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On the identification of a single body immersed in a Navier-Stokes fluid

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In this work we consider the inverse problem of the identification of a single rigid body immersed in a fluid governed by the stationary Navier-Stokes equations. It is assumed that friction forces are known on a part of the outer boundary. We first prove a uniqueness result. Then, we establish a formula for the observed friction forces, at first order, in terms of the deformation of the rigid body. In some particular situations, this provides a strategy that could be used to compute approximations to the solution of the inverse problem. In the proofs we use unique continuation and regularity results for the Navier-Stokes equations and domain variation techniques.

1 Introduction and main results

Let $\Omega \subset \mathbb{R}^N$ be a bounded connected open set ($N = 2$ or $N = 3$) whose boundary $\partial\Omega$ is of class $W^{2,\infty}$. Let γ be a nonempty open subset of $\partial\Omega$ and let us denote by 1_γ the characteristic function of γ.

We will consider the following family of subsets of Ω, where D^* is a fixed nonempty set:

$$\mathscr{D} = \{\, D \subset \Omega : D \text{ is a simply connected open set, } \partial D \text{ is of class } W^{2,\infty},\ D \subset\subset D^* \subset\subset \Omega \,\}.$$

In this paper we will deal with the following inverse problem:

Given φ and α in appropriate spaces, find a set $D \in \mathscr{D}$ such that a solution (u, p) of the Navier-Stokes problem

$$\begin{cases} -\nu\Delta u + (u\cdot\nabla)u + \nabla p = 0, \quad \nabla\cdot u = 0 & \text{in } \Omega\setminus\overline{D},\\ u = \varphi & \text{on } \partial\Omega,\\ u = 0 & \text{on } \partial D,\end{cases} \tag{1.1}$$

satisfies the additional condition

$$\sigma(u,p)\cdot n \equiv (-p\, Id + 2\nu\, e(u))\cdot n = \alpha \quad \text{on} \quad \gamma. \tag{1.2}$$

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In (1.2), Id is the identity matrix, $\nu > 0$ is a constant (the kinematic viscosity of the fluid) and $e(u)$ is the linear strain tensor, given by

$$e(u) = \frac{1}{2}(\nabla u + {}^t\nabla u).$$

The interpretation of problem (1.1)–(1.2) is the following. We assume that a stationary Newtonian viscous fluid fills an unknown domain $\Omega \setminus \overline{D}$ at rest. The velocity φ on the outer boundary $\partial\Omega$ is given and we are able to measure on a part of $\partial\Omega$ the *normal stresses* $\alpha = \sigma(u,p) \cdot n$, i.e. the force exerted by the fluid. Then the question is whether we can determine D from Ω, φ and α. From the practical viewpoint, we try to compute the shape of a body around which a real fluid flows from measurements performed far from the body.

A related problem was considered in Alvarez *et al.* [6]. Another similar but more simple problem has been analyzed in Kavian [19]. There, instead of (1.1), one has

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \setminus \overline{D}, \\ u = \varphi & \text{on } \partial\Omega, \\ u = 0 & \text{on } \partial D, \end{cases}$$

and the role of the additional information (1.2) is replaced by

$$\frac{\partial u}{\partial n} = \alpha \quad \text{on} \quad \gamma.$$

Other problems of this kind have been studied by several other authors [1, 2, 3, 4, 5, 7, 10, 11, 13, 20, 21].

Concerning the direct problem associated to (1.1), i.e. the determination of (u,p) (and then α) from Ω, D and φ, we have the following result.

Theorem 1.1 *Assume that $D \in \mathscr{D}$ and $\varphi \in C^1(\partial\Omega)^N$ satisfies*

$$\int_{\partial\Omega} \varphi \cdot n \, d\Gamma = 0. \tag{1.3}$$

- *For each $\nu > 0$, (1.1) possesses at least one solution (u,p) that belongs to $H^1(\Omega \setminus \overline{D})^N \times L^2(\Omega \setminus \overline{D})$ and satisfies*

$$||u||_{H^1(\Omega\setminus\overline{D})} \leqslant \frac{C}{\nu}(\nu + 1)||\varphi||_{C^1(\partial\Omega)}, \tag{1.4}$$

  where C only depends on Ω and D^*.
- *There exists $\nu_1 = \nu_1(\Omega, D^*, \|\varphi\|_{C^1}) > 0$ such that, for $\nu > \nu_1$, the solution of (1.1) is unique (p is unique up to a constant) and belongs to $W^{1,r}(\Omega\setminus\overline{D})^N \times L^r(\Omega\setminus\overline{D})$ for all $r \in [1,+\infty)$.*
- *Furthermore, the solutions of (1.1) satisfy $\sigma(u,p)\cdot n \in W^{-1/r,r}(\partial\Omega)$ for all finite r.*

This result is essentially well known, with a number of classical references for results of this kind are [14, 22, 23, 28]. However, for completeness we will sketch the proof in Appendix A.

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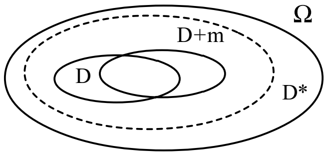

FIGURE 1. Deformations of D.

In the sequel, we will always assume that $\varphi \in C^1(\partial\Omega)^N$ satisfies (1.3) and $\nu > \nu_1$. Accordingly, for each $D \in \mathscr{D}$, we can speak of the unique solution (u, p) of (1.1).

In the context of the inverse problem (1.1)–(1.2), the first property we will analyze is *uniqueness*. Thus, let D^1 and D^2 be two sets in $\mathscr{D}$ and let us consider the Navier-Stokes system

$$\begin{cases} -\nu\Delta u^i + (u^i \cdot \nabla)u^i + \nabla p^i = 0, \quad \nabla \cdot u^i = 0 & \text{in } \Omega \setminus \overline{D^i}, \\ u^i = \varphi & \text{on } \partial\Omega, \\ u^i = 0 & \text{on } \partial D^i, \end{cases} \tag{1.5}$$

for $i = 1$ and $i = 2$. We have the following uniqueness result:

Theorem 1.2 *Assume that* $\varphi \in C^1(\partial\Omega)^N$ *satisfies* (1.3), φ *does not vanish identically and* $\nu > \nu_1$. *Let* D^1 *and* D^2 *be two sets in* $\mathscr{D}$, *let* (u^i, p^i) *be the solution of* (1.5) *and let us set* $\alpha^i = \sigma(u^i, p^i) \cdot n$ *for* $i = 1, 2$. *Assume that*

$$\alpha^1 = \alpha^2 \quad on \quad \gamma. \tag{1.6}$$

Then $D^1 = D^2$.

For the proof of this result, we will adapt an argument that can be found for instance in Andrieux *et al.* [7] and Canuto & Kavian [12]. To this end, an appropriate unique continuation property for Stokes-like systems will be required. Notice that the unique continuation property we need is *local* in the sense that we do not know the behaviour of the solution on the whole boundary; see more details in § 2.

We shall also be concerned by the way $\sigma \cdot n$ depends on (small) perturbations of D and some related consequences. In order to represent the deformations of a set $D \in \mathscr{D}$, let us introduce

$$\mathscr{W}_\varepsilon = \{\, m \in W^{2,\infty}(\mathbb{R}^N; \mathbb{R}^N) \;:\; \|m\|_{W^{2,\infty}} \leqslant \varepsilon, \quad m = 0 \;\text{ in }\; \Omega \setminus D^* \,\},$$

where $\varepsilon > 0$ is small enough. For each $m \in \mathscr{W}_\varepsilon$, we define a new domain $D + m$ (see Figure 1) by

$$D + m = \{\, z \in \mathbb{R}^N \;:\; z = x + m(x),\; x \in D \,\}.$$

It is then known that, if ε is small enough, for any $D \in \mathscr{D}$ and any $m \in \mathscr{W}_\varepsilon$, one has again $D + m \in \mathscr{D}$ [27].

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For each $m \in \mathscr{W}_\varepsilon$, let us consider the "perturbed" Navier-Stokes system

$$\begin{cases} -\nu\Delta u(m) + (u(m)\cdot\nabla)u(m) + \nabla p(m) = 0 & \text{in } \Omega\setminus(\overline{D+m}), \\ \nabla\cdot u(m) = 0 & \text{in } \Omega\setminus(\overline{D+m}), \\ u(m) = \varphi & \text{on } \partial\Omega, \\ u(m) = 0 & \text{on } \partial(D+m). \end{cases} \tag{1.7}$$

Thanks to Theorem 1.1, there exists exactly one solution $(u(m), p(m))$ of (1.7) that belongs to $W^{1,r}(\Omega\setminus(\overline{D+m}))^N \times L^r(\Omega\setminus(\overline{D+m}))$ and satisfies $\sigma(u(m), p(m))\cdot n \in W^{-1/r,r}(\partial\Omega)$ for all finite r.

Our aim is to deduce an identity of the form

$$\sigma(u(m), p(m))\cdot n - \sigma(u(0), p(0))\cdot n = Lm + o(m) \quad \text{on} \quad \gamma,$$

where L is a linear operator and

$$o(m)\|m\|^{-1}_{W^{2,\infty}} \to 0 \quad \text{as} \quad \|m\|_{W^{2,\infty}} \to 0. \tag{1.8}$$

In the sequel, for simplicity of the notation, the couple $(u(0), p(0))$ will be simply denoted by (u, p).

