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On higher differentiability of solutions of parabolic systems with discontinuous coefficients and (p, q)-growth

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We consider weak solutions $u : \Omega_T \to \mathbb{R}^N$ to parabolic systems of the type

$$u_t - \operatorname{div} a(x, t, Du) = 0 \quad \text{in } \Omega_T = \Omega \times (0, T),$$

where the function $a(x, t, \xi)$ satisfies (p, q)-growth conditions. We give an a priori estimate for weak solutions in the case of possibly discontinuous coefficients. More precisely, the partial maps $x \mapsto a(x, t, \xi)$ under consideration may not be continuous, but may only possess a Sobolev-type regularity. In a certain sense, our assumption means that the weak derivatives $D_x a(\cdot, \cdot, \xi)$ are contained in the class $L^\alpha(0, T; L^\beta(\Omega))$, where the integrability exponents α, β are coupled by

$$\frac{p(n + 2) - 2n}{2\alpha} + \frac{n}{\beta} = 1 - \kappa$$

for some $\kappa \in (0, 1)$. For the gap between the two growth exponents we assume

$$2 \leqslant p < q \leqslant p + \frac{2\kappa}{n + 2}.$$

Under further assumptions on the integrability of the spatial gradient, we prove a result on higher differentiability in space as well as the existence of a weak time derivative $u_t \in L^{p/(q-1)}_{\mathrm{loc}}(\Omega_T)$. We use the corresponding a priori estimate to deduce the existence of solutions of Cauchy–Dirichlet problems with the mentioned higher differentiability property.

Keywords: Parabolic systems; higher differentiability; discontinuous coefficients

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1. Introduction and statement of the results

We consider parabolic systems of the type

$$u_t - \operatorname{div} a(x, t, Du) = 0 \quad \text{in } \Omega_T = \Omega \times (0, T), \tag{1.1}$$

where the function $a(x,t,\xi)$ satisfies (p,q)-growth conditions in the gradient variable and is possibly discontinuous in the x-variable. The exponents that we consider satisfy $2 \leqslant p < q \leqslant p+1$. A simple model case of the systems that we have in mind is given by

$$u_t - \operatorname{div}\,(|Du|^{p-2}Du + b(x)c(t)|Du|^{q-2}Du) = 0 \quad \text{in } \Omega_T = \Omega \times (0,T),$$

with a Sobolev coefficient $b(x)$, and a merely measurable coefficient $c(t)$.

For the case $q=p$, there is a wide literature on the regularity properties of weak solutions $u \in L^p(0,T;W^{1,p}(\Omega,\mathbb{R}^N))$ of (1.1). We refer to [**12**, **14**] for an overview of results on regularity, partial regularity, higher differentiability and integrability properties of the spatial gradients. In the case of p-Laplacian systems with coefficients differentiable in the spatial variable, it is known that higher differentiability of solutions holds in the sense that the non-linear expressions $V_p(Du) := (\mu^2 + |Du|^2)^{(p-2)/4}Du$ of the first spatial derivatives are weakly differentiable with $D(V_p(Du)) \in L^2_{\mathrm{loc}}(\Omega_T)$. The corresponding regularity property for the time derivative is $u_t \in L^{p/(p-1)}_{\mathrm{loc}}(\Omega_T)$. We refer to Duzaar, Mingione and Steffen in [**14**] for the precise results, see also [**2**, **13**, **31**]. We point out that under the weaker assumption of Hölder continuous coefficients, their methods imply only a higher differentiability result on a fractional Sobolev scale. For coefficients with weaker regularity properties than Hölder continuity, the known regularity results include Calderón–Zygmund type integrability results for the spatial gradient under a VMO-type condition on $x \mapsto a(x,t,\xi)$ [**1**, **14**, **32**], and a partial Hölder regularity result of the form $u \in C^{0;\alpha,\alpha/2}_{\mathrm{loc}}(\Omega_T \setminus \Sigma_u)$ with $|\Sigma_u| = 0$, in the case of merely continuous coefficients that might not even be Dini-continuous [**3**]. In our recent article [**20**], we found that a Sobolev-type condition on the coefficients is sufficient to establish the existence of higher derivatives. This result was inspired by the earlier papers [**9**, **19**, **22**, **29**, **30**] in the elliptic setting.

In the present paper, we deal with the case of a non-standard growth condition introduced by Marcellini [**27**, **28**] in the elliptic context and later on widely studied by many authors, see e.g. [**7**, **8**, **10**, **15**–**18**]. In the parabolic case, in contrast to the elliptic setting, not much is known for the non-standard (p,q)-growth condition. We refer to [**4**, **5**, **34**, **35**] and references therein for some results about regularity in parabolic problems with (p,q)-growth. As far as we know, the case of discontinuous coefficients has not been exploited yet.

Our aim is to combine a non-standard growth assumption with coefficients possessing only a weak Sobolev-type regularity with respect to the x-variable. In order to make precise our assumptions on the coefficients, we use a characterization of Sobolev functions due to DeVore and Sharpley [**11**] and assume

$$|a(x,t,\xi) - a(y,t,\xi)| \leqslant |x-y|[\gamma(x,t) + \gamma(y,t)](\mu^2 + |\xi|^2)^{(q-1)/2}, \tag{1.2}$$

for some $\gamma : \Omega_T \to [0,\infty)$, which plays the role of the derivative $D_x a$, see also [**24**]. Note that (1.2) is a weak form of continuity since the function γ may blow up at some points. On the function γ we impose the anisotropic integrability assumption

$$\begin{cases} \gamma \in L^\alpha(0,T;L^\beta(\Omega)) & \text{if } p < \alpha < \infty,\ n < \beta < \infty, \\ \gamma \in C^0([0,T];L^\beta(\Omega)) & \text{if } \alpha = \infty, \end{cases} \tag{1.3}$$

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that determines the Sobolev regularity of the coefficients, where $p < \alpha \leqslant +\infty$ and $n \leqslant \beta < +\infty$ are related by

$$\frac{p(n+2)-2n}{2\alpha} + \frac{n}{\beta} = 1 - \kappa, \tag{1.4}$$

for a fixed $\kappa \in (0, 1)$. We note that in the special case $p = 2$, this condition takes the simpler form

$$\frac{2}{\alpha} + \frac{n}{\beta} = 1 - \kappa.$$

For a non-negative function γ satisfying (1.3) and for some exponents $2 \leqslant p \leqslant q < p + 1$, we consider a measurable function $a : \Omega_T \times \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ for which

$$\xi \mapsto a(x, t, \xi) \quad \text{is differentiable for a.e. } (x, t) \in \Omega_T.$$

Moreover, for all $x, y \in \Omega$, $t \in (0, T)$ and all $\xi, \bar{\xi} \in \mathbb{R}^{Nn}$, we impose the following (p, q)-growth conditions

$$|a(x,t,\xi)| + (\mu^2 + |\xi|^2)^{1/2} |\partial_\xi a(x,t,\xi)| \leqslant L(\mu^2 + |\xi|^2)^{(q-1)/2}, \tag{1.5}$$

$$\langle \partial_\xi a(x,t,\xi)\bar{\xi}, \bar{\xi} \rangle \geqslant \nu(\mu^2 + |\xi|^2)^{(p-2)/2} |\bar{\xi}|^2, \tag{1.6}$$

$$|a(x,t,\xi) - a(y,t,\xi)| \leqslant |x - y| \left[\gamma(x,t) + \gamma(y,t)\right] (\mu^2 + |\xi|^2)^{(q-1)/2}, \tag{1.7}$$

for constants $0 < \nu \leqslant L$, $\mu \in [0, 1]$ and $\gamma : \Omega_T \to [0, \infty)$ satisfying (1.3). We consider weak solutions, in the sense of definition 2.1, to

$$\begin{cases} u_t - \operatorname{div} a(x, t, Du) = 0 & \text{in } \Omega_T = \Omega \times (0, T), \\ u = g & \text{on } \partial_{\text{par}} \Omega_T, \end{cases} \tag{1.8}$$

with the initial and boundary data satisfying

$$\begin{cases} g \in L^{p(q-1)/(p-1)}(0, T; W^{1,p(q-1)/(p-1)}(\Omega_T, \mathbb{R}^N)) \cap C^0([0, T]; L^2(\Omega, \mathbb{R}^N)), \\ \partial_t g \in L^{p'}(0, T; W^{-1,p'}(\Omega_T, \mathbb{R}^N)). \end{cases} \tag{1.9}$$

under additional integrability assumptions on the spatial gradient. More precisely, we assume that $Du \in L^{p+4/n}_{\text{loc}}(\Omega_T)$ and $Du \in L^\delta\text{-}L^\sigma_{\text{loc}}(\Omega_T, \mathbb{R}^N)$ for the exponents δ, σ satisfying the conditions (1.12) below (see § 2 for the intuitive notation $L^\delta\text{-}L^\sigma$). We note that the resulting estimate will not depend on the $L^{p+4/n}$-norm or on the $L^\delta\text{-}L^\sigma$-norm of the gradients. The a priori estimate we will obtain is stated in the following

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THEOREM 1.1. *Let $u \in L^p(0,T;W^{1,p}(\Omega,\mathbb{R}^N))$ be a weak solution of system (1.1), under the assumptions (1.5), (1.6), (1.7), where*

$$2 \leqslant p < q \leqslant p + \frac{2\kappa}{n+2}. \tag{1.10}$$

Moreover, suppose that $\gamma \in L^\alpha\text{-}L^\beta(\Omega_T)$ with the exponents α, β from (1.4) and assume

$$Du \in L^{p+4/n}_{\mathrm{loc}}(\Omega_T) \quad \text{and} \quad Du \in L^\delta\text{-}L^\sigma_{\mathrm{loc}}(\Omega_T,\mathbb{R}^N) \tag{1.11}$$

for the exponents δ and σ given by

$$\delta := \frac{p(2-p+\alpha)-\alpha(p-2)(q-p)}{\alpha(p-q+1)-p} \quad \text{and} \quad \sigma := \frac{\beta p(2-p+\alpha)-\alpha\beta(p-2)(q-p)}{\beta(\alpha(p-q+1)-p+2)-2\alpha}. \tag{1.12}$$

There exists a constant $\gamma_0 = \gamma_0(n,N,p,q,\nu,L,\alpha,\beta,K) > 0$ such that, if

$$\int_{Q_{R_0}(z_0)} (\mu^2 + |Du|^2)^{p/2}\,\mathrm{d}z < K \quad \text{and} \quad \|\gamma(x,t)\|_{L^\alpha\text{-}L^\beta(Q_{R_0}(z_0))} \leqslant \gamma_0 \tag{1.13}$$

for some parabolic cylinder $Q_{R_0}(z_0) \Subset \Omega_T$, then we have $D(V_p(Du)) \in L^2_{\mathrm{loc}}(Q_{R_0}(z_0))$ and $\partial_t u \in L^{p/(q-1)}_{\mathrm{loc}}(Q_{R_0}(z_0))$ with the estimates

$$\begin{aligned} \sup_{t_0-(R/2)^2<t<t_0} &\fint_{B_{R/2}(x_0)} |Du(x,t)|^2 \mathrm{d}x + R^2 \fint_{Q_{R/2}(z_0)} |D(V_p(Du))|^2\,\mathrm{d}z \\ &\leqslant c\left(\fint_{Q_R(z_0)} (1+|Du|^p)\,\mathrm{d}z\right)^{1+(q-p)/\theta} \end{aligned} \tag{1.14}$$

and

$$R^{p/(q-1)} \fint_{Q_{R/4}} |\partial_t u|^{p/(q-1)}\,\mathrm{d}z \leqslant c\left(\fint_{Q_R} (1+|Du|^p)\,\mathrm{d}z\right)^{1+(p(q-p))/(2(q-1)\theta)} \tag{1.15}$$

for any radius $R \in (0,R_0)$, with constants $c = c(n,N,p,q,\nu,L,\alpha,\beta)$ and $\theta = 1-(n+2)(q-p)/2$.

REMARK 1.2. In the case $\alpha = \infty$, we interpret the exponents δ, σ from (1.12) as the respective limits when $\alpha \to \infty$.

As an application of the a priori estimate in theorem 1.1, we prove the existence of at least one solution which exhibits a higher differentiability property. Note that, since the uniqueness is not guaranteed in problems with (p,q)-growth and (x,t)-dependence, we can not expect this regularity for an arbitrary solution. For related existence results, see e.g. [**5**, **6**].

In order to specify the dependence of the maximal radius in the following result, we abbreviate

$$M_g := \int_{\Omega_T} (1+|Dg|)^{p(q-1)/(p-1)}\,\mathrm{d}z + \|g\|^2_{L^\infty\text{-}L^2} + \|\partial_t g\|^{p'}_{L^{p'}-W^{-1,p'}}. \tag{1.16}$$

Then, referring to § 2 for the definition of the space C_w, we have:

THEOREM 1.3. *Assume (1.4)–(1.7), where p and q are related by*

$$2 \leqslant p < q \leqslant p + \frac{2\kappa}{n+2}, \tag{1.17}$$

and consider Cauchy–Dirichlet data as in (1.9).

