

# *Decomposable theories*

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## Abstract

We present in this paper a general algorithm for solving first-order formulas in particular theories called *decomposable theories*. First of all, using special quantifiers, we give a formal characterization of decomposable theories and show some of their properties. Then, we present a general algorithm for solving first-order formulas in any decomposable theory  $T$ . The algorithm is given in the form of five rewriting rules. It transforms a first-order formula  $\varphi$ , which can possibly contain free variables, into a conjunction  $\phi$  of solved formulas easily transformable into a Boolean combination of existentially quantified conjunctions of atomic formulas. In particular, if  $\varphi$  has no free variables then  $\phi$  is either the formula *true* or  $\neg$ *true*. The correctness of our algorithm proves the completeness of the decomposable theories. Finally, we show that the theory  $\mathcal{T}$  of finite or infinite trees is a decomposable theory and give some benchmarks realized by an implementation of our algorithm, solving formulas on two-partner games in  $\mathcal{T}$  with more than 160 nested alternated quantifiers.

**KEYWORDS:** Logical first-order formula, Complete theory, Rewriting rules, Theory of trees

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## 1 Introduction

The algebra of (possibly) infinite trees plays a fundamental role in computer science: it is a model for composed data known as *record* in Pascal or *structure* in C. The construction operation corresponds to the creation of a new record, i.e. of a cell containing elementary information possibly followed by  $n$  cells, each one pointing to a record. Infinite trees correspond to a circuit of pointers.

As early as 1976, Huet gave an algorithm for unifying infinite terms, that is solving equations in that algebra (Huet 1976). Clark proposed a complete axiomatization of the equality theory, also called Clark equational theory CET, and gave intuitions about a complete axiomatization of the theory of finite trees (Clark 1978). Courcelle has studied the properties of infinite trees in the scope of recursive program schemes (Courcelle 1983; Courcelle 1986). Colmerauer has described the execution of Prolog II, III and IV programs in terms of solving equations and disequations in that algebra (Colmerauer 1984; Colmerauer 1990; Benhamou 1996).

M. Maher has axiomatized all the cases by complete first-order theories (Maher 1988), i.e. he has introduced the theory  $\mathcal{T}$  of finite or infinite trees having an infinite set  $F$  of functional symbols. It is this theory which has been the starting point of our works. After having studied its properties, we have created a new class of complete theories that we call *decomposable theories* and have shown that a lot of theories used in fundamental computer science are decomposable. We can cite for example: the theory of finite trees, of infinite trees, of finite or infinite trees (Djelloul 2006a), of additive rational or real numbers with addition and subtraction, of linear dense order without endpoints, of ordered additive rational or real numbers with addition, subtraction and a linear dense order relation without endpoints, of the combination of trees and ordered additive rational or real numbers (Djelloul 2005b), of the construction of trees on an ordered set (Djelloul 2005a), of the extension into trees of first-order theories (Djelloul 2006b) and many other combinations of fundamental theories.

T. Dao whose works focused on the theory of finite or infinite trees has given a first version of a general algorithm solving first order formulas in finite or infinite trees (Dao 2000) using a basic simplification of quantified conjunctions of tree atomic formulas. Unfortunately, this simplification holds only in the theory of finite or infinite trees and can not be used in theories having completely different properties, such as the theory of additive rational or real numbers. We have then generalized this result by introducing the term *decomposable theories* (Djelloul 2005a; Djelloul 2005b) and by showing that in each decomposable theory  $T$ , every quantified conjunction of atomic formulas can be decomposed into three embedded sequences of quantifications having very particular properties, which can be expressed with the help of three special quantifiers denoted by  $\exists?$ ,  $\exists!$ ,  $\exists_{\infty}^{\Psi(u)}$  and called *at-most-one*, *exactly-one*, *infinite*. While the quantifiers  $\exists?$ ,  $\exists!$  are just convenient notations already used in other works, the new quantifier  $\exists_{\infty}^{\Psi(u)}$ , one of the essential keys of this class of theories, expresses a property which is not expressible at the first-order level.

