|_{L^{2/(2-q)}}^{2/(2-q)} \leqslant J_\lambda(u) \leqslant (1+\lambda)J^{1/(1+\lambda)}(u) + \frac{2-q}{2q}\lambda\|f\|_{L^{2/(2-q)}}^{2/(2-q)}.$$

(iii) For each $u \in \Sigma$, by (i), (ii) and

$$\sup_{t\geqslant 0} J^{1/(1\pm\lambda)}(tu) = J^{1/(1\pm\lambda)}(t^{1/(1\pm\lambda)}(u)u) = (1\pm\lambda)^{2/(2^*-2)}J_0(t_0u)$$

we then obtain

$$(1-\lambda)^{2^*/(2^*-2)}J_0(t_0u) - \frac{2-q}{2q}\lambda\|f\|_{L^{2/(2-q)}}^{2/(2-q)} \leqslant J_\lambda(t^-u) \leqslant (1+\lambda)^{2^*/(2^*-2)}J_0(t_0u) + \frac{2-q}{2q}\lambda\|f\|_{L^{2/(2-q)}}^{2/(2-q)}.$$

□

3. Existence of a ground-state solution

First of all, recall the definition of the numbers α_λ, α_λ^+ and α_λ^- before lemma 2.7. Next, we use an idea in Tarantello [10] to show the existence of a $(\mathrm{PS})_{\alpha_\lambda}$-sequence and a $(\mathrm{PS})_{\alpha_\lambda^-}$-sequence in $H_0^1(\Omega)$ for J_λ.

Proposition 3.1.

(i) *For $\lambda \in (0, \Lambda)$, there is a $(\mathrm{PS})_{\alpha_\lambda}$-sequence $\{u_n\} \subset \boldsymbol{M}_\lambda$ in $H_0^1(\Omega)$ for J_λ.*

(ii) *For $\lambda \in (0, \frac{1}{2}q\Lambda)$, we have that $\alpha_\lambda^- > 0$ (lemma 2.7) and there is a $(\mathrm{PS})_{\alpha_\lambda^-}$-sequence $\{u_n\} \subset \boldsymbol{M}_\lambda^-$ in $H_0^1(\Omega)$ for J_λ.*

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Proof. The proof is similar to [12, proposition 9]. □

THEOREM 3.2. *Assume that f satisfies (f1). If $\lambda \in (0, \Lambda)$, then there exists at least one positive ground-state solution u_λ of equation $(E_{\lambda f})$ in Ω. Moreover, we have that $u_\lambda \in \boldsymbol{M}_\lambda^+$ and $J_\lambda(u_\lambda) = \alpha_\lambda = \alpha_\lambda^+$.*

Proof. By proposition 3.1(i), there is a minimizing sequence $\{u_n\} \subset \boldsymbol{M}_\lambda$ for J_λ such that $J_\lambda(u_n) = \alpha_\lambda + o_n(1)$ and $J'_\lambda(u_n) = o_n(1)$ in $H^{-1}(\Omega)$. Since J_λ is coercive on $\boldsymbol{M}_\lambda$, $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Then there exist a subsequence $\{u_n\}$ and $u_\lambda \in H_0^1(\Omega)$ such that $u_n \rightharpoonup u_\lambda$ weakly in $H_0^1(\Omega)$, $u_n \to u_\lambda$ a.e. in Ω, $u_n \to u_\lambda$ strongly in $L^s(\Omega)$ for any $1 \leqslant s < 2^*$. It is easy to see that u_λ is a solution of equation $(E_{\lambda f})$ in Ω. First, we claim that u_λ is positive. By (2.2 a) and $u_\lambda \in \boldsymbol{M}_\lambda$, we get

$$\lambda \int_\Omega f(z)(u_n^+)^q = \frac{q(2^*-2)}{2(2^*-q)}\|u_n\|_H^2 - \frac{2^*q}{2^*-q}J_\lambda(u_n) \quad \text{for each } n \in \mathbb{N}.$$

Letting $n \to \infty$, we deduce that

$$\lambda \int_\Omega f(z)(u_\lambda^+)^q \geqslant -\frac{2^*q}{2^*-q}\alpha_\lambda > 0.$$

Thus, u_λ is a non-zero solution of equation $(E_{\lambda f})$ in Ω. By lemma 2.3, u is positive. Next, we want to show that $u_n \to u_\lambda$ strongly in $H_0^1(\Omega)$ and $J_\lambda(u_\lambda) = \alpha_\lambda$. Since $u_\lambda \in \boldsymbol{M}_\lambda$, by (2.2 a) and the Fatou lemma, we have

$$\begin{aligned}
\alpha_\lambda \leqslant J_\lambda(u_\lambda) &= \frac{1}{N}\|u_\lambda\|_H^2 - \lambda\left(\frac{2^*-q}{2^*q}\right)\int_\Omega f(z)(u_\lambda^+)^q \\
&\leqslant \liminf_{n\to\infty}\left(\frac{1}{N}\|u_n\|_H^2 - \lambda\left(\frac{2^*-q}{2^*q}\right)\int_\Omega f(z)(u_n^+)^q\right) \\
&\leqslant \liminf_{n\to\infty} J_\lambda(u_n) = \alpha_\lambda.
\end{aligned}$$