Then there exists a weak solution $u \in g + L^p(0,T;W_0^{1,p}(\Omega,\mathbb{R}^N)) \cap C_w([0,T]; L^2(\Omega,\mathbb{R}))$ of the Cauchy–Dirichlet problem (1.8), which satisfies

$$D(V_p(Du)) \in L^2_{\mathrm{loc}}(\Omega_T) \quad \textit{and} \quad \partial_t u \in L^{p/(q-1)}_{\mathrm{loc}}(\Omega_T).$$

Moreover, there exist a radius $R_0 = R_0(n,N,p,q,\nu,L,\alpha,\beta,\gamma(\cdot),M_g) > 0$ and a constant $c = c(n,N,p,q,\nu,L,\alpha,\beta)$ such that

$$\begin{aligned} \sup_{t_0-(R/2)^2<t<t_0} & ⨍_{B_{R/2}} |Du(x,t)|^2\mathrm{d}x + R^2 ⨍_{Q_{R/2}} |D(V_p(Du))|^2\,\mathrm{d}z \\ &\leqslant c\left(⨍_{Q_R} (1+|Du|^p)\,\mathrm{d}z\right)^{1+(q-p)/\theta} \end{aligned} \tag{1.18}$$

and

$$R^{p/(q-1)} ⨍_{Q_{R/4}} |\partial_t u|^{p/(q-1)}\,\mathrm{d}z \leqslant c\left(⨍_{Q_R} (1+|Du|^p)\,\mathrm{d}z\right)^{1+(p(q-p))/(2(q-1)\theta)} \tag{1.19}$$

hold for every parabolic cylinder $Q_R(z_0) \subset Q_{R_0}(z_0) \Subset \Omega_T$, with

$$\theta := 1 - \frac{n+2}{2}(q-p). \tag{1.20}$$

By virtue of the interpolation inequality stated in lemma 2.6 below, the estimate (1.18) implies an integrability estimate for the first spatial derivative that we state in the following.

COROLLARY 1.4. *Under the assumptions of theorem 1.3, we have $Du \in L^\delta\text{-}L^\sigma_{\mathrm{loc}}(\Omega_T)$ for any exponents δ, σ such that $p < \delta < +\infty$, $2 < \sigma < np/(n-2)$ and*

$$\frac{2}{\sigma} + \frac{p+4/n-2}{\delta} = 1. \tag{1.21}$$

Furthermore, with the radius $R_0 > 0$ and the parameter $\theta \in (0,1)$ from theorem 1.3, we have the estimate

$$R^{-n}\|Du(x,t)\|^{2\delta/(\delta-p+2)}_{L^\delta\text{-}L^\sigma(Q_{R/4}(z_0))} \leqslant c\left(⨍_{Q_R} (1+|Du|^p)\,\mathrm{d}z\right)^{1+(q-p)/\theta} \tag{1.22}$$

for every parabolic cylinder $Q_R(z_0) \subset Q_{R_0}(z_0) \Subset \Omega_T$, where $c = c(n,N,p,q,\nu,L,\alpha,\beta)$, and θ is as in (1.20).

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REMARK 1.5. In particular, in the estimate (1.22) we can choose $\delta = \sigma = p + 4/n$, which gives $Du \in L^{p+4/n}_{\mathrm{loc}}(\Omega_T)$ with the local estimate

$$\left(\fint_{Q_{R/4}(z_0)} |Du|^{p+4/n}\,\mathrm{d}z\right)^{n/(n+2)} \leqslant c\left(\fint_{Q_R} (1+|Du|^p)\,\mathrm{d}z\right)^{1+(q-p)/\theta}$$

with a constant $c = c(n, N, p, q, \nu, L, \alpha, \beta)$ and for every parabolic cylinder $Q_R(z_0) \subset Q_{R_0}(z_0) \Subset \Omega_T$. The parameter θ in some sense measures the distance of $q - p$ from the critical bound $2/(n + 2)$. In particular, it holds $\theta \downarrow 0$ when $q - p \uparrow 2/(n + 2)$.

We conclude this section with some technical details of the proofs. The difference quotient method leads to integrals of the type

$$\int_{Q_\rho} \gamma^2 |Du|^p\,\mathrm{d}x\,\mathrm{d}t,$$

due to the Sobolev-type assumption on the coefficients. In order to bound this integral from above, we need an additional integrability assumption of the type $Du \in L^\delta\text{-}L^\sigma(Q_{R_0})$, for exponents δ, σ that are determined — in an anything but obvious way — by the given exponents α, β (cf. (1.12)). We note that once the desired higher differentiability is established, this kind of integrability follows from the interpolation result from lemma 2.6. Moreover, in order to re-absorb certain integrals that arise during the estimates, we rely on the second additional assumption

$$\|\gamma(x,t)\|_{L^\alpha\text{-}L^\beta(Q_{R_0})} \leqslant \gamma_0 \tag{1.23}$$

for a sufficiently small parameter $\gamma_0 > 0$. This can always be achieved by localizing in the domain. We point out that in the case $\alpha = \infty$ and $\beta > n$, our assumption $\gamma \in C^0([0,T]; L^\beta(\Omega))$ is crucial in order to achieve (1.23) on a sufficiently small cylinder. Finally, we note that the unbalanced growth of the parabolic system (1.8) leads to inequalities that do not possess the preferable homogeneity. This is the point where the first assumption from (1.13) can be exploited in order to compensate for the inhomogeneity of the estimates.

For the construction of the solutions in theorem 1.3, we consider regularized problems of the type

$$\partial_t u_\varepsilon - \operatorname{div}\ a(x,t,Du_\varepsilon) - \varepsilon\operatorname{div}\left(|Du_\varepsilon|^{q-2}Du_\varepsilon\right) = 0,$$

whose higher differentiability follows by our preceding paper [20]. This allows us to apply the a priori estimate from theorem 1.1 and obtain estimates which are independent of ε. The interpolation lemma 2.6 guarantees the higher integrability of the spatial gradient leading to the locally strong convergence. The passage to the limit in the parabolic system is therefore legitimate, and we obtain a solution with the asserted regularity properties.

2. Preliminaries

2.1. Notation and elementary lemmas

In this section, we fix the notations used throughout the article and collect some preliminary results that will prove useful in the sequel.

We shall denote by Ω a bounded domain in $\mathbb{R}^n$ and abbreviate $\Omega_T = \Omega \times (0, T)$, with $T > 0$. For points in space-time, we will frequently use abbreviations like $z = (x, t)$ or $z_0 = (x_0, t_0)$, for spatial variables $x, x_0 \in \mathbb{R}^n$ and times $t, t_0 \in \mathbb{R}$. We write $B_\rho(x_0) \subset \mathbb{R}^n$ for the open ball of radius $\rho > 0$ and center $x_0 \in \mathbb{R}^n$. Moreover, we use the notation

$$Q_\rho(z_0) := B_\rho(x_0) \times (t_0 - \rho^2, t_0), \quad z_0 = (x_0, t_0) \in \mathbb{R}^n \times \mathbb{R},\ \rho > 0,$$

for backward parabolic cylinders. For the standard scalar product on the space $\mathbb{R}^{Nn}$ of $N \times n$ matrices, we write $\langle \cdot, \cdot \rangle$, in contrast to the Euclidean scalar product on $\mathbb{R}^N$, which we denote by '$\cdot$'.

For integrability exponents $1 \leqslant \delta, \sigma \leqslant \infty$, we use the short-hand notation

$$L^\delta\text{-}L^\sigma(\Omega_T, \mathbb{R}^k) := L^\delta(0, T; L^\sigma(\Omega, \mathbb{R}^k)).$$

For the corresponding norm, we write

$$\|u\|_{L^\delta\text{-}L^\sigma(\Omega_T)} := \left[\int_0^T \left(\int_\Omega |u|^\sigma \, \mathrm{d}x \right)^{\delta/\sigma} \mathrm{d}t \right]^{1/\delta}. \tag{2.1}$$

Definition 2.1. *For exponents $2 \leqslant p \leqslant q \leqslant p+1$, a function $u \in g + L^p(0, T; W_0^{1,p}(\Omega; \mathbb{R}^N)) \cap C_w([0, T]; L^2(\Omega, \mathbb{R}^N))$ is a weak solution of the Cauchy–Dirichlet problem (1.8) if $u(\cdot, 0) = g(\cdot, 0)$ and*

$$\int_{\Omega_T} (u\varphi_t - \langle a(x, t, Du), D\varphi \rangle) \, \mathrm{d}z = 0$$

for every $\varphi \in C_0^\infty(\Omega_T; \mathbb{R}^N)$.

Here, we used the notation $C_w([0, T]; L^2(\Omega, \mathbb{R}^N))$ for the space of maps $u \in L^\infty(0, T; L^2(\Omega, \mathbb{R}^N))$ that are continuous with respect to the weak L^2-topology.

We will denote by c a general constant that may vary on different occasions, even within the same line of estimates. Relevant dependencies on parameters and special constants will be suitably emphasized using parentheses or subscripts.

We use the customary notation

$$V_p(\xi) := (\mu^2 + |\xi|^2)^{(p-2)/4} \xi \quad \text{for all } \xi \in \mathbb{R}^k.$$

Since the value of $\mu \in [0, 1]$ is fixed throughout the article, we omit the dependence on μ in the notation.

The next two lemmas contain technical inequalities that will be useful for our aims.

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Lemma 2.2 ([21, lemma 2.2]). *For any $p \geqslant 2$ we have*

$$c^{-1}(\mu^2+|\xi|^2+|\eta|^2)^{(p-2)/2}|\xi-\eta|^2 \leqslant |V_p(\xi)-V_p(\eta)|^2 \leqslant c(\mu^2+|\xi|^2+|\eta|^2)^{(p-2)/2}|\xi-\eta|^2$$

for any $\xi,\eta \in \mathbb{R}^k$ and a constant $c=c(p)>0$.

Lemma 2.3 ([21, lemma 2.1]). *For any $p \geqslant 2$ we have*

$$c^{-1}(\mu^2+|\xi|^2+|\eta|^2)^{(p-2)/2} \leqslant \int_0^1 (\mu^2+|\xi+s(\eta-\xi)|^2)^{(p-2)/2}\,\mathrm{d}s \leqslant (\mu^2+|\xi|^2+|\eta|^2)^{(p-2)/2}$$

for any $\xi,\eta \in \mathbb{R}^k$ and a constant $c=c(p)>0$.

In conclusion, we recall the following well-known iteration lemma that has important applications in the so-called hole filling method, cf. [23, lemma 6.1, p. 191].

Lemma 2.4. *For $R_0<R_1$, consider a bounded function $f:[R_0,R_1]\to[0,\infty)$ with*

$$f(r) \leqslant \vartheta f(\rho)+\frac{A}{(\rho-r)^\alpha}+\frac{B}{(\rho-r)^\beta}+C \quad \text{for all } R_0<r<\rho<R_1,$$

where A,B,C and $\alpha>\beta$ denote non-negative constants and $\vartheta\in(0,1)$. Then we have

$$f(R_0) \leqslant c(\alpha,\vartheta)\left(\frac{A}{(R_1-R_0)^\alpha}+\frac{B}{(R_1-R_0)^\alpha}+C\right).$$

2.2. A higher differentiability result under standard growth conditions

For later use, we recall a previous result from [20, theorem 1.1] on higher differentiability of solutions to parabolic systems with standard growth conditions. It will be needed in the approximation argument to justify the application of the a priori estimate. We restate the result from [20] with q in place of p since we will apply it for this exponent. The mentioned result is concerned with weak solutions to systems of the form

$$\partial_t u-\operatorname{div} a(x,t,Du)=0 \quad \text{on } \Omega_T, \tag{2.2}$$

under the standard q-growth conditions

$$|a(x,t,\xi)|+(\mu^2+|\xi|^2)^{1/2}|\partial_\xi a(x,t,\xi)| \leqslant L(\mu^2+|\xi|^2)^{(q-1)/2}, \tag{2.3}$$

$$\langle \partial_\xi a(x,t,\xi)\bar\xi,\bar\xi\rangle \geqslant \nu(\mu^2+|\xi|^2)^{(q-2)/2}|\bar\xi|^2, \tag{2.4}$$

for constants $0<\nu\leqslant L$, $\mu\in[0,1]$, where the x-dependence satisfies a Sobolev-type condition of the form

$$|a(x,t,\xi)-a(y,t,\xi)| \leqslant |x-y|\,[\gamma(x,t)+\gamma(y,t)]\,(\mu^2+|\xi|^2)^{(q-1)/2}, \tag{2.5}$$

for a function $\gamma \in L^\alpha(0,T;L^\beta(\Omega))$ for $q < \alpha < \infty$ and $n < \beta < \infty$ with

$$\frac{q(n+2)-2n}{2\alpha} + \frac{n}{\beta} = 1. \tag{2.6}$$

THEOREM 2.5 ([**20**, theorem 1.1]). *Let $u \in L^q(0,T;W^{1,q}(\Omega,\mathbb{R}^N))$ be a weak solution of system (2.2) under the assumptions (2.3)–(2.6). Moreover, for a fixed positive constant K, assume that*

$$\int_{\Omega_T} (\mu^2 + |Du(x,t)|^2)^{q/2}\,\mathrm{d}z < K. \tag{2.7}$$

Then $V_q(Du) := (\mu^2 + |Du|^2)^{(q-2)/4} Du$ satisfies $D(V_q(Du)) \in L^2_{\mathrm{loc}}(\Omega_T)$ and there exists a radius $R_0 = R_0(n,N,q,\nu,L,\alpha,\beta,\gamma(\cdot),K)$ such that

$$\begin{aligned}\sup_{t_0-(R/2)^2<t<t_0} &\fint_{B_{R/2}(x_0)} |Du(x,t)|^2 \mathrm{d}x + R^2 \fint_{Q_{R/2}(z_0)} |D(V_q(Du))|^2\,\mathrm{d}z \\ &\leqslant c \fint_{Q_R(z_0)} (\mu^q + |Du|^2 + |Du|^q)\,\mathrm{d}z\end{aligned}$$

holds for every parabolic cylinder $Q_R(z_0) \subset Q_{R_0}(z_0) \Subset \Omega_T$, with a constant $c = c(n,N,q,\nu,L,\alpha,\beta)$.

