Canad. Math. Bull. Vol. 64 (2), 2021, pp. 264–281
http://dx.doi.org/10.4153/S0008439520000430

Growth of frequently hypercyclic functions for some weighted Taylor shifts on the unit disc

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Abstract. For any $\alpha \in \mathbb{R}$, we consider the weighted Taylor shift operators T_α acting on the space of analytic functions in the unit disc given by $T_\alpha : H(\mathbb{D}) \to H(\mathbb{D})$,

$$f(z) = \sum_{k\geq 0} a_k z^k \mapsto T_\alpha(f)(z) = a_1 + \sum_{k\geq 1}\left(1+\frac{1}{k}\right)^\alpha a_{k+1} z^k.$$

We establish the optimal growth of frequently hypercyclic functions for T_α in terms of L^p averages, $1 \leq p \leq +\infty$. This allows us to highlight a critical exponent.

1 Introduction

Let $\mathbb{D}$ denote the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$ of the complex plane. If $0 < r < 1$ and f is an analytic function in $\mathbb{D}$ (*i.e.*, $f \in H(\mathbb{D})$), we set

$$M_p(r, f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{1/p} (1 \leq p < \infty),$$
$$M_\infty(r, f) = \sup_{0\leq t\leq 2\pi} |f(re^{it})|.$$

In the same spirit, for any holomorphic polynomial q, let us define, for all $p \geq 1$,

$$\|q\|_p = \left(\frac{1}{2\pi}\int_0^{2\pi} |q(e^{i\theta})|^p d\theta\right)^{1/p} \quad \text{and} \quad \|q\|_\infty = \sup_{0\leq t\leq 2\pi} |q(e^{it})|.$$

Moreover, for all $p > 1$, p' will stand for the exponent conjugate to p, *i.e.*, $\frac{1}{p} + \frac{1}{p'} = 1$. For any f in $H(\mathbb{D})$ and any z in $\mathbb{D}$, we will write $f(z) = \sum_{k\geq 0} a_k z^k$. For all $\alpha \in \mathbb{R}$, we consider the sequence $(\omega_n(\alpha))$ defined as follows:

$$\omega_0(\alpha) = 1 \quad \text{and} \quad \omega_n(\alpha) = \left(1+\frac{1}{n}\right)^\alpha \text{ for } n \geq 1.$$

We deal with the following (polynomial) weighted Taylor shifts acting on $H(\mathbb{D})$ endowed with the topology of uniform convergence on compact subsets:

Received by the editors November 29, 2019; revised June 2, 2020.
Published online on Cambridge Core June 11, 2020.
The first author was partly supported by the grant ANR-17-CE40-0021 of the French National Research Agency ANR (project Front).
AMS subject classification: **47A16**, 47B38, 30B30.
Keywords: Frequently hypercyclic operator, Taylor shift, boundary behavior.

$$T_\alpha : H(\mathbb{D}) \longrightarrow H(\mathbb{D}), \quad f(z) = \sum_{k\geq 0} a_k z^k \longmapsto T_\alpha(f)(z) = \sum_{k\geq 0} \omega_k(\alpha) a_{k+1} z^k.$$

Notice that T_0 is the classical Taylor shift acting on $H(\mathbb{D})$ [4]. In [14], the author shows that the operator T_0 is frequently hypercyclic. Actually, for any $\alpha \in \mathbb{R}$, T_α is frequently hypercyclic. Let us recall that an operator $T : X \to X$, where X is a Fréchet space, is said to be *frequently hypercyclic* if there is a vector $x \in X$ such that for every non-empty open set $U \subset X$, the lower density of the set $N(x, U) := \{n \in \mathbb{N} : T^n x \in U\}$ is positive (*i.e.*, $\underline{d}(N(x,U)) = \liminf_{N\to+\infty} \#\{1 \leq n \leq N : n \in N(x,U)\}/N > 0$). Such a vector x is called *frequently hypercyclic*. This natural new concept of hypercyclicity was introduced in 2004 by Bayart and Grivaux [1, 2] and has been the subject of many developments since then. We refer the reader to [3] or [10] and the references therein. In particular, the following result is a crucial tool used to exhibit a lot of examples of frequently hypercyclic operators.

Theorem 1.1 (Frequent hypercyclicity criterion) *Let T be an operator on a separable Fréchet space X. If there is a dense subset X_0 of X and a map $S : X_0 \to X_0$ such that, for any $x \in X_0$,*

(i) *$\sum_{n\geq 0} T^n x$ converges unconditionally,*
(ii) *$\sum_{n\geq 0} S^n x$ converges unconditionally,*
(iii) *$TSx = x$,*

then T is frequently hypercyclic.

In our context, a straightforward application of the frequent hypercyclicity criterion yields the frequent hypercyclicity of the operators T_α for all $\alpha \in \mathbb{R}$. Indeed, let X_0 be the set of polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$. Clearly, the set X_0 is a countable and dense subset in $H(\mathbb{D})$. Now we apply Theorem 1.1. First observe that condition (i) is clearly satisfied, since for any polynomial f, $T_\alpha^n f = 0$ if the integer n is strictly greater than the degree of f. Moreover, if we consider the map $S_\alpha : X_0 \to X_0$, given by $\sum_{k=0}^d b_k z^k \mapsto \sum_{k=0}^d b_k(\omega_k(\alpha))^{-1} z^{k+1}$, condition (iii) is obviously satisfied. Finally, for $f(z) = \sum_{k=0}^d b_k z^k$, for all $0 < r < 1$, we easily get

$$\sup_{|z|\leq r} |S_\alpha^n f(z)| \leq (n+1)^{|\alpha|} \frac{r^n}{1-r} \max_{j=0,\dots,d} |b_j|,$$

which gives condition (ii). Thus, the operators T_α are frequently hypercyclic for all $\alpha \in \mathbb{R}$.

Now we are interested in the growth of frequently hypercyclic functions for the Taylor shifts $T_\alpha, \alpha \in \mathbb{R}$. Such results have been achieved in the case of frequently hypercyclic functions for the differentiation operator on $H(\mathbb{C})$. We refer the reader to [5, 6] and the references therein. Recently, in [11], the authors obtained the optimal growth in terms of average L^p-norms of frequently hypercyclic functions $f \in H(\mathbb{D})$ for the Taylor shift T_0. The result is stated as follows.

Theorem 1.2 ([11]) *We have the following assertions.*

(i) *Given $2 \leq p \leq +\infty$, there is a frequently hypercyclic function $f \in H(\mathbb{D})$ for the Taylor shift T_0 satisfying the following estimate: there exists $C > 0$ such that for every $0 < r < 1$,*

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$$M_p(r,f) \le \frac{C}{\sqrt{1-r}}.$$

This estimate is optimal: every frequently hypercyclic function $f \in H(\mathbb{D})$ for the Taylor shift T_0 satisfies

$$\liminf_{r\to 1}(\sqrt{1-r}M_p(r,f)) > 0.$$

(ii) *Given $1 < p < 2$, there is a frequently hypercyclic function $f \in H(\mathbb{D})$ for the Taylor shift T_0 satisfying the following estimate: there exists $C > 0$ such that for every $0 < r < 1$,*

$$M_p(r,f) \le \frac{C}{(1-r)^{1/p'}}.$$

This estimate is optimal: every frequently hypercyclic function $f \in H(\mathbb{D})$ for the Taylor shift T_0 satisfies

$$\liminf_{r\to 1}((1-r)^{1/p'}M_p(r,f)) > 0.$$

A version of this statement in the case $p = 1$ was also given. A natural question that arises is the following: what is the slowest possible growth for a frequently hypercyclic vector f for the operator T_α? Here, we exhibit a critical exponent from which the growth of frequently hypercyclic functions changes from polynomial to logarithmic behavior, and then becomes bounded on the whole unit disc. Summing up, the following theorem is our main result.