It follows that $J_\lambda(u_\lambda) = \alpha_\lambda$ and $\lim_{n\to\infty}\|u_n\|_H^2 = \|u_\lambda\|_H^2$. Applying the Brézis–Lieb lemma, we obtain

$$\|u_n - u_\lambda\|_H^2 = \|u_n\|_H^2 - \|u_\lambda\|_H^2 + o_n(1) = o_n(1),$$

that is, $u_n \to u_\lambda$ strongly in $H_0^1(\Omega)$. Finally, we claim that $u_\lambda \in \boldsymbol{M}_\lambda^+$. On the contrary, assume that

$$u_\lambda \in \boldsymbol{M}_\lambda^- (\boldsymbol{M}_\lambda^0 = \varnothing \text{ for } \lambda \in (0, \Lambda)).$$

Since

$$\lambda \int_\Omega f(z)(u_\lambda^+)^q > 0,$$

by lemma 2.6, there exist positive numbers $t_0^+ < t_{\max} < t_0^- = 1$ such that $t_0^+ u_\lambda \in \boldsymbol{M}_\lambda^+$, $t_0^- u_\lambda \in \boldsymbol{M}_\lambda^-$ and

$$J_\lambda(t_0^+ u_\lambda) < J_\lambda(t_0^- u_\lambda) = J_\lambda(u_\lambda) = \alpha_\lambda,$$

which is a contradiction. Hence, $u_\lambda \in \boldsymbol{M}_\lambda^+$ and $J_\lambda(u_\lambda) = \alpha_\lambda = \alpha_\lambda^+$. □

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4. Existence of multiple solutions

In this section, we use the idea of category to prove theorem 4.13. Initially, we want to show that J_λ satisfies the $(\mathrm{PS})_\beta$-condition in $H_0^1(\Omega)$ for $\beta \in (-\infty, (1/N)S^{N/2} - C_0\lambda^{2/(2-q)})$, where C_0 is defined in the following lemma.

LEMMA 4.1. *Assume that f satisfies (f1). If $\{u_n\}$ is a $(\mathrm{PS})_\beta$-sequence in $H_0^1(\Omega)$ for J_λ with $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$, then $J'_\lambda(u) = 0$ in $H^{-1}(\Omega)$ and there is a constant $C_0 = C_0(N, q, S, |\Omega|, |f|_\infty) > 0$ such that $J_\lambda(u) \geqslant -C_0\lambda^{2/(2-q)}$.*

Proof. Since $\{u_n\}$ is a $(\mathrm{PS})_\beta$-sequence in $H_0^1(\Omega)$ for J_λ with $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$, it is easy to check that $J'_\lambda(u) = 0$ in $H^{-1}(\Omega)$. Then we have $\langle J'_\lambda(u), u\rangle = 0$, that is,

$$\|u\|_H^2 - \lambda \int_\Omega f(z)(u^+)^q = \int_\Omega (u^+)^{2^*}.$$

Thus, by (2.2 b) and the Young inequality ($p_1 = 2/q$ and $p_2 = 2/(2-q)$)

$$\begin{aligned} J_\lambda(u) &\geqslant \frac{1}{N}\|u\|_H^2 - \lambda\left(\frac{2^* - q}{2^* q}\right)|\Omega|^{(2^*-q)/2^*} S^{-q/2}\|u\|_H^q |f|_\infty \\ &\geqslant \frac{1}{N}\|u\|_H^2 - \frac{1}{N}\|u\|_H^2 - C_0\lambda^{2/(2-q)} \\ &= -C_0\lambda^{2/(2-q)}, \end{aligned}$$

where $C_0 = C_0(N, q, S, |\Omega|, |f|_\infty) > 0$. □

LEMMA 4.2. *Assume that f satisfies (f1). Then J_λ satisfies the $(\mathrm{PS})_\beta$-condition in $H_0^1(\Omega)$ for $\beta \in (-\infty, (1/N)S^{N/2} - C_0\lambda^{2/(2-q)})$, where $C_0 > 0$ is given in lemma 4.1.*

Proof. Let $\{u_n\}$ be a $(\mathrm{PS})_\beta$-sequence in $H_0^1(\Omega)$ for J_λ such that $J_\lambda(u_n) = \beta + o_n(1)$ and $J'_\lambda(u_n) = o_n(1)$ in $H^{-1}(\Omega)$. Similarly to the proof of theorem 3.2, we have that there exist a subsequence $\{u_n\}$ and non-negative $u \in H_0^1(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$, $u_n \to u$ a.e. in Ω, $u_n \to u$ strongly in $L^s(\Omega)$ for any $1 \leqslant s < 2^*$. Then we get

$$\left.\begin{aligned} \lambda \int_\Omega f(z)(u_n^+)^q &= \lambda \int_\Omega f(z) u^q + o_n(1) && (\because \Omega \text{ is bounded in } \mathbb{R}^N), \\ \|u_n - u\|_H^2 &= \|u_n\|_H^2 - \|u\|_H^2 + o_n(1) && (\because H_0^1(\Omega) \text{ is a Hilbert space}), \\ \int_\Omega [(u_n - u)^+]^{2^*} &= \int_\Omega (u_n^+)^{2^*} - \int_\Omega u^{2^*} + o_n(1) && (\because \text{Brézis–Lieb lemma}). \end{aligned}\right\} \tag{4.1}$$

By lemma 4.1, $J'_\lambda(u) = 0$ in $H^{-1}(\Omega)$. Since $J_\lambda(u_n) = \beta + o_n(1)$ and $J'_\lambda(u_n) = o_n(1)$ in $H^{-1}(\Omega)$, by (4.1), we deduce that

$$\tfrac{1}{2}\|u_n - u\|_H^2 - \frac{1}{2^*}\int_\Omega [(u_n - u)^+]^{2^*} = \beta - J_\lambda(u) + o_n(1) \tag{4.2}$$

and

$$\|u_n - u\|_H^2 - \int_\Omega [(u_n - u)^+]^{2^*} = o_n(1).$$

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Now, we may assume that

$$\|u_n - u\|_H^2 \to l \quad \text{and} \quad \int_\Omega [(u_n - u)^+]^{2^*} \to l \quad \text{as } n \to \infty. \tag{4.3}$$