2.3. An interpolation inequality

The higher differentiability can be exploited by means of the following interpolation inequality to derive an integrability estimate for the first spatial derivatives. The proof can be found in [**20**, proposition 3.1]. Note that for $q = s = p + 4/n$ it reduces to lemma 5.3 in [**14**].

LEMMA 2.6. *Let $u : \Omega_T \to \mathbb{R}^N$ be a function with $Du \in L^\infty(0,T;L^2(\Omega,\mathbb{R}^{Nn}))$ and $D(V_p(Du)) \in L^2(\Omega_T)$. Then for exponents δ, σ with $p < \delta < \infty$, $2 < \sigma < np/(n-2)$ and*

$$\frac{2}{\sigma} + \frac{p + 4/n - 2}{\delta} = 1, \tag{2.8}$$

it holds $|Du| \in L^\delta\text{-}L^\sigma_{\mathrm{loc}}(\Omega_T)$, with

$$\begin{aligned}\int_{t_0-\rho^2}^{t_0} &\left(\int_{B_{\lambda\rho}(x_0)} |Du|^\sigma \mathrm{d}x\right)^{\delta/\sigma} \mathrm{d}t \\ &\leqslant c \sup_{t_0-\rho^2<t<t_0} \left(\int_{B_\rho(x_0)} (\mu + |Du|)^2\,\mathrm{d}x\right)^{(\delta-p)/2} \\ &\quad\times \int_{Q_\rho(z_0)} \left(|DV_p(Du)|^2 + \frac{(\mu+|Du|)^p}{\rho^2(1-\lambda)^2}\right)\mathrm{d}z\end{aligned} \tag{2.9}$$

for every $0 < \lambda < 1$ and for every parabolic cylinder $Q_\rho(z_0) \Subset \Omega_T$. Here, the constant c depends only on n, N, p, δ and σ.

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3. Proof of the a priori estimate

For the proof of Theorem 1.1, we employ the notation

$$\begin{aligned}\tau_h F(x,t) \equiv \tau_{h,i} F(x,t) &:= F(x+he_i,t) - F(x,t),\\ \Delta_h F(x,t) \equiv \Delta_{h,i} F(x,t) &:= \frac{F(x+he_i,t)-F(x,t)}{h},\end{aligned} \tag{3.1}$$

for any $F \in L^1_{\mathrm{loc}}(\Omega_T, \mathbb{R}^N)$ and $i = 1, \ldots, n$, $|h| > 0$.

We divide the proof of the theorem in two parts that are given in the following two subsections. The first subsection is devoted to the proof of (1.14), while the second contains the proof of (1.15).

3.1. The second spatial derivatives

Since u is a weak solution of the parabolic system (1.1), we have

$$\int_{\Omega_T} u \cdot \partial_t \varphi - \langle a(x,t,Du), D\varphi\rangle \,\mathrm{d}z = 0,$$

for every test function $\varphi \in C_0^\infty(\Omega_T)$. Replacing φ with $\tau_{-h}\varphi$ and performing a discrete integration by parts, we get

$$\int_{\Omega_T} \tau_h u \cdot \partial_t \varphi - \langle \tau_h(a(x,t,Du)), D\varphi\rangle \,\mathrm{d}z = 0,$$

for sufficiently small $|h| > 0$. Now we fix smooth symmetric mollifiers $\psi_{1,\epsilon} \in C_0^\infty(B_1(0))$ and $\psi_{2,\epsilon} \in C_0^\infty((-1,1))$, $\epsilon > 0$ and abbreviate $F_\epsilon := \psi_{1,\epsilon}(x)\psi_{2,\epsilon}(t) * F$ for the mollification of a function $F : \Omega_T \to \mathbb{R}^k$. Replacing φ by φ_ϵ in the previous equation and using the symmetry of the mollifiers, we deduce

$$\int_{\Omega_T} (\tau_h u)_\epsilon \cdot \partial_t \varphi - \langle \tau_h(a(x,t,Du))_\epsilon, D\varphi\rangle \,\mathrm{d}z = 0, \tag{3.2}$$

for $\epsilon > 0$ small enough. Now, we choose a testing function of the form $\varphi = \Phi \cdot (\tau_h u)_\epsilon$, for a smooth function Φ with compact support. An integration by parts yields

$$\begin{aligned}&-\frac{1}{2}\int_{\Omega_T} |(\tau_h u)_\epsilon|^2 \partial_t \Phi \,\mathrm{d}z + \int_{\Omega_T} \Phi \langle (\tau_h a(x,t,Du))_\epsilon, D(\tau_h u)_\epsilon\rangle \,\mathrm{d}z\\ &\quad = -\int_{\Omega_T} \langle (\tau_h a(x,t,Du))_\epsilon, D\Phi \otimes (\tau_h u)_\epsilon\rangle \,\mathrm{d}z.\end{aligned} \tag{3.3}$$

At this point, we note that the assumptions (1.11) and (1.10) imply $Du \in L^{p+4/n}(\Omega_T) \subset L^q(\Omega_T)$. In view of the q-growth condition (1.5), it is therefore legitimate to pass to the limit as $\epsilon \downarrow 0$ in the preceding integrals. In this way, we derive

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the identity

$$
\begin{aligned}
&-\frac{1}{2}\int_{\Omega_T}|\tau_h u|^2\partial_t\Phi\,\mathrm{d}z+\int_{\Omega_T}\Phi\langle\tau_h a(x,t,Du),D\tau_h u\rangle\,\mathrm{d}z\\
&\quad=-\int_{\Omega_T}\langle\tau_h a(x,t,Du),D\Phi\otimes\tau_h u\rangle\,\mathrm{d}z. \qquad (3.4)
\end{aligned}
$$

A standard approximation argument yields the same identity for any $\Phi\in W_0^{1,\infty}(\Omega_T)$. Now, we consider a parabolic cylinder $Q_\rho(z_0)\subset Q_{R_0}(z_0)\subset\Omega_T$ for $R_0\in(0,1]$ to be chosen later. For a fixed time $t_1\in(t_0-\rho^2,t_0)$ and $\Delta\in(0,t_0-t_1)$, we choose $\Phi(x,t)=\bar\chi(t)\chi(t)\eta^2(x)$ with $\chi\in W^{1,\infty}((0,T),[0,1])$, $\chi\equiv 0$ on $(0,t_0-\rho^2)$ and $\partial_t\chi\geqslant 0$, $\eta\in C_0^\infty(B_\rho(x_0),[0,1])$, and with the Lipschitz continuous function $\bar\chi:(0,T)\to\mathbb{R}$ defined by

$$
\bar\chi(t)=\begin{cases}1 & \text{if } t\leqslant t_1,\\ \text{affine} & \text{if } t_1<t<t_1+\Delta,\\ 0 & \text{if } t\geqslant t_1+\Delta.\end{cases} \qquad (3.5)
$$

With such a choice, equation (3.4) becomes

$$
\begin{aligned}
&-\frac{1}{2}\int_{\Omega_T}|\tau_h u|^2\eta^2(x)\chi(t)\partial_t\bar\chi(t)\,\mathrm{d}z-\frac{1}{2}\int_{\Omega_T}|\tau_h u|^2\eta^2(x)\bar\chi(t)\partial_t\chi(t)\,\mathrm{d}z\\
&\quad+\int_{\Omega_T}\bar\chi(t)\chi(t)\eta^2(x)\langle\tau_h a(x,t,Du),D\tau_h u\rangle\,\mathrm{d}z\\
&\quad=-2\int_{\Omega_T}\langle\tau_h a(x,t,Du),\bar\chi(t)\chi(t)\eta(x)\nabla\eta(x)\otimes\tau_h u\rangle\,\mathrm{d}z. \qquad (3.6)
\end{aligned}
$$

Letting $\Delta\to 0$ in the previous equality, we get

$$
\begin{aligned}
&\frac{1}{2}\int_{B_\rho(x_0)}\chi(t_1)\eta^2(x)|\tau_h u(x,t_1)|^2\mathrm{d}x+\int_{Q^{t_1}}\chi(t)\eta^2(x)\langle\tau_h a(x,t,Du),D\tau_h u\rangle\,\mathrm{d}z\\
&\quad=-2\int_{Q^{t_1}}\chi(t)\eta(x)\langle\tau_h a(x,t,Du),\nabla\eta\otimes\tau_h u\rangle\,\mathrm{d}z+\frac{1}{2}\int_{Q^{t_1}}\partial_t\chi\,\eta^2(x)|\tau_h u|^2\,\mathrm{d}z, \qquad (3.7)
\end{aligned}
$$

for almost every $t_1\in(0,T)$, where we abbreviated $Q^{t_1}=B_\rho(x_0)\times(t_0-\rho^2,t_1)$. Decomposing now

$$
\begin{aligned}
\tau_h a(x,t,Du)&=[a(x+he_i,t,Du(x+he_i,t))-a(x+he_i,t,Du(x,t))]\\
&\quad+[a(x+he_i,t,Du(x,t))-a(x,t,Du(x,t))]\\
&=:\mathcal{A}_h+\mathcal{B}_h, \qquad (3.8)
\end{aligned}
$$

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equation (3.7) can be re-written as follows:

$$\begin{aligned}
&\frac{1}{2}\int_{B_\rho(x_0)} \chi(t_1)\eta^2(x)|\tau_h u(x,t_1)|^2 \mathrm{d}x + \int_{Q^{t_1}} \chi(t)\eta^2(x)\langle \mathcal{A}_h, D\tau_h u\rangle \,\mathrm{d}z \\
&= -2\int_{Q^{t_1}} \chi(t)\eta(x)\langle \mathcal{A}_h, \nabla\eta\otimes\tau_h u\rangle \,\mathrm{d}z - 2\int_{Q^{t_1}} \chi(t)\eta(x)\langle \mathcal{B}_h, \nabla\eta\otimes\tau_h u\rangle \,\mathrm{d}z \\
&\quad + \frac{1}{2}\int_{Q^{t_1}} \partial_t\chi\,\eta^2|\tau_h u|^2 \,\mathrm{d}z - \int_{Q^{t_1}} \chi(t)\eta^2(x)\langle \mathcal{B}_h, D\tau_h u\rangle \,\mathrm{d}z \\
&=: I_1 + I_2 + I_3 + I_4. \qquad (3.9)
\end{aligned}$$

Observe that

$$\mathcal{A}_h = \int_0^1 \partial_\xi a(x+he_i, t, Du(x,t) + s\tau_h Du)\,\tau_h Du \,\mathrm{d}s. \qquad (3.10)$$

By virtue of assumption (1.6) and lemma 2.3, this yields that

$$\begin{aligned}
\langle \mathcal{A}_h, \tau_h Du\rangle &\geqslant c_p\nu(\mu^2 + |Du|^2 + |Du(x+he_i,t)|^2)^{(p-2)/2}|\tau_h Du|^2 \\
&= c_p\nu\mathcal{D}(h)^{(p-2)/2}|\tau_h Du|^2, \qquad (3.11)
\end{aligned}$$

where we set

$$\mathcal{D}(h) := \mu^2 + |Du(x+he_i,t)|^2 + |Du(x,t)|^2. \qquad (3.12)$$

The estimate (3.11) implies that the left-hand side of (3.9) can be controlled from below as follows

$$\begin{aligned}
&\frac{1}{2}\int_{B_\rho(x_0)} \chi(t_1)\eta^2(x)|\tau_h u(x,t_1)|^2 \mathrm{d}x + \int_{Q^{t_1}} \chi(t)\eta^2(x)\langle \mathcal{A}_h, D\tau_h u\rangle \,\mathrm{d}z \\
&\quad \geqslant \frac{1}{2}\int_{B_\rho(x_0)} \chi(t_1)\eta^2(x)|\tau_h u(x,t_1)|^2 \mathrm{d}x \\
&\qquad + c_p\nu\int_{Q^{t_1}} \chi(t)\eta^2(x)\mathcal{D}(h)^{(p-2)/2}|\tau_h Du|^2 \,\mathrm{d}z. \qquad (3.13)
\end{aligned}$$

Now, by assumption (1.7) and setting

$$\Gamma(h) := \gamma(x+he_i,t) + \gamma(x,t), \qquad (3.14)$$

we deduce that

$$\begin{aligned}
|I_4| &\leqslant \int_{Q_\rho} \chi(t)\eta^2(x)|\mathcal{B}_h|\,|D\tau_h u| \,\mathrm{d}z \\
&\leqslant |h|\int_{Q_\rho} \chi(t)\eta^2(x)\Gamma(h)(\mu^2 + |Du(x,t)|^2)^{(q-1)/2}|D\tau_h u| \,\mathrm{d}z \\
&\leqslant |h|\int_{Q_\rho} \chi(t)\eta^2(x)\Gamma(h)\mathcal{D}(h)^{(q-1)/2}|D\tau_h u| \,\mathrm{d}z \\
&\leqslant \varepsilon\int_{Q_\rho} \chi(t)\eta^2(x)\mathcal{D}(h)^{(p-2)/2}|D\tau_h u|^2 \,\mathrm{d}z + c_\varepsilon h^2\int_{Q_\rho} \Gamma(h)^2\mathcal{D}(h)^{(2q-p)/2} \,\mathrm{d}z, \qquad (3.15)
\end{aligned}$$

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where we have used the fact $0 \leqslant \chi, \eta \leqslant 1$ and Young's inequality, for some constant $\varepsilon \in (0,1)$ that will be chosen later. At this point, we distinguish between the two cases $\alpha \in (p,\infty)$ and $\alpha = \infty$, in which we shall use different arguments to estimate the second integral in (3.15).