We have the following result:

Theorem 1.3 *Assume that $\varphi \in C^1(\partial\Omega)^N$ satisfies* (1.3)*, φ does not vanish identically and $\nu > \nu_1$. Assume that $D \in \mathscr{D}$ and $m \in \mathscr{W}_\varepsilon$ and let $(u(m), p(m))$ and (u, p) be the solutions of* (1.7) *and* (1.1)*, respectively. Then we have*

$$\sigma(u(m), p(m))\cdot n - \sigma(u, p)\cdot n = \sigma(u'(m), p'(m))\cdot n + o(m) \quad \text{on} \quad \gamma, \tag{1.9}$$

where $o(m)$ satisfies (1.8) *and $(u'(m), p'(m))$ is the solution of the linear problem*

$$\begin{cases} -\nu\Delta u'(m) + (u'(m)\cdot\nabla)u + (u\cdot\nabla)u'(m) + \nabla p'(m) = 0, & \text{in } \Omega\setminus\overline{D}, \\ \nabla\cdot u'(m) = 0, & \text{in } \Omega\setminus\overline{D}, \\ u'(m) + (m\cdot\nabla)u \in H^1_0(\Omega\setminus\overline{D})^N. \end{cases} \tag{1.10}$$

If, furthermore, φ is regular enough, for any $\overline{\psi} \in C^2(\overline{\gamma})^N$ satisfying

$$\int_\gamma \overline{\psi}\cdot n\, d\Gamma = 0, \tag{1.11}$$

we also have

$$\begin{cases} \langle \sigma(u(m), p(m))\cdot n - \sigma(u, p)\cdot n, \overline{\psi}\, 1_\gamma\rangle_{\partial\Omega} \\ \quad = -\nu \displaystyle\int_{\partial D} (m\cdot n)\frac{\partial u}{\partial n}\cdot\frac{\partial \psi}{\partial n}\, d\Gamma + o(m), \end{cases} \tag{1.12}$$

where $\langle\cdot,\cdot\rangle_{\partial\Omega}$ denotes the duality product in $W^{-1/r,r}(\partial\Omega)^N \times W^{1/r,r'}(\partial\Omega)^N$ and (ψ, π) is the solution of the adjoint system

$$\begin{cases} -\nu\Delta\psi - (\nabla u)^t\psi - (u\cdot\nabla)\psi + \nabla\pi = 0, \quad \nabla\cdot\psi = 0 & \text{in } \Omega\setminus\overline{D}, \\ \psi = \overline{\psi}\, 1_\Gamma & \text{on } \partial\Omega, \\ \psi = 0 & \text{on } \partial D. \end{cases} \tag{1.13}$$

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The first part of this result, i.e. the identity (1.9), was proved in Bello *et al.* [9] (see Theorem 5 therein). We will present the proof of the second part in § 5. Our main tools will be domain variation techniques [9, 25, 27] and Green's formula.

Notice that the boundary conditions on $u'(m)$ are implicitly given in (1.10) by imposing $u'(m) + (m \cdot \nabla)u$ to belong to $H_0^1(\Omega \setminus \overline{D})^N$.

An immediate consequence of Theorem 1.3 is the following:

Corollary 1.4 *Let the assumptions of Theorem 1.3 hold and suppose that φ is regular enough and $m = \lambda n + m'$ on ∂D, where $\lambda \in \mathbb{R}$ and $(m', n) = 0$. Then, if $\overline{\psi}$ satisfies* (1.11) *and*

$$\int_{\partial D} \frac{\partial u}{\partial n} \cdot \frac{\partial \psi}{\partial n} \, d\Gamma \neq 0,$$

we have

$$\lambda = -\frac{\langle \sigma(u(m), p(m)) \cdot n - \sigma(u, p) \cdot n, \, \overline{\psi} \, 1_\gamma \rangle}{\nu \displaystyle\int_{\partial D} \frac{\partial u}{\partial n} \cdot \frac{\partial \psi}{\partial n} \, d\Gamma} + o(m).$$

Remark 1.5 Notice that, in view of (1.9)–(1.10), for each $m \in \mathscr{W}_\varepsilon$ we can compute the local derivative $(u'(m), p'(m))$ and thus the difference $\sigma(u(m), p(m)) \cdot n - \sigma(u, p) \cdot n$ on γ up to second-order perturbations. On the other hand, we see from (1.12) that the same quantity can be easily computed using (ψ, π), which is independent of m.

Remark 1.6 Assume that φ is regular enough and we have already computed a first regular approximation $\tilde{D}$ to the solution of our inverse problem. Then, the associated solution $(\tilde{u}, \tilde{p})$ and consequently $\tilde{\alpha}|_\gamma \equiv \sigma(\tilde{u}, \tilde{p}) \cdot n|_\gamma$ are known. Our goal now is to compute a new (and possibly better) approximation of the form $\tilde{D} + m \equiv \tilde{D} + \lambda n + m'$, where $(m', n) = 0$ and $\lambda \in \mathbb{R}$. From (1.12), for each $\overline{\psi}$ as in Corollary 1.4, we can write

$$\langle \sigma(u(m), p(m)) \cdot n - \tilde{\alpha}, \overline{\psi} \, 1_\gamma \rangle = -\nu \lambda \int_{\partial D} \frac{\partial \tilde{u}}{\partial n} \cdot \frac{\partial \psi}{\partial n} \, ds + o(\lambda),$$

where (ψ, π) is the solution of (1.13). So, the "good" strategy is to choose λ, if possible, according to the formula

$$\lambda = -\frac{\langle \alpha - \tilde{\alpha}, \overline{\psi} \, 1_\gamma \rangle}{\nu \displaystyle\int_{\partial D} \frac{\partial \tilde{u}}{\partial n} \cdot \frac{\partial \psi}{\partial n} \, ds}. \tag{1.14}$$

Indeed, this is a way to ensure that, the projections of $\sigma(u(m), p(m)) \cdot n|_\gamma$ and $\alpha|_\gamma$ in the direction of $\overline{\psi}$ coincide, at least at first order.

Remark 1.7 More generally, starting from an already computed candidate $\tilde{D}$ to the solution of problem (1.1)–(1.2), let us try to determine a better candidate of the form $\tilde{D} + m$, where $m \cdot n|_{\partial \tilde{D}} \in M$ and M is a finite dimensional space. Let $\{f_1, \ldots, f_d\}$ be a basis of M. Then we can write

$$m \cdot n|_{\partial \tilde{D}} = \sum_{i=1}^{d} a_i f_i \, ,$$

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for some a_i to be determined. Let us introduce d linearly independent functions $\overline{\psi}_i \in C^2(\overline{\gamma})^N$ satisfying (1.11). Using again (1.12), we see now that

$$\langle \sigma(u(m), p(m)) \cdot n - \tilde{\alpha}, \overline{\psi}_j 1_\gamma \rangle = -v \sum_{i=1}^{d} a_i \int_{\partial D} f_i \frac{\partial \tilde{u}}{\partial n} \cdot \frac{\partial \psi_j}{\partial n} \, ds + o(m). \tag{1.15}$$

Consequently, a strategy to compute the coefficients a_i is to solve (if possible) the system of equations

$$\begin{cases} \displaystyle\sum_{i=1}^{d} \left(\int_{\partial D} f_i \frac{\partial \tilde{u}}{\partial n} \cdot \frac{\partial \psi_j}{\partial n} \, ds \right) a_i = -\frac{1}{v} \langle \alpha - \tilde{\alpha}, \overline{\psi}_j 1_\gamma \rangle, \\ 1 \leqslant j \leqslant d, \end{cases}$$

where, for each j, (ψ_j, π_j) is the solution of (1.13) corresponding to $\overline{\psi}_j$. A more detailed analysis of the performance of this method and its application to the numerical solution of the inverse problem (1.1)–(1.2) is under study and will appear in the near future.

The rest of this paper is organized as follows. In § 2, we will give the formulation of a unique continuation property and we will deduce some consequences, needed for the proof of Theorem 1.2. Theorems 1.2 is proved in § 3. In § 4, we present some comments on other related (but different) inverse problems. § 1.3 is devoted to the proof of Theorem 5. Finally, Appendix A deals with the proof of Theorem 1.1 (as well as other technical results) and in Appendix B we give a sketch of the proof of the unique continuation property we have mentioned above.

2 A unique continuation property

In this section, we will present a unique continuation property which will be used in the proof of Theorem 1.2. Let $G \subset \mathbb{R}^N$ be a bounded connected open set ($N = 2$ or $N = 3$) whose boundary ∂G is of class $W^{1,\infty}$. Then we have the following result:

Lemma 2.1 *Let $\mathcal{O} \subset G$ be a nonempty open set. Assume that $a \in L^\infty(G)^N$, $b \in L^\infty(G)^N$ and $\nabla \cdot a = \nabla \cdot b = 0$ in G. Then any solution $(y, q) \in H^1_{\text{loc}}(G)^N \times L^2_{\text{loc}}(G)$ of the linear system*

$$\begin{cases} -v\Delta y + (a \cdot \nabla) y + (y \cdot \nabla) b + \nabla q = 0 & \text{in } G, \\ \nabla \cdot y = 0 & \text{in } G, \end{cases} \tag{2.1}$$

satisfying $y = 0$ in $\mathcal{O}$ is zero everywhere, i.e. satisfies $y \equiv 0$ in G and $q \equiv$ Const. in G.

The proof of this lemma is similar, but not identical, to the proof of Proposition 1.1 in Fabre & Lebeau [16] (where $b \equiv 0$, i.e. the authors do not include terms of the form $(v \cdot \nabla)b$). For clarity, we give a sketch of the proof of Lemma 2.1 in Appendix B.