Applying the Sobolev inequality, we obtain

$$\|u_n - u\|_H^2 \geqslant S\|(u_n - u)^+\|_{L^{2^*}}^2.$$

Then $l \geqslant S l^{(N-2)/N}$. If $l \neq 0$ ($l \geqslant S^{N/2}$), by lemma 4.1, (4.2) and (4.3), we have that

$$\beta = \left(\frac{1}{2} - \frac{1}{2^*}\right) l + J_\lambda(u) \geqslant \frac{1}{N} S^{N/2} - C_0 \lambda^{2/(2-q)},$$

which is a contradiction. Hence, $l = 0$, that is, $u_n \to u$ strongly in $H_0^1(\Omega)$. □

Recall that the best Sobolev constant S is defined as

$$S = \inf_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\|\nabla u\|_{L^2}^2}{\|u\|_{L^{2^*}}^2}.$$

It is well known that

$$U(z) = \frac{[N(N-2)]^{(N-2)/4}}{[1 + |z|^2]^{(N-2)/2}}$$

solves $-\Delta u = u^{2^*-1}$ in $\mathbb{R}^N$ and $\|\nabla U\|_{L^2}^2 = \|U\|_{L^{2^*}}^{2^*} = S^{N/2}$. Let $z_0 \in B^N(0;\rho_0)$ and let $\eta \in C_0^\infty(\Omega)$ be a cut-off function such that $0 \leqslant \eta \leqslant 1$, $|\nabla \eta| \leqslant C$ and $\eta(z) = 1$ for $|z| < 2\rho_0$ and $\eta(z) = 0$ for $|z| > 3\rho_0$. We define

$$u_\varepsilon(z) = \varepsilon^{(2-N)/2} \eta(z) U\left(\frac{z - z_0}{\varepsilon}\right) = \frac{c_1 \varepsilon^{(N-2)/2} \eta(z)}{[\varepsilon^2 + |z - z_0|^2]^{(N-2)/2}},$$

where $c_1 = [N(N-2)]^{(N-2)/4}$.

From now on, we assume that $< N/(N-2) < q < 2$ and $N > 4$.

Lemma 4.3. *Assume that f satisfies (f1) and (f2). There then exists a number $0 < \Lambda^* < \frac{1}{2} q \Lambda$ such that if $\lambda \in (0, \Lambda^*)$, then*

$$\sup_{t \geqslant 0} J_\lambda(t u_\varepsilon) < \frac{1}{N} S^{N/2} - C_0 \lambda^{2/(2-q)} \quad \text{uniformly in } z_0 \in B^N(0;\rho_0),$$

where $\varepsilon \leqslant \rho_0$, and $C_0 > 0$ is given in lemma 4.1. In particular,

$$0 < \alpha_\lambda^- < \frac{1}{N} S^{N/2} - C_0 \lambda^{2/(2-q)} \quad \text{for any } \lambda \in (0, \Lambda^*).$$

Proof. First, we consider the functional $I \colon H_0^1(\Omega) \to \mathbb{R}^N$ defined by

$$I(u) = \tfrac{1}{2}\|u\|_H^2 - \frac{1}{2^*} \int_\Omega |u|^{2^*}.$$

Step 1. Show that

$$\sup_{t \geqslant 0} I(t u_\varepsilon) \leqslant \frac{1}{N} S^{N/2} + O(\varepsilon^{N-2}).$$

It is known (see [4], [6, lemma 3.2], [9] or [11, lemma 1.46]) that

$$\|u_\varepsilon\|^2_{L^{2^*}} = \|U\|^2_{L^{2^*}(\mathbb{R}^N)} + O(\varepsilon^N), \tag{4.4}$$

$$\|\nabla u_\varepsilon\|^2_{L^2} = \|\nabla U\|^2_{L^2(\mathbb{R}^N)} + O(\varepsilon^{N-2}). \tag{4.5}$$

Moreover, for $N/(N-2) < q < 2$ and $N > 4$,

$$\|u_\varepsilon\|^2_{L^2} \geqslant c\varepsilon^\theta + O(\varepsilon^{N-2}), \quad \text{where } \theta = N - \tfrac{1}{2}(N-2)q.$$

Using (4.4) and (4.5), we obtain that

$$\begin{aligned} \frac{\|\nabla u_\varepsilon\|^2_{L^2}}{\|u_\varepsilon\|^2_{L^{2^*}}} - S &= \frac{\|\nabla U\|^2_{L^2(\mathbb{R}^N)} + O(\varepsilon^{N-2})}{\|U\|^2_{L^{2^*}(\mathbb{R}^N)} + O(\varepsilon^N)} - \frac{\|\nabla U\|^2_{L^2(\mathbb{R}^N)}}{\|U\|^2_{L^{2^*}(\mathbb{R}^N)}} \\ &= \frac{\|U\|^2_{L^{2^*}(\mathbb{R}^N)} O(\varepsilon^{N-2}) - \|\nabla U\|^2_{L^2(\mathbb{R}^N)} O(\varepsilon^N)}{(\|U\|^2_{L^{2^*}(\mathbb{R}^N)} + O(\varepsilon^N))\|U\|^2_{L^{2^*}(\mathbb{R}^N)}} \\ &= O(\varepsilon^{N-2}). \end{aligned} \tag{4.6}$$