We start with the case $\alpha \in (p,\infty)$. Let us first state some bounds for our parameters that will justify the applications of Hölder's inequality below. As a consequence of (1.17) and (1.4) we have

$$q - p \leqslant \frac{2\kappa}{n+2} < \kappa \quad \text{and} \quad \alpha \geqslant \frac{p(n+2)-2n}{2(1-\kappa)} \geqslant \frac{p}{1-\kappa}, \tag{3.16}$$

where we used $p \geqslant 2$ in the last step. This implies in particular

$$\alpha(p-q+1) > p. \tag{3.17}$$

Next, we note that because of $p \geqslant 2$, the coupling (1.4) implies

$$\frac{1}{\beta} \leqslant \frac{1-\kappa}{n} - \frac{p}{\alpha n}.$$

Using this bound and (3.16), we deduce

$$2 - p + \alpha - \alpha(q-p) > 2\frac{\alpha}{\beta} > 0. \tag{3.18}$$

Now, we estimate the last integral in (3.15) by Hölder's inequality with exponents $\alpha/(\alpha(p-q+1)-p+2)$ and $\alpha/(p-2+\alpha q-\alpha p)$ as follows. We note that the Hölder exponents are greater than 1 because of (3.17) and $q \geqslant p \geqslant 2$. In this way, we deduce

$$\begin{aligned}
&\int_{Q_\rho} \Gamma(h)^2 \mathcal{D}(h)^{(2q-p)/2}\,\mathrm{d}z \\
&\quad = \int_{Q_\rho} \Gamma(h)^2 \mathcal{D}(h)^{(\alpha(2q-p)-p(p-2+\alpha q-\alpha p))/2\alpha} \mathcal{D}(h)^{(p(p-2+\alpha q-\alpha p)/2\alpha}\,\mathrm{d}z \\
&\quad \leqslant \left(\int_{Q_\rho} \Gamma(h)^{2\alpha/(\alpha(p-q+1)-p+2)} \right. \\
&\qquad \left. \times \mathcal{D}(h)^{(\alpha(2q-p)-p(p-2+\alpha q-\alpha p))/(2\alpha(p-q+1)-2p+4)}\,\mathrm{d}z \right)^{(\alpha(p-q+1)-p+2)/\alpha} \\
&\qquad \times \left(\int_{Q_\rho} \mathcal{D}(h)^{p/2}\,\mathrm{d}z \right)^{(p-2+\alpha q-\alpha p)/\alpha} \\
&\quad \leqslant c(K) \left(\int_{Q_\rho} \Gamma(h)^{2\alpha/(\alpha(p-q+1)-p+2)} \right. \\
&\qquad \left. \times \mathcal{D}(h)^{(\alpha(2q-p)-p(p-2+\alpha q-\alpha p))/(2\alpha(p-q+1)-2p+4)}\,\mathrm{d}z \right)^{(\alpha(p-q+1)-p+2)/\alpha}
\end{aligned}$$

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with the constant K from assumption (1.13). Next, we first use Hölder's inequality in space with exponents $\beta[\alpha(p-q+1)-p+2]/2\alpha$ and $\beta[\alpha(p-q+1)-p+2]/(\beta[\alpha(p-q+1)-p+2]-2\alpha)$, and then in time with exponents $(\alpha(p-q+1)-p+2)/2$ and $(\alpha(p-q+1)-p+2)/(\alpha(p-q+1)-p)$, which is legitimate because of (3.17) and (3.18). In this way, we deduce

$$\begin{aligned}
&\int_{Q_\rho} \Gamma(h)^{(2\alpha)/(\alpha(p-q+1)-p+2)} \mathcal{D}(h)^{(\alpha(2q-p)-p(p-2+\alpha q-\alpha p))/(2\alpha(p-q+1)-2p+4)}\,\mathrm{d}z \\
&= \int_{t_0-\rho^2}^{t_0} \left(\int_{B_\rho(x_0)} \Gamma(h)^{2\alpha/(\alpha(p-q+1)-p+2)} \right. \\
&\qquad \left. \times \mathcal{D}(h)^{(\alpha(2q-p)-p(p-2+\alpha q-\alpha p))/(2\alpha(p-q+1)-2p+4)}\,\mathrm{d}x \right) \mathrm{d}t \\
&\leqslant \int_{t_0-\rho^2}^{t_0} \left[\left(\int_{B_\rho(x_0)} \Gamma(h)^\beta\,\mathrm{d}x \right)^{2\alpha/(\beta[\alpha(p-q+1)-p+2])} \right. \\
&\quad \left. \times \left(\int_{B_\rho(x_0)} \mathcal{D}(h)^{\sigma/2}\,\mathrm{d}x \right)^{(\beta[\alpha(p-q+1)-p+2]-2\alpha)/(\beta[\alpha(p-q+1)-p+2])} \right] \mathrm{d}t \\
&\leqslant \left[\int_{t_0-\rho^2}^{t_0} \left(\int_{B_\rho(x_0)} \Gamma(h)^\beta\,\mathrm{d}x \right)^{\alpha/\beta} \mathrm{d}t \right]^{2/(\alpha(p-q+1)-p+2)} \\
&\quad \cdot \left[\int_{t_0-\rho^2}^{t_0} \left(\int_{B_\rho(x_0)} \mathcal{D}(h)^{\sigma/2}\,\mathrm{d}x \right)^{\delta/\sigma} \mathrm{d}t \right]^{(\alpha(p-q+1)-p)/(\alpha(p-q+1)-p+2)}.
\end{aligned}$$

Joining the two preceding estimates, we arrive at

$$\begin{aligned}
\int_{Q_\rho} \Gamma(h)^2 \mathcal{D}(h)^{(2q-p)/2}\,\mathrm{d}z &\leqslant c(K) \|\Gamma(h)\|^2_{L^\alpha\text{-}L^\beta(Q_\rho)} \|\mathcal{D}(h)^{1/2}\|^{\delta[\alpha(p-q+1)-p]/\alpha}_{L^\delta\text{-}L^\sigma(Q_\rho)} \\
&= c(K) \|\Gamma(h)\|^2_{L^\alpha\text{-}L^\beta(Q_\rho)} \|\mathcal{D}(h)^{1/2}\|^{2\delta/(\delta-p+2)}_{L^\delta\text{-}L^\sigma(Q_\rho)} \qquad (3.19)
\end{aligned}$$

where we used that

$$\frac{\alpha(p-q+1)-p}{\alpha} = \frac{2}{\delta-p+2}.$$

Now we turn our attention to the case $\alpha=\infty$, in which we have $\beta = n/(1-\kappa)$,

$$\delta = \frac{p-(p-2)(q-p)}{p-q+1} \quad \text{and} \quad \sigma = \frac{\beta(p-(p-2)(q-p))}{\beta(p-q+1)-2}.$$

We note that $q-p \leqslant 2\kappa/(n+2) < \kappa$ implies $\sigma<\infty$. We apply Hölder's inequality with exponents $1/(p-q+1)$ and $1/(q-p)$ to the last integral in (3.15), with the result

$$\begin{aligned}
&\int_{Q_\rho} \Gamma(h)^2 \mathcal{D}(h)^{(2q-p)/2}\,\mathrm{d}z \\
&\quad = \int_{Q_\rho} \Gamma(h)^2 \mathcal{D}(h)^{(2q-p-p(q-p))/2} \mathcal{D}(h)^{p(q-p)/2}\,\mathrm{d}z \\
&\quad \leqslant \left(\int_{Q_\rho} \Gamma(h)^{2/(p-q+1)} \mathcal{D}(h)^{(2q-p-p(q-p))/(2(p-q+1))}\,\mathrm{d}z\right)^{p-q+1} \\
&\qquad \times \left(\int_{Q_\rho} \mathcal{D}(h)^{p/2}\,\mathrm{d}z\right)^{q-p} \\
&\quad \leqslant c(K) \left(\int_{Q_\rho} \Gamma(h)^{2/(p-q+1)} \mathcal{D}(h)^{\delta/2}\,\mathrm{d}z\right)^{p-q+1},
\end{aligned}$$

where we used assumption (1.13) in the last step. To the last integral, we apply Hölder's inequality in space with exponents $\beta(p-q+1)/2$ and $(\beta(p-q+1)/(\beta(p-q+1)-2) = \sigma/\delta$ to deduce

$$\begin{aligned}
&\int_{Q_\rho} \Gamma(h)^{2/(p-q+1)} \mathcal{D}(h)^{\delta/2}\,\mathrm{d}z \\
&\quad = \int_{t_0-\rho^2}^{t_0} \left(\int_{B_\rho(x_0)} \Gamma(h)^{\beta}\,\mathrm{d}x\right)^{2/(\beta(p-q+1))} \left(\int_{B_\rho(x_0)} \mathcal{D}(h)^{\sigma/2}\,\mathrm{d}x\right)^{\delta/\sigma} \mathrm{d}t \\
&\quad \leqslant \|\Gamma(h)\|^{2/(p-q+1)}_{L^\infty\text{-}L^\beta(Q_\rho)} \|\mathcal{D}(h)^{1/2}\|^{\delta}_{L^\delta\text{-}L^\sigma(Q_\rho)}.
\end{aligned}$$

Combining both estimates, we deduce

$$\begin{aligned}
\int_{Q_\rho} \Gamma(h)^2 \mathcal{D}(h)^{(2q-p)/2}\,\mathrm{d}z &\leqslant c(K) \|\Gamma(h)\|^2_{L^\infty\text{-}L^\beta(Q_\rho)} \|\mathcal{D}(h)^{1/2}\|^{\delta(p-q+1)}_{L^\delta\text{-}L^\sigma(Q_\rho)} \\
&= c(K) \|\Gamma(h)\|^2_{L^\alpha\text{-}L^\beta(Q_\rho)} \|\mathcal{D}(h)^{1/2}\|^{2\delta/(\delta-p+2)}_{L^\delta\text{-}L^\sigma(Q_\rho)}, \qquad (3.20)
\end{aligned}$$

which establishes (3.19) also in the borderline case $\alpha = \infty$. Plugging the estimate (3.19), respectively (3.20), into (3.15) we get

$$\begin{aligned}
|I_4| \leqslant \varepsilon \int_{Q_\rho} &\chi(t)\eta^2(x)\mathcal{D}(h)^{(p-2)/2} |D\tau_h u|^2\,\mathrm{d}z \\
&+ c_\varepsilon(K) h^2 \|\Gamma(h)\|^2_{L^\alpha\text{-}L^\beta(Q_\rho)} \|\mathcal{D}(h)^{1/2}\|^{2\delta/(\delta-p+2)}_{L^\delta\text{-}L^\sigma(Q_\rho)}. \qquad (3.21)
\end{aligned}$$

The estimate (3.19), respectively (3.20), can also be used to bound the second integral from (3.9) as in the following. Indeed, by assumption (1.7) and Young's

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inequality we have that

$$
\begin{aligned}
|I_2| &\leqslant 2|h| \int_{Q_\rho} \chi(t)\eta(x)|\nabla\eta|\Gamma(h)(\mu^2+|Du(x,t)|^2)^{(q-1)/2}|\tau_h u|\,\mathrm{d}z \\
&\leqslant \int_{Q_\rho} \chi(t)|\nabla\eta|^2\mathcal{D}(h)^{(p-2)/2}|\tau_h u|^2\,\mathrm{d}z \\
&\quad + h^2\int_{Q_\rho}\chi(t)\eta^2(x)\Gamma(h)^2\mathcal{D}(h)^{(2q-p)/2}\,\mathrm{d}z
\end{aligned}
$$

and hence, by using (3.19), or (3.20), we conclude that

$$
\begin{aligned}
|I_2| &\leqslant \int_{Q_\rho} \chi(t)|\nabla\eta|^2\mathcal{D}(h)^{(p-2)/2}|\tau_h u|^2\,\mathrm{d}z \\
&\quad + c_\varepsilon(K)h^2\|\Gamma(h)\|^2_{L^\alpha\text{-}L^\beta(Q_\rho)}\|\mathcal{D}(h)^{1/2}\|^{2\delta/(\delta-p+2)}_{L^\delta\text{-}L^\sigma(Q_\rho)}.
\end{aligned} \tag{3.22}
$$

It remains to estimate I_1. Assumption (1.5) and the equality (3.10) yield

$$
\begin{aligned}
|\mathcal{A}_h| &\leqslant L|\tau_h Du(x,t)|\int_0^1(\mu^2+|Du(x,t)+s\tau_h Du(x,t)|^2)^{(q-2)/2}\,\mathrm{d}s \\
&\leqslant cL\mathcal{D}(h)^{(q-2)/2}|\tau_h Du(x,t)|,
\end{aligned} \tag{3.23}
$$

where we have used lemma 2.3 in the last step. Applying Young's and Hölder's inequalities, we thus deduce

$$
\begin{aligned}
|I_1| &\leqslant cL\int_{Q_\rho}\chi(t)\eta(x)|\nabla\eta|\mathcal{D}(h)^{(q-2)/2}|\tau_h Du||\tau_h u|\,\mathrm{d}z \\
&\leqslant \varepsilon\int_{Q_\rho}\chi(t)\eta^2(x)\mathcal{D}(h)^{(p-2)/2}|\tau_h Du|^2\,\mathrm{d}z \\
&\quad + c_\varepsilon\int_{Q_\rho}\chi(t)|\nabla\eta|^2\mathcal{D}(h)^{(2q-p-2)/2}|\tau_h u|^2\,\mathrm{d}z \\
&\leqslant \varepsilon\int_{Q_\rho}\chi(t)\eta^2(x)\mathcal{D}(h)^{(p-2)/2}|\tau_h Du|^2\,\mathrm{d}z \\
&\quad + c_\varepsilon\left(\int_{Q_\rho}\chi(t)|\nabla\eta|^2\mathcal{D}(h)^{(2q-p)/2}\,\mathrm{d}z\right)^{(2q-p-2)/(2q-p)} \\
&\quad \times\left(\int_{Q_\rho}\chi(t)|\nabla\eta|^2|\tau_h u|^{2q-p}\,\mathrm{d}z\right)^{2/(2q-p)}.
\end{aligned} \tag{3.24}
$$