As a consequence of Lemma 2.1, we obtain the following result:

Corollary 2.2 *Let $\Gamma \subset \partial G$ be a nonempty open set. Assume that $a \in L^\infty(G)^N$, $b \in L^\infty(G)^N$ and $\nabla \cdot a = \nabla \cdot b = 0$ in G. Then any solution $(y, q) \in H^1(G)^N \times L^2(G)$ of (2.1) that satisfies $y = 0$ on ∂G and $\sigma(y, q) \cdot n = 0$ on Γ is zero everywhere.*

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Proof Let us fix a point $x_0 \in \Gamma$ and a number $r > 0$ such that

$$\overline{B}(x_0;r) \cap \partial G \subset \Gamma .$$

Here, $B(x_0;r)$ (resp. $\overline{B}(x_0;r)$) stands for the open (resp. closed) ball centered at x_0 of radius r. Then we have

$$\sigma(y,q)\cdot n = 0 \quad \text{on} \quad \overline{B}(x_0;r) \cap \partial G$$

and

$$y = 0 \quad \text{on} \quad \overline{B}(x_0;r) \cap \partial G.$$

Let us set

$$G' = G \cup B(x_0;r).$$

Then we can define the couple $(\tilde{y},\tilde{q}) \in H^1(G') \times L^2(G')$ by extending by zero (y,q) to the whole set G', i.e. by setting

$$(\tilde{y},\tilde{q})(x) = \begin{cases} (y,q), & \text{in} \ \ G, \\ (0,0), & \text{in} \ \ B(x_0;r) \cap G^c. \end{cases}$$

In this way, we obtain a solution $(\tilde{y},\tilde{q})$ of (2.1) in G' which vanishes in $B(x_0;r)\cap G^c \subset G'$. By applying Lemma 2.1, we deduce that $\tilde{y} = 0$ in G' and $\tilde{q} \equiv$ Const. in G'. In particular, we obtain that y vanishes in G. □

3 Proof of Theorem 1.2

Let D^1 and D^2 be two different open sets in $\mathscr{D}$, let (u^i,p^i) be the solution of the system

$$\begin{cases} -\nu\Delta u^i + (u^i\cdot\nabla)u^i + \nabla p^i = 0, \quad \nabla\cdot u^i = 0 & \text{in } \Omega \setminus \overline{D^i}, \\ u^i = \varphi & \text{on } \partial\Omega, \\ u^i = 0 & \text{on } \partial D^i \end{cases} \tag{3.1}$$

and let us set $\alpha^i = \sigma(u^i,p^i)\cdot n$ for $i = 1,2$.

Assume that (1.6) holds. Let us consider the open sets $D^1 \cup D^2 \in \mathscr{D}$ and $\mathscr{O}^0 = \Omega \setminus \overline{D^1 \cup D^2}$. Let $\mathscr{O}$ be the unique connected component of $\mathscr{O}^0$ whose boundary contains $\partial\Omega$ (recall that D^1 and D^2 are subset of D^*) and let us introduce

$$v = u^1 - u^2 \quad \text{and} \quad \pi = p^1 - p^2.$$

Then $(v,\pi) \in H^1(\mathscr{O})^N \times L^2(\mathscr{O})$ and verifies

$$\begin{cases} -\nu\Delta v + (u^1\cdot\nabla)v + (v\cdot\nabla)u^2 + \nabla\pi = 0, \quad \nabla\cdot v = 0 & \text{in } \mathscr{O}, \\ v = 0 & \text{on } \partial\Omega, \\ \sigma(v,\pi)\cdot n = 0 & \text{on } \gamma. \end{cases}$$

We now apply the unique continuation result of Corollary 2.2 and we deduce that

$$v = 0 \quad \text{in} \quad \mathscr{O},$$

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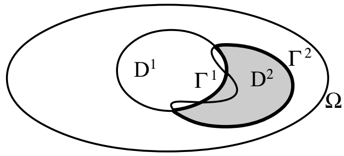

FIGURE 2. Shaded is $D^3 \setminus \overline{D^1}$.

that is to say,

$$u^1 = u^2 \quad \text{in} \quad \mathcal{O}. \tag{3.2}$$

For instance, let us assume that $D^2 \setminus \overline{D^1}$ is nonempty and let us put $D^3 = D^2 \cup ((\Omega \setminus \overline{D^1}) \cap (\Omega \setminus \overline{\mathcal{O}}))$. By hypothesis, $D^3 \setminus \overline{D^1}$ is nonempty. Moreover, $\partial(D^3 \setminus \overline{D^1}) = \Gamma^1 \cup \Gamma^2$, where $\Gamma^1 = \partial(D^3 \setminus \overline{D^1}) \cap \partial D^1$ and $\Gamma^2 = \partial(D^3 \setminus \overline{D^1}) \cap \partial D^2$ (see Figure 2).

In view of (3.1) and (3.2), the couple (u^1, p^1) satisfies

$$\begin{cases} -\nu\Delta u^1 + (u^1 \cdot \nabla)u^1 + \nabla p^1 = 0, \quad \nabla \cdot u^1 = 0 & \text{in } D^3 \setminus \overline{D^1}, \\ u^1 = u^2 = 0 & \text{on } \Gamma^2, \\ u^1 = 0 & \text{on } \Gamma^1. \end{cases}$$

Of course, this implies $u^1 = 0$ in $D^3 \setminus \overline{D^1}$. Consequently, from Lemma 2.1 we deduce that $u^1 = 0$ in $\Omega \setminus \overline{D^1}$, which is impossible because $u^1 = \varphi$ on $\partial\Omega$ and φ is not identically zero. This implies that $D^2 \setminus \overline{D^1}$ is the empty set.

We can prove in the same way that the set $D^1 \setminus \overline{D^2}$ is empty. Therefore, $D^1 = D^2$. □

4 Some comments on other inverse problems for Stokes systems

In this section we shall consider other interesting related inverse problems. For them, it will be seen that the previous uniqueness result is more difficult to establish. First, let us simply change the information (1.2) by

$$\nabla p = \beta \quad \text{on} \quad \gamma, \tag{4.1}$$

where β is given. It is maybe difficult in practice to get an observation like (4.1) for the real flow of a fluid. However, it will be seen below that the related inverse problem is meaningful.

The new inverse problem is the following:

Given φ and β in appropriate spaces, find a set $D \in \mathcal{D}$ such that a solution (u, p) of the Navier-Stokes problem (1.1) *satisfies the additional condition* (4.1).

This is more complicated then (1.1)–(1.2). To clarify this claim, let us discuss uniqueness in the context of the similar but simpler linear Stokes system

$$\begin{cases} -\nu\Delta u + \nabla p = 0, \quad \nabla \cdot u = 0 & \text{in } \Omega \setminus \overline{D}, \\ u = \varphi & \text{on } \partial\Omega, \\ u = 0 & \text{on } \partial D, \end{cases} \tag{4.2}$$

together with the additional information (4.1). We have the following:

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Proposition 4.1 *Assume that $N = 2$. If $\partial\Omega$ contains a non-empty open segment γ', then we have uniqueness for the inverse problem* (4.2), (4.1).

Proof First, notice that it can be assumed without loss of generality that γ' is a vertical segment. Otherwise, it suffices to perform the change of variables

$$x' = Rx + b, \quad u'(x') = Ru(R^T(x' - b)), \quad p'(x') = p(R^T(x' - b)),$$

where R is an appropriate rotation matrix and b is an appropriate point in $\mathbb{R}^2$.

Let us argue as in the proof of Theorem 1.2. Thus, let D^1 and D^2 be two different open sets in $\mathscr{D}$, let (u^i, p^i) be the solution of the system

$$\begin{cases} -\nu\Delta u^i + \nabla p^i = 0, \quad \nabla \cdot u^i = 0 & \text{in } \Omega \setminus \overline{D^i}, \\ u^i = \varphi & \text{on } \partial\Omega, \\ u^i = 0 & \text{on } \partial D^i \end{cases} \tag{4.3}$$

and let us assume that

$$\nabla p^1 = \nabla p^2 \quad \text{on} \quad \gamma.$$

Let us introduce $u = u^1 - u^2$ and $p = p^1 - p^2$. Let us set $G^0 = \Omega \setminus \overline{D^1 \cup D^2}$ and let G be the unique connected component of G^0 whose boundary contains $\partial\Omega$. We have

$$\begin{cases} -\nu\Delta u + \nabla p = 0, \quad \nabla \cdot u = 0 & \text{in } G, \\ u = 0 & \text{on } \partial\Omega, \\ \nabla p = 0 & \text{on } \gamma. \end{cases} \tag{4.4}$$

Applying the divergence operator to the first equation in (4.4) and taking into account that $\nabla \cdot u = 0$ in G, we see that $\Delta p = 0$ in G and $\nabla p = 0$ on γ. Therefore, from the well known unique continuation property of the Laplace operator, we deduce that p is a constant in G.

Thus, $u = u^1 - u^2$ satisfies:

$$\begin{cases} \Delta u = 0, \quad \nabla \cdot u = 0 & \text{in } G, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.5}$$

Let G' be a simply connected neighbourhood of γ'. Since $\nabla \cdot u = 0$ in $G \cap G'$, there exists a function $\psi \in H^2(G \cap G')$ such that u is the curl of ψ, i.e.

$$u = \nabla \times \psi = (\partial_2\psi, -\partial_1\psi).$$

Thanks to (4.5), the following holds:

$$\begin{cases} \Delta(\nabla \times \psi) = 0 & \text{in } G \cap G', \\ \partial_1\psi = \partial_2\psi = 0 & \text{on } \gamma'. \end{cases} \tag{4.6}$$

In particular, $\Delta(\partial_2\psi) = 0$ in $G \cap G'$, $\partial_2\psi = 0$ on γ' and $\partial_1(\partial_2\psi) = \partial_2(\partial_1\psi) = 0$ on γ'. Therefore, in view of the unique continuation property, we deduce that $\partial_2\psi = 0$ in $G \cap G'$, so that we have $u = (0, u_2(x_1))$ in $G \cap G'$.