On the other hand, we wish to be able to extract from the definition of decomposable theory a general algorithm for solving first-order formulas in any decomposable theory  $T$ . For that, we have given an efficient algorithm for solving first-order formulas in finite or infinite trees from which we have deduced a general algorithm for solving first-order formulas in any decomposable theory  $T$  (Djelloul 2006a). Note that the first part<sup>1</sup> of Djelloul (2006a) was a joint work with Dao in which we improved the algorithm of (Dao 2000) and presented interesting benchmarks on finite or infinite trees with high performances. By solving a formula  $\varphi$  (with or without free variables) in a decomposable theory  $T$ , we mean to transform  $\varphi$  into a conjunction  $\phi$  of solved formulas, which is equivalent to  $\varphi$  in  $T$ , does not contain new free variables and such that: (1) either  $\phi$  is the formula *true*, thus  $\varphi$  is always true in  $T$ , (2) or  $\phi$  is the formula *-true*, thus  $\varphi$  is always false in  $T$ , (3) or  $\phi$  has at least one free variable and is easily transformable into a Boolean combination

<sup>1</sup> The algorithm for solving first-order formulas in finite or infinite trees.

of existentially quantified conjunctions of atomic formulas. In particular, if  $\phi$  has no free variables then  $\phi$  is either the formula *true* or  $\neg$ *true*.

Recently, we have also shown that an extension of the model of Prolog III and IV is possible by allowing the user to incorporate universal and existential quantifiers to Prolog clauses and to solve any first-order formula, with or without free variables, in a combination of trees and first-order theories (Djelloul 2006b). For that, we have first given an automatic way to combine any first-order theory  $T$  with the theory of finite or infinite trees. Note that the two theories can have non-joint signatures. Then, using the definition of decomposable theories, we have established simple conditions on  $T$  and only on  $T$  to get a decomposable combination and thus a complete combination. These extended theories have an interesting power of expressiveness and allow us to model complex problems with first-order formulas in a combination of trees and other first-order theories. We can cite for example the works of Colmerauer (Colmerauer 1990) who has described the execution of Prolog III using a combination of trees and rational numbers with addition, subtraction and linear dense order relation. A full proof of the decomposability of this hybrid theory can be found in detail in Djelloul (2005b).

The paper is organized in five sections followed by a conclusion. This introduction is the first section. The second one introduces the needed elements of first-order logic and ends with a sufficient condition for the completeness of any first-order theory. We have built this condition using a syntactic analysis of the general structure of first-order formulas.

In Section 3, we present the vectorial quantifiers  $\exists?$ ,  $\exists!$ ,  $\exists_{\infty}^{\Psi(u)}$  and show some of their properties. We also give a formal definition of *decomposable theories* and show their completeness using the sufficient condition of completeness defined in section 2. If  $T$  is decomposable, we show that each formula is equivalent in  $T$  to a Boolean combination of basic formulas and give a sufficient condition so that  $T$  accepts full elimination of quantifiers. We end this section with two examples of simple decomposable theories: a simple extension of the Clark equational theory CET (Clark 1978) and the theory of rational or real numbers with addition and subtraction.

In Section 4, we present our algorithm of resolution in any decomposable theory  $T$ , given in the form of a set of five rewriting rules. The conjunction  $\phi$  of solved formulas obtained from an initial formula  $\varphi$  is equivalent to  $\varphi$  in  $T$  and does not have new free variables. In particular, if  $\varphi$  has no free variables then  $\phi$  is either the formula *true* or  $\neg$ *true*. The correctness of our algorithm is another proof of completeness of the decomposable theories.

Finally, we show in Section 5 that the theory  $\mathcal{T}$  of finite or infinite trees is a decomposable theory and end with examples and benchmarks done by an implementation of our algorithm solving formulas on two-partner games in  $\mathcal{T}$  with more than 160 nested alternated quantifiers. We compare our results with those of Djelloul (2006a), Dao (2000) and Colmerauer (2003) where a dedicated algorithm for solving finite or infinite tree constraints has been given. We show that we have competitive results even if our algorithm is general and holds for any decomposable theory  $T$ . This is a detailed full version with full proofs of our works

on decomposable theories (Djelloul 2005b; Djelloul 2006a). The infinite quantifier, the properties of the vectorial quantifiers, the class of the decomposable theories and the algorithm of resolution in any decomposable theory are our contributions in all these works. The proof of decomposability of the theory of equality and the theory of additive rational or real numbers as well as the benchmarks on decomposable theories are our main contributions in this paper.

## 2 Formal preliminaries

### 2.1 Expression

We are given once and for all, an infinite countable set  $V$  of *variables* and the set  $L$  of *logical* symbols:

$$=, \text{true}, \text{false}, \neg, \wedge, \vee, \rightarrow, \leftrightarrow, \forall, \exists, (, ).$$

We are also given once and for all, a *signature*  $S$ , i.e. a set of symbols partitioned into two subsets: the set of *function* symbols and the set of *relation* symbols. To each element  $s$  of  $S$  is linked a non-negative integer called *arity* of  $s$ . An  $n$ -ary symbol is a symbol with arity  $n$ . A 0-ary function symbol is called *constant*.