Collecting estimates (3.9), (3.13), (3.21), (3.22) and (3.24), and taking the supremum over $t_1 \in (t_0 - (\lambda\rho)^2, t_0)$, we obtain

$$
\begin{aligned}
&\sup_{t_o-(\lambda\rho)^2<t_1<t_0} \frac{1}{2}\int_{B_\rho(x_0)} \chi(t_1)\eta^2(x)|\tau_h u(x,t_1)|^2 \mathrm{d}x \\
&\quad + c_p\nu \int_{Q_\rho} \chi(t)\eta^2(x)\mathcal{D}(h)^{(p-2)/2}|\tau_h Du|^2 \,\mathrm{d}z \\
&\leqslant 4\varepsilon \int_{Q_\rho} \chi(t)\eta^2(x)\mathcal{D}(h)^{(p-2)/2}|\tau_h Du|^2 \,\mathrm{d}z \\
&\quad + c_\varepsilon \int_{Q_\rho} \chi(t)|\nabla\eta|^2\mathcal{D}(h)^{(p-2)/2}|\tau_h u|^2 \,\mathrm{d}z \\
&\quad + c_\varepsilon(K)h^2\|\Gamma(h)\|^2_{L^\alpha\text{-}L^\beta(Q_\rho)}\|\mathcal{D}(h)^{1/2}\|^{2\delta/(\delta-p+2)}_{L^\delta\text{-}L^\sigma(Q_\rho)} + \frac{1}{2}\int_{Q_\rho} \partial_t\chi\,\eta^2|\tau_h u|^2 \,\mathrm{d}z \\
&\quad + c_\varepsilon \left(\int_{Q_\rho} \chi(t)|\nabla\eta|^2\mathcal{D}(h)^{(2q-p)/2}\,\mathrm{d}z\right)^{(2q-p-2)/(2q-p)} \\
&\quad \times \left(\int_{Q_\rho} \chi(t)|\nabla\eta|^2|\tau_h u|^{2q-p}\,\mathrm{d}z\right)^{2/(2q-p)}.
\end{aligned}
$$

At this point, we choose the parameter $\varepsilon = c_p\nu/8$, which enables us to reabsorb the first integral from the right-hand side into the left-hand side to get

$$
\begin{aligned}
&\sup_{t_0-(\lambda\rho)^2<t_1<t_0} \frac{1}{2}\int_{B_\rho(x_0)} \chi(t_1)\eta^2(x)|\tau_h u(x,t_1)|^2 \mathrm{d}x \\
&\quad + \frac{c_p\nu}{2} \int_{Q_\rho} \chi(t)\eta^2(x)\mathcal{D}(h)^{(p-2)/2}|\tau_h Du|^2 \,\mathrm{d}z \\
&\leqslant c \int_{Q_\rho} \chi(t)|\nabla\eta|^2\mathcal{D}(h)^{(p-2)/2}|\tau_h u|^2 \,\mathrm{d}z + \frac{1}{2}\int_{Q_\rho} \partial_t\chi\,\eta^2|\tau_h u|^2 \,\mathrm{d}z \\
&\quad + c(K)h^2\|\Gamma(h)\|^2_{L^\alpha\text{-}L^\beta(Q_\rho)}\|\mathcal{D}(h)^{1/2}\|^{2\delta/(\delta-p+2)}_{L^\delta\text{-}L^\sigma(Q_\rho)} \\
&\quad + c \left(\int_{Q_\rho} \chi(t)|\nabla\eta|^2\mathcal{D}(h)^{(2q-p)/2}\,\mathrm{d}z\right)^{(2q-p-2)/(2q-p)} \\
&\quad \times \left(\int_{Q_\rho} \chi(t)|\nabla\eta|^2|\tau_h u|^{2q-p}\,\mathrm{d}z\right)^{2/(2q-p)}.
\end{aligned}
\tag{3.25}
$$

Here and in the remainder of the proof, we write c for constants that depend at most on $n, N, p, q, \nu, L, \alpha$ and β, and $c(K)$ for constants that may additionally depend on K. For any parameter $1/2 < \lambda < 1$ we choose a cut-off function $\eta \in C_0^\infty(B_\rho(x_0), [0,1])$ with $\eta \equiv 1$ on $B_{\lambda\rho}(x_0)$ and $|\nabla\eta| \leqslant 2/(\rho(1-\lambda))$ on $B_\rho(x_0)$. For the cut-off function in time, we choose the piecewise affine function $\chi : (0,T) \to [0,1]$ with $\chi(t) \equiv 0$ on $(0, t_0 - \rho^2)$, $\chi(t) \equiv 1$ on $(t_0 - (\lambda\rho)^2, T)$ and

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$\partial_t\chi \equiv 1/(\rho^2(1-\lambda^2))$ on $(t_0-\rho^2, t_0-(\lambda\rho)^2)$. We divide both sides of (3.25) by h^2. In view of the properties of χ and η, this yields the estimate

$$\begin{aligned}
&\sup_{t_0-(\lambda\rho)^2<t<t_0}\int_{B_{\lambda\rho}(x_0)}|\Delta_h u(x,t)|^2\mathrm{d}x + \int_{Q_{\lambda\rho}}\mathcal{D}(h)^{(p-2)/2}|\Delta_h Du|^2\,\mathrm{d}z \\
&\quad\leqslant \frac{c}{\rho^2(1-\lambda)^2}\int_{Q_\rho}\mathcal{D}(h)^{(p-2)/2}|\Delta_h u|^2\,\mathrm{d}z + \frac{c}{\rho^2(1-\lambda^2)}\int_{Q_\rho}|\Delta_h u|^2\,\mathrm{d}z \\
&\qquad + c(K)\|\Gamma(h)\|^2_{L^\alpha\text{-}L^\beta(Q_\rho)}\|\mathcal{D}(h)^{1/2}\|^{2\delta/(\delta-p+2)}_{L^\delta\text{-}L^\sigma(Q_\rho)} \\
&\qquad + \frac{c}{\rho^2(1-\lambda)^2}\left(\int_{Q_\rho}\mathcal{D}(h)^{(2q-p)/2}\,\mathrm{d}z\right)^{(2q-p-2)/(2q-p)} \\
&\qquad \times\left(\int_{Q_\rho}|\Delta_h u|^{2q-p}\,\mathrm{d}z\right)^{2/(2q-p)},
\end{aligned} \tag{3.26}$$

with the difference quotients Δ_h as defined in (3.1). Since lemma 2.2 implies

$$\mathcal{D}(h)^{(p-2)/2}|\Delta_h Du|^2 \geqslant \frac{1}{c}|\Delta_h(V_p(Du))|^2,$$

we infer the following bound from (3.26) by letting $h\to 0$:

$$\begin{aligned}
&\sup_{t_0-(\lambda\rho)^2<t<t_0}\int_{B_{\lambda\rho}(x_0)}|Du(x,t)|^2\mathrm{d}x + \int_{Q_{\lambda\rho}}|D(V_p(Du))|^2\,\mathrm{d}z \\
&\quad\leqslant \frac{c}{\rho^2(1-\lambda)^2}\int_{Q_\rho}(\mu^p+|Du|^2+|Du|^p)\,\mathrm{d}z \\
&\qquad + c(K)\|\gamma(x,t)\|^2_{L^\alpha\text{-}L^\beta(Q_\rho)}\|Du(x,t)\|^{2\delta/(\delta-p+2)}_{L^\delta\text{-}L^\sigma(Q_\rho)} \\
&\qquad + \frac{c}{\rho^2(1-\lambda)^2}\int_{Q_\rho}|Du(x,t)|^{2q-p}\,\mathrm{d}z.
\end{aligned}$$

We bound the last integral in the previous estimate first by Hölder's inequality with exponents $2/(2-n(q-p))$ and $2/(n(q-p))$ and then by Young's inequality with exponents $2/(2-(n+2)(q-p))$ and $2/((n+2)(q-p))$, which is possible by virtue of our assumptions on q. This provides us with the estimate

$$\begin{aligned}
&\frac{c}{\rho^2(1-\lambda)^2}\int_{Q_\rho}|Du|^{2q-p}\,\mathrm{d}z \\
&\quad\leqslant \frac{c}{\rho^2(1-\lambda)^2}\left(\int_{Q_\rho}|Du|^p\,\mathrm{d}z\right)^{1-(q-p)n/2}\left(\int_{Q_\rho}|Du|^{p+\frac{4}{n}}\,\mathrm{d}z\right)^{(q-p)n/2} \\
&\quad\leqslant \varepsilon\left(\int_{Q_\rho}|Du|^{p+4/n}\,\mathrm{d}z\right)^{n/(n+2)} + \left(\frac{c_\varepsilon}{\rho^2(1-\lambda)^2}\right)^{2/(2-(n+2)(q-p))} \\
&\qquad\times\left(\int_{Q_\rho}|Du|^p\,\mathrm{d}z\right)^{(2-n(q-p))/(2-(n+2)(q-p))}
\end{aligned}$$

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$$= \varepsilon \left(\int_{Q_\rho} |Du|^{p+4/n} \, \mathrm{d}z \right)^{n/(n+2)} + \left(\frac{c_\varepsilon}{\rho^2(1-\lambda)^2} \right)^{1/\theta} \times \left(\int_{Q_\rho} |Du|^p \, \mathrm{d}z \right)^{1+(q-p)/\theta}$$

for any $\varepsilon \in (0,1)$, where we used the abbreviation $\theta := 1-(n+2)(q-p)/2$ in the last step. In view of assumption (1.13), we arrive at the estimate

$$\begin{aligned} &\sup_{t_0-(\lambda\rho)^2<t<t_0} \int_{B_{\lambda\rho}(x_0)} |Du(x,t)|^2 \mathrm{d}x + \int_{Q_{\lambda\rho}} |D(V_p(Du))|^2 \, \mathrm{d}z \\ &\leqslant \frac{c}{\rho^2(1-\lambda)^2} \int_{Q_\rho} (\mu^p + |Du|^2 + |Du|^p) \, \mathrm{d}z + \left(\frac{c_\varepsilon}{\rho^2(1-\lambda)^2} \right)^{1/\theta} \\ &\quad \times \left(\int_{Q_\rho} |Du|^p \, \mathrm{d}z \right)^{1+(q-p)/\theta} \\ &\quad + c(K) \|\gamma(x,t)\|^2_{L^\alpha\text{-}L^\beta(Q_\rho)} \|Du(x,t)\|^{2\delta/(\delta-p+2)}_{L^\delta\text{-}L^\sigma(Q_\rho)} \\ &\quad + \varepsilon \left(\int_{Q_\rho} |Du|^{p+4/n} \, \mathrm{d}z \right)^{n/(n+2)} \\ &=: \mathrm{RHS}, \end{aligned} \tag{3.27}$$

where RHS denotes the right-hand side. Now, note that from assumptions (1.4) and (1.10) we have

$$\frac{2}{\sigma} + \frac{p+4/n-2}{\delta} \geqslant 1.$$

Since $\delta > p$, we can find a $\sigma_* \in (\sigma, np/(n-2))$ with

$$\frac{2}{\sigma_*} + \frac{p+4/n-2}{\delta} = 1.$$

Therefore, using the interpolation inequality of lemma 2.6 with $\lambda\rho$ in place of ρ, and with the exponents δ and σ_*, we deduce

$$\begin{aligned} \|Du(x,t)\|^\delta_{L^\delta\text{-}L^\sigma(Q_{\lambda^2\rho})} &\leqslant c \int_{t_0-(\lambda\rho)^2}^{t_0} \left(\int_{B_{\lambda^2\rho}} |Du|^{\sigma_*} \mathrm{d}x \right)^{\delta/\sigma_*} \mathrm{d}t \\ &\leqslant c \sup_{t_0-\lambda^2\rho^2<t<t_0} \left(\int_{B_{\lambda\rho}} (\mu + |Du|)^2 \, \mathrm{d}x \right)^{(\delta-p)/2} \\ &\quad \times \int_{Q_{\lambda\rho}} (|D(V_p(Du))|^2 + \frac{(\mu+|Du|)^p}{\rho^2(1-\lambda)^2}) \, \mathrm{d}z, \end{aligned}$$

for all $0 < \rho \leqslant R_0 \leqslant 1$, with a constant $c = c(n, N, p, q, \delta, \sigma_*) = c(n, N, p, q, \alpha, \beta)$. Bounding the last two integrals with the help of (3.27) and taking into account the

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definition of RHS, we arrive at

$$\|Du(x,t)\|^{2\delta/(\delta-p+2)}_{L^\delta\text{-}L^\sigma(Q_{\lambda^2\rho})} \leqslant c\cdot \mathrm{RHS}. \tag{3.28}$$

Similarly, choosing $\delta=\sigma=p+4/n$ in the interpolation lemma 2.6, we infer

$$\int_{Q_{\lambda^2\rho}} |Du|^{p+4/n}\,\mathrm{d}z \leqslant c \sup_{t_0-\lambda^2\rho^2<t<t_0}\left(\int_{B_{\lambda\rho}}(\mu+|Du|)^2\,\mathrm{d}x\right)^{2/n} \times \int_{Q_{\lambda\rho}}(|D(V_p(Du))|^2+\frac{(\mu+|Du|)^p}{\rho^2(1-\lambda)^2})\,\mathrm{d}z,$$

which by virtue of (3.27) implies

$$\left(\int_{Q_{\lambda^2\rho}}|Du|^{p+4/n}\,\mathrm{d}z\right)^{n/(n+2)} \leqslant c\cdot\mathrm{RHS}. \tag{3.29}$$