Theorem 1.3 *Let $\alpha \in \mathbb{R}$. The following assertions hold:*

(i) *For any $2 \le p < +\infty$, there is a frequently hypercyclic function f in $H(\mathbb{D})$ for T_α satisfying the following estimates: there exists $C > 0$ such that for every $0 < r < 1$,*

$$M_p(r,f) \le \begin{cases} C(1-r)^{\alpha-\frac{1}{2}} & \text{if } \alpha < 1/2,\\ C\sqrt{|\log(1-r)|} & \text{if } \alpha = 1/2,\\ C & \text{if } \alpha > 1/2.\end{cases}$$

These estimates are optimal: every frequently hypercyclic function f in $H(\mathbb{D})$ for T_α is bounded from below by the corresponding previous estimate depending on α.

(ii) *For any $1 < p < 2$, there is a frequently hypercyclic function f in $H(\mathbb{D})$ for T_α satisfying the following estimates: there exists $C > 0$ such that for every $0 < r < 1$,*

$$M_p(r,f) \le \begin{cases} C(1-r)^{\alpha-\frac{1}{p'}} & \text{if } \alpha < 1/p',\\ C|\log(1-r)|^{\frac{1}{p}} & \text{if } \alpha = 1/p',\\ C & \text{if } \alpha > 1/p'.\end{cases}$$

These estimates are optimal: every frequently hypercyclic function f in $H(\mathbb{D})$ for T_α is bounded from below by the corresponding previous estimate depending on α.

(iii) *There is a frequently hypercyclic function f in $H(\mathbb{D})$ for T_α satisfying the following estimates: there exists $C > 0$ such that for every $0 < r < 1$,*

$$M_\infty(r,f) \leq \begin{cases} C(1-r)^{\alpha-\frac{1}{2}} & \text{if } \alpha < 1/2, \\ C|\log(1-r)| & \text{if } \alpha = 1/2, \\ C & \text{if } \alpha > 1/2. \end{cases}$$

For $\alpha \neq 1/2$, these estimates are optimal: every frequently hypercyclic function f in $H(\mathbb{D})$ for T_α is bounded from below by the corresponding previous estimate depending on α.

Finally, we deal with the case $p = 1$ to obtain a similar statement that extends the result for the Taylor shift T_0 given in [11] to the weighted Taylor shifts T_α, for all $\alpha \in \mathbb{R}$. We refer the reader to Proposition 4.1 and Theorem 4.4.

The paper is organized as follows: in Section 2, the minimum growth of frequently hypercyclic functions for T_α is established. To do this, we use several well-known inequalities, such as the Jensen inequality, the Hardy–Littlewood inequality, or the Hausdorff–Young inequality. In Section 3, inspired by the ideas of [6], using Rudin–Shapiro polynomials, we will show by a constructive proof that these minimum estimates of the growth of frequently hypercyclic functions are attained. In Section 4, we deal with the case $p = 1$.

Throughout the paper, whenever A and B depend on some parameters, we will use the notation $A \lesssim B$ (resp. $A \gtrsim B$) to mean $A \leq CB$ (resp. $A \geq CB$) for some constant $C > 0$ that does not depend on the involved parameters apart from p and α.

2 On the Growth of the Frequently Hypercyclic Functions

Let $\alpha \in \mathbb{R}$. We establish some results concerning the blowing-up in terms of L^p-norms of a frequently hypercyclic function in $H(\mathbb{D})$ for the weighted shift operator T_α. We begin with the case $2 \leq p \leq +\infty$.

Proposition 2.1 *Let $2 \leq p \leq +\infty$, $\alpha \in \mathbb{R}$ and $f \in H(\mathbb{D})$. Assume that f is a frequently hypercyclic vector for T_α. Then, for all $0 < r < 1$, the following estimates hold:*

$$M_p(r,f) \gtrsim \begin{cases} (1-r)^{\alpha-\frac{1}{2}} & \text{if } \alpha < 1/2, \\ \sqrt{|\log(1-r)|} & \text{if } \alpha = 1/2, \\ 1 & \text{if } \alpha > 1/2. \end{cases}$$

Proof We write $f(z) = a_0 + \sum_{k\geq 1} a_k k^{-\alpha} z^k$. Since f is frequently hypercyclic, there exists an increasing subsequence $(n_k) \subset \mathbb{N}$ with positive lower density such that, for all $k \geq 1$, $|T_\alpha^{n_k} f(0) - 3/2| = |a_{n_k} - 3/2| < 1/2$, which implies that $|a_{n_k}| \geq 1$. Moreover, since the sequence (n_k) has positive lower density, there exists $C > 1$ such that $k \leq n_k \leq Ck$. Let us consider a positive integer $l \geq 1$. For any $r \in [1-2^{-l}, 1-2^{-(l+1)}]$, Jensen's inequality and Parseval's Theorem imply

$$[M_p(r,f)]^2 \geq \left[M_p(1-2^{-l},f)\right]^2 \geq \sum_{k\geq 1} \frac{|a_k|^2}{k^{2\alpha}}(1-2^{-l})^{2k}.$$

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Thus, we deduce

$$[M_p(r,f)]^2 \geq \sum_{k\geq 1} \frac{|a_{n_k}|^2}{n_k^{2\alpha}} \left(1-2^{-l}\right)^{2n_k} \geq \sum_{k\geq 1} \frac{\left(1-2^{-l}\right)^{2Ck}}{(Ck)^{2\alpha}} \gtrsim \sum_{k=1}^{2^l} k^{-2\alpha}$$
$$\gtrsim \begin{cases} 1 & \text{if } \alpha > \frac{1}{2}, \\ 2^{(1-2\alpha)l} & \text{if } \alpha < \frac{1}{2}, \\ \log(2^l) & \text{if } \alpha = \frac{1}{2}. \end{cases}$$

To conclude, it suffices to observe that for all $l \geq 1$, if we have $r \in [1-2^{-l}, 1-2^{-(l+1)}]$, then $1-r \leq 2^{-l} \leq 2(1-r)$. Thus, we deduce for any $\frac{1}{2} < r < 1$,

$$M_p(r,f) \gtrsim \begin{cases} 1 & \text{if } \alpha > \frac{1}{2}, \\ (1-r)^{\alpha-\frac{1}{2}} & \text{if } \alpha < \frac{1}{2}, \\ \sqrt{|\log(1-r)|} & \text{if } \alpha = \frac{1}{2}. \end{cases}$$ ∎

Using a similar method, we obtain the following estimates for the case $p \in (1,2)$.

Proposition 2.2 *Let $1 < p < 2$, $\alpha \in \mathbb{R}$ and $f \in H(\mathbb{D})$. Assume that f is a frequently hypercyclic vector for T_α. Then, for all $0 < r < 1$, the following estimates hold:*

$$M_p(r,f) \gtrsim \begin{cases} (1-r)^{\alpha-\frac{1}{p'}} & \text{if } \alpha < 1/p', \\ |\log(1-r)|^{\frac{1}{p}} & \text{if } \alpha = 1/p', \\ 1 & \text{if } \alpha > 1/p', \end{cases}$$

with $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof We write $f(z) = a_0 + \sum_{k\geq 1} a_k k^{-\alpha} z^k$. Since f is frequently hypercyclic, there exists an increasing subsequence $(n_k) \subset \mathbb{N}$ with positive lower density such that, for all $k \geq 1$, $|T_\alpha^{n_k} f(0) - 3/2| = |a_{n_k} - 3/2| < 1/2$, which implies that $|a_{n_k}| \geq 1$. Moreover, since the sequence (n_k) has positive lower density, there exists $C > 1$ such that $k \leq n_k \leq Ck$. Let us consider a positive integer $l \geq 1$.