Since

$$\max_{t\geqslant 0}\left(\frac{1}{2}at^2 - \frac{b}{2^*}t^{2^*}\right) = \frac{1}{N}\left(\frac{a}{b^{2/2^*}}\right)^{N/2} \quad \text{for any } a > 0 \text{ and } b > 0,$$

by (4.6), we deduce that

$$\sup_{t\geqslant 0} I(tu_\varepsilon) = \frac{1}{N}\left(\frac{\|\nabla u_\varepsilon\|^2_{L^2}}{\|u_\varepsilon\|^2_{L^{2^*}}}\right)^{N/2} \leqslant \frac{1}{N}S^{N/2} + O(\varepsilon^{N-2}).$$

Step 2. Choose a positive number $\Lambda_1 < \frac{1}{2}q\Lambda$ such that

$$\frac{1}{N}S^{N/2} - C_0\lambda^{2/(2-q)} > 0 \quad \text{for any } \lambda \in (0, \Lambda_1).$$

By (f1), we get

$$J_\lambda(tu_\varepsilon) \leqslant \tfrac{1}{2}t^2\|\nabla u_\varepsilon\|^2_{L^2} \quad \text{for all } t \geqslant 0.$$

Using (4.5), $\{u_\varepsilon\}$ is uniformly bounded in $H^1_0(\Omega)$ for $0 < \varepsilon \leqslant 1$. Since J_λ is continuous in $H^1_0(\Omega)$, there exists $t_0 > 0$ (independent of ε) such that

$$\sup_{0\leqslant t\leqslant t_0} J_\lambda(tu_\varepsilon) < \frac{1}{N}S^{N/2} - C_0\lambda^{2/(2-q)} \quad \text{for any } \lambda \in (0, \Lambda_1) \text{ and } 0 < \varepsilon \leqslant 1.$$

Applying the results in step 1 and (f2), we have that for $N/(N-2) < q < 2$ and $N > 4$,

$$\begin{aligned} \sup_{t\geqslant t_0} J_\lambda(tu_\varepsilon) &= \sup_{t\geqslant t_0}\left[I(tu_\varepsilon) - \frac{t^q}{q}\lambda\int_\Omega f(z)u^q_\varepsilon\,\mathrm{d}z\right] \\ &\leqslant \frac{1}{N}S^{N/2} + O(\varepsilon^{N-2}) - \frac{t_0^q}{q}\lambda d_0\int_{B^N(z_0;\rho_0)} u^q_\varepsilon\,\mathrm{d}z \end{aligned}$$

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$$\leqslant \frac{1}{N} S^{N/2} + O(\varepsilon^{N-2}) - \frac{t_0^q}{q} \lambda d_0 \int_{B^N(0;\rho_0)} \frac{\varepsilon^{(N-2)q/2}}{(\varepsilon^2 + \rho_0^2)^{[(N-2)q]/2}} \, \mathrm{d}z$$

$$\leqslant \frac{1}{N} S^{N/2} + O(\varepsilon^{N-2}) - \frac{t_0^q}{q} \lambda^{1+\theta} \bar{d}_0 C(N, \rho_0),$$

where $\theta = N - \frac{1}{2}(N-2)q$. Choose a positive number $[(2-q)\theta]/q < \tau < N - 2 - \theta$ such that $\tau + \theta < N - 2$ and $\tau + \theta < 2\tau/(2-q)$. Let $\lambda = \varepsilon^\tau$. Then there exists a number $\Lambda_2 > 0$ such that

$$O(\varepsilon^{N-2}) - \frac{t_0^q}{q} \lambda^{1+\theta} \bar{d}_0 C(N, \rho_0) < -C_0 \lambda^{2/(2-q)} \quad \text{for any } \lambda \in (0, \Lambda_2).$$

Let $\Lambda^* = \min\{\Lambda_1, \rho_0^{[(2-q)(N-2)]/2}, \Lambda_2\} > 0$. We obtain

$$\sup_{t \geqslant 0} J_\lambda(t u_\varepsilon) < \frac{1}{N} S^{N/2} - C_0 \lambda^{2/(2-q)}.$$

Moreover, since

$$\int_\Omega f(z) u_\varepsilon^q > 0,$$

by lemma 2.6(ii), there exists $t_\varepsilon^- = t_\varepsilon^-(u_\varepsilon) > 0$ such that $t_\varepsilon^- u_\varepsilon \in \boldsymbol{M}_\lambda^-$ and

$$\alpha_\lambda^- \leqslant J_\lambda(t_\varepsilon^- u_\varepsilon) \leqslant \sup_{t \geqslant 0} J_\lambda(t u_\varepsilon) < \frac{1}{N} S^{N/2} - C_0 \lambda^{2/(2-q)} \quad \text{for any } \lambda \in (0, \Lambda^*).$$

□

We need the Palais–Smale decomposition lemma to prove lemma 4.5. Recall that the best Sobolev constant S is independent of the domain and is never achieved except when $\Omega = \mathbb{R}^N$.