Combining (3.28) and (3.29) and keeping in mind the definition of RHS from (3.27), we deduce

$$\begin{aligned}
&\|Du(x,t)\|^{2\delta/(\delta-p+2)}_{L^\delta\text{-}L^\sigma(Q_{\lambda^2\rho})} + \left(\int_{Q_{\lambda^2\rho}}|Du|^{p+4/n}\,\mathrm{d}z\right)^{n/(n+2)}\\
&\leqslant \frac{c}{\rho^2(1-\lambda)^2}\int_{Q_\rho}(\mu^p+|Du|^2+|Du|^p)\,\mathrm{d}z\\
&\quad+\left(\frac{c_\varepsilon}{\rho^2(1-\lambda)^2}\right)^{1/\theta}\left(\int_{Q_\rho}|Du|^p\,\mathrm{d}z\right)^{1+(q-p)/\theta}\\
&\quad+c(K)\|\gamma(x,t)\|^2_{L^\alpha\text{-}L^\beta(Q_\rho)}\|Du(x,t)\|^{2\delta/(\delta-p+2)}_{L^\delta\text{-}L^\sigma(Q_\rho)}\\
&\quad+c\varepsilon\left(\int_{Q_\rho}|Du|^{p+4/n}\,\mathrm{d}z\right)^{n/(n+2)}.
\end{aligned}$$

As above, the constant c depends at most on $n, N, p, q, \nu, L, \alpha$ and β, while the constant $c(K)$ may additionally depend on K. At this stage, we choose the constants $\gamma_0\in(0,1)$ and $\varepsilon\in(0,1)$ in such a way that

$$\gamma_0^2=\frac{1}{2c(K)} \quad\text{and}\quad c\varepsilon=\frac{1}{2}.$$

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Note that the choice of ε depends only on $n, N, p, \nu, L, \alpha, \beta$, and the choice of γ_0 additionally on K. Then we obtain, in view of assumption (1.13)

$$\begin{aligned}
&\|Du(x,t)\|^{2\delta/(\delta-p+2)}_{L^\delta\text{-}L^\sigma(Q_{\lambda^2\rho})} + \left(\int_{Q_{\lambda^2\rho}} |Du|^{p+4/n}\,\mathrm{d}z\right)^{n/(n+2)} \\
&\quad\leqslant \frac{1}{2}\left[\|Du(x,t)\|^{2\delta/(\delta-p+2)}_{L^\delta\text{-}L^\sigma(Q_\rho)} + \left(\int_{Q_\rho} |Du|^{p+4/n}\,\mathrm{d}z\right)^{n/(n+2)}\right] \\
&\qquad + \frac{c}{(\rho-\lambda^2\rho)^2}\int_{Q_R} (\mu^p + |Du|^2 + |Du|^p)\,\mathrm{d}z \\
&\qquad + \left(\frac{c}{(\rho-\lambda^2\rho)^2}\right)^{1/\theta}\left(\int_{Q_R} |Du|^p\,\mathrm{d}z\right)^{1+(q-p)/\theta}
\end{aligned}$$

for every $\rho < R < R_0$, where we used $1-\lambda^2 \leqslant 2(1-\lambda)$. Exploiting the previous estimate for any λ with $3/4 < \lambda^2 < 1$, we infer from the iteration lemma 2.4

$$\begin{aligned}
&\|Du(x,t)\|^{2\delta/(\delta-p+2)}_{L^\delta\text{-}L^\sigma(Q_{3R/4})} + \left(\int_{Q_{3R/4}} |Du|^{p+4/n}\,\mathrm{d}z\right)^{n/(n+2)} \\
&\quad\leqslant \frac{c}{R^2}\int_{Q_R} (\mu^p + |Du|^2 + |Du|^p)\,\mathrm{d}z + \left(\frac{c}{R^2}\right)^{1/\theta}\left(\int_{Q_R} |Du|^p\,\mathrm{d}z\right)^{1+(q-p)/\theta} \\
&\quad= cR^n \fint_{Q_R} (\mu^p + |Du|^2 + |Du|^p)\,\mathrm{d}z + cR^n\left(\fint_{Q_R} |Du|^p\,\mathrm{d}z\right)^{1+(q-p)/\theta}. \qquad (3.30)
\end{aligned}$$

Now, we use (3.27) with $\rho = 3R/4$ and $\lambda = 2/3$ and estimate the right-hand side of (3.27) by the preceding estimate. This yields the bound

$$\begin{aligned}
&\sup_{t_0-(R/2)^2<t<t_0} \int_{B_{R/2}} |Du(x,t)|^2\mathrm{d}x + \int_{Q_{R/2}} |D(V_p(Du))|^2\,\mathrm{d}z \\
&\quad\leqslant cR^n \fint_{Q_R} \left(\mu^p + |Du|^p + |Du|^2\right)\,\mathrm{d}z + cR^n\left(\fint_{Q_R} |Du|^p\,\mathrm{d}z\right)^{1+(q-p)/\theta}. \qquad (3.31)
\end{aligned}$$

Dividing both sides by R^n and estimating $\mu^p + |Du|^2 \leqslant c(p)(1+|Du|^p)$, we arrive at the asserted estimate (1.14).

3.2. The time derivative

It remains to establish the estimate (1.15) for the time derivative. To this end, we recall that we have

$$\Delta_h(a(x,t,Du)) = \frac{1}{h}\mathcal{A}_h + \frac{1}{h}\mathcal{B}_h, \qquad (3.32)$$

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with $\mathcal{A}_h$ and $\mathcal{B}_h$ as introduced in (3.8). From (3.23) and (1.7), we have

$$\begin{aligned}\left|\frac{1}{h}\mathcal{A}_h\right|^{p/(q-1)} &\leqslant c\mathcal{D}(h)^{((q-2)p)/(2(q-1))}|\Delta_h Du|^{p/(q-1)}\\ &= c\mathcal{D}(h)^{(p(p-2))/(4(q-1))}|\Delta_h Du|^{p/(q-1)}\\ &\quad\times \mathcal{D}(h)^{p(p-2+2(q-p))/(4(q-1))} \end{aligned}\tag{3.33}$$

and

$$\begin{aligned}\left|\frac{1}{h}\mathcal{B}_h\right|^{p/(q-1)} &\leqslant c\Gamma(h)^{p/(q-1)}(\mu^2+|Du|^2)^{p/2} \leqslant c\Gamma(h)^{p/(q-1)}\mathcal{D}(h)^{p/2}\\ &= c\Gamma(h)^{p/(q-1)}\mathcal{D}(h)^{p(2q-p)/(4(q-1))}\mathcal{D}(h)^{p(p-2)/(4(q-1))}, \end{aligned}\tag{3.34}$$

where $\mathcal{D}(h)$ and $\Gamma(h)$ are defined in (3.12) and (3.14), respectively. Next, we integrate the inequality (3.33) over the parabolic cylinder $Q_{R/4}$ and use Hölder's inequality with exponents $2(q-1)/p$ and $2(q-1)/(p-2+2(q-p))$. In this way, we deduce

$$\begin{aligned}\fint_{Q_{R/4}}\left|\frac{1}{h}\mathcal{A}_h\right|^{p/(q-1)}\,\mathrm{d}z &\leqslant c\left(\fint_{Q_{R/4}}\mathcal{D}(h)^{(p-2)/2}|\Delta_h Du|^2\,\mathrm{d}z\right)^{p/(2(q-1))}\\ &\quad\times\left(\fint_{Q_{R/4}}\mathcal{D}(h)^{p/2}\,\mathrm{d}z\right)^{(p-2+2(q-p))/(2(q-1))}\\ &\leqslant c\left(\fint_{Q_{R/4}}|\Delta_h(V_p(Du))|^2\,\mathrm{d}z\right)^{p/(2(q-1))}\\ &\quad\times\left(\fint_{Q_{R/4}}\mathcal{D}(h)^{p/2}\,\mathrm{d}z\right)^{(p-2+2(q-p))/(2(q-1))}, \end{aligned}\tag{3.35}$$

where the last estimate follows from lemma 2.2. Similarly, integrating the inequality (3.34) and applying Hölder's inequality with the exponents $2(q-1)/p$, $2(q-1)/(p-2)$ and $(q-1)/(q-p)$ in the case $p>2$, respectively, with exponents $q-1$ and $(q-1)/(q-2)$ if $p=2$, we infer

$$\begin{aligned}\fint_{Q_{R/4}}\left|\frac{1}{h}\mathcal{B}_h\right|^{p/(q-1)}\,\mathrm{d}z &\leqslant c\left(\fint_{Q_{R/4}}\Gamma(h)^2\mathcal{D}(h)^{(2q-p)/2}\,\mathrm{d}z\right)^{p/(2(q-1))}\\ &\quad\times\left(\fint_{Q_{R/4}}\mathcal{D}(h)^{p/2}\,\mathrm{d}z\right)^{(p-2)/(2(q-1))}. \end{aligned}\tag{3.36}$$

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From (3.32), (3.35) and (3.36), it follows

$$\begin{aligned}
&\fint_{Q_{R/4}} |\Delta_h(a(x,t,Du))|^{p/(q-1)}\,\mathrm{d}z \\
&\quad\leqslant c\left(\fint_{Q_{R/4}} |\Delta_h(V_p(Du))|^2\,\mathrm{d}z\right)^{p/(2(q-1))} \\
&\qquad\times\left(\fint_{Q_{R/4}} \mathcal{D}(h)^{p/2}\,\mathrm{d}z\right)^{(p-2+2(q-p))/(2(q-1))} \\
&\qquad+c\left(\fint_{Q_{R/4}} \Gamma(h)^2\mathcal{D}(h)^{(2q-p)/2}\,\mathrm{d}z\right)^{p/(2(q-1))} \\
&\qquad\times\left(\fint_{Q_{R/4}} \mathcal{D}(h)^{p/2}\,\mathrm{d}z\right)^{(p-2)/(2(q-1))}.
\end{aligned}\tag{3.37}$$

By using (3.19), respectively (3.20), we infer

$$\begin{aligned}
&\fint_{Q_{R/4}} |\Delta_h(a(x,t,Du))|^{p/(q-1)}\,\mathrm{d}z \\
&\quad\leqslant c\left(\fint_{Q_{R/4}} |\Delta_h V_p(Du)|^2\,\mathrm{d}z\right)^{p/(2(q-1))} \\
&\qquad\times\left(\fint_{Q_{R/4}} \mathcal{D}(h)^{p/2}\,\mathrm{d}z\right)^{(p-2+2(q-p))/(2(q-1))} \\
&\qquad+c(K)R^{-(n+2)p/(2(q-1))}(\|\Gamma(h)\|^2_{L^\alpha\text{-}L^\beta(Q_{R/4})} \\
&\qquad\times\|\mathcal{D}(h)^{1/2}\|^{2\delta/(\delta-p+2)}_{L^\delta\text{-}L^\sigma(Q_{R/4})})^{p/(2(q-1))} \\
&\qquad\times\left(\fint_{Q_{R/4}} \mathcal{D}(h)^{p/2}\,\mathrm{d}z\right)^{(p-2)/(2(q-1))},
\end{aligned}$$

where α,β are the exponents in (1.4) and δ,σ have been defined in (1.12). Letting $h\to 0$, we deduce

$$\begin{aligned}
&\left(\fint_{Q_{R/4}} |D(a(x,t,Du))|^{p/(q-1)}\,\mathrm{d}z\right)^{2(q-1)/p} \\
&\quad\leqslant c\fint_{Q_{R/4}} |D(V_p(Du))|^2\,\mathrm{d}z\left(\fint_{Q_{R/4}} (\mu^2+|Du|^2)^{p/2}\,\mathrm{d}z\right)^{(p-2+2(q-p))/p}
\end{aligned}$$

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$$+\frac{c(K)}{R^{n+2}}\|\gamma(x,t)\|^2_{L^\alpha\text{-}L^\beta(Q_{R/4})}(R^n+\|Du(x,t)\|^{2\delta/(\delta-p+2)}_{L^\delta\text{-}L^\sigma(Q_{R/4})}$$
$$\times\left(\fint_{Q_{R/4}}(\mu^2+|Du|^2)^{p/2}\,\mathrm{d}z\right)^{(p-2)/p}. \tag{3.38}$$

Assumption (1.13) implies

$$c(K)\|\gamma(x,t)\|^2_{L^\alpha\text{-}L^\beta(Q_{R_0})}\leqslant c(K)\gamma_0^2<1$$

if we diminish the value of γ_0 if necessary. We estimate the right-hand side of the estimate (3.38) by means of (3.30) and (3.31). Since $(p-2+2(q-p))/p > (p-2)/p$ we obtain

$$\left(\fint_{Q_{R/4}}|D(a(x,t,Du))|^{p/(q-1)}\,\mathrm{d}z\right)^{2(q-1)/p}$$
$$\leqslant\frac{c}{R^2}(\fint_{Q_R}(1+|Du|^p)\,\mathrm{d}z)^{1+(q-p)/\theta}\left(\fint_{Q_R}(1+|Du|^p)\,\mathrm{d}z\right)^{(p-2+2(q-p))/p}$$
$$=\frac{c}{R^2}\left(\fint_{Q_R}(1+|Du|^p)\,\mathrm{d}z\right)^{2(q-1)/p+(q-p)/\theta}.$$

From system (1.8) we thereby deduce that

$$R^{\frac{p}{q-1}}\fint_{Q_{R/4}}|\partial_t u|^{p/(q-1)}\,\mathrm{d}z\leqslant c\left(\fint_{Q_R}(1+|Du|^p)\,\mathrm{d}z\right)^{1+p(q-p)/(2(q-1)\theta)}$$

with a constant $c=c(n,N,p,q,\nu,L,\alpha,\beta)$. This establishes the claimed estimate (1.15) and concludes the proof of theorem 1.1.