As usual, an *expression* is a word on  $L \cup S \cup V$  which is either a *term*, i.e. of one of the two forms:

$$x, ft_1 \dots t_n, \quad (1)$$

or a *formula*, i.e. of one of the eleven forms:

$$\begin{aligned} s = t, rt_1 \dots t_n, \text{true}, \text{false}, \\ \neg\varphi, (\varphi \wedge \psi), (\varphi \vee \psi), (\varphi \rightarrow \psi), (\varphi \leftrightarrow \psi), \\ (\forall x \varphi), (\exists x \varphi). \end{aligned} \quad (2)$$

In (1),  $x$  is taken from  $V$ ,  $f$  is an  $n$ -ary function symbol taken from  $S$  and the  $t_i$ 's are shorter terms. In (2),  $s, t$  and the  $t_i$ 's are terms,  $r$  is an  $n$ -ary relation symbol taken from  $S$  and  $\varphi$  and  $\psi$  are shorter formulas. The set of the expressions forms a *first-order language with equality*.

The formulas of the first line of (2) are known as *atomic*, and *flat* if they are of one of the following forms:

$$\text{true}, \text{false}, x_0 = x_1, x_0 = fx_1 \dots x_n, rx_1 \dots x_n,$$

where all the  $x_i$ 's are possibly non-distinct variables taken from  $V$ ,  $f$  is an  $n$ -ary function symbol taken from  $S$  and  $r$  is an  $n$ -ary relation symbol taken from  $S$ . An *equation* is a formula of the form  $s = t$  with  $s$  and  $t$  terms.

An occurrence of a variable  $x$  in a formula is *bound* if it occurs in a sub-formula of the form  $(\forall x \varphi)$  or  $(\exists x \varphi)$ . It is *free* in the contrary case. The *free variables of a formula* are those which have at least one free occurrence in this formula. A *proposition* or a *sentence* is a formula without free variables. If  $\varphi$  is a formula, then we denote by  $\text{var}(\varphi)$  the set of the free variables of  $\varphi$ .

The syntax of the formulas being constraining, we allowed ourselves to use infix notations for the binary symbols and to add and remove brackets when there are no ambiguities.

We do not distinguish two formulas which can be made equal using the following transformations of sub-formulas:

$$\begin{aligned}\varphi \wedge \varphi &\Longrightarrow \varphi, & \varphi \wedge \psi &\Longrightarrow \psi \wedge \varphi, & (\varphi \wedge \psi) \wedge \phi &\Longrightarrow \varphi \wedge (\psi \wedge \phi), \\ \varphi \wedge \text{true} &\Longrightarrow \varphi, & \varphi \vee \text{false} &\Longrightarrow \varphi.\end{aligned}$$

If  $I$  is the set  $\{i_1, \dots, i_n\}$ , we call *conjunction* of formulas and write  $\bigwedge_{i \in I} \varphi_i$ , each formula of the form  $\varphi_{i_1} \wedge \varphi_{i_2} \wedge \dots \wedge \varphi_{i_n} \wedge \text{true}$ . In particular, for  $I = \emptyset$ , the conjunction  $\bigwedge_{i \in I} \varphi_i$  is reduced to *true*. We denote by *FL* the set of the conjunctions of flat formulas. We denote by *AT* the set of the conjunctions of atomic formulas. A set  $\Psi$  of formulas is *closed under conjunction* if for each formula  $\varphi \in \Psi$  and each formula  $\phi \in \Psi$ , the formula  $\varphi \wedge \phi$  belongs to  $\Psi$ . All these considerations will be useful for the algorithm of resolution given in section 4.

## 2.2 Model

A *model* is a couple  $M = (\mathcal{M}, \mathcal{F})$ , where:

- $\mathcal{M}$ , the *universe* or *domain* of  $M$ , is a **nonempty set** disjoint from  $S$ , its elements are called *individuals* of  $M$ ;
- $\mathcal{F}$  is a family of operations and relations in the set  $\mathcal{M}$ , subscripted by the elements of  $S$  and such that:
  - for every  $n$ -ary function symbol  $f$  taken from  $S$ ,  $f^M$  is an  $n$ -ary operation in  $\mathcal{M}$ , i.e. an application from  $\mathcal{M}^n$  in  $\mathcal{M}$ . In particular, when  $f$  is a constant,  $f^M$  belongs to  $\mathcal{M}$ ;
  - for every  $n$ -ary relation symbol  $r$  taken from  $S$ ,  $r^M$  is an  $n$ -ary relation in  $\mathcal{M}$ , i.e. a subset of  $\mathcal{M}^n$ .