Case $\alpha \neq \frac{1}{p'}$: from the Hausdorff–Young inequality (we refer the reader to [7]), we get, for any r in $[1-2^{-l}, 1-2^{-(l+1)}]$,

$$[M_p(r,f)]^{p'} \geq \left[M_p(1-2^{-l},f)\right]^{p'} \geq \sum_{k\geq 1} \frac{|a_k|^{p'}}{k^{\alpha p'}} (1-2^{-l})^{kp'}$$
$$\geq \sum_{k\geq 1} \frac{|a_{n_k}|^{p'}}{n_k^{\alpha p'}} (1-2^{-l})^{n_k p'}.$$

Thus, we have

$$[M_p(r,f)]^{p'} \geq \sum_{k\geq 1} \frac{\left(1-2^{-l}\right)^{Ckp'}}{(Ck)^{\alpha p'}} \gtrsim \sum_{k=1}^{2^l} k^{-\alpha p'} \gtrsim \begin{cases} 1 & \text{if } \alpha > \frac{1}{p'} \\ 2^{(1-\alpha p')l} & \text{if } \alpha < \frac{1}{p'}. \end{cases}$$

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Hence, we obtain

$$M_p(r,f) \gtrsim \begin{cases} 1 & \text{if } p'\alpha > 1, \\ (1-r)^{\alpha - \frac{1}{p'}} & \text{if } p'\alpha < 1. \end{cases}$$

Case $\alpha = \frac{1}{p'}$: we use a Hardy–Littewood inequality [7, Theorem 6.2] to obtain for any $r \in [1-2^{-l}, 1-2^{-(l+1)}]$,

$$[M_p(r,f)]^p \geq \left[M_p(1-2^{-l},f)\right]^p \geq \sum_{k\geq 1} |a_k|^p k^{-\frac{p}{p'}} (1+k)^{p-2} \left(1-2^{-l}\right)^{kp}.$$

Thus, we have

$$\left[M_p(r,f)\right]^p \geq \left[M_p(1-2^{-l},f)\right]^p \gtrsim \sum_{k\geq 1} \frac{|a_{n_k}|^p}{n_k} \left(1-2^{-l}\right)^{n_k p}$$
$$\gtrsim \sum_{k\geq 1} \frac{\left(1-2^{-l}\right)^{Ckp}}{Ck} \gtrsim \sum_{k=1}^{2^l} \frac{1}{k} \gtrsim \log(2^l).$$

Hence, we obtain

$$M_p(r,f) \gtrsim |\log(1-r)|^{\frac{1}{p}}. \qquad \blacksquare$$

3 Proof of the Main Result

3.1 Definitions and Notation

The proof of Theorem 1.3 follows the construction given in [6]. In particular, we use the so-called Rudin–Shapiro polynomials (combined with the de la Vallée–Poussin polynomials), which have coefficients ±1 (or bounded by 1) and an optimal growth of sup-norm (*ultra-flat* polynomials). Let us recall [6, Lemma 2.1], which records the result of Rudin–Shapiro [12].

Lemma 3.1

(i) *For each* $N \geq 1$, *there is a trigonometric polynomial* $p_N = \sum_{k=0}^{N-1} \varepsilon_{N,k} e^{ik\theta}$ *where* $\varepsilon_{N,k} = \pm 1$ *for all* $0 \leq k \leq N-1$, *with at least half of the coefficients being* $+1$ *and with*

$$\|p_N\|_p \leq 5\sqrt{N} \text{ for } p \in [2, +\infty].$$

(ii) *For each* $N \geq 1$, *there is a trigonometric polynomial* $p_N^* = \sum_{k=0}^{N-1} a_{N,k} e^{ik\theta}$ *where* $|a_{N,k}| \leq 1$ *for all* $0 \leq k \leq N-1$, *with at least* $\lfloor \frac{N}{4} \rfloor$ *coefficients being* $+1$ *and with*

$$\|p_N^*\|_p \leq 3N^{1/p'} \text{ for } p \in [1,2].$$

We keep the notation adopted in [6]. For any given polynomial q with $q(z) = \sum_{j=0}^{d} b_j z^j$ with $b_d \neq 0$, we denote $d = \deg(q)$ and $\|q\|_{\ell^1} = \sum_{j=0}^{d} |b_j|$. We set $2\mathbb{N} = \bigcup_{k\geq 1} \mathcal{A}_k$ where for any $k \geq 1$, $\mathcal{A}_k = \left\{2^k(2j-1); j \in \mathbb{N}\right\}$. Denote by $\mathcal{P}$ the countable

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set of polynomials with rational coefficients and let us also consider pairs (q, l) with $q \in \mathcal{P}$ and $l \in \mathbb{N}$ satisfying $\|q\|_{\ell^1} \leq l$. Let us consider an enumeration (q_k) of $\mathcal{P}$ and a sequence (l_k) tending to $+\infty$ such that $\|q_k\| \leq l_k$. Clearly, (q_k) is a dense set in $H(\mathbb{D})$. Hence, for any $k \geq 1$, we set $d_k = \deg(q_k)$, and we have

$$\|q_k\|_{\ell^1} \leq l_k \text{ for every } k \geq 1.$$

In our context, for any $\alpha \in \mathbb{R}$, we will also need to modify the family of polynomials (q_k) as follows: for any positive integer $k \geq 1$, we set $\widetilde{q_k}(z) = \sum_{j=0}^{d_k} j^\alpha b_j^{(k)} z^j$ (we omit the dependance on α in order to keep some readable notation).

Let α be a real number and let $p \in (1, \infty]$. For all integers $n \geq 0$, we set $I_n = \{2^n, \ldots, 2^{n+1} - 1\}$. Next, for $k \geq 1$, let us define the integers

$$\alpha_k = 1 + \left\lfloor \max\left(l_k^2 d_k^{2\max(\alpha,0)}, d_k + \max(3, 3+\alpha) l_k^2 + \max(\alpha, 0) l_k \log(1 + d_k)\right)\right\rfloor,$$
$$\alpha_k^* = 1 + \left\lfloor \max\left(l_k^{p'} d_k^{p'\max(\alpha,0)}, d_k + \max(3, 3+\alpha) l_k^2 + \max(\alpha, 0) l_k \log(1 + d_k)\right)\right\rfloor.$$

We set $f_\alpha = \sum_{n\geq 0} P_{n,\alpha}$ where the blocks $(P_{n,\alpha})$ are polynomials defined as follows, using Rudin–Shapiro polynomials given by Lemma 3.1,

$$P_{n,\alpha}(z) = \begin{cases} 0 & \text{if } n \text{ is odd}, \\ 0 & \text{if } n \in \mathcal{A}_k \text{ and } 2^{n-1} < \alpha_k, \\ z^{2^n} Q_n(z) & \text{if } n \in \mathcal{A}_k \text{ and } 2^{n-1} \geq \alpha_k, \end{cases} \tag{3.1}$$

with for $n \in \mathcal{A}_k$,

$$Q_n(z) = \sum_{j \in I_n} j^{-\alpha} c_{j-2^n}^{(k)} z^{j-2^n},$$

where the sequence $(c_j^{(k)})$ denotes the sequence of the coefficients of the polynomial given by $z \mapsto p_{\lfloor \frac{2^{n-1}}{\alpha_k} \rfloor}(z^{\alpha_k}) \tilde{q}_k(z)$.