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But this implies that $u \equiv 0$ in G. Indeed, from unique continuation applied to the functions u_1 and $\partial_2 u_2$, we first deduce that u is of the form $u = (0, u_2(x_1))$ in G. Since $u = 0$ on $\partial\Omega$, there exists a nonempty open set where $u = 0$. Consequently, $u \equiv 0$ in G.

Now, arguing as in the last part of the proof of Theorem 1.2, it is easy to conclude that $D^1 = D^2$. □

From the previous proof, we see that uniqueness holds for the inverse problem (4.2), (4.1) if Ω and any couple of sets D^1 and D^2 satisfy the following unique continuation property: if $u \in H^1(G)^2$ and (4.5) holds, then we necessarily have $u \equiv 0$ in G.

Since this is true whenever $\partial\Omega$ contains a nonempty open segment, we have the following generic result:

Corollary 4.2 *Let $\Omega \subset \mathbb{R}^2$ be a nonempty bounded open set. For any $\varepsilon > 0$ there exists another open set Ω_ε with $\Omega_\varepsilon \supset \Omega$ and $|\Omega_\varepsilon \setminus \Omega| \leqslant \varepsilon$, such that the inverse problem* (4.2), (4.1) *in Ω_ε satisfies uniqueness.*

However, there exist open sets Ω and sets D^1 and D^2 such that (4.5) does not imply $u \equiv 0$ in G. For instance, this happens when Ω is a ball. Indeed, let us assume that Ω is a ball of radius R_0 and let us construct a non-trivial function $\psi = \tilde{\psi}(r)$ such that the corresponding

$$u = \nabla \times \psi = \frac{\tilde{\psi}'(r)}{r}(x_2, -x_1)$$

satisfies (4.5). Taking into account that $\tilde{\psi}(r)$ is axially symmetric, we easily see that it suffices to find nonconstant solutions of

$$\begin{cases} \tilde{\psi}''' + \dfrac{\tilde{\psi}''}{r} - \dfrac{\tilde{\psi}'}{r^2} = 0 \quad \text{in } (0, R_0), \\ \tilde{\psi}'(R_0) = 0. \end{cases} \tag{4.7}$$

Thus, we can take for instance

$$\tilde{\psi}(r) = a\left(\frac{r^2}{2} - R_0^2 \log r\right),$$

where $a \in \mathbb{R}$, $a \neq 0$.

When Ω is a ball, the uniqueness of (4.2), (4.1) is, to our knowledge, an open question. Thus, we see that even for $N = 2$ only partial results are known concerning the uniqueness of this inverse problem.

When $N = 3$, the situation is much more interesting and, obviously, less understood. Of course, for the Navier-Stokes system (1.1) together with (4.1), the uniqueness of the associated inverse problem remains open (for $N = 2$ and $N = 3$). Apparently, this is a nontrivial and rather difficult question.

Let us now present some ideas concerning similar evolution problems. Let $T > 0$ be given and let us consider the following inverse problem:

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Given the nonzero functions φ, u^0 and α in appropriate spaces, find a set $D \in \mathscr{D}$ such that the solution (u, p) of

$$\begin{cases} u_t - \nu\Delta u + \nabla p = 0, \quad \nabla \cdot u = 0 & \text{in } (\Omega \setminus \overline{D}) \times (0, T), \\ u = \varphi & \text{on } \partial\Omega \times (0, T), \\ u = 0 & \text{on } \partial D \times (0, T), \\ u(x, 0) = u^0(x) & \text{in } \Omega \setminus \overline{D}, \end{cases} \tag{4.8}$$

satisfies the additional condition

$$\sigma(u, p) \cdot n = \alpha \quad \text{on} \quad \gamma \times (0, T). \tag{4.9}$$

We can try to follow the arguments in the proof of Theorem 1.2 in order to get uniqueness for (4.8), (4.9). Thus, let D^1, D^2 be two different open sets in $\mathscr{D}$ and let (u^i, p^i) be a solution of (4.8) associated to $D = D^i$ for $i = 1, 2$. Let us assume that

$$\sigma(u^1, p^1) \cdot n = \sigma(u^2, p^2) \cdot n = \alpha \quad \text{on} \quad \gamma \times (0, T).$$

Let us set $u = u^1 - u^2$ and $p = p^1 - p^2$. Then, again setting $G_0 = \Omega \setminus \overline{D^1 \cup D^2}$ and G the unique connected component of G_0 whose boundary contains $\partial\Omega$, we have that the couple (u, p) satisfies

$$\begin{cases} u_t - \nu\Delta u + \nabla p = 0, \quad \nabla \cdot u = 0 & \text{in } G \times (0, T), \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ \sigma(u, p) \cdot n = 0 & \text{on } \gamma \times (0, T). \end{cases} \tag{4.10}$$

Therefore, in view of the unique continuation property given in Fabre [15] we have $u = 0$ in $G \times (0, T)$, whence $u^1 = u^2$ in $G \times (0, T)$.

Now, let us assume that $D^2 \setminus \overline{D^1}$ is nonempty and let us introduce $D^3 = D^2 \cup ((\Omega \setminus \overline{D^1}) \cap (\Omega \setminus \overline{\mathcal{O}}))$. By hypothesis, $D^3 \setminus \overline{D^1}$ is nonempty. Moreover, $\partial(D^3 \setminus \overline{D^1}) = \Gamma^1 \cup \Gamma^2$, where $\Gamma^1 = \partial(D^3 \setminus \overline{D^1}) \cap \partial D^1$ and $\Gamma^2 = \partial(D^3 \setminus \overline{D^1}) \cap \partial D^2$. Then we find that

$$\begin{cases} u^1_t - \nu\Delta u^1 + \nabla p^1 = 0, \quad \nabla \cdot u^1 = 0 & \text{in } (D^3 \setminus \overline{D^1}) \times (0, T), \\ u^1 = 0 & \text{on } (\Gamma^1 \cup \Gamma^2) \times (0, T), \\ u^1(x, 0) = u^0(x) & \text{in } D^3 \setminus \overline{D^1}. \end{cases} \tag{4.11}$$

This is not in contradiction with the fact that $u^1 = \varphi$ on $\partial\Omega \times (0, T)$. But it suggests to consider a new inverse problem:

Given the nonzero functions φ, $\bar{\varphi}$, u^0, α and $\bar{\alpha}$ in appropriate spaces with $\varphi \neq \bar{\varphi}$, find a set $D \in \mathscr{D}$ such that

(1) *The solution (u, p) of* (4.8) *associated with φ satisfies*

$$\sigma(u, p) \cdot n = \alpha \quad \text{on} \quad \gamma \times (0, T). \tag{4.12}$$

(2) *The solution $(\bar{u}, \bar{p})$ of* (4.8) *associated with $\bar{\varphi}$ satisfies*

$$\sigma(\bar{u}, \bar{p}) \cdot n = \bar{\alpha} \quad \text{on} \quad \gamma \times (0, T). \tag{4.13}$$

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We have the following:

Proposition 4.3 *For the inverse problem* (4.2), (4.12), (4.12), *one has uniqueness.*

Proof Again, let D^1 and D^2 be as above and let (u^i, p^i) and $(\bar{u}^i, \bar{p}^i)$ be the solutions of (4.8) for $D = D^i$ associated to φ and $\bar{\varphi}$, respectively. Then, arguing as before we find that $u^1 = u^2$ and $\bar{u}^1 = \bar{u}^2$ in $G \times (0, T)$, where G is the unique connected component of $G_0 = \Omega \setminus \overline{D^1 \cup D^2}$ whose boundary touches $\partial\Omega$.

Now, let us once more assume that $D^2 \setminus \overline{D^1}$ is nonempty and let us introduce $D^3 = D^2 \cup ((\Omega \setminus \overline{D^1}) \cap (\Omega \setminus \overline{\mathcal{O}}))$. By hypothesis, $D^3 \setminus \overline{D^1}$ is nonempty. Moreover, $\partial(D^3 \setminus \overline{D^1}) = \Gamma^1 \cup \Gamma^2$, where $\Gamma^1 = \partial(D^3 \setminus \overline{D^1}) \cap \partial D^1$ and $\Gamma^2 = \partial(D^3 \setminus \overline{D^1}) \cap \partial D^2$. In $(D^3 \setminus \overline{D^1}) \times (0, T)$ we have that both (u^1, p^1) and $(\bar{u}^1, \bar{p}^1)$ solve (4.11). Therefore, thanks to the uniqueness of solution of (4.11) and the unique continuation property from Fabre [15], we must have

$$u^1 \equiv \bar{u}^1 \quad \text{in} \quad (\Omega \setminus \overline{D^1}) \times (0, T).$$

This implies $\varphi = \bar{\varphi}$, which is an absurd.

Arguing as before, this proves that $D^1 = D^2$. □

We can also consider the inverse problem associated with the time-dependent Navier-Stokes system

$$\begin{cases} u_t + (u \cdot \nabla)u - \nu \Delta u + \nabla p = 0, \quad \nabla \cdot u = 0 & \text{in } (\Omega \setminus \overline{D}) \times (0, T), \\ u = \varphi & \text{on } \partial\Omega \times (0, T), \\ u = 0 & \text{on } \partial D \times (0, T), \\ u(x, 0) = u^0(x) & \text{in } \Omega \setminus \overline{D}, \end{cases} \tag{4.14}$$

together with (4.12) and (4.13). In this setting, we can prove a uniqueness result of the same kind provided we have the unique continuation property for the linearized Navier-Stokes system and the uniqueness of solution of the nonlinear Navier-Stokes equations. This is the case when the boundaries of Ω and D and the data φ and u^0 are regular enough and either $N = 2$ or u^0 is small enough.