4. Existence of regular solutions

In this section, we will give the proof of theorem 1.3 by constructing a sequence of approximating problems for which we are allowed to apply the a priori estimate from theorem 1.1. For the passage to the limit we will apply techniques developed in [**5**] for the proof of existence of variational solutions of problems with (p,q)-growth.

Proof of Theorem 1.3. *Regularization.* For some sequence $\varepsilon_k\in(0,1]$ with $\varepsilon_k\downarrow 0$, we define regularized functions a_k as

$$a_k(x,t,\xi):=a(x,t,\xi)+\varepsilon_k|\xi|^{q-2}\xi\quad\text{for }\xi\in\mathbb{R}^{Nn},\ (x,t)\in\Omega_T.$$

As a consequence of the assumptions (1.5), (1.6) and (1.7) on a, we have

$$|a_k(x,t,\xi)|+(\mu^2+|\xi|^2)^{1/2}|D_\xi a_k(x,t,\xi)|\leqslant(L+c(q))(\mu^2+|\xi|^2)^{(q-1)/2}, \tag{4.1}$$
$$\langle D_\xi a_k(x,t,\xi)\bar\xi,\bar\xi\rangle\geqslant\nu(\mu^2+|\xi|^2)^{(p-2)/2}|\bar\xi|^2+c(q)\varepsilon_k|\xi|^{q-2}|\bar\xi|^2, \tag{4.2}$$
$$|a_k(x,t,\xi)-a_k(y,t,\xi)|\leqslant|x-y|[\gamma(x,t)+\gamma(y,t)](\mu^2+|\xi|^2)^{(q-1)/2}, \tag{4.3}$$

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for each $k \in \mathbb{N}$, any $(x,t) \in \Omega_T$ and any $\xi, \bar{\xi} \in \mathbb{R}^{Nn}$. In particular, the regularized functions a_k satisfy standard q-growth and ellipticity conditions. Since, furthermore, the initial and boundary data satisfy (1.9), classical theory [**26**] yields the existence of unique solutions $u_k \in L^q(0,T;W^{1,q}(\Omega,\mathbb{R}^N))$ of the Cauchy–Dirichlet problems

$$\begin{cases} \partial_t u_k - \operatorname{div} a_k(x,t,Du_k) = 0 & \text{in } \Omega_T, \\ u_k = g & \text{on } \partial_{\mathrm{par}}\Omega_T, \end{cases} \tag{4.4}$$

for any $k \in \mathbb{N}$.

Energy bounds. As a direct consequence of the parabolic system in (4.4) and assumption (1.9) on the boundary values, we have

$$\partial_t g,\ \partial_t u_k \in L^{q'}\left(0,T;W^{-1,q'}\left(\Omega,\mathbb{R}^N\right)\right).$$

Hence, for any $t_0 \in (0,T)$ we may use $\varphi = (u_k - g)\chi_{(0,t_0)} \in L^q(0,T;W_0^{1,q}(\Omega,\mathbb{R}^N))$ as test function in (4.4), with the result

$$\begin{aligned} \mathrm{I} + \mathrm{II} &:= \int_0^{t_0} \langle \partial_t(u_k - g), u_k - g\rangle_{W^{-1,q'}}\,\mathrm{d}t \\ &\quad + \int_{\Omega_{t_0}} \langle a_k(x,t,Du_k) - a_k(x,t,Dg), Du_k - Dg\rangle\,\mathrm{d}z \\ &= -\int_0^{t_0} \langle \partial_t g, u_k - g\rangle_{W^{-1,q'}}\,\mathrm{d}t - \int_{\Omega_{t_0}} \langle a_k(x,t,Dg), Du_k - Dg\rangle\,\mathrm{d}z \\ &=: \mathrm{III} + \mathrm{IV}, \end{aligned} \tag{4.5}$$

where $\langle\cdot,\cdot\rangle_{W^{-1,q'}}$ denotes the duality pairing between $W^{-1,q'}(\Omega,\mathbb{R}^N)$ and $W_0^{1,q}(\Omega,\mathbb{R}^N)$. The first integral can be re-written in a standard way (cf. [**33**, proposition III.1.2]) as

$$\begin{aligned} \mathrm{I} &= \int_0^{t_0} \langle \partial_t(u_k - g), u_k - g\rangle_{W^{-1,q'}}\,\mathrm{d}t \\ &= \frac{1}{2}\int_0^{t_0} \partial_t \|u_k - g\|^2_{L^2(\Omega)}\,\mathrm{d}t = \frac{1}{2}\int_{\Omega\times\{t_0\}} |u_k - g|^2\,\mathrm{d}x, \end{aligned} \tag{4.6}$$

where we used the initial condition $u_k = g$ at $t = 0$ according to (4.4). For the estimate of the second integral on the left-hand side of (4.5), we use (4.2) to infer

$$\begin{aligned} \mathrm{II} &= \int_{\Omega_{t_0}} \langle a_k(x,t,Du_k) - a_k(x,t,Dg), Du_k - Dg\rangle\,\mathrm{d}z \\ &\geqslant c(p)\nu \int_{\Omega_{t_0}} (\mu^2 + |Du_k|^2 + |Dg|^2)^{(p-2)/2} |Du_k - Dg|^2\,\mathrm{d}z \\ &\geqslant c(p)\nu \int_{\Omega_{t_0}} |Du_k - Dg|^p\,\mathrm{d}z. \end{aligned} \tag{4.7}$$

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For the estimate of III, we use the fact $\partial_t g \in L^{p'}(0,T;W^{-1,p'}(\Omega,\mathbb{R}^N))$, which is a consequence of (1.9). This allows us to estimate

$$\begin{aligned} \mathrm{III} &\leqslant \int_0^{t_0} \|Du_k - Dg\|_{L^p(\Omega)} \|\partial_t g\|_{W^{-1,p'}(\Omega)} \,\mathrm{d}t \\ &\leqslant \frac{c(p)\nu}{4} \int_{\Omega_{t_0}} |Du_k - Dg|^p \,\mathrm{d}z + c(p,\nu) \|\partial_t g\|^{p'}_{L^{p'}-W^{-1,p'}(\Omega_T)}. \end{aligned} \tag{4.8}$$

Finally, using (4.1) and Young's inequality, we deduce

$$\begin{aligned} \mathrm{IV} &\leqslant c(q,L) \int_{\Omega_{t_0}} (\mu^2 + |Dg|^2)^{(q-1)/2} |Du_k - Dg| \,\mathrm{d}z \\ &\leqslant \frac{c(p)\nu}{4} \int_{\Omega_{t_0}} |Du_k - Dg|^p \,\mathrm{d}z + c(p,q,\nu,L) \int_{\Omega_{t_0}} (\mu + |Dg|)^{p(q-1)/(p-1)} \,\mathrm{d}z. \end{aligned} \tag{4.9}$$

We note that the last integral is finite by (1.9). Plugging (4.6), (4.7), (4.8) and (4.9) into (4.5) and reabsorbing the integral involving the spatial derivatives, we arrive at

$$\begin{aligned} &\frac{1}{2} \int_{\Omega \times \{t_0\}} |u_k - g|^2 \,\mathrm{d}x + \frac{c(p)\nu}{2} \int_{\Omega_{t_0}} |Du_k - Dg|^p \,\mathrm{d}z \\ &\quad \leqslant c(p,\nu) \|\partial_t g\|^{p'}_{L^{p'}-W^{-1,p'}(\Omega_T)} + c(p,q,\nu,L) \int_{\Omega_T} (\mu + |Dg|)^{p(q-1)/(p-1)} \,\mathrm{d}z \end{aligned} \tag{4.10}$$

for every $t_0 \in [0,T]$. Taking the supremum over $t_0 \in [0,T]$, we arrive at the energy bound

$$\sup_{t\in[0,T]} \int_{\Omega\times\{t\}} |u_k|^2 \,\mathrm{d}x + \int_{\Omega_T} |Du_k|^p \,\mathrm{d}z \leqslant c(p,q,\nu,L) M_g, \tag{4.11}$$

for every $k \in \mathbb{N}$, where we used the abbreviation (1.16). Moreover, applying Poincaré's inequality to $(u_k - g)(\cdot,t)$ for a.e. $t \in (0,T)$, we deduce

$$\begin{aligned} \int_{\Omega_T} |u_k|^p \,\mathrm{d}z &\leqslant c(n,p,\mathrm{diam}(\Omega)) \int_{\Omega_T} |Du_k - Dg|^p \,\mathrm{d}z + c(p) \int_{\Omega_T} |g|^p \,\mathrm{d}z \\ &\leqslant c(n,p,q,\nu,L,\mathrm{diam}(\Omega)) M_g + c(p) \int_{\Omega_T} |g|^p \,\mathrm{d}z, \end{aligned} \tag{4.12}$$

independently of $k \in \mathbb{N}$. Combining the energy bounds (4.11) and (4.12) with the Gagliardo–Nirenberg interpolation inequality, we infer

$$\begin{aligned} \int_{\Omega_T} |u_k|^{p(n+2)/n} \,\mathrm{d}z &\leqslant c \left(\sup_{t\in[0,T]} \int_{\Omega\times\{t\}} |u_k|^2 \,\mathrm{d}x \right)^{p/n} \int_{\Omega_T} (|Du_k|^p + |u_k|^p) \,\mathrm{d}z \\ &\leqslant c(n,p,q,\nu,L,\mathrm{diam}(\Omega), M_g, \|g\|_{L^p}), \end{aligned} \tag{4.13}$$

for any $k \in \mathbb{N}$. In view of the preceding energy bounds, after passing to a subsequence we can achieve weak convergence to some limit map $u \in g +$

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$L^p(0,T;W_0^{1,p}(\Omega,\mathbb{R}^N))$ in the sense

$$\begin{cases} Du_k \rightharpoonup Du & \text{in } L^p(\Omega_T,\mathbb{R}^{Nn}), \\ u_k \rightharpoonup u & \text{in } L^{p(n+2)/n}(\Omega_T,\mathbb{R}^N), \end{cases} \tag{4.14}$$

in the limit $k\to\infty$.

Weak continuity in time. Testing the weak formulation of the parabolic system (4.4) with a function $\varphi\in C_0^\infty(\Omega_T)$ and exploiting the growth condition (4.1), we infer

$$\begin{aligned} \left|\int_{\Omega_T} u_k\partial_t\varphi\,\mathrm{d}z\right| &\leqslant c(q,L)\int_{\Omega_T}(1+|Du_k|)^{q-1}|D\varphi|\,\mathrm{d}z \\ &\leqslant c(q,L)|\operatorname{spt}\varphi|^{(p+1-q)/p}\left(\int_{\Omega_T}(1+|Du_k|)^p\,\mathrm{d}z\right)^{(q-1)/p} \\ &\quad\times\|D\varphi\|_{L^\infty(\Omega_T)} \\ &\leqslant c|\operatorname{spt}\varphi|^{(p+1-q)/p}\|\varphi\|_{L^\infty-W^{\ell,2}(\Omega_T)} \end{aligned}$$

for $\ell>(n+2)/2$, by the energy estimate (4.11) and Sobolev's embedding, applied on the time slices $\Omega\times\{t\}$ for $t\in(0,T)$. The constant in the last line depends at most on p,q,ν,L,M_g and $\operatorname{diam}(\Omega)$. Now we choose a testing function of the form $\varphi(x,t)=\chi_\delta(t)\psi(x)$, where $\psi\in C_0^\infty(\Omega)$ and χ_δ are suitable smooth functions that approximate $\chi_{(t_1,t_2)}$ for times $0\leqslant t_1<t_2\leqslant T$. Letting $\delta\downarrow 0$, the preceding estimate implies

$$\left|\int_\Omega(u_k(x,t_2)-u_k(x,t_1))\psi(x)\,\mathrm{d}x\right|\leqslant c|t_2-t_1|^{(p+1-q)/p}\|\psi\|_{W^{\ell,2}(\Omega)}$$

for every $k\in\mathbb{N}$. This implies $u_k\in C^0([0,T];W^{-\ell,2}(\Omega,\mathbb{R}^N))$ with a uniform estimate

$$\|u_k(\cdot,t_2)-u_k(\cdot,t_1)\|_{W^{-\ell,2}(\Omega)}\leqslant c|t_2-t_1|^{(p+1-q)/p}$$

for any $t_1,t_2\in[0,T]$. Since, furthermore, the sequence u_k is bounded in the space $L^\infty(0,T;L^2(\Omega,\mathbb{R}^N))$ by (4.11), a compactness result from [**25**] implies

$$u_k(\cdot,t)\rightharpoonup u(\cdot,t)\quad\text{weakly in } L^2(\Omega,\mathbb{R}^N),\text{ for every } t\in[0,T], \tag{4.15}$$

and the limit map satisfies $u\in C_w([0,T];L^2(\Omega,\mathbb{R}^N))$. A detailed proof of this compactness result can be found in [**5**, theorem A.2].