Lemma 4.4 (PS decomposition lemma for J_0). *Let $\{u_n\}$ be a $(\mathrm{PS})_\beta$-sequence in $H_0^1(\Omega)$ for J_0. Then there exist a subsequence $\{u_n\}$, a non-negative integer l, sequences $\{z_n^i\}_{n=1}^\infty$ in Ω, $\epsilon_n^i > 0$, functions u in $H_0^1(\Omega)$ and $w^i \neq 0$ in $H^1(\mathbb{R}^N)$ for $1 \leqslant i \leqslant l$ such that*

$$\frac{1}{\epsilon_n^i} \operatorname{dist}(z_n^i, \partial\Omega) \to \infty \quad \text{as } n \to \infty,$$

$$-\Delta u = |u|^{2^*-2} u^+ \quad \text{in } \Omega,$$

$$-\Delta w^i = |w^i|^{2^*-2} (w^i)^+ \quad \text{in } \mathbb{R}^N,$$

$$u_n = u + \sum_{i=1}^{l} \left(\frac{1}{\epsilon_n^i} \right)^{(N-2)/2} w^i \left(\frac{z - z_n^i}{\epsilon_n^i} \right) + o_n(1) \quad \text{strongly in } H^1(\mathbb{R}^N),$$

$$J_0(u_n) = J_0(u) + \sum_{i=1}^{l} J_0(w^i) + o_n(1).$$

Proof. See [9, theorem 3.1] and [11, theorem 8.13]. □

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LEMMA 4.5. *There exists a number $\delta_0 > 0$ such that if $u \in \boldsymbol{M}_0(\Omega)$ and $J_0(u) \leqslant \alpha_0(\Omega) + \delta_0 = (1/N)S^{N/2} + \delta_0$, then*

$$\int_{\mathbb{R}^N} \frac{z}{|z|}|\nabla u|^2 \, \mathrm{d}z \neq \mathbf{0}.$$

Proof. Assuming the contrary, there exists a sequence $\{u_n\}$ in $\boldsymbol{M}_0(\Omega)$ such that $J_0(u_n) = \alpha_0(\Omega) + o_n(1)$ as $n \to \infty$ and

$$\int_{\mathbb{R}^N} \frac{z}{|z|}|\nabla u_n|^2 \, \mathrm{d}z = \mathbf{0} \quad \text{for all } n.$$

Using the argument in [5, p. 156], we have that every minimizing sequence in $\boldsymbol{M}_0(\Omega)$ of $\alpha_0(\Omega)$ is a $(\mathrm{PS})_{\alpha_0(\Omega)}$-sequence in $H_0^1(\Omega)$ for J_0. Thus, $\{u_n\}$ is a $(\mathrm{PS})_{\alpha_0}$-sequence in $H_0^1(\Omega)$ for J_0. We know that

$$\alpha_0(\Omega) = \alpha_0(\mathbb{R}^N) = \frac{1}{N}S^{N/2},$$

and $\inf_{v \in \boldsymbol{M}_0(\Omega)} J_0(v) = \alpha_0(\Omega)$ is not achieved. Now, applying the Palais–Smale decomposition lemma (lemma 4.4), we have that there exist sequences $\epsilon_n > 0$ and $\{z_n\} \subset \Omega$ such that

$$\frac{1}{\epsilon_n} \operatorname{dist}(z_n, \partial\Omega) \to \infty \quad \text{as } n \to \infty$$

and

$$u_n(z) = \left(\frac{1}{\epsilon_n}\right)^{(N-2)/2} U\left(\frac{z - z_n}{\epsilon_n}\right) + o_n(1) \quad \text{strongly in } H^1(\mathbb{R}^N),$$

where U is the positive solution of equation (1.1) in $\mathbb{R}^N$. Since Ω is a bounded domain and $\{z_n\} \subset \Omega$, there is a subsequence $\{\epsilon_n\}$ such that $\epsilon_n \to 0$. Suppose the subsequence $z_n/|z_n| \to z_0$ as $n \to \infty$, where z_0 is a unit vector in $\mathbb{R}^N$. Then, by the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}
\mathbf{0} &= \int_{\mathbb{R}^N} \frac{z}{|z|}|\nabla u_n|^2 \, \mathrm{d}z \\
&= \int_{\mathbb{R}^N} \frac{\epsilon_n z + z_n}{|\epsilon_n z + z_n|}|\nabla U|^2 \, \mathrm{d}z + o_n(1) \\
&= S^{N/2} z_0 + o(1),
\end{aligned}$$

which is a contradiction. □

From now on, we assume that $\lambda \in (0, \frac{1}{2}q\Lambda)$. Using the results of lemma 2.6, let $K_\lambda(u) = J_\lambda(t_u^- u) = \sup_{t \geqslant 0} J_\lambda(tu)$ for each $u \in H_0^1(\Omega) \setminus \{0\}$. For $c \in \mathbb{R}$, define

$$[K_\lambda \leqslant c] = \{u \in \Sigma \mid K_\lambda(u) \leqslant c\},$$

where

$$\Sigma = \{u \in H_0^1(\Omega) \mid u_+ \not\equiv 0 \text{ and } \|u\|_{H^1} = 1\}.$$

Then we have the following lemma.

LEMMA 4.6.

(i) $K_\lambda \in C^1(\Sigma, \mathbb{R})$ *and*

$$\langle K'_\lambda(u), \varphi \rangle = t_u^- \langle J'_\lambda(t_u^- u), \varphi \rangle \tag{4.7}$$

for all $\varphi \in T_u\Sigma = \{\varphi \in H_0^1(\Omega) \mid \langle \varphi, u \rangle = 0\}$.

(ii) $u \in \Sigma$ *is a critical point of* $K_\lambda(u)$ *if and only if* $t_u^- u \in H_0^1(\Omega)$ *is a critical point of* J_λ.