We also set $f_\alpha^* = \sum_{n\geq 0} P_{n,\alpha}^*$ where the blocks $(P_{n,\alpha}^*)$ are polynomials defined as follows, using the de la Vallée–Poussin polynomials given by Lemma 3.1,

$$P_{n,\alpha}^*(z) = \begin{cases} 0 & \text{if } n \text{ is odd}, \\ 0 & \text{if } n \in \mathcal{A}_k \text{ and } 2^{n-1} < \alpha_k^*, \\ z^{2^n} Q_n^*(z) & \text{if } n \in \mathcal{A}_k \text{ and } 2^{n-1} \geq \alpha_k^*, \end{cases} \tag{3.2}$$

with, for $n \in \mathcal{A}_k$,

$$Q_n^*(z) = \sum_{j \in I_n} j^{-\alpha} c_{j-2^n}^{(k)} z^{j-2^n},$$

where the sequence $(c_j^{(k)})$ denotes the sequence of the coefficients of the polynomial given by $z \mapsto p^*_{\lfloor \frac{2^{n-1}}{\alpha_k^*} \rfloor}(z^{\alpha_k^*}) \widetilde{q_k}(z)$.

A combination of Lemma 3.2 (resp. Lemma 3.3) below with the triangle inequality will ensure that the function f_α (resp. f_α^*) belongs to $H(\mathbb{D})$. Observe that, if we denote the polynomial $z \mapsto p_{\lfloor \frac{2^{n-1}}{\alpha_k} \rfloor}(z^{\alpha_k})$ (resp. $z \mapsto p^*_{\lfloor \frac{2^{n-1}}{\alpha_k} \rfloor}(z^{\alpha_k^*})$) by g_k (resp. g_k^*), we have, for

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all $1 \leq p \leq +\infty$, $\|g_k\|_p = \|p_{\lfloor \frac{2^{n-1}}{\alpha_k} \rfloor}\|_p$ (resp. $\|g_k^*\|_p = \|p^*_{\lfloor \frac{2^{n-1}}{\alpha_k^*} \rfloor}\|_p$). Finally, for any integer n, let us denote $(\phi_n(k))$ the sequence defined as follows

$$\phi_n(k) = \begin{cases} k^{-\alpha} & \text{if } k \in I_n, \\ 0 & \text{otherwise.} \end{cases}$$

3.2 Growth of Specific Functions: the L^p–case for $1 < p < +\infty$

In this subsection, for all $1 < p < +\infty$, we are going to construct specific functions $f \in H(\mathbb{D})$ and we shall study their growth. We will appeal to a strong form of the Marcinkiewicz Multiplier Theorem (we refer the reader to [8]) to deal with all the cases except for the critical cases (*i.e.*, $\alpha = \frac{1}{2}$ when $p \geq 2$ and $\alpha = \frac{1}{p'}$ when $1 < p < 2$) where we will use a Paley–Littlewood decomposition or a method of interpolation to conclude. We begin this section with some useful lemmas.

Lemma 3.2 *For any* $2 \leq p < +\infty$, *any* $0 < r < 1$ *and any* $n \in \mathbb{N}$, *we have*

$$M_p(r, P_{n,\alpha}) \lesssim 2^{n(\frac{1}{2}-\alpha)} r^{2^n}.$$

Proof Let $2 \leq p < +\infty$. Let n be a positive integer. Without loss of generality, we can assume that n belongs to the set $\mathcal{A}_k$ for some $k \geq 1$. Let r be in $(0,1)$. Since $r \mapsto M_p(r, \cdot)$ is increasing, we get

$$M_p(r, P_{n,\alpha}) \leq r^{2^n} \|Q_n\|_p.$$

Then the polynomial Q_n can be viewed as a trigonometric polynomial obtained by an abstract convolution operator on $\mathbb{T}$, given by $(c_k)_{k\geq 0} \mapsto (\phi_n(j) c^{(k)}_{j-2^n})_{j\geq 0}$ (where $(c_j^{(k)})$ denotes the sequence of the coefficients of the polynomial $p_{\lfloor \frac{2^{n-1}}{\alpha_k} \rfloor} \widetilde{q_k}$). Now, we are going to apply the Marcinkiewicz Multiplier Theorem [8, Theorem 8.2 p. 148]. To do this, observe that we have, for any $l \geq 1$,

$$\sup_{j\in I_l} |\phi_n(j)| \leq \sup_{j\in I_n} |\phi_n(j)| \lesssim 2^{-n\alpha},$$
$$\sup_l \sum_{j\in I_l} |\phi_n(j+1) - \phi_n(j)| \leq \sum_{j\in I_n} |\phi_n(j+1) - \phi_n(j)| \lesssim 2^{-n\alpha}.$$

Hence, taking into account the choice of α_k and Lemma 3.1, we get

$$\begin{aligned} \|Q_n\|_p &\lesssim 2^{-n\alpha} \|p_{\lfloor \frac{2^{n-1}}{\alpha_k} \rfloor}\|_p \|\tilde{q}_k\|_\infty \\ &\lesssim 2^{-n\alpha} \sqrt{\frac{2^n}{\alpha_k}}\, l_k (1+d_k)^{\max(\alpha,0)} \\ &\lesssim 2^{n(\frac{1}{2}-\alpha)}. \end{aligned}$$

Finally, we obtain the desired estimate

$$M_p(r, P_{n,\alpha}) \lesssim 2^{n(\frac{1}{2}-\alpha)} r^{2^n}.$$ ∎

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Lemma 3.3 *For any* $1 < p < 2$, *any* $0 < r < 1$, *and any* $n \in \mathbb{N}$, *we have*

$$M_p(r, P^*_{n,\alpha}) \lesssim 2^{n(\frac{1}{p'}-\alpha)} r^{2^n}.$$

Proof The proof is similar to that of Lemma 3.2. Let $1 < p < 2$ and n be a positive integer. Without loss of generality, we can assume that n belongs to the set $\mathcal{A}_k$ for some $k \geq 1$. Let r be in $(0,1)$. Again, combining the Marcinkiewicz Multiplier Theorem with Lemma 3.1 and the choice of α^*_k, we obtain

$$\|Q^*_n\|_p \lesssim 2^{-n\alpha} \|p^*_{\lfloor \frac{2^n-1}{\alpha^*_k} \rfloor}\|_p \|\tilde{q}_k\|_\infty \lesssim 2^{-n\alpha} \Big(\frac{2^n}{\alpha^*_k}\Big)^{\frac{1}{p'}} l_k (1+d_k)^{\max(\alpha,0)} \lesssim 2^{n(\frac{1}{p'}-\alpha)}.$$

Finally, using (3.2), we deduce the desired estimate. ∎

Now we are ready to obtain the rate of growth of the aforementioned functions f_α and f^*_α. We begin with the case $2 \leq p < +\infty$.