Application of the a priori estimate. As a consequence of (4.1), (4.2) and (4.3), the function a_k satisfies the q-growth conditions (2.3) to (2.5) of theorem 2.5 with $c(q)\varepsilon_k$ instead of ν and $L+c(q)$ instead of L. Moreover, we infer from (1.17) and (1.4) that we have

$$\frac{q(n+2)-2n}{2\alpha}+\frac{n}{\beta}\leqslant\frac{p(n+2)-2n}{2\alpha}+\frac{n}{\beta}+\frac{\kappa}{\alpha}\leqslant 1-\kappa+\frac{\kappa}{\alpha}<1.$$

Hence, we can find smaller exponents $\alpha_0\in[q,\alpha)$ and $\beta_0\in[n,\beta)$ for which (2.6) holds true. This enables us to apply theorem 2.5 — with these exponents

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α_0 and β_0 — to the solutions u_k of the regularized problems. From the mentioned theorem, we deduce

$$D(V_q(Du_k)) \in L^2_{\mathrm{loc}}\left(\Omega_T, \mathbb{R}^{Nn^2}\right) \quad \text{and} \quad Du_k \in L^\infty_{\mathrm{loc}}(0,T;L^2_{\mathrm{loc}}(\Omega,\mathbb{R}^{Nn})),$$

for any $k \in \mathbb{N}$. However, the corresponding estimates provided by theorem 2.5 are not independent of k. For the exponents δ, σ defined in (1.12), we have

$$\frac{2}{p+4/n} + \frac{q+4/n-2}{p+4/n} \geqslant 1 \quad \text{and} \quad \frac{2}{\sigma} + \frac{q+4/n-2}{\delta} \geqslant 1,$$

so that the interpolation lemma 2.6 with q in place of p and Hölder's inequality imply

$$Du_k \in L^{p+4/n}_{\mathrm{loc}}(\Omega_T) \quad \text{and} \quad Du_k \in L^\delta_{\mathrm{loc}}(0,T;L^\sigma_{\mathrm{loc}}(\Omega,\mathbb{R}^N)).$$

Therefore, assumption (1.11) of the a priori estimate is satisfied. Moreover, the first part of (1.13) is satisfied for some constant $K = K(p,q,\nu,L,M_g)$ as a consequence of the energy bound (4.11). Because of the absolute continuity of the integral and the definition (2.1), we can find a radius $R_0 > 0$, only depending on $n, N, p, q, \nu, L, \alpha, \beta, \gamma(\cdot)$ and K, so that

$$\|\gamma\|_{L^\alpha\text{-}L^\beta(Q_{R_0}(z_0))} \leqslant \gamma_0 \tag{4.16}$$

holds for any cylinder $Q_{R_0}(z_0) \subset \Omega_T$, where $\gamma_0 = \gamma_0(n,N,p,q,\nu,L,\alpha,\beta,K)$ denotes the constant from theorem 1.1. We note that in the case $\alpha = \infty$ and $\beta = n/(1-\kappa)$, we rely on the assumption $\gamma \in C^0([0,T];L^\beta(\Omega))$, which enables us to choose R_0 so small that (4.16) holds true. Since, furthermore, the assumptions (1.5), (1.6), (1.7) are satisfied for a_k and $L + c(q)$ in place of L by (4.1), (4.2) and (4.3), we are in a position to apply the a priori estimates from theorem 1.1 to the sequence u_k. This yields the uniform estimates

$$\begin{aligned} &\sup_{t_0-(R/2)^2<t<t_0} \fint_{B_{R/2}(x_0)} |Du_k(x,t)|^2 \mathrm{d}x + R^2 \fint_{Q_{R/2}(z_0)} |D(V_p(Du_k))|^2 \,\mathrm{d}z \\ &\qquad \leqslant c \left(\fint_{Q_R(z_0)} (1+|Du_k|^p) \,\mathrm{d}z \right)^{1+(q-p)/\theta} \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} &R^{p/(q-1)} \fint_{Q_{R/4}(z_0)} |\partial_t u_k|^{p/(q-1)} \,\mathrm{d}z \\ &\qquad \leqslant c \left(\fint_{Q_R(z_0)} (1+|Du_k|^p) \,\mathrm{d}z \right)^{1+p(q-p)/(2(q-1)\theta)} \end{aligned} \tag{4.18}$$

for every $k \in \mathbb{N}$, provided $R \in (0, R_0)$. In the above estimates, the constant c depends only on $n, N, p, q, \nu, L, \alpha$ and β, and θ is given by $\theta = 1 - (n+2)(q-p)/2$.

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By virtue of (4.17), the interpolation lemma 2.6 yields the integrability estimate

$$\left(\fint_{Q_{R/4}(z_0)} |Du_k|^{p+4/n}\,\mathrm{d}z\right)^{n/(n+2)} \leqslant c\left(\fint_{Q_R(z_0)} (1+|Du_k|^p)\,\mathrm{d}z\right)^{1+(q-p)/\theta}. \tag{4.19}$$

Locally strong convergence. Our next goal is the proof of the strong convergence

$$Du_k \to Du \quad \text{in } L^p(Q_0,\mathbb{R}^N), \text{ as } k\to\infty, \tag{4.20}$$

for any parabolic subcylinder $Q_0 = \mathcal{O}\times(t_1,t_2) \Subset \Omega_T$. This will enable us to pass to the limit in (4.17), which will conclude the proof of the theorem. We choose another subdomain Q_1 with $Q_0 \Subset Q_1 \Subset \Omega_T$ and a non-negative cut-off function $\varphi \in C_0^\infty(Q_1)$ with $\varphi \equiv 1$ on Q_0. First, we note that because of (4.18), (4.19) and $q < p+4/n$, a covering argument yields

$$\sup_{k\in\mathbb{N}}\left(\|\partial_t u_k\|_{L^{\frac{p}{q-1}}(Q_1)} + \|Du_k\|_{L^q(Q_1)}\right) < \infty. \tag{4.21}$$

Consequently, we have weak convergence

$$Du_k \rightharpoondown Du \quad \text{in } L^q(Q_1,\mathbb{R}^{Nn}), \text{ as } k\to\infty. \tag{4.22}$$

and

$$\partial_t u_k \rightharpoondown \partial_t u \quad \text{in } L^{p/(q-1)}(Q_1,\mathbb{R}^{Nn}), \text{ as } k\to\infty. \tag{4.23}$$

Moreover, from Rellich's theorem, we infer

$$u_k \to u \quad \text{in } L^{p/(q-1)}(Q_1), \text{ as } k\to\infty.$$

Since the sequence u_k is bounded in the space $L^{p(n+2)/n}(\Omega_T)$ by (4.13) and $q < p(n+2)/n$, an interpolation argument yields

$$u_k \to u \quad \text{in } L^q(Q_1), \text{ as } k\to\infty. \tag{4.24}$$

Moreover, we deduce from (4.21) that also the weak limit u satisfies

$$\partial_t u \in L^{p/(q-1)}(Q_1,\mathbb{R}^N) \quad \text{and} \quad Du \in L^q(Q_1,\mathbb{R}^{Nn}). \tag{4.25}$$

As a consequence, we have $\varphi(u_k-u) \in L^q(0,T;W_0^{1,q}(\Omega,\mathbb{R}^N))$. Moreover, we note that $\varphi(u_k-u) \in L^{p(n+2)/n}(\Omega_T)$ by (4.13), and that $q-p < 2/(n+2)$ implies $p(n+2)/n > p/(p+1-q) = (p/(q-1))'$. Hence, it is legitimate to use $\varphi(u_k-u)$

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as test function in equation (4.4), which yields

$$
\begin{aligned}
\mathrm{I_k} + \mathrm{II_k} &:= \int_{\Omega_T} \partial_t(u_k - u)\cdot \varphi(u_k - u)\,\mathrm{d}z \\
&\quad + \int_{\Omega_T} \langle a_k(x,t,Du_k) - a_k(x,t,Du), D(\varphi(u_k - u))\rangle\,\mathrm{d}z \\
&= -\int_{\Omega_T} \partial_t u\cdot\varphi(u_k - u)\,\mathrm{d}z - \int_{\Omega_T}\langle a_k(x,t,Du), D(\varphi(u_k - u)\rangle\,\mathrm{d}z \\
&=: \mathrm{III_k} + \mathrm{IV_k}.
\end{aligned} \tag{4.26}
$$

The first integral can be re-written as

$$
\mathrm{I_k} = \frac{1}{2}\int_{\Omega_T}\partial_t|u_k - u|^2\varphi\,\mathrm{d}z = -\frac{1}{2}\int_{\Omega_T}|u_k - u|^2\partial_t\varphi\,\mathrm{d}z \to 0 \tag{4.27}
$$

as $k\to\infty$, because of $\operatorname{spt}\varphi \Subset Q_1 \Subset \Omega_T$ and (4.24). We estimate the second integral in (4.26) by means of (4.1), (4.2) and lemma 2.3. This leads us to

$$
\begin{aligned}
\mathrm{II_k} &\geqslant c(p)\nu\int_{\Omega_T}\varphi(\mu^2 + |Du|^2 + |Du_k|^2)^{(p-2)/2}|Du_k - Du|^2\,\mathrm{d}z \\
&\quad - c(q,L)\int_{\Omega_T}|D\varphi|(\mu^2 + |Du|^2 + |Du_k|^2)^{(q-1)/2}|u_k - u|\,\mathrm{d}z \\
&\geqslant c(p)\nu\int_{\Omega_T}\varphi|Du - Du_k|^p\,\mathrm{d}z - c(q,L) \\
&\quad \times\int_{\Omega_T}|D\varphi|(1 + |Du|^2 + |Du_k|^2)^{(q-1)/2}|u_k - u|\,\mathrm{d}z.
\end{aligned} \tag{4.28}
$$

Because of $\varphi\partial_t u \in L^{p/(q-1)}(\Omega_T)$ and $p(n+2)/n > (p/(q-1))'$, the weak convergence (4.14) implies

$$
\lim_{k\to\infty}\mathrm{III}_k = 0. \tag{4.29}
$$

Finally, the growth condition (4.1) and the fact $a_k(x,t,\xi) - a(x,t,\xi) = \varepsilon_k|\xi|^{q-2}\xi$ yield

$$
\begin{aligned}
\mathrm{IV}_k &= -\int_{\Omega_T}\varphi\langle a(x,t,Du), D(u_k - u)\rangle\,\mathrm{d}z \\
&\quad + \int_{\Omega_T}\varphi\langle a(x,t,Du) - a_k(x,t,Du), D(u_k - u)\rangle\,\mathrm{d}z \\
&\quad - \int_{\Omega_T}\langle a_k(x,t,Du), D\varphi\otimes(u_k - u)\rangle\,\mathrm{d}z
\end{aligned}
$$

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$$\leqslant - \int_{\Omega_T} \varphi\langle a(x,t,Du), D(u_k - u)\rangle \,\mathrm{d}z + \varepsilon_k \int_{\Omega_T} \varphi|Du|^{q-1}\left(|Du_k| + |Du|\right)\mathrm{d}z + c(q,L)\int_{\Omega_T} (\mu^2 + |Du|^2)^{(q-1)/2}|D\varphi|\,|u_k - u|\,\mathrm{d}z, \tag{4.30}$$

which goes to 0 as $k \to \infty$, as a consequence of the weak convergence (4.22), the uniform bound (4.21), and the strong convergence (4.24). Using (4.27), (4.28), (4.29) and (4.30) in (4.26), we arrive at

$$\begin{aligned} c(p)\nu \int_{Q_0} |Du - Du_k|^p \,\mathrm{d}z &\leqslant c(p)\nu \int_{\Omega_T} \varphi|Du - Du_k|^p \,\mathrm{d}z \\ &\leqslant \mathrm{III}_k + \mathrm{IV}_k - \mathrm{I}_k + c(q,L)\int_{\Omega_T} |D\varphi|(1 + |Du| + |Du_k|)^{q-1}|u_k - u|\,\mathrm{d}z \\ &\to 0 \end{aligned} \tag{4.31}$$

in the limit $k \to \infty$, where we used the energy bound (4.21) and (4.24) for the convergence of the last integral. This implies the asserted local strong convergence (4.20), for every $Q_0 \Subset \Omega_T$. In particular, keeping in mind lemma 2.2, we deduce

$$V_p(Du_k) \to V_p(Du) \quad \text{in } L^2(Q_0), \text{ as } k \to \infty,$$

for every subcylinder $Q_0 = \mathcal{O} \times (t_1, t_2) \Subset \Omega_T$. Recalling the local bounds (4.17) and (4.18), we deduce

$$\begin{cases} Du_k \overset{*}{\rightharpoonup} Du & \text{weakly}* \text{ in } L^\infty(t_1, t_2; L^2(\mathcal{O}, \mathbb{R}^N)), \\ D(V_p(Du_k)) \rightharpoonup D(V_p(Du)) & \text{weakly in } L^2(Q_0), \\ \partial_t u_k \rightharpoonup \partial_t u & \text{weakly in } L^{\frac{p}{q-1}}(Q_0), \end{cases}$$

in the limit $k \to \infty$. Using lower semicontinuity of the respective norms with respect to the above convergences, we pass to the limit in the bounds (4.17) and (4.18). This yields the asserted estimates (1.18) and (1.19). Finally, it remains to show that the limit map u is a weak solution to system (1.8). To this end, we fix a testing function $\varphi \in C_0^\infty(\Omega_T, \mathbb{R}^N)$ and use it in the systems (4.4), with the result

$$\int_{\Omega_T} \left(u_k\varphi_t - \langle a(x,t,Du_k), D\varphi\rangle - \varepsilon_k\langle |Du_k|^{q-2}Du_k, D\varphi\rangle\right)\mathrm{d}z = 0$$

for every $k \in \mathbb{N}$. By the strong convergence (4.20) with $Q_0 = \operatorname{spt}\varphi$, the growth condition (1.5), and $q \leqslant p + 1$, we can pass to the limit $k \to \infty$ in the last equation to infer

$$\int_{\Omega_T} \left(u\varphi_t - \langle a(x,t,Du), D\varphi\rangle\right)\mathrm{d}z = 0.$$

This completes the proof of Theorem 1.3. □

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