Proof. (i) For $u \in \Sigma$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} J_\lambda(tu)\bigg|_{t=t_u^-} = 0 \quad \text{and} \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2} J_\lambda(tu)\bigg|_{t=t_u^-} < 0$$

(see the proof of lemma 2.6). Then, using the implicit function theorem, we obtain that $t_u^- \in C^1(\Sigma, (0,\infty))$. Therefore,

$$K_\lambda(u) = J_\lambda(t_u^- u) \in C^1(\Sigma, \mathbb{R}).$$

Since $t_u^- u \in \boldsymbol{M}_\lambda$, we can obtain $\langle J'_\lambda(t_u^- u), u \rangle = 0$. Thus,

$$\begin{aligned} \langle K'_\lambda(u), \varphi \rangle &= \langle J'_\lambda(t_u^- u), t_u^- \varphi \rangle + \langle J'_\lambda(t_u^- u), \langle (t_u^-)', \varphi \rangle u \rangle \\ &= t_u^- \langle J'_\lambda(t_u^- u), \varphi \rangle \quad \text{for all } \varphi \in T_u\Sigma. \end{aligned}$$

(ii) By (i), $K'_\lambda(u) = 0$ if and only if $\langle J'_\lambda(t_u^- u), \varphi \rangle = 0$ for all $\varphi \in T_u\Sigma$. Since $H_0^1(\Omega)$ is a Hilbert space and $\langle J'_\lambda(t_u^- u), u \rangle = 0$, this is equivalent to $J'_\lambda(t_u^- u) = 0$ in $H^{-1}(\Omega)$. □

LEMMA 4.7. *Assume that* f *satisfies (f1). There exists a number* $\Lambda_0 \in (0, \Lambda^*)$ *such that if* $0 < \lambda \leqslant \Lambda_0$, *then*

$$\int_{\mathbb{R}^N} \frac{z}{|z|} |\nabla u|^2 \, \mathrm{d}z \neq \mathbf{0} \quad \textit{for any } u \in \left[K_\lambda < \frac{1}{N} S^{N/2} - C_0 \lambda^{2/(2-q)} \right],$$

where $C_0 > 0$ *is given in lemma 4.1.*

Proof. By lemma 4.3, the set

$$\left[K_\lambda < \frac{1}{N} S^{N/2} - C_0 \lambda^{2/(2-q)} \right]$$

is non-empty. For any $u \in [K_\lambda < (1/N)S^{N/2} - C_0\lambda^{2/(2-q)}]$, we have $u \in \Sigma$, $t_u^- u \in \boldsymbol{M}_\lambda$ and

$$J_\lambda(t_u^- u) < \frac{1}{N} S^{N/2} - C_0 \lambda^{2/(2-q)}.$$

By lemma 2.8, for $\lambda \in (0,1)$, we get

$$J_0(t_0 u) \leqslant (1-\lambda q)^{-N/2} \left[J_\lambda(t_u^- u) + \frac{2-q}{2} \lambda \|f\|^{2/(2-q)}_{L^{2/(2-q)}} \right],$$

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where $t_0 u \in \boldsymbol{M}_0$. Hence, there exists $\Lambda_0 \in (0, \Lambda^*)$ such that if $0 < \lambda \leqslant \Lambda_0$, then

$$J_0(t_0 u) \leqslant \frac{1}{N} S^{N/2} + \delta_0 = \alpha_0(\Omega) + \delta_0.$$

By lemma 4.5, we obtain

$$\int_{\mathbb{R}^N} \frac{z}{|z|} |\nabla(t_0 u)|^2 \, dz \neq \mathbf{0} \quad \text{or} \quad \int_{\mathbb{R}^N} \frac{z}{|z|} |\nabla u|^2 \, dz \neq \mathbf{0}.$$

□

We shall attempt to show that, for a sufficiently small $\sigma > 0$,

$$\operatorname{cat}\left(\left[K_\lambda < \frac{1}{N} S^{N/2} - C_0 \lambda^{2/(2-q)}\right]\right) \geqslant 2. \tag{4.8}$$

To prove (4.8), we need some preliminaries. Recall the definition of the Lyusternik–Schnirelman category.

Definition 4.8.

(i) For a topological space X, we say a non-empty, closed subset $A \subset X$ is contractible to a point in X if and only if there exists a continuous mapping

$$\eta \colon [0,1] \times A \to X$$

such that, for some $x_0 \in X$,

$$\eta(0, x) = x \qquad \text{for all } x \in A$$

and

$$\eta(1, x) = x_0 \quad \text{for all } x \in A.$$

(ii) We define

$$\operatorname{cat}(X) = \min\left\{ k \in \mathbb{N} \mid \exists \text{ closed subsets } A_1, \dots, A_k \subset X : \right.$$
$$\left. A_j \text{ is contractible to a point in } X,\ \forall j \text{ and } \bigcup_{j=1}^{k} A_j = X \right\}.$$

When there do not exist finitely many closed subsets $A_1, \dots, A_k \subset X$ such that A_j is contractible to a point in X for all j and $\bigcup_{j=1}^{k} A_j = X$, we say $\operatorname{cat}(X) = \infty$.

We need the following two lemmas.

Lemma 4.9. *Suppose that X is a Hilbert manifold and $\Psi \in C^1(X, \mathbb{R})$. Assume that there exist $c_0 \in \mathbb{R}$ and $k \in \mathbb{N}$ and*

(i) *$\Psi(x)$ satisfies the $(\mathrm{PS})_c$-condition for $c \leqslant c_0$,*

(ii) *$\operatorname{cat}(\{x \in X \mid \Psi(x) \leqslant c_0\}) \geqslant k$.*

Then $\Psi(x)$ has at least k critical points in $\{x \in X; \Psi(x) \leqslant c_0\}$.