Lemma 3.4 *Let* $2 \leq p < +\infty$. *For all* $0 < r < 1$, *the following estimates hold:*

$$M_p(r, f_\alpha) \lesssim \begin{cases} (1-r)^{\alpha-\frac{1}{2}} & \textit{if } \alpha < 1/2, \\ \sqrt{|\log(1-r)|} & \textit{if } \alpha = 1/2, \\ 1 & \textit{if } \alpha > 1/2. \end{cases}$$

Proof Let $2 \leq p < +\infty$. Let us recall that $f_\alpha = \sum_{n\geq 0} P_{n,\alpha}$. First, we deal with the case $\alpha > \frac{1}{2}$. Combining Lemma 3.2 with the triangle inequality, we obtain, for any r in $(0,1)$,

$$M_p(r, f_\alpha) \leq \sum_{n\geq 0} M_p(r, P_{n,\alpha}) \lesssim \sum_{n\geq 0} 2^{n(\frac{1}{2}-\alpha)} r^{2^n} \lesssim 1.$$

Then we treat the case $\alpha < \frac{1}{2}$. Combining Lemma 3.2 with the triangle inequality and the integral comparison test, we get, for any r in $(0,1)$,

$$M_p(r, f_\alpha) \leq \sum_{n\geq 0} M_p(r, P_{n,\alpha}) \lesssim \sum_{n\geq 0} 2^{n(\frac{1}{2}-\alpha)} r^{2^n} \lesssim (1-r)^{\alpha-\frac{1}{2}}.$$

Finally, for the case $\alpha = \frac{1}{2}$, we have by a corollary of the Paley–Littlewood decomposition (we refer the reader to [8, Theorem 5.3.1]) and the integral comparison test $\forall 0 < r < 1$,

$$M_p(r, f_\alpha) \lesssim \Big(\sum_{n\geq 0} \big[M_p(r, P_{n,\alpha})\big]^2 \Big)^{\frac{1}{2}} \lesssim \Big(\sum_{n\geq 0} r^{2^n} \Big)^{\frac{1}{2}}$$
$$\lesssim \sqrt{|\log(1-r)|}.$$
∎

Next we deal with the case $1 < p < 2$. The following result holds.

Lemma 3.5 *Let* $1 < p < 2$. *For all* $0 < r < 1$, *the following estimates hold:*

$$M_p(r, f^*_\alpha) \lesssim \begin{cases} (1-r)^{\alpha-\frac{1}{p'}} & \textit{if } \alpha < 1/p', \\ |\log(1-r)|^{\frac{1}{p}} & \textit{if } \alpha = 1/p', \\ 1 & \textit{if } \alpha > 1/p'. \end{cases}$$

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Proof Let $1 < p < 2$. Let us recall that $f_\alpha^* = \sum_{n\geq 0} P_{n,\alpha}^*$. Then, applying Lemma 3.3, we obtain by the triangle inequality, for any $0 < r < 1$,

$$M_p(r, f_\alpha^*) \leq \sum_{n\geq 0} M_p(r, P_{n,\alpha}^*) \lesssim \sum_{n\geq 0} 2^{n(\frac{1}{p'}-\alpha)} r^{2^n}.$$

Therefore, using a straightforward estimate or the integral comparison test, we get, for any $0 < r < 1$,

$$M_p(r, f_\alpha^*) \lesssim \begin{cases} 1 & \text{if } p'\alpha > 1, \\ (1-r)^{\alpha - \frac{1}{p'}} & \text{if } p'\alpha < 1. \end{cases}$$

Finally, we deal with the case $\alpha = \frac{1}{p'}$: first we have by the classical L^p Bernstein inequality

$$M_p(r, (P_{n,\alpha}^*)') \lesssim 2^n M_p(r, P_{n,\alpha}^*) \lesssim 2^n r^{2^n}.$$

By an easy calculation, we deduce

$$M_p(r, (f_\alpha^*)') \lesssim \sum_{n\geq 0} 2^n r^{2^n} \lesssim \frac{1}{1-r}.$$

Hence, [9, Theorem 2] gives the following conclusion, for any $0 < r < 1$,

$$M_p(r, f_\alpha^*) \lesssim |\log(1-r)|^{\frac{1}{p}}.$$

■

3.3 Growth of Specific Functions: the L^∞ case

This part is based on the use of a weighted Bernstein inequality for the case $\alpha \leq 0$. Then we proceed to an induction based on a fine control of derivatives, since the multipliers we are dealing with are extremely localized for the case $\alpha \geq 0$. Nonetheless, to deal with the critical case given by $\alpha = \frac{1}{2}$, we need to exploit the flatness of the Rudin–Shapiro polynomials.

We begin this section with some useful estimates of the sup norm of the aforementioned polynomials $P_{n,\alpha}$.

Lemma 3.6 *For any $\alpha \leq 0$ and any integer $n \geq 0$, we have*

$$M_\infty(r, P_{n,\alpha}) \lesssim 2^{n(\frac{1}{2}-\alpha)} r^{2^n}.$$

Proof Let $\alpha \leq 0$. Let n be an integer such that n belongs to $\mathcal{A}_k$ for some $k \geq 1$. Then, using the form of the polynomial $P_{n,\alpha}$, Lemma 3.1, and a fractional Bernstein's inequality (see [13, Theorem 19.10 and Remark 19.5]), we obtain

$$M_\infty(r, P_{n,\alpha}) \lesssim 2^{-n\alpha} r^{2^n} \sqrt{\frac{2^n}{\alpha_k}} l_k.$$

Finally, the choice of α_k ensures that

$$M_\infty(r, P_{n,\alpha}) \lesssim 2^{-n(\alpha - \frac{1}{2})} r^{2^n}.$$

■

Then we complete the previous proof to obtain the following lemma.

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Lemma 3.7 *For any $\alpha \in \mathbb{R}$ and any integer $n \geq 0$, we have*

$$M_\infty(r, P_{n,\alpha}) \lesssim 2^{n(\frac{1}{2}-\alpha)} r^{2^n}.$$

Proof In the case $\alpha \leq 0$, the proof comes from Lemma 3.6. Assume now that $\alpha > 0$. First, we deal with the case $0 < \alpha \leq 1$. We keep the notation of Section 3.1. Let $n \geq 1$ be an integer and assume that n belongs to $\mathcal{A}_k$ for some $k \geq 1$. Then we write

$$P_{n,\alpha}(z) = \sum_{j \in I_n} \frac{u_j}{j^\alpha} z^j \quad \text{and} \quad P'_{n,\alpha}(z) = \sum_{j \in I_n} \frac{u_j}{j^{\alpha-1}} z^{j-1}.$$

Since $\alpha - 1 \leq 0$, it follows from the proof of Lemma 3.6 that

$$M_\infty(r, P'_{n,\alpha}) \lesssim 2^{-n((\alpha-1)-\frac{1}{2})} r^{2^n-1}.$$

We deduce, for any $\theta \in \mathbb{R}$ and any $0 < r < 1$,

$$\begin{aligned} |P_{n,\alpha}(re^{i\theta})| = \left| \int_0^r e^{i\theta} P'_{n,\alpha}(te^{i\theta}) dt \right| &\leq \int_0^r |P'_{n,\alpha}(te^{i\theta})| dt \\ &\lesssim 2^{-n((\alpha-1)-\frac{1}{2})} \int_0^r t^{2^n-1} dt \\ &\lesssim 2^{-n(\alpha-\frac{1}{2})} r^{2^n}. \end{aligned}$$

Finally, we obtain

$$M_\infty(r, P_{n,\alpha}) \lesssim 2^{n(\frac{1}{2}-\alpha)} r^{2^n}.$$

The general case (given by $\alpha \geq 0$) follows by a straightforward induction on $l \geq 0$, which is the order of differentiation with $l < \alpha \leq l + 1$. ∎

Now we are ready to estimate the rate of growth of the aforementioned function f_α.