5 Proof of Theorem 1.3

To prove the equality (1.9), we apply the domain variation techniques introduced in Murat & Simon [24, 25] and Simon [27], and particularized in Bello *et al.* [9] to Navier-Stokes systems. Notice that the main difficulty in seeing that the mapping $m \mapsto (u(m), p(m))$ is *differentiable* stems from the fact that $u(m)$ and $p(m)$ are functions defined for $x \in \Omega \setminus (\overline{D + m})$, a domain that depends on m. The right way to proceed is as follows:

- First, we introduce a suitable change of variables, we rewrite the equations satisfied by $(u(m), p(m))$ in a fixed domain $\Omega \setminus \overline{D}$ and we prove the existence of the derivative of the transported variable $(u(m), p(m)) \circ (\mathrm{Id} + m)$. This leads to the definition of the *total derivative* of $(u(m), p(m))$ at 0:

$$(\dot{u}(m), \dot{p}(m)) = \lim_{t \to 0} \frac{(u(tm), p(tm)) \circ (\mathrm{Id} + m) - (u, p)}{t}.$$

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- Then, we prove the existence of the *local derivative* $(u'(m), p'(m))$ of the mapping $m \mapsto (u(m), p(m))$, which is defined as follows: For any open set $\omega \subset\subset \Omega \setminus \overline{D}$, we put

$$(u'(m), p'(m))|_\omega = \lim_{t\to 0} \frac{(u(tm), p(tm))|_\omega - (u,p)|_\omega}{t}.$$

From [9], we have the following result:

Lemma 5.1 *Assume that* $\nu > \nu_1$. *Then*

- *The mapping* $m \mapsto (u(m), p(m)) \circ (Id + m)$, *which is defined in* $\mathscr{W}_\varepsilon$ *and takes values in* $H^1(\Omega\setminus\overline{D})^N \times L^2(\Omega\setminus\overline{D})$, *is differentiable at* 0, *with (total) derivative denoted by* $(\dot{u}(m), \dot{p}(m))$. *That is to say, there exists a linear continuous mapping* $m \mapsto (\dot{u}(m), \dot{p}(m))$ *such that*

$$(u(m), p(m)) \circ (Id + m) - (u,p) = (\dot{u}(m), \dot{p}(m)) + o(m), \tag{5.1}$$

  where $o(m)$ *satisfies* (1.8).
- *For each* $\omega \subset\subset \Omega \setminus \overline{D}$, *the mapping* $m \mapsto (u(m), p(m))|_\omega$, *which is defined in* $\mathscr{W}_\varepsilon$ *and takes values in* $H^1(\omega)^N \times L^2(\omega)$, *is differentiable at* 0. *In other words,* $m \mapsto (u(m), p(m))$ *is locally differentiable. The local derivative at* 0 *in the direction* m *is denoted by* $(u'(m), p'(m))$.
- *Furthermore,* $(u'(m), p'(m))$ *is the unique solution of the linear system* (1.10) *and*

$$(\dot{u}(m), \dot{p}(m)) = (u'(m), p'(m)) + (m \cdot \nabla)(u,p), \tag{5.2}$$

  where $(u,p) = (u(0), p(0))$.

In view of (5.1) and (5.2), taking into account that $m(x) = 0$ in a neighborhood of $\partial\Omega$, we find that

$$\sigma(u(m), p(m)) \cdot n - \sigma(u,p) \cdot n = \sigma(u'(m), p'(m)) \cdot n + o(m) \quad \text{on } \gamma.$$

This proves (1.9).

Let us now assume that φ is regular enough. Then the solution (u,p) of (1.1) satisfies $(u,p) \in H^2(\Omega \setminus \overline{D}) \times H^1(\Omega \setminus \overline{D})$ and, thanks to the fact that

$$u'(m) + (m \cdot \nabla)u \in H_0^1(\Omega \setminus \overline{D})^N,$$

we have:

$$u'(m) = 0 \quad \text{on} \quad \partial\Omega \quad \text{and} \quad u'(m) = -(m \cdot n)\frac{\partial u}{\partial n} \quad \text{on} \quad \partial D;$$

see Remark A.1 in Appendix A.

Let $\overline{\psi} \in C^2(\overline{\gamma})^N$ satisfy (1.11) and let (ψ, π) be the associated solution of (1.13). By multiplying (1.10) by ψ and integrating by parts in $\Omega \setminus \overline{D}$, using Green's formula, we easily

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find that

$$0 = \int_{\Omega\setminus\overline{D}} (-\nu\Delta u'(m) + (u'(m)\cdot\nabla)u + (u\cdot\nabla)u'(m) + \nabla p'(m))\,\psi\,dx$$
$$= \int_{\Omega\setminus\overline{D}} u'(m)(-\nu\Delta\psi - (\nabla u)^t\psi - (u\cdot\nabla)\psi + \nabla\pi)\,dx$$
$$+ \langle\sigma(u'(m), p'(m))\cdot n, \overline{\psi}\,1_\gamma\rangle_{\partial\Omega} - \int_{\partial D} u'(m)\cdot(\sigma(\psi,\pi)\cdot n)\,ds.$$

Consequently,

$$\langle\sigma(u'(m), p'(m))\cdot n, \overline{\psi}\,1_\gamma\rangle_{\partial\Omega} = \int_{\partial D} u'(m)(\sigma(\psi,\pi)\cdot n)\,ds = -\int_{\partial D}(m\cdot n)\frac{\partial u}{\partial n}\cdot(\sigma(\psi,\pi)\cdot n)\,ds.$$

On the boundary ∂D, since u vanishes, we have

$$\frac{\partial u}{\partial n}\cdot n = \sum_{i=1}^{N}\frac{\partial u_i}{\partial n}n_i = \sum_{i=1}^{N}\partial_i u_i = \nabla\cdot u = 0.$$

On the other hand, since ψ also vanishes on ∂D,

$$e(\psi)\cdot n = \frac{1}{2}\sum_{i=1}^{N}(\partial_i\psi_j + \partial_j\psi_i)n_i = \frac{1}{2}\frac{\partial\psi}{\partial n} + \frac{1}{2}(\nabla\cdot\psi)n,$$

whence

$$\sigma(\psi,\pi)\cdot n = \nu\frac{\partial\psi}{\partial n} + \nu(\nabla\cdot\psi)n - \pi n.$$

Then,

$$\frac{\partial u}{\partial n}\cdot(\sigma(\psi,\pi)\cdot n) = \nu\frac{\partial u}{\partial n}\cdot\frac{\partial\psi}{\partial n} \quad \text{on} \quad \partial D$$

and we obtain that

$$\langle\sigma(u'(m), p'(m))\cdot n, \overline{\psi}\,1_\gamma\rangle_{\partial\Omega} = -\nu\int_{\partial D}(m\cdot n)\frac{\partial u}{\partial n}\cdot\frac{\partial\psi}{\partial n}\,ds. \qquad (5.3)$$

Now, using (5.3) in (1.9) and taking into account that $\sigma(u(m), p(m))\cdot n$ and $\sigma(u,p)\cdot n$ belong to $W^{-1/r,r}(\partial\Omega)^N$, we get (1.12).

This ends the proof of Theorem 1.3.

6 Conclusions

We have considered the inverse problem of the identification of a single rigid body immersed in a Navier-Stokes fluid when friction forces are known on a part of the outer boundary. We have proved a uniqueness result, we have established a formula that provides the observed friction forces, at first order, in terms of the deformation of the rigid body and we have presented a strategy that can be used to compute appropriate approximations to the solution.

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We have also considered other similar inverse problems (see §4). For some of them, several open questions have been stated.

Appendix A Some technical results

For completeness, in this section we will present a sketch of the proof of Theorem 1.1, which provides existence, uniqueness and regularity properties of the solution of (1.1). For the proof we will use the standard Galerkin method and some properties of Sobolev spaces.

As mentioned above, there are many classical references for these questions [14, 22, 23, 28].