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Proof. See [2, theorem 2.3]. □

LEMMA 4.10. *Let $N \geqslant 1$, $S^{N-1} = \{z \in \mathbb{R}^N \mid |z| = 1\}$ and let X be a topological space. Suppose that there are two continuous maps*

$$F\colon S^{N-1} \to X, \qquad G\colon X \to S^{N-1}$$

such that $G \circ F$ is homotopic to the identity map of S^{N-1}, that is, there exists a continuous map $\zeta\colon [0,1] \times S^{N-1} \to S^{N-1}$ such that

$$\begin{aligned} \zeta(0,z) &= (G \circ F)(z) \quad \text{for each } z \in S^{N-1}, \\ \zeta(1,z) &= z \qquad\qquad\quad \text{for each } z \in S^{N-1}. \end{aligned}$$

Then $\operatorname{cat}(X) \geqslant 2$.

Proof. See [1, lemma 2.5]. □

In the following section, assume that f satisfies (f1) and (f2). Let $\lambda = \varepsilon^{\tau}$ (lemma 4.3) and $\bar{z} \in S^{N-1}$. Then

$$u_\varepsilon(z) = \frac{c_1 \varepsilon^{(N-2)/2} \eta(z)}{[\varepsilon^2 + |z - \rho_0 \bar{z}/2|^2]^{(N-2)/2}} \in H_0^1(\Omega),$$

where $c_1 = [N(N-2)]^{(N-2)/4}$. By lemma 2.6(ii), there exists a unique number $t^-(\varepsilon, \bar{z}) > 0$ such that $t^-(\varepsilon, \bar{z}) u_\varepsilon \in \boldsymbol{M}_\lambda^-(\Omega)$. We define a map $F_\varepsilon\colon S^{N-1} \to H_0^1(\Omega)$ by

$$F_\varepsilon(\bar{z})(z) = \frac{t^-(\varepsilon, \bar{z}) u_\varepsilon(z)}{\|t^-(\varepsilon, \bar{z}) u_\varepsilon(z)\|_{H^1}} \quad \text{for } \bar{z} \in S^{N-1}.$$

Then we have the following lemma.

LEMMA 4.11. *There exists $\sigma(\varepsilon) > 0$ such that*

$$F_\varepsilon(S^{N-1}) \subset \left[K_\lambda \leqslant \frac{1}{N} S^{N/2} - C_0 \lambda^{2/(2-q)} - \sigma(\varepsilon)\right] \quad \textit{for any } \lambda \in (0, \Lambda^*).$$

Proof. Since there exists $t^-(\varepsilon, \bar{z}) > 0$ such that $t^-(\varepsilon, \bar{z}) u_\varepsilon \in \boldsymbol{M}_\lambda^-(\Omega)$, and by the definition of K_λ, we obtain that there exists $s(\varepsilon, \bar{z}) > 0$ such that

$$K_\lambda\left(\frac{t^-(\varepsilon, \bar{z}) u_\varepsilon(z)}{\|t^-(\varepsilon, \bar{z}) u_\varepsilon(z)\|_{H^1}}\right) = J_\lambda\left(s(\varepsilon, \bar{z}) \frac{t^-(\varepsilon, \bar{z}) u_\varepsilon(z)}{\|t^-(\varepsilon, \bar{z}) u_\varepsilon(z)\|_{H^1}}\right),$$

where $s(\varepsilon, \bar{z}) = \|t^-(\varepsilon, \bar{z}) u_\varepsilon(z)\|_{H^1}$. By lemma 4.3, there exists $\Lambda^* > 0$ such that for any $\lambda \in (0, \Lambda^*)$ we have

$$J_\lambda(t^-(\varepsilon, \bar{z}) u_\varepsilon) \leqslant \sup_{t \geqslant 0} J_\lambda(t u_\varepsilon) < \frac{1}{N} S^{N/2} - C_0 \lambda^{2/(2-q)}$$

uniformly in $\bar{z} \in S^{N-1}$. Thus, the conclusion holds. □

Applying lemma 4.7 for $\lambda \in (0, \Lambda_0)$, we obtain

$$\int_{\mathbb{R}^N} \frac{z}{|z|} |\nabla u|^2 \, \mathrm{d}z \neq \mathbf{0} \quad \text{for any } u \in \left[K_\lambda < \frac{1}{N} S^{N/2} - C_0 \lambda^{2/(2-q)}\right].$$

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Now, we define

$$G\colon \left[K_\lambda < \frac{1}{N}S^{N/2} - C_0\lambda^{2/(2-q)}\right] \to S^{N-1}$$

by

$$G(u) = \int_{\mathbb{R}^N} \frac{z}{|z|}|\nabla u|^2\,\mathrm{d}z\biggl(\biggl|\int_{\mathbb{R}^N} \frac{z}{|z|}|\nabla u|^2\,\mathrm{d}z\biggr|\biggr)^{-1}.$$

LEMMA 4.12. *For any $\lambda \in (0, \Lambda_0)$, the map*

$$G \circ F_\varepsilon\colon S^{N-1} \to S^{N-1}$$

is homotopic to the identity, where $\lambda = \varepsilon^\tau$ (lemma 4.3).

