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COMMENTS ON A PAPER OF R. A. BRUALDI

BY
G. BONGIOVANNI, D. P. BOVET AND A. CERIOLI

ABSTRACT. R. A. Brualdi [1] presents a construction yielding matrices whose Birkhoff representation consists of the maximum number of permutation matrices and having $0(2^{n^2})$ line sum. In this note a counterexample to such a construction is given. Furthermore, a new construction is presented, yielding matrices with lower line sums.

1. **Preliminaries.** An $n \times n$ matrix $D = [d_{ij}]$ is *doubly stochastic* provided $d_{ij} \geqq 0$ $(i, j = 1, \dots, n)$, and the sum of the entries in each row and in each column of D is equal to one.

An $n \times n$ matrix Q is called *quasi doubly stochastic* (qds for short) if each one of its elements is a nonnegative integer and there exist a doubly stochastic matrix D and an integer s, such that $Q = sD$.

The number s is called the *line sum* of Q.

Let Ω_n be the set of $n \times n$ doubly stochastic matrices. It is well known that Ω_n is the convex hull of the $n \times n$ permutation matrices.

A representation of a doubly stochastic matrix D as a convex combination of permutation matrices P_i, obtained by applying the Birkhoff algorithm, is called a *Birkhoff representation* of D (see [1] for the details of the Birkhoff algorithm).

A Birkhoff representation of a doubly stochastic matrix D is said to have *length t* if the number of permutation matrices in it equals t, i.e. if D is written as:

$$D = \sum_{i=1}^{t} \eta_i P_i, \quad \text{with } 0 < \eta_i \leqq 1.$$

The extension of Birkhoff representation to qds matrices is done as follows. Let Q be a qds matrix with line sum s; a Birkhoff representation of Q is a linear combination of permutation matrices P_i obtained by applying the Birkhoff algorithm, such that:

$$Q = \sum_{i=1}^{t} \theta_i P_i, \quad \text{with } \sum_{i=1}^{t} \theta_i = s \text{ and } \theta_i \in \mathbf{N}.$$

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A position (i, j) of a matrix A is *positive* when $a_{ij} > 0$, *null* otherwise.

Let U_A denote the set of positive positions of a matrix A.

An $n \times n$ matrix A has *total support* if, for each positive position (u, v) of A, there exists a permutation matrix P such that $(u, v) \in U_P \subset U_A$.

An $n \times n$ matrix A is *fully indecomposable* if it does not contain any $r \times s$ zero submatrix with $r + s = n$.

It is well known that a fully indecomposable matrix has total support.

A set S of positive positions of a matrix A is *separable* [2] if, for each position $(u, v) \in S$, a permutation matrix P exists such that $U_P \subset U_A$ and $U_P \cap S = \{(u, v)\}$.

A set S of positive positions of a matrix A is *strongly stable* [2] if, for each permutation matrix P such that $U_P \subset U_A$, $U_P \cap S$ contains at most one position of S.

For the sake of brevity, a position (i_m, j_m) will also be referred to as τ_m.

Let $G = [g_{ij}]$ be an $n \times n$ doubly stochastic matrix (qds matrix with line sum s) and let $G_{(0,1)}$ the matrix obtained from G by replacing each positive entry of G with a one.

Let $\sigma(G)$ be the number of positive entries of G, and let $\mathscr{F}(G)$ be the smallest face of Ω_n containing G (containing $(1/s)G$).

Suppose that $G_{(0,1)}$ is fully indecomposable and each Birkhoff representation of G has length not less than $L(G) = \sigma(G) - 2n + 2 = \dim \mathscr{F}(G) + 1$. Then G is said to be *hard*.

2. **A counterexample to Brualdi's construction.** The goal of the construction given by Brualdi [1] for hard qds matrices is to produce, for each face $\mathscr{F}$ of Ω_n, a qds hard matrix Q.

In this section a counterexample to the construction is presented. Namely, a qds matrix A is obtained following Brualdi's rules for which a Birkhoff representation having length less than $\dim \mathscr{F}(A) + 1$ exists. The counterexample consists of a 4×4 matrix with $\sigma(A) = n^2 = 16$, so that $\dim \mathscr{F}(A) = \sigma(A) - 2_n + 1 = 9$, and of a Birkhoff decomposition of A having length $L(A) = 9 < \dim \mathscr{F}(A) + 1$.

The counterexample is illustrated by figures 1 and 2. The matrix of figure 1 is a symbolic representation of the construction utilized to build A. The shaded positions are those of the tree matrix T. The numbers contained in each position identify the permutation matrices for which such a position is a positive position: for example, the positive positions of permutation matrix P_1 are (1, 2), (2, 1), (3, 3), (4, 4). The ordering of the positive positions of $A - T$ is chosen as follows: each position has an index in the ordering which is equal to the minimum number contained in that position.