Proof of Theorem 1.1

1. Existence: Assume that $D \in \mathscr{D}$ and $\varphi \in C^1(\partial\Omega)^N$ is such that $\int_{\partial\Omega} \varphi \cdot n \, ds = 0$. For simplicity, the usual norms in the space $L^2(\Omega \setminus \overline{D^*})^N$, $H^1(\Omega \setminus \overline{D^*})^N, \ldots$ will be respectively denoted by $||\cdot||_{L^2}$, $||\cdot||_{H^1}, \ldots$

For any given regular domain $\mathcal{O} \subset \mathbb{R}^N$, let us set

$$V(\mathcal{O}) = \{ v \in H_0^1(\mathcal{O})^N \; : \; \nabla \cdot v = 0 \}.$$

Then, for every $\alpha > 0$, there exists Φ_α^*, with $\Phi_\alpha^* \in H^1(\Omega \setminus \overline{D^*})^N$, that satisfies

$$\begin{cases} \nabla \cdot \Phi_\alpha^* = 0 & \text{in } \Omega \setminus \overline{D^*}, \\ \Phi_\alpha^* = \varphi & \text{on } \partial\Omega, \\ \Phi_\alpha^* = 0 & \text{on } \partial D^* \end{cases}$$

$$||\Phi_\alpha^*||_{H^1(\Omega\setminus\overline{D^*})} \leqslant C(\Omega, D^*)||\varphi||_{C^1(\partial\Omega)}$$

and

$$\left| \int_{\Omega\setminus\overline{D^*}} (u \cdot \nabla)\Phi_\alpha^* \cdot u \, dx \right| \leqslant \alpha \|u\|^2_{H^1(\Omega\setminus\overline{D^*})}$$

for all $u \in V(\Omega \setminus \overline{D^*})$ (see [17, 18]). Let us take

$$\Phi_\alpha = \begin{cases} \Phi_\alpha^* & \text{in } \Omega \setminus \overline{D^*}, \\ 0 & \text{in the rest.} \end{cases} \tag{A 1}$$

Then we have $\Phi_\alpha \in H^1(\Omega \setminus \overline{D})^N$,

$$\left| \int_\Omega (u \cdot \nabla)\Phi_\alpha \cdot u \, dx \right| \leqslant \alpha \|u\|^2_{H^1} \tag{A 2}$$

and

$$\|\Phi_\alpha\|_{H^1} \leqslant C(\Omega, D^*)\|\varphi\|_{C^1(\partial\Omega)} . \tag{A 3}$$

Let us introduce F with $F = \nu\Delta\Phi_\alpha - (\Phi_\alpha \cdot \nabla)\Phi_\alpha$. Then

$$\|F\|_{H^{-1}(\Omega\setminus\overline{D})} \leqslant C(\Omega, D^*)(\nu + 1)\|\varphi\|_{C^1} . \tag{A 4}$$

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We will look for a solution u of the system (1.1) of the form

$$u = w + \Phi_\alpha.$$

Observe that the couple (w, p) must satisfy

$$\begin{cases} -\nu\Delta w + (w\cdot\nabla)\Phi_\alpha + (\Phi_\alpha\cdot\nabla)w + (w\cdot\nabla)w + \nabla p = F & \text{in } \Omega\setminus\overline{D}, \\ \nabla\cdot w = 0 & \text{in } \Omega\setminus\overline{D}, \\ w = 0 & \text{on } \partial\Omega\cup\partial D. \end{cases} \tag{A 5}$$

Thus, it will be sufficient to show that there exists a positive constant α such that the nonlinear system (A 5) possesses at least one weak solution, more precisely, a couple (w, p) that belongs to $V(\Omega\setminus\overline{D})^N \times L^2(\Omega\setminus\overline{D})$ and satisfies the partial differential equations in (A 5) in the weak or distributional sense. To this end, a standard Galerkin method can be used.

Let $\{v_1, v_2, \ldots\}$ be a basis of $V(\Omega\setminus\overline{D})^N$. We set $V_n = [v_1, \ldots, v_n]$ (the space spanned by v_i for $1 \leqslant i \leqslant n$). Then the n-th approximated problem is the following:

$$\begin{cases} \nu(\nabla w_n, \nabla v) + ((w_n\cdot\nabla)\Phi_\alpha + (\Phi_\alpha\cdot\nabla)w_n + (w_n\cdot\nabla)w_n\,, v) = \langle F, v\rangle_{H^{-1}} \\ \forall v \in V_n\,, \quad w_n \in V_n\,. \end{cases} \tag{A 6}$$

As usual, in order to obtain the existence result, the key point is to prove *a priori* estimates on the approximated solutions $\{w_n\}_{n\geqslant 1}$. Taking $v = w_n$ in (A 6) we have

$$\nu\|\nabla w_n\|_{L^2}^2 = \int_{\Omega\setminus\overline{D}} (w_n\cdot\nabla)\Phi_\alpha\cdot w_n\,dx + \langle F, w_n\rangle_{H^{-1}}.$$

Thanks to (A 2), we deduce that

$$\nu\|\nabla w_n\|_{L^2}^2 \leqslant \alpha||\nabla w_n||_{L^2}^2 + \frac{\nu}{2}||\nabla w_n||_{L^2}^2 + \frac{1}{2\nu}||F||_{H^{-1}}^2\,. \tag{A 7}$$

Now, let us take $\alpha = \dfrac{\nu}{4}$. Then, from (A 7), we easily obtain

$$\|\nabla w_n\|_{L^2} \leqslant \frac{2}{\nu^2}||F||_{H^{-1}}^2 \tag{A 8}$$

and, consequently, w_n is uniformly bounded in $V(\Omega\setminus\overline{D})^N$. In a classical way, this proves the existence of a solution (w, p) of (A 5) that belongs to $V(\Omega\setminus\overline{D})^N \times L^2(\Omega\setminus\overline{D})$.

Finally, from (A 8) and (A 4) we deduce that

$$||\nabla u||_{L^2(\Omega\setminus\overline{D})} \leqslant \frac{C(\Omega, D^*)}{\nu}(\nu+1)||\varphi||_{C^1(\partial\Omega)}\,.$$

Obviously, this proves (1.4). Therefore, (A 5) possesses at least one solution with the desired regularity and estimate.

2. Uniqueness: Let us assume that there exist two solutions (u^1, p^1) and (u^2, p^2) of (1.1) that satisfy (1.4). Let us set

$$u = u^1 - u^2 \quad\text{and}\quad p = p^1 - p^2.$$

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We have that (u, p) satisfies the following system:

$$\begin{cases} -\nu\Delta u + (u\cdot\nabla)u^1 + (u^2\cdot\nabla)u + \nabla p = 0 & \text{in } \Omega\setminus\overline{D}, \\ \nabla\cdot u = 0 & \text{in } \Omega\setminus\overline{D}, \\ u = 0 & \text{on } \partial\Omega\cup\partial D. \end{cases} \tag{A 9}$$

By multiplying the first equation of (A 9) by u and integrating in $\Omega\setminus\overline{D}$, we get

$$\nu\|\nabla u\|_{L^2}^2 = -\int_{\Omega\setminus\overline{D}} (u\cdot\nabla)u^1\cdot u\,dx.$$

Therefore, we have

$$\nu\|\nabla u\|_{L^2}^2 \leqslant C\|\nabla u^1\|_{L^2}\|u\|_{L^3}\|u\|_{L^6} \leqslant C(\Omega, D^*)\|\nabla u^1\|_{L^2}\|\nabla u\|_{L^2}^2. \tag{A 10}$$

But, in view of the estimates (1.4) satisfied by u^i, we have

$$||\nabla u^1||_{L^2} \leqslant \frac{C(\Omega, D^*)}{\nu}(\nu+1)||\varphi||_{C^1(\partial\Omega)}.$$

Combining this inequality and (A 10), we see that

$$\|\nabla u\|_{L^2}^2 \leqslant \frac{(\nu+1)}{\nu^2}C(\Omega, D^*)\|\varphi\|_{C^1(\partial\Omega)}\|\nabla u\|_{L^2}^2.$$

Consequently, if $\nu > \nu_1$ for some $\nu_1 = \nu_1(\Omega, D^*, \|\varphi\|_{C^1(\partial\Omega)}) > 0$, we necessarily have

$$\|\nabla u\|_{L^2}^2 \leqslant 0.$$

This proves the uniqueness of (u, p) (of course, p is unique up to a constant).

Now, notice that we can easily repeat the arguments in the previous point with a function Φ_α^* that satisfies $\Phi_\alpha^* \in W^{1,r}(\Omega\setminus\overline{D})^N$ for all $r \in [1, +\infty)$. As a consequence, the solution (w, p) of (A 5) satisfies

$$-\nu\Delta w + (w\cdot\nabla)w + \nabla p = \tilde{F},$$

with

$$\tilde{F} = \nu\Delta\Phi_\alpha - (\Phi_\alpha\cdot\nabla)\Phi_\alpha - (w\cdot\nabla)\Phi_\alpha + (\Phi_\alpha\cdot\nabla)w \in W^{-1,r}(\Omega\setminus\overline{D})^N$$

for all $r \in [1, +\infty)$. From standard regularity properties of the Navier-Stokes system (see for instance [26]), we deduce that $w \in W_0^{1,r}(\Omega\setminus\overline{D})^N$ and $p \in L^r(\Omega\setminus\overline{D})$.

Therefore, if $\nu > \nu_1$, (1.1) possesses exactly one solution (u, p) with this regularity.

3. Regularity of $\sigma(u, p)\cdot n$*:* Let us assume that $\nu > \nu_1$ and let us see that $\sigma(u, p)\cdot n \in W^{-1/r,r}(\partial\Omega)$ for all $r \in [1, +\infty)$.

Let us fix r. In view of well known results, it suffices to prove that $\sigma(u, p) \in L^r(\Omega\setminus\overline{D})^{N\times N}$ and $\nabla\cdot\sigma(u, p) \in L^r(\Omega\setminus\overline{D})^N$; for instance, see [8].

But this is very easy to check. Indeed, we have $(u, p) \in W^{1,r}(\Omega\setminus\overline{D})^N \times L^r(\Omega\setminus\overline{D})$ and consequently $\sigma(u, p) \in L^r(\Omega\setminus\overline{D})^{N\times N}$. On the other hand, $\nabla\cdot\sigma(u, p) = (u\cdot\nabla)u$, whence we also have $\nabla\cdot\sigma(u, p) \in L^r(\Omega\setminus\overline{D})^N$. □

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Remark A.1 Let us now assume that φ is sufficiently smooth. For instance, let us suppose that $\varphi \in C^2(\partial\Omega)^N$. Then the weak solutions of (1.1) are in fact strong solutions, that is to say, they satisfy $(u,p) \in W^{2,r}(\Omega \setminus \overline{D})^N \times W^{1,r}(\Omega \setminus \overline{D})$ for all $r \in [1,+\infty)$, with appropriate estimates. Indeed, we can assume in this case that $\Phi \in W^{2,r}(\Omega \setminus \overline{D})^N$. Then $w = u - \Phi$ is, together with p, a solution of a Stokes problem

$$\begin{cases} -\nu\Delta w + \nabla p = \tilde{F}, \quad \nabla\cdot w = 0 & \text{in } \Omega\setminus\overline{D}, \\ w = 0 & \text{on } \partial\Omega \cup \partial D, \end{cases}$$

where $\tilde{F} \in L^r(\Omega \setminus \overline{D})^N$ and $\partial(\Omega \setminus \overline{D})$ is regular enough. In view of the $W^{2,r}$-regularity theory for Stokes problems (cf. [8] and the references therein), we have $(w,p) \in W^{2,r}(\Omega \setminus \overline{D})^N \times W^{1,r}(\Omega \setminus \overline{D})$. Obviously, this provides the same regularity for (u,p).