Lemma 3.8 *For all $0 < r < 1$, the following estimates hold:*

$$M_\infty(r, f_\alpha) \lesssim \begin{cases} (1-r)^{\alpha-\frac{1}{2}} & \text{if } \alpha < 1/2, \\ |\log(1-r)| & \text{if } \alpha = 1/2, \\ 1 & \text{if } \alpha > 1/2. \end{cases}$$

Proof Let us recall that $f_\alpha = \sum_{n\geq 0} P_{n,\alpha}$. Taking into account Lemma 3.7, we obtain—along the same lines as the proof of Lemma 3.4— on one hand, for $\alpha > \frac{1}{2}$ and for any r in $(0,1)$,

$$M_\infty(r, f_\alpha) \leq \sum_{n\geq 0} M_\infty(r, P_{n,\alpha}) \lesssim \sum_{n\geq 0} 2^{n(\frac{1}{2}-\alpha)} r^{2^n} \lesssim 1,$$

on the other hand, for $\alpha < \frac{1}{2}$ and for any r in $(0,1)$,

$$M_\infty(r, f_\alpha) \leq \sum_{n\geq 0} M_\infty(r, P_{n,\alpha}) \lesssim \sum_{n\geq 0} 2^{n(\frac{1}{2}-\alpha)} r^{2^n} \lesssim (1-r)^{\alpha-\frac{1}{2}}.$$

Finally the case $\alpha = 1/2$ is easy: Lemma 3.7 implies

$$M_\infty(r, f_{\frac{1}{2}}) \leq \sum_{n\geq 0} M_\infty(r, P_{n,\frac{1}{2}}) \lesssim \sum_{n\geq 0} r^{2^n},$$

and the estimate $\sum_{n\geq 0} r^{2^n} \lesssim |\log(1-r)|$ gives the conclusion. ∎

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3.4 Frequent Hypercyclicity

In this subsection, we are going to prove that the functions f_α and f_α^* given by Section 3.1 are frequently hypercyclic vectors for the weighted Taylor shift operator $T_\alpha, \alpha \in \mathbb{R}$. The construction of these elements and the proof of this crucial fact are an adaptation of the ideas given by Drasin and Saksman in the case of Mac–Lane differentiation operator [6].

Lemma 3.9 *We keep the notation of Section 3.1. For $p \geq 2$ (resp. $1 < p < 2$), the function f_α (resp. f_α^*) is a frequently hypercyclic vector for the operator T_α.*

Proof Since the proof for f_α^* is very similar, we only prove the frequent hypercyclicity of f_α for the operator T_α. We do not repeat the details for f_α^*: it suffices to replace α_k and $p_{\lfloor \frac{2^{n-1}}{\alpha_k} \rfloor}$ by α_k^* and $p^*_{\lfloor \frac{2^{n-1}}{\alpha_k^*} \rfloor}$ and to adapt the proof.

Let k be a large enough integer. Let us consider $n \in \mathcal{A}_k$ such that $2^{n-1} \geq \alpha_k$. We consider $\mathcal{B}_n$ the set of s in $[2^n, 2^{n+1}] \cap \mathbb{N}$ such that the coefficient z^s in the polynomial $z^{2^n} p_{\lfloor \frac{2^{n-1}}{\alpha_k} \rfloor}(z^{\alpha_k})$ is equal to 1. We denote by $T_k = \left\{ s : s \in \mathcal{B}_n,\ n \in \mathcal{A}_k,\ 2^{n-1} \geq \alpha_k \right\}$. According to Lemma 3.1, we have

$$\#\mathcal{B}_n \geq \frac{2^n}{10\alpha_k} = \frac{\#(\{2^n, \ldots, 2^{n+1}-1\})}{10\alpha_k} = \frac{\#(\{2^n, \ldots, 2^{n+2}-1\})}{30\alpha_k}.$$

Since the elements of $\mathcal{A}_k$ are in arithmetic progression, we finally obtain that T_k has positive lower density.

Then let α be a real number and let k be in $\mathbb{N}$. Now let us consider $s \in \mathcal{B}_n$ with $n \in \mathcal{A}_k$ satisfying $2^{n-1} \geq \alpha_k$. We are going to prove that

$$\sup_{|z| = 1 - \frac{1}{l_k}} |T_\alpha^s(f_\alpha)(z) - q_k(z)| \lesssim \frac{1}{l_k},$$

provided that k is chosen large enough. This will allow us to obtain the frequent hypercyclicity property of T_α using the properties of the enumeration (q_k, l_k). Therefore, to do this, we choose an even integer $n \geq 1$ and $s \in \mathcal{B}_n$. Let us write $s = 2^n + m\alpha_k$ for some $m \in \{0, \ldots, \lfloor \frac{2^{n-1}}{\alpha_k} \rfloor\}$ with

$$z^{2^n} p_{\lfloor \frac{2^{n-1}}{\alpha_k} \rfloor}(z^{\alpha_k}) = \cdots + \mathbf{1} z^s + \cdots.$$

By the choice of α_k, the next block of coefficients is dissociated from the present one because of the condition $\alpha_k + d_k < 2\alpha_k$. Hence, using the form of the polynomials $\tilde{q}_k$ and the definition of T_α, we have that the first d_k Taylor coefficients of $T_\alpha^s(f_\alpha)$ are precisely those of q_k. Notice again by the choice of α_k (or α_k^*) that the Taylor coefficients of $T_\alpha^s(f_\alpha)$ of index j with $s + d_k + 1 \leq j \leq s + (\max(3, 3+\alpha))\, l_k^2 + \max(\alpha, 0) l_k \log(d_k)$ are null. Moreover, since n is an even integer, by the construction of each block P_n, all the coefficients having indexes lying in $[2^{n+1}, 2^{n+2}-1]$ are also null. Indeed, the last index that belongs to $\mathcal{B}_n$ is bounded from above by $2^n + 2^{n-1} + d_k < 2^{n+1}$ (by the choice of n such that $2^n \geq 2\alpha_k \geq 2d_k$). Therefore, we can write

$$T_\alpha^s(f_\alpha)(z) - q_k(z) = \sum_{j_k \leq j \leq 2^{n+1}-1} \frac{c_j^{(k)}}{(j-s)^\alpha} z^{j-s} + \sum_{j=2^{n+2}}^{+\infty} \frac{c_j^{(k)}}{(j-s)^\alpha} z^{j-s} := S_1(z) + S_2(z),$$

where $j_k = s + (\max(3, 3+\alpha))\, l_k^2 + \max(\alpha, 0) l_k \log(1 + d_k)$. By the construction of f_α, we observe for any $j \in [j_k, 2^{n+1} - 1] \cap \mathbb{N}$ that all the coefficients $c_j^{(k)}$ coincide with the coefficients of $\tilde{q}_k$ possibly multiplied by some coefficient bounded by 1. Therefore, we obtain, for any $j \in [j_k, 2^{n+1} - 1] \cap \mathbb{N}$, the following estimate:

$$|c_j^{(k)}| \leq \|\tilde{q}_k\|_{\ell_1} \leq d_k^{\max(\alpha,0)} l_k.$$

Hence, we get, by the triangle inequality and from the basic inequality $1 - t \leq e^{-t}$ for $t > 0$,

$$\begin{aligned} \sup_{|z|=1-\frac{1}{l_k}} |S_1(z)| &\leq d_k^{\max(\alpha,0)} l_k \sum_{j \geq j_k} \max((j-s)^{-\alpha}, 1)\Big(1 - \frac{1}{l_k}\Big)^{j-s} \\ &\leq d_k^{\max(\alpha,0)} l_k \sum_{j \geq j_k - s} \max(j^{-\alpha}, 1) e^{-\frac{j}{l_k}}. \end{aligned}$$

For $\alpha > 0$, thanks to the inequality $t/3 \leq 1 - e^{-t}$, for $0 < t < 1$, we get

$$\sup_{|z|=1-\frac{1}{l_k}} |S_1(z)| \leq d_k^\alpha l_k \sum_{j \geq j_k - s} e^{-\frac{j}{l_k}} \leq \frac{d_k^\alpha l_k}{1 - e^{-1/l_k}} e^{-(3+\alpha) l_k - \alpha \log(1+d_k)} \leq \frac{3}{l_k}.$$

In the same spirit, for $\alpha \leq 0$, we get, for all k large enough,

$$\sup_{|z|=1-\frac{1}{l_k}} |S_1(z)| \leq l_k \sum_{j \geq j_k - s} j^{-\alpha} e^{-\frac{j}{l_k}} \lesssim l_k^2 (3 l_k^2)^{-\alpha} e^{-3 l_k} \lesssim \frac{1}{l_k},$$

using an integral comparison test.