Let  $M = (\mathcal{M}, \mathcal{F})$  be a model. An *M-expression*  $\varphi$  is an expression built on the signature  $S \cup \mathcal{M}$  instead of  $S$ , by considering the elements of  $\mathcal{M}$  as 0-ary function symbols. If for each free variable  $x$  of  $\varphi$ , we replace each free occurrence of  $x$  by a same element in  $\mathcal{M}$ , we get an *M-expression* called *instantiation* or *valuation* of  $\varphi$  by individuals of  $M$ .

If  $\varphi$  is an *M-formula*, we say that  $\varphi$  is *true in M* and we write

$$M \models \varphi, \tag{3}$$

if for any instantiation  $\varphi'$  of  $\varphi$  by individuals of  $M$ , the set  $\mathcal{M}$  has the property expressed by  $\varphi'$ , when we interpret the function and relation symbols of  $\varphi'$  by the corresponding functions and relations of  $M$  and when we give to the logical symbols their usual meaning.

### Remark 2.2.1

For every *M-formula*  $\varphi$  without free variables, one and only one of the following properties holds:  $M \models \varphi$ ,  $M \models \neg\varphi$ .

Let us finish this sub-section by a convenient notation. Let  $\bar{x} = x_1 \dots x_n$  be a word on  $V$  and let  $\bar{i} = i_1 \dots i_n$  be a word on  $\mathcal{M}$  or  $V$  of the same length as  $\bar{x}$ . If  $\varphi(\bar{x})$  and  $\phi$  are two *M-formulas*, then we denote by  $\varphi(\bar{i})$ , respectively  $\phi_{\bar{x} \leftarrow \bar{i}}$ , the *M-formula* obtained by replacing in  $\varphi(\bar{x})$ , respectively in  $\phi$ , each free occurrence of  $x_j$  by  $i_j$

### 2.3 Theory

A *theory* is a (possibly infinite) set of propositions called *axioms*. We say that the model  $M$  is a *model of*  $T$ , if for each element  $\varphi$  of  $T$ ,  $M \models \varphi$ . If  $\varphi$  is a formula, we write

$$T \models \varphi,$$

if for each model  $M$  of  $T$ ,  $M \models \varphi$ . We say that the formulas  $\varphi$  and  $\psi$  are *equivalent in*  $T$  if  $T \models \varphi \leftrightarrow \psi$ .

Let  $T$  be a theory. A set  $\Psi$  of formulas is called  $T$ -closed if:

- $\Psi \subseteq AT$ ,
- $\Psi$  is closed under conjunction,
- every flat formula  $\varphi$  is equivalent in  $T$  to a formula which belongs to  $\Psi$  and does not contain other free variables than those of  $\varphi$ .

The sets  $AT$  and  $FL$  are  $T$ -closed in any theory  $T$ . This notion of  $T$ -closed set is useful when we need to transform formulas of  $FL$  into formulas which belong to  $\Psi$ . The transformation of normalized formulas into working formulas defined at Section 4.2 illustrates this notion.

A theory  $T$  is *complete* if for every proposition  $\varphi$ , one and only one of the following properties holds:  $T \models \varphi$ ,  $T \models \neg\varphi$ .

Let us now present a sufficient condition for the completeness of any first-order theory. We will use the abbreviation wnfv for “*without new free variables*”. A formula  $\varphi$  is equivalent to a wnfv formula  $\psi$  in  $T$  means that  $T \models \varphi \leftrightarrow \psi$  and  $\psi$  does not contain other free variables than those of  $\varphi$ .

#### Property 2.3.1

A theory  $T$  is complete if there exists a set of formulas, called *basic formulas*, such that:

1. every flat formula is equivalent in  $T$  to a wnfv Boolean combination of basic formulas,
2. every basic formula without free variables is equivalent in  $T$ , either to *true* or to *false*,
3. every formula of the form

$$\exists x ((\bigwedge_{i \in I} \varphi_i) \wedge (\bigwedge_{i \in I'} \neg\varphi_i)), \quad (4)$$

where the  $\varphi_i$ 's are basic formulas, is equivalent in  $T$  to a wnfv Boolean combination of basic formulas.