Appendix B Sketch of the proof of Lemma 2.1

As we already have mentioned, the proof of Lemma 2.1 is based on the ideas and the arguments in Fabre & Lebeau [16]. The main difference is that, in Fabre & Lebeau [16], $b \equiv 0$ and the proof does not hold in our case.

The proof will be composed of four steps. First, we recall an appropriate local Carleman inequality from Fabre & Lebeau [16]. Then, using this Carleman inequality, we prove the result of Lemma 2.1 but in a ball and for potentials a and b with sufficiently small L^∞ norms. Next, in the third step we will show the result in small balls and, finally, we will conclude the proof.

Step 1: A local Carleman inequality.

Lemma B.1 *Let $U \subset \mathbb{R}^N$ be an open set, $K \subset U$ a nonempty compact set, $a_{jk} \in C^\infty(\mathbb{R}^N)$ for $1 \leqslant j \leqslant s$, $1 \leqslant k \leqslant N$ and $\varphi \in \mathcal{D}(\mathbb{R}^N)$. Let us set*

$$L_1 f = \sum_{j=1}^{s}\sum_{k=1}^{N} a_{jk}\partial_k f_j \quad \forall f = (f_1,\dots,f_s) \in L^2(U)^s$$

and

$$a_0(x,\xi) = \sum_{j=1}^{N}\left(\xi_j^2 - (\partial_j\varphi(x))^2\right), \quad b_0(x,\xi) = 2\sum_{j=1}^{N}\xi_j\,\partial_j\varphi(x) \quad \forall (x,\xi) \in U \times \mathbb{R}^N$$

and let us assume that φ satisfies the following property:

$$\begin{cases} \nabla\varphi \text{ does not vanish in } U\text{; furthermore,} \\ \exists\, C_0 > 0 \text{ such that } \partial_\xi a_0(x,\xi)\cdot\partial_x b_0(x,\xi) - \partial_x a_0(x,\xi)\cdot\partial_\xi b_0(x,\xi) \geqslant C_0 \\ \text{for all } (x,\xi) \in U \times \mathbb{R}^N \text{ such that } a_0(x,\xi) = b_0(x,\xi) = 0. \end{cases} \tag{B 1}$$

Then, there exist constants $C > 0$ and $h_1 > 0$ such that, for any couple $(y,F) \in H_0^1(U) \times L^2(U)^s$ satisfying $\mathrm{supp}\,(y) \cup \mathrm{supp}\,(F) \subset K$ *and $\Delta y - L_1 F \in L^2(U)$ and any $h \in (0,h_1)$, one*

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has:

$$\int_K e^{2\varphi/h}(|y|^2 + h^2|\nabla y|^2)\,dx \leqslant C \int_K e^{2\varphi/h}(h|F|^2 + h^3|\Delta y - L_1 F|^2)\,dx. \tag{B 2}$$

Step 2: A unique continuation property for small coefficients.

We will deduce here the result of Lemma 2.1 but for potentials with sufficiently small norm.

Let us consider (2.1) in $B(0;2)$, where $B(0;r)$ will denote the open ball of radius $r > 0$ centred at the origin:

$$\begin{cases} -\nu\Delta v + (a\cdot\nabla)v + (v\cdot\nabla)b + \nabla q = 0 & \text{in } B(0;2), \\ \nabla\cdot v = 0 & \text{in } B(0;2). \end{cases} \tag{B 3}$$

We have the following result, which is a modified version of Lemma 3.1 in Fabre & Lebeau [16] (where $b \equiv 0$):

Lemma B.2 *Assume that* $a \in L^\infty(B(0;2))^N$, $b \in L^\infty(B(0;2))^N$ *and* $\nabla\cdot a = \nabla\cdot b = 0$ *in* $B(0;2)$. *Then there exists* $\epsilon > 0$ *such that, if*

$$\|a\|_\infty \leqslant \epsilon \quad \textit{and} \quad \|b\|_\infty \leqslant \epsilon,$$

any solution $(v,q) \in H^1(B(0;2))^N \times L^2(B(0;2))$ *of* (B 3) *satisfying* $v = 0$ *in* $B(0;1)$ *is zero everywhere.*

Proof of Lemma B.2 Let $(v,q) \in H^1(B(0;2))^N \times L^2(B(0;2))$ be a solution of (B 3) satisfying $v = 0$ in $B(0;1)$. Since $q \equiv$ Const. in $B(0;1)$, it is not restrictive to assume that it also vanishes in $B(0;1)$. Let us notice that Lemma B.1 can be applied in this context for some appropriate choices of U, K, L_1 and φ.

Indeed, let us choose $\varepsilon > 0$ and let us set

$$K = \left\{x \in \mathbb{R}^N \;:\; \frac{3}{4} \leqslant |x| \leqslant 2 - \varepsilon\right\}, \quad U = \left\{x \in \mathbb{R}^N \;:\; \frac{1}{2} < |x| < 2\right\}.$$

For $\delta > 4$, let $\varphi \in \mathscr{D}(\mathbb{R}^N)$ be such that

$$\varphi(x) = e^{-\delta|x|^2} \quad \forall\, x \in \overline{B}(0;2). \tag{B 4}$$

Arguing as in Fabre & Lebeau [16] we get

$$\partial_\xi a_0(x,\xi)\cdot\partial_x b_0(x,\xi) - \partial_x a_0(x,\xi)\cdot\partial_\xi b_0(x,\xi) \geqslant 16\delta^3 e^{-3\delta/4}\left(\frac{\delta}{4} - 1\right).$$

Therefore, (B 1) is satisfied by this function φ in this open set U. Now, let us introduce a function $\zeta \in \mathscr{D}(\overset{\circ}{K})$ such that

$$\zeta = 1 \quad \text{in} \quad 1 - \varepsilon \leqslant |x| \leqslant 2 - 2\varepsilon$$

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and we set

$$\tilde{v} = \zeta v, \quad \tilde{q} = \zeta q, \tag{B 5}$$

where $(v, q) \in H^1(B(0;2))^N \times L^2(B(0;2))$ is a solution of (B 3). Obviously, we have $(\tilde{v}, \tilde{q}) \in H_0^1(\overset{\circ}{K})^N \times L^2(\overset{\circ}{K})$.

Using (B 3), we obtain that

$$-\nu\Delta\tilde{v} + \nabla\tilde{q} + \nabla\cdot(\tilde{v}b) = -(a\cdot\nabla)\,\tilde{v} + J_1 \quad \text{in } \overset{\circ}{K}, \tag{B 6}$$

where $J_1 \in L^2(\overset{\circ}{K})$ is given by

$$J_1 = b(v\cdot\nabla)\zeta - 2\nu\nabla\zeta\cdot\nabla v - \nu v\Delta\zeta + (a\cdot\nabla\zeta)v + q\nabla\zeta, \tag{B 7}$$

since $\nabla\cdot v = 0$ in U. We take the divergence in the first equation of (B 3) and we deduce

$$\Delta q = -\nabla\cdot((a\cdot\nabla)v + (v\cdot\nabla)b) = -\nabla\cdot((a\cdot\nabla)v + (\nabla v)\,b), \tag{B 8}$$

where we have used that

$$\partial_i(v_j\partial_j b_i) = \partial_j(\partial_i v_j b_i),$$

which is a consequence of the identities $\nabla\cdot v = \nabla\cdot b = 0$. Then, taking into account (B 8) we deduce that $\tilde{q}$ satisfies

$$\Delta\tilde{q} + \nabla\cdot((a\cdot\nabla)\tilde{v}) + \nabla\cdot((\nabla\tilde{v})b) = J_2 \quad \text{in } \overset{\circ}{K}, \tag{B 9}$$

with $J_2 \in L^2(\overset{\circ}{K})$ given by

$$\begin{aligned} J_2 = {} & (a\cdot\nabla)v\cdot\nabla\zeta + \nabla\cdot((a\cdot\nabla\zeta)v) + \nabla\zeta\cdot((b\cdot\nabla)v) \\ & + \nabla\cdot((b\cdot\nabla\zeta)v) + 2\nabla\zeta\cdot\nabla q + q\Delta\zeta. \end{aligned} \tag{B 10}$$

So, we are ready now to apply Lemma B.1. In fact, we will do this twice. More precisely, let us first take $s = N+1$,

$$L_1 f = -\frac{1}{\nu}\partial_k f - \frac{1}{\nu}\sum_{j=1}^{N}\partial_j f_j \quad \forall f = (f_0, f_1, \ldots, f_N) \in L^2(U)^{N+1},$$

$y = \tilde{v}_k$ and $F = (\tilde{q}, \tilde{v}_1 b_k, \ldots, \tilde{v}_N b_k)$. Thanks to (B 6), we have $(y, F) \in H_0^1(U) \times L^2(U)$, $\Delta y - L_1 F \in L^2(U)$ and $\operatorname{supp}(y) \cup \operatorname{supp}(F) \subset K$. Applying Lemma B.1 we deduce that there exist $C > 0$ and $h_1 > 0$ such that, for any $h \in (0, h_1)$, the following holds:

$$\begin{aligned} \int_K e^{2\varphi/h}(|\tilde{v}|^2 + h^2|\nabla\tilde{v}|^2)\,dx \leqslant Ch\int_K e^{2\varphi/h}(|\tilde{q}|^2 + |b\tilde{v}|^2)\,dx \\ + Ch^3\int_K e^{2\varphi/h}|(a\cdot\nabla)\tilde{v}|^2\,dx + \; Ch^3\int_K e^{2\varphi/h}|J_1|^2\,dx \end{aligned}$$

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with $J_1 \in L^2(\overset{\circ}{K})$ given by (B 7). Thus, we also have

$$\int_K e^{2\varphi/h}(|\tilde{v}|^2 + h^2|\nabla\tilde{v}|^2)\,dx \leqslant Ch\int_K e^{2\varphi/h}|\tilde{q}|^2\,dx + Ch^3\int_K e^{2\varphi/h}|J_1|^2\,dx \qquad \text{(B 11)}$$

for sufficiently small h, more precisely, for $0 < h < h_2 := \min(h_1, C(\|a\|_\infty^{-2} + \|b\|_\infty^{-2})$.