Proof. Recall that

$$u_\varepsilon(z) = \varepsilon^{(2-N)/2}\eta(z)U\biggl(\frac{z - \rho_0\bar{z}/2}{\varepsilon}\biggr),$$

where $c_1 = [N(N-2)]^{(N-2)/4}$ and $\bar{z} \in S^{N-1}$. Define

$$\zeta(\theta, \bar{z})\colon [0,1] \times S^{N-1} \to S^{N-1}$$

by

$$\zeta(\theta,\bar{z}) = \begin{cases} G\biggl(\dfrac{(1-2\theta)t^-(\varepsilon,\bar{z})u_\varepsilon(z) + 2\theta u_\varepsilon(z)}{\|(1-2\theta)t^-(\varepsilon,\bar{z})u_\varepsilon(z) + 2\theta u_\varepsilon(z)\|_{H^1}}\biggr) & \text{for } \theta \in [0, \tfrac{1}{2}), \\ G\biggl(\dfrac{u_{2(1-\theta)\varepsilon}(z)}{\|u_{2(1-\theta)\varepsilon}(z)\|_{H^1}}\biggr) & \text{for } \theta \in [\tfrac{1}{2}, 1), \\ \bar{z} & \text{for } \theta = 1. \end{cases}$$

We need to show that $\lim_{\theta\to 1^-}\zeta(\theta,\bar{z}) = \bar{z}$ and

$$\lim_{\theta\to(1/2)^-}\zeta(\theta,\bar{z}) = G\biggl(\frac{u_\varepsilon(z)}{\|u_\varepsilon(z)\|_{H^1}}\biggr).$$

(a) $\lim_{\theta\to 1^-}\zeta(\theta,\bar{z}) = \bar{z}$: for $\frac{1}{2} < \theta < 1$, since

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{z}{|z|}\biggl|\nabla\biggl[\eta(z)U\biggl(\frac{z-\rho_0\bar{z}/2}{2(1-\theta)\varepsilon}\biggr)\biggr]\biggr|^2\,\mathrm{d}z \\ &= \int_{\mathbb{R}^N} \frac{[2(1-\theta)\varepsilon]z + \rho_0\bar{z}/2}{|[2(1-\theta)\varepsilon]z + \rho_0\bar{z}/2|}|\nabla U(z)|^2\,\mathrm{d}z + o(1) \\ &= S^{N/2}\bar{z} + o(1) \quad \text{as } \theta \to 1^-,\end{aligned}$$

we have

$$\lim_{\theta\to 1^-}\zeta(\theta,\bar{z}) = \bar{z}.$$

(b) By the continuity of G, it is easy to check that

$$\lim_{\theta\to(1/2)^-}\zeta(\theta,\bar{z}) = G\biggl(\frac{u_\varepsilon(z)}{\|u_\varepsilon(z)\|_{H^1}}\biggr).$$

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Thus, $\zeta(\theta, \bar{z}) \in C([0,1] \times S^{N-1}, S^{N-1})$ and

$$\begin{aligned} \zeta(0, \bar{z}) &= G(F_\varepsilon(\bar{z})) \quad \text{for all } \bar{z} \in S^{N-1}, \\ \zeta(1, \bar{z}) &= \bar{z} \qquad\qquad \text{for all } \bar{z} \in S^{N-1}, \end{aligned}$$

provided that $\lambda \in (0, \Lambda_0)$. This completes the proof. □

THEOREM 4.13. *Assume that f satisfies (f1) and (f2) and that $N/(N-2) < q < 2$ for $N > 4$. For $\lambda \in (0, \Lambda_0)$, $J_\lambda(u)$ has at least two critical points in*

$$\left[K_\lambda < \frac{1}{N} S^{N/2} - C_0 \lambda^{2/(2-q)}\right].$$

Moreover, there exist at least three positive solutions of equation $(E_{\lambda f})$ in Ω.

Proof. Applying lemmas 4.10 and 4.12, we have, for $\lambda \in (0, \Lambda_0)$,

$$\operatorname{cat}\left(\left[K_\lambda \leqslant \frac{1}{N} S^{N/2} - C_0 \lambda^{2/(2-q)} - \sigma(\varepsilon)\right]\right) \geqslant 2.$$

Next, we need to show that K_λ satisfies the $(\mathrm{PS})_\beta$-condition for

$$0 < \beta \leqslant \frac{1}{N} S^{N/2} - C_0 \lambda^{2/(2-q)} - \sigma(\varepsilon).$$

Let $\{u_n\} \subset \Sigma$ satisfy $K_\lambda(u_n) = \beta + o_n(1)$ and

$$\begin{aligned} \|K_\lambda'(u_n)\|_{T_{u_n}^{-1}\Sigma} &= \sup\{\langle K_\lambda'(u_n), \varphi\rangle \mid \varphi \in T_{u_n}\Sigma \text{ and } \|\varphi\|_{H^1} = 1\} \\ &= o_n(1) \quad \text{as } n \to \infty. \end{aligned}$$

Since $K_\lambda(u_n) = J_\lambda(t_n u_n) = \beta + o_n(1)$ as $n \to \infty$ and $t_n u_n \in \boldsymbol{M}_\lambda^-(\Omega)$, we get that

$$t_n^2 = c + o_n(1) \quad \text{for some } c > 0.$$

Using (4.7) and $\langle J_\lambda'(t_n u_n), u_n\rangle = 0$, we obtain that

$$\|J_\lambda'(t_n u_n)\|_{H^{-1}} = \frac{1}{t_n} \|K_\lambda'(u_n)\|_{T_{u_n}^{-1}\Sigma} = o_n(1) \quad \text{as } n \to \infty.$$

By lemma 4.2, K_λ satisfies the $(\mathrm{PS})_\beta$-condition for

$$0 < \beta \leqslant \frac{1}{N} S^{N/2} - C_0 \lambda^{2/(2-q)} - \sigma(\varepsilon).$$

Now, we apply lemma 4.9, to obtain that K_λ has at least two critical points in

$$\left[K_\lambda < \frac{1}{N} S^{N/2} - C_0 \lambda^{2/(2-q)}\right].$$

Moreover, by lemmas 4.6(ii) and 2.3 and theorem 3.2, there are at least three positive solutions of equation $(E_{\lambda f})$ in Ω. □

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