Now we deal with the sum $S_2(z)$. Combining the growth of f_α given by Section 3.3 with Cauchy formula's along the radii $r = 1 - 1/j$, we get, for any $j \geq 1$,

$$|c_j^{(k)}| \leq C_p j^\alpha \max(j^{\frac{1}{2}-\alpha+\varepsilon}, 1) \lesssim j^{\max(\frac{1}{2}+\varepsilon, \alpha)},$$

where $C_p > 0$ only depends on p and $0 < \varepsilon \ll 1$ (which allows us to take into account the logarithmic factor for the case $\alpha = \frac{1}{2}$). Since we have both $2^{n+1} \geq 4\alpha_k \geq 4\max(3, 3+\alpha) l_k^2$, which implies $2^{n+1} \geq 4\alpha_k \geq 4\max(1, \alpha) l_k^2$, and $\{j - s : j \geq 2^{n+2}\} \subset \{j \geq 2^{n+1}\}$, we obtain, provided that k is large enough again,

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$$\sup_{|z|=1-\frac{1}{l_k}} |S_2(z)| \lesssim \sum_{j\geq 2^{n+1}} j^{\max(-\alpha,0)} (j+s)^{\max(\frac{1}{2}+\varepsilon,\alpha)} \left(1-\frac{1}{l_k}\right)^j$$
$$\lesssim \sum_{j\geq 2^{n+1}} j^{\max(\frac{1}{2}+\varepsilon,\frac{1}{2}+\varepsilon-\alpha,\alpha)} e^{-\frac{j}{l_k}}$$
$$\lesssim \int_{4\max(1,\alpha) l_k^2}^{+\infty} t^{\max(\frac{1}{2}+\varepsilon,\frac{1}{2}+\varepsilon-\alpha,\alpha)} e^{-\frac{t}{l_k}} dt.$$

Taking into consideration the different cases, an easy calculation leads to the following inequality:

$$\sup_{|z|=1-\frac{1}{l_k}} |S_2(z)| \lesssim l_k^{\beta(\alpha)+1} l_k^{\beta(\alpha)} e^{-4\max(1,\alpha) l_k} \lesssim \frac{1}{l_k},$$

with $\beta(\alpha) = \max(1/2+\varepsilon, 1/2+\varepsilon-\alpha, \alpha)$. This finishes the proof. ∎

3.5 Final Step

Now we are ready to prove Theorem 1.3. To do this, it suffices to combine Lemma 3.9 with:

- Proposition 2.1 and Lemma 3.4, in the case $2 \leq p < +\infty$;
- Proposition 2.1 and Lemma 3.8, in the case $p = +\infty$;
- Proposition 2.2 and Lemma 3.5, in the case $1 < p < 2$.

Remark 3.10 For $p = +\infty$ and $\alpha = \frac{1}{2}$, there is still a question: what is the optimal growth of a frequently hypercyclic vector for $T_{\frac{1}{2}}$? According to Proposition 2.1, such an element f satisfies, for any $0 < r < 1$, $M_\infty(r, f) \gtrsim \sqrt{|\log(1-r)|}$. On the other hand, Lemmas 3.8 and 3.9 ensure that there exists frequently hypercyclic vector f for $T_{\frac{1}{2}}$ such that, for any $0 < r < 1$, $M_\infty(r, f) \lesssim |\log(1-r)|$.

4 The Case $p = 1$

To conclude, let us consider the case $p = 1$. First it is easy to obtain the following estimates. The proof follows the ideas of Proposition 2.1. The case $\alpha = 0$ was already known [11].

Proposition 4.1 *Let $\alpha \in \mathbb{R}$ and $f \in H(\mathbb{D})$. Assume that f is a frequently hypercyclic vector for T_α. Then, for all $0 < r < 1$, the following estimates hold:*

$$M_1(r, f) \gtrsim \begin{cases} (1-r)^\alpha & \text{if } \alpha < 0, \\ -\log(1-r) & \text{if } \alpha = 0, \\ 1 & \text{if } \alpha > 0. \end{cases}$$

Proof We write $f(z) = a_0 + \sum_{k\geq 1} a_k k^{-\alpha} z^k$. We argue as in the beginning of the proof of Proposition 2.1 to find an increasing subsequence $(n_k) \subset \mathbb{N}$ satisfying $k \leq$

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$n_k \le Ck$ with $C > 1$ such that for all k, $|a_{n_k}| \ge 1$. Let us consider a positive integer $l \ge 1$. For any $r \in [1 - 2^{-l}, 1 - 2^{-(l+1)}]$, we apply Hardy's inequality to obtain

$$\begin{aligned} M_1(r,f) &\ge M_1(1-2^{-l},f) \ge \sum_{k\ge 1} \frac{|a_k|}{k^{\alpha+1}} \left(1-2^{-l}\right)^k \\ &\ge \sum_{k\ge 1} \frac{|a_{n_k}|}{n_k^{\alpha+1}} \left(1-2^{-l}\right)^{n_k} \ge \sum_{k\ge 1} \frac{\left(1-2^{-l}\right)^{Ck}}{(Ck)^{\alpha+1}} \\ &\gtrsim \sum_{k=1}^{2^l} k^{-\alpha-1} \gtrsim \begin{cases} (1-r)^{\alpha} & \text{if } \alpha < 0, \\ -\log(1-r) & \text{if } \alpha = 0, \\ 1 & \text{if } \alpha > 0. \end{cases} \end{aligned}$$ ∎

On the other hand, for $\alpha = 0$, we established the following result (see [11]), which gives in some sense the optimality of the estimate of Proposition 4.1. In the following, for any positive integer l, the notation $\log_l$ will stand for the iterated function $\log \circ \cdots \circ \log$ where log appears l times.

Theorem 4.2 ([11]) *For any $l \ge 1$, there is a frequently hypercyclic function $f \in H(\mathbb{D})$ for T satisfying the following estimate: there exists $C > 0$ such that for every $0 < r < 1$ sufficiently large,*

$$M_1(r,f) \le C\left|\log(1-r)\right|\log_l\left(-\log(1-r)\right).$$

For $\alpha \neq 0$, we are going to combine the ideas of [11] with those of Subsection 3.3 to obtain a similar result. Let $l \ge 1$. We also set $f_\alpha^* = \sum_{n\ge 0} P_{n,\alpha}^*$ where the blocks $(P_{n,\alpha}^*)$ are polynomials defined as in (3.2), with

$$\alpha_k^* = 1 + \left\lfloor \max\left(u_k, d_k + \max(3, 3+\alpha)l_k^2 + \max(\alpha,0)l_k \log(1+d_k)\right)\right\rfloor,$$

where u_k is chosen such that $\log_{l+1}(\alpha_k^*) \ge l_k$.