Notice that J_1 is independent of h and has the same support than $\nabla\zeta$. This we will use below.

To get a suitable estimates for the first term in the right hand side of (B 11), let us use again Lemma B.1. This time, we take $s = N$,

$$L_1 f = -\nabla\cdot f \quad \forall f = (f_1, \ldots, f_N) \in L^2(U)^N,$$

$y = \tilde{q}$ and $F = (a\cdot\nabla)\tilde{v} + (\nabla\tilde{v})b$. In view of (B 5) and (B 9), we have $(y, F) \in H_0^1(U)\times L^2(U)^N$, $\Delta y - L_1 F \in L^2(U)$ and $\operatorname{supp}(y)\cup\operatorname{supp}(F)\subset K$. Therefore, thanks to Lemma B.1 there exist $C > 0$ and $h_3 > 0$ such that

$$\begin{cases} \displaystyle\int_K e^{2\varphi/h}(|\tilde{q}|^2 + h^2|\nabla\tilde{q}|^2)\,dx \leqslant Ch\int_K e^{2\varphi/h}(|(a\cdot\nabla)\tilde{v}|^2 + |(\nabla\tilde{v})b|^2)\,dx \\ \displaystyle\quad + Ch^3\int_K e^{2\varphi/h}|J_2|^2\,dx \end{cases} \qquad \text{(B 12)}$$

for any $h \in (0, h_3)$, where $J_2 \in L^2(\overset{\circ}{K})$ is given by $(B\ 10)$.

Taking into account the inequality (B 12) in (B 11) we deduce that there exists a positive constant R_0 such that

$$\begin{cases} \displaystyle\int_K e^{2\varphi/h}(|\tilde{v}|^2 + h^2|\nabla\tilde{v}|^2)\,dx \leqslant R_0 h^2(\|a\|_\infty^2 + \|b\|_\infty^2)\int_K e^{2\varphi/h}|\nabla\tilde{v}|^2\,dx \\ \displaystyle\quad + \int_K e^{2\varphi/h}(h^3|J_1|^2 + h^4|J_2|^2)\,dx \end{cases} \qquad \text{(B 13)}$$

for any $h \in (0, h_4)$ with $h_4 = \min(h_2, h_3)$, where J_1 and J_2 are respectively given by (B 7) and (B 10). Notice that h_4 can be chosen in the form:

$$h_4 = C\min(1, \|a\|_\infty^{-2}, \|b\|_\infty^{-2}).$$

Let us now assume that

$$\|a\|_\infty \leqslant \epsilon \quad \text{and} \quad \|b\|_\infty \leqslant \epsilon, \quad \text{where} \quad \epsilon := \frac{1}{2\sqrt{R_0}}.$$

Then, we deduce from (B 13) that

$$\int_K e^{2\varphi/h}(|\tilde{v}|^2 + h^2|\nabla\tilde{v}|^2)\,dx \leqslant C\int_K e^{2\varphi/h}(h^3|J_1|^2 + h^4|J_2|^2)\,dx \qquad \text{(B 14)}$$

for any $h \in (0, h_4)$.

Finally, to conclude the proof, we will argue as in Fabre & Lebeau [16].

Since J_1 and J_2 (respectively given by (B 7) and (B 10)) have the same support than $\nabla\zeta$, we obtain that J_1 and J_2 vanish outside the ring $2 - 2\varepsilon \leqslant |x| \leqslant 2$.

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We have from (B 4) that φ is a radially decreasing positive function in U, so we have

$$\int_K e^{2\varphi/h}(h^3|J_1|^2 + h^4|J_2|^2)\,dx \leqslant e^{\frac{2\varphi(2-2\varepsilon)}{h}} \int_K (h^3|J_1|^2 + h^4|J_2|^2\,dx. \tag{B 15}$$

On the other hand, we also have

$$\begin{aligned}\int_K e^{2\varphi/h}(|\tilde v|^2 + h^2|\nabla\tilde v|^2)\,dx &\geqslant \int_{1\leqslant|x|\leqslant 2-3\varepsilon} e^{2\varphi/h}(|\tilde v|^2 + h^2|\nabla\tilde v|^2)\,dx\\ &\geqslant e^{\frac{2\varphi(2-3\varepsilon)}{h}} \int_{1\leqslant|x|\leqslant 2-3\varepsilon} (|\tilde v|^2 + h^2|\nabla\tilde v|^2)\,dx.\end{aligned} \tag{B 16}$$

Combining (B 14), (B 15) and (B 16), the following is found:

$$\int_{1\leqslant|x|\leqslant 2-3\varepsilon} (|\tilde v|^2 + h^2|\nabla\tilde v|^2)\,dx \leqslant Ch^3 e^{\frac{2}{h}(\varphi(2-2\varepsilon)-\varphi(2-3\varepsilon))} \int_U |J_3|^2\,dx, \tag{B 17}$$

where $J_3 \in L^2(\overset{\circ}{K})$ is independent of h. Using that $\varphi(2-3\varepsilon) - \varphi(2-2\varepsilon) > 0$ and passing to the limit in (B 17) as $h \to 0$, we get

$$\tilde v = 0 \quad \text{in} \quad 1 \leqslant |x| \leqslant 2 - 3\varepsilon.$$

As (B 5) shows, we have $\tilde v = \zeta v$ and, since $\zeta = 1$ in $1 \leqslant |x| \leqslant 2 - 3\varepsilon$, we finally deduce that $v = 0$ in $B(0; 2 - 3\varepsilon)$. Since $\varepsilon > 0$ is arbitrarily small, we finally deduce that v vanish identically. □

Step 3: A unique continuation for small balls.
We can now deduce a result similar to Lemma B.2 for not necessarily small coefficients but in a small ball. More precisely, arguing as in the proof of Lemma 3.2 in [16], we obtain from Lemma B.2 the following:

Lemma B.3 *Let G be an open connected such that $x_0 \in G$. Assume that $a \in L^\infty(G)^N$, $b \in L^\infty(G)^N$ and $\nabla \cdot a = \nabla \cdot b = 0$ in G. There exists $r_0 > 0$ such that, if $0 < r < r_0$, any solution $(v, q) \in H^1(G)^N \times L^2(G)$ of (2.1) satisfying $v = 0$ in $B(x_0; r)$ vanishes in $B(x_0; 2r)$. Furthermore, r_0 can be chosen as follows:*

$$r_0 = \min\left(\frac{\epsilon}{\|a\|_\infty}, \frac{\epsilon}{\|b\|_\infty}, \frac{\rho}{2}\right), \tag{B 18}$$

where ϵ is the constant given in Lemma B.2 and ρ is such that $\overline{B}(0; \rho) \subset G$.

Step 4: Conclusion.
To achieve the proof of Lemma 2.1, we consider a solution (v, q) of (2.1) satisfying $v = 0$ in Ω.

We assume that $\overline{B}(x_0; \rho_0) \subset \Omega$ and let $\bar x$ be another point in G. There exists $\widetilde\gamma \in C^\infty([0, 1])$ with $\widetilde\gamma(0) = x_0$, $\widetilde\gamma(1) = \bar x$ and such that $\widetilde\gamma(t) \in G$ for all $t \in [0, 1]$. Let $\bar U \subset\subset G$ be a bounded

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open neighbourhood of $\widetilde{\gamma}([0,1])$. There exists $\rho_1 \in (0, \rho_0]$ such that $\overline{B}(x;\rho_1) \subset U$ for all $x \in \widetilde{\gamma}([0,1])$. Let us set

$$r_0 = \min\left(\frac{\epsilon}{\|a\|_\infty}, \frac{\epsilon}{\|b\|_\infty}, \frac{\rho_1}{2}\right).$$

In view of Lemma B.3, for $r \in (0, r_0)$ and any $x \in \widetilde{\gamma}([0,1])$, the equalities $v = 0$ and $\eta = 0$ in $B(x;r)$ imply $v = 0$ in $B(x;2r)$.

We fix now r with $0 < r < r_0$. It is then clear that

$$\sup\{\, t \in [0,1] : u = 0 \;\text{ in }\; B(\gamma(\tau);r) \;\; \forall \tau \leqslant t \,\} = 1.$$

Hence, $v = 0$ in $B(\bar{x};r)$. □

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