The matrices of figure 2 represent the successive steps of the Birkhoff algorithm. The first matrix is A, and the k^{th} matrix $(k > 1)$ is obtained from the $(k - 1)^{\text{th}}$ matrix by subtracting the permutation matrix whose positive positions

4 3 1	10 6 2	5	9 8 7
8 7 5 2	9 3 1	6 4	10
10 9	7 5 4	8 2 1	6 3
6	8	10 9 7 3	5 4 2 1

Figure 1 — The construction of the 4 × 4 counterexample.

704	273	32	14
300	642	80	1
3	104	772	144
16	4	139	864

1

703	273	32	14
300	642	80	0
3	104	771	144
16	3	139	864

2

703	273	32	11
300	642	77	0
0	104	771	144
16	0	139	864

3

703	273	32	0
300	642	66	0
0	93	771	144
5	0	139	864

4

703	273	27	0
300	637	66	0
0	93	771	139
0	0	139	864

5

564	273	27	0
300	498	66	0
0	93	771	0
0	0	0	864

6

564	273	0	0
273	498	66	0
0	66	771	0
0	0	0	837

7

564	0	0	0
0	498	66	0
0	66	498	0
0	0	0	564

8

498	0	0	0
0	498	0	0
0	0	498	0
0	0	0	498

9

Figure 2 — The Birkhoff algorithm applied to the counterexample of figure 1.

are the shaded ones in the $(k - 1)^{\text{th}}$ matrix; the coefficient by which the permutation matrix is multiplied is the number contained in the thick-bordered position(s) of the $(k - 1)^{\text{th}}$ matrix.

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3. **A construction for hard qds matrices.** It is now shown how to construct hard matrices whose line sum is of order $0(2^{n \cdot \log(n)})$.

Let J_n be the $n \times n$ matrix having all entries equal to 1, and let P, P' be two permutation matrices such that $U_P \cap U_{P'} = \phi$, and $P + P'$ is fully indecomposable.

Then, $\sigma(P + P') = 2n$ and the set $V = U_{J_n - (P+P')}$ is a separable set of positive positions of J_n.

For each $(i, j) \in V$ let R_{ij} be a permutation matrix such that:

$$E_{ij} \leqq R_{ij} \leqq P + P' + E_{ij}.$$

The matrix:

$$Q = (n - 1)^{n-1}(P + P') + \sum_{(i,j) \in V} (n - 1)^{i-1} R_{ij}$$

is hard, and its line sum is:

$$s \leqq (n + 1)(n - 1)^{n-1} - 1.$$

In fact, let $\sum_{i=1}^{t} \theta_i P_i$ be a Birkhoff representation of the qds matrix Q.

Let σ_i be the number of positive positions of the i^{th} row of Q.

Note that the $\sigma_i - 2$ positive positions of the i^{th} row of Q not belonging to $U_{(P+P')}$ contain an entry with value $(n - 1)^{i-1}$.

Let:

$$q(k) = \sum_{j=1}^{k} (\sigma_j - 2) = \sum_{j=1}^{k} \sigma_j - 2k \quad (k = 1, 2, \ldots, n).$$

Let $t(h)$ be the minimum number such that, for at least one reordering of the matrices $P_1, \ldots, P_t$, the matrix:

$$Q_h = Q - \sum_{i=1}^{t(h)} \theta_i P_i$$

has a zero in each position of the first h rows not belonging to $U_{(P+P')}$.

It is now shown by induction that $t(n) \geqq q(n)$.

This is trivial for $n = 1$: $t(1) \geqq \sigma_1 - 2 = q(1)$, since the positive positions of any line form a strongly stable set.

Suppose that $t(k) \geqq q(k)$. The sum Θ_k of the coefficients of the $t(k)$ permutation matrices is:

$$\Theta_k = \sum_{i=1}^{t(k)} \theta_i = \sum_{j=1}^{k} (\sigma_j - 2)(n - 1)^{j-1} \leqq \sum_{j=1}^{k} (n - 2)(n - 1)^{j-1}$$
$$= (n - 1)^k - 1.$$

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The positive positions of the $(k + 1)^{\text{th}}$ row of Q not belonging to $U_{(P+P')}$ initially contain an entry with value $(n - 1)^k$; thus, all of the n positions of the $(k + 1)^{\text{th}}$ row of Q_k are positive. Consequently,

$$t(k + 1) \geqq t(k) + (\sigma_{k+1} - 2) \geqq \sum_{i=1}^{k+1} \sigma_i - 2(k + 1) = q(k + 1).$$

It follows that:

$$t(n) \geqq q(n) = \sum_{j=1}^{n} \sigma_j - 2n = \sigma(Q) - 2n.$$

Since the positive positions of $P + P'$ initially contain an entry with value $(n - 1)^{n-1}$, at least two more permutation matrices are needed to complete the decomposition.

In conclusion:

$$t \geqq q(n) + 2 = \sigma(Q) - 2n + 2,$$

and the matrix Q is hard.

References

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Dipartimento di Matematica
Universita' degli Studi di Roma "La Sapienza"
Piazzale Aldo Moro 5
1-00185 Roma – Italy

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