Lemma 4.3 *We have for any $\alpha \in \mathbb{R}$, for all $n \in \mathbb{N}$, and for any $r \in (0,1)$,*

$$M_1(r, P_{n,\alpha}^*) \lesssim r^{2^n} 2^{-n\alpha} l_k.$$

Proof We argue as in the proof of Lemmas 3.6 and 3.7. First, let $\alpha \le 0$. Then, by the maximum principle, we can write

$$\begin{aligned} M_1(r, P_{n,\alpha}^*) &\le \frac{r^{2^n}}{2\pi} \int_0^{2\pi} \left|Q_n^*(re^{it})\right| dt \\ &= \frac{r^{2^n}}{2\pi} \int_0^{2\pi} \Big| \sum_{j\in I_n} j^{-\alpha} c_{j-2^n}^{(k)} (re^{it})^{j-2^n} \Big| dt. \end{aligned} \tag{4.1}$$

Let us consider the trigonometric polynomials

$$U_{n,\alpha}(e^{i\theta}) = \sum_{j\in I_n} j^{-\alpha} c_{j-2^n}^{(k)} r^{j-2^n} e^{ij\theta} \quad \text{and} \quad V_n(e^{i\theta}) = \sum_{j\in I_n} c_{j-2^n}^{(k)} r^{j-2^n} e^{ij\theta}.$$

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Observe that

(4.2)
$$\sum_{j\in I_n} j^{-\alpha} c^{(k)}_{j-2^n} (re^{i\theta})^{j-2^n} = e^{-i2^n\theta} U_{n,\alpha}(e^{i\theta}) \quad \text{and} \quad q^*_{n,k}(e^{i\theta}) = e^{-i2^n\theta} V_n(e^{i\theta}).$$

From (4.1), (4.2), and the fractional Bernstein's inequality (see [13, Theorem 19.10 and Remark 19.5]), we get

$$M_1(r, P^*_{n,\alpha}) \le r^{2^n} \|U_{n,\alpha}\|_1 \lesssim r^{2^n} 2^{-(n+1)\alpha} \|V_n\|_1 = r^{2^n} 2^{-(n+1)\alpha} \|q^*_{n,k}\|_1.$$

Thus, using Lemma 3.1, we obtain

$$M_1(r, P^*_{n,\alpha}) \lesssim r^{2^n} 2^{-n\alpha} l_k. \tag{4.3}$$

Now let us consider the case $\alpha \in (0,1]$. Since we have, for any $r \in (0,1)$,

$$\begin{aligned} |P^*_{n,\alpha}(re^{i\theta})| &= \left| \int_0^r e^{i\theta} \left(P^*_{n,\alpha}\right)'(te^{i\theta}) dt \right| \\ &\le \int_0^r |\left(P^*_{n,\alpha}\right)'(te^{i\theta})| dt = \int_0^r t^{-1} |P^*_{n,\alpha-1}(te^{i\theta})| dt, \end{aligned}$$

we deduce, by integrating this inequality and using (4.3) with $\alpha - 1 \le 0$,

$$M_1(r, P^*_{n,\alpha}) \lesssim \int_0^r t^{-1} M_1(t, P^*_{n,\alpha-1}) dt \lesssim l_k 2^{-n(\alpha-1)} \int_0^r t^{2^n-1} dt \lesssim r^{2^n} 2^{-n\alpha} l_k.$$

The general case follows by an easy induction on m with $m < \alpha \le m+1$. ∎

We are ready to prove in a certain sense the optimality of Proposition 4.1.

Theorem 4.4 *Let $\alpha \in \mathbb{R}$. For any $l \ge 1$, there is a frequently hypercyclic function $f^*_\alpha \in H(\mathbb{D})$ for T satisfying the following estimate: there exists $C > 0$ such that for every $0 < r < 1$ sufficiently large*

$$M_1(r, f) \le \begin{cases} C(1-r)^\alpha \log_l(-\log(1-r)) & \text{if } \alpha < 0, \\ C\,|\log(1-r)|\log_l(-\log(1-r)) & \text{if } \alpha = 0, \\ C & \text{if } \alpha > 0. \end{cases}$$

Proof The case $\alpha = 0$ is given by Theorem 4.2. In the following, for $l \ge 1$, we denote by A_l the smallest integer so that for all $n \ge A_l$, $\log_l(n) \ge \log_{l+1}(\alpha_k)$ (with $2^{n-1} \ge \alpha_k$). Then, for any j large enough, the use of Lemma 4.3 and the inequality $(1-t) \le e^{-t}$, for $0 \le t \le 1$, leads to

$$\begin{aligned} M_1\left(1 - \frac{1}{2^j}, f^*_\alpha\right) &\le \sum_{n\ge 1} M_1\left(1 - \frac{1}{2^j}, P^*_{n,\alpha}\right) \\ &\lesssim \sum_{\substack{n\ge 0,\\ 2^{n-1}\ge \alpha^*_k}} \left(1 - \frac{1}{2^j}\right)^{2^{n-1}} 2^{-n\alpha} l_k \\ &\le \log_{l+1}(\alpha^*_k) A_l 2^{A_l \max(0,-\alpha)} + \sum_{n\ge A_l} e^{-2^{n-j-1}} 2^{-n\alpha} \log_l(n). \end{aligned}$$

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For $\alpha > 0$, the last sum is bounded by

$$\sum_{n\geq A_l} e^{-2^{n-j-1}} 2^{-n\alpha} \log_l(n) \leq \sum_{n\geq A_l} 2^{-n\alpha} \log_l(n) \lesssim 1.$$

For $\alpha < 0$, we can write

$$\begin{aligned}\sum_{n\geq A_l} e^{-2^{n-j-1}} 2^{-n\alpha} \log_l(n) &\leq \sum_{n=A_l}^{j+1} e^{-2^{n-j-1}} 2^{-n\alpha} \log_l(n) \\ &\quad + \sum_{n\geq j+2} e^{-2^{n-j-1}} 2^{-n\alpha} \log_l(n) \\ &\leq \log_l(j+1) \sum_{n=1}^{j+1} 2^{-n\alpha} \\ &\quad + 2^{-(j+1)\alpha} \sum_{m\geq 1} e^{-2^m} 2^{-m\alpha} \log_l(m+j+1) \\ &\leq \log_l(j+1) 2^{-(j+1)\alpha} \\ &\quad + 2^{-(j+1)\alpha} \log_l(j+1) \sum_{m\geq 1} e^{-2^m} 2^{-m\alpha}(m+1) \\ &\lesssim 2^{-j\alpha} \log_l(j).\end{aligned}$$

We deduce

$$M_1(1-\frac{1}{2^j}, f_\alpha^*) \lesssim \begin{cases} \log_l(j) 2^{-j\alpha} & \text{if } \alpha < 0, \\ 1 & \text{if } \alpha > 0. \end{cases}$$

Hence, we obtain for any $1-\frac{1}{2^j} \leq r < 1-\frac{1}{2^{j+1}}$ (with j large enough),

$$\begin{aligned} M_1(r, f_\alpha^*) \leq M_1\Big(1-\frac{1}{2^{j+1}}, f_\alpha^*\Big) &\lesssim \begin{cases} \log_l(j) 2^{-j\alpha} & \text{if } \alpha < 0, \\ 1 & \text{if } \alpha > 0. \end{cases} \\ &\lesssim \begin{cases} \log_l(-\log(1-r))(1-r)^\alpha & \text{if } \alpha < 0, \\ 1 & \text{if } \alpha > 0. \end{cases}\end{aligned}$$

To obtain the frequent hypercyclicity of the vector f_α^* for the weighted Taylor shift T_α, it suffices to mimic the proof of Lemma 3.9. ∎

Remark 4.5 For $p = 1$ and $\alpha \leq 0$, the following problems remain open: does there exist a frequently hypercyclic function $g_\alpha \in H(\mathbb{D})$ for the Taylor shift T_α such that $\limsup_{r\to 1}((1-r)^{-\alpha} M_1(r, g_\alpha)) < +\infty$ if $\alpha < 0$ or $\limsup_{r\to 1}\left(\frac{M_1(r,g_0)}{-\log(1-r)}\right) < +\infty$ if $\alpha = 0$?

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