#### Proof

Let  $\Phi$  be the set of all the formulas which are equivalent in  $T$  to a wnfv Boolean combination of basic formulas.

Let us show first that every formula  $\psi$  belongs to  $\Phi$ . Let us make a proof by induction on the syntactic structure of  $\psi$ . Without losing generalities we can restrict ourselves to the cases where  $\psi$  contains only flat formulas and the following logical

symbols<sup>2</sup>:  $\neg, \wedge, \exists$ . If  $\psi$  is a flat formula, then  $\psi \in \Phi$  according to the first condition of the property. If  $\psi$  is of the form  $\neg\varphi_1$  or  $\varphi_1 \wedge \varphi_2$ , with  $\varphi_1, \varphi_2 \in \Phi$ , then  $\psi \in \Phi$  according to the definition of  $\Phi$ . If  $\psi$  is of the form  $\exists x \varphi$ , with  $\varphi \in \Phi$ , then according to the definition of  $\Phi$ , the formula  $\varphi$  is equivalent to a wnfv formula  $\varphi'$ , which is a Boolean combination of basic formulas  $\varphi_{ij}$ . Without losing generalities we can suppose that  $\varphi'$  is of the form

$$\varphi' = \bigvee_{i \in I} ((\bigwedge_{j \in J} \varphi_{ij}) \wedge (\bigwedge_{j \in J'} \neg \varphi_{ij})). \quad (5)$$

By distributing the existential quantifier, the formula  $\exists x \varphi'$  is equivalent in  $T$  to

$$\bigvee_{i \in I} (\exists x ((\bigwedge_{j \in J} \varphi_{ij}) \wedge (\bigwedge_{j \in J'} \neg \varphi_{ij}))), \quad (6)$$

which, according to the third condition of the property, belongs to  $\Phi$ . Thus the formula  $\exists x \varphi$ , i.e.  $\psi$ , belongs to  $\Phi$ .

Let now  $\psi$  be a proposition. According to what we have just shown  $\psi \in \Phi$ . Thus, the formula  $\psi$  is equivalent in  $T$  to a Boolean combination of basic formulas without free variables. According to the second condition of the property, one and only one of the following properties holds:  $T \models \psi$ ,  $T \models \neg\psi$ . Thus  $T$  is a complete theory.  $\square$

This sufficient condition is interesting in the sense that it reasons on the syntactic structure of first-order formulas. Informally, the basic formulas are generally formulas of the form  $\exists \bar{x} \alpha$  with  $\alpha \in AT$ . We will use this sufficient condition in Section 3.3 to show the completeness of the decomposable theories.

#### Corollary 2.3.2

If  $T$  satisfies the three conditions of Property 2.3.1 then every formula is equivalent in  $T$  to a wnfv Boolean combination of basic formulas.

This corollary is a consequence of the proof of Property 2.3.1 in which we have shown that if  $\Phi$  is the set of all the formulas which are equivalent in  $T$  to a wnfv Boolean combination of basic formulas then every formula  $\psi$  belongs to  $\Phi$ .

### 3 Decomposable theory

#### 3.1 Vectorial quantifiers

Let  $M$  be a model and let  $T$  be a theory. Let  $\bar{x} = x_1 \dots x_n$  and  $\bar{y} = y_1 \dots y_n$  be two words on  $V$  of the same length. Let  $\phi, \varphi$  and  $\varphi(\bar{x})$  be  $M$ -formulas. We write

$$\begin{aligned} \exists \bar{x} \varphi & \quad \text{for } \exists x_1 \dots \exists x_n \varphi, \\ \forall \bar{x} \varphi & \quad \text{for } \forall x_1 \dots \forall x_n \varphi, \\ \exists ? \bar{x} \varphi(\bar{x}) & \quad \text{for } \forall \bar{x} \forall \bar{y} \varphi(\bar{x}) \wedge \varphi(\bar{y}) \rightarrow \bigwedge_{i \in \{1, \dots, n\}} x_i = y_i, \\ \exists ! \bar{x} \varphi & \quad \text{for } (\exists \bar{x} \varphi) \wedge (\exists ? \bar{x} \varphi). \end{aligned}$$

<sup>2</sup> Because each atomic formula is equivalent in the empty theory to a wnfv quantified conjunction of flat formulas and each formula is equivalent in the empty theory to a wnfv formula which contains only the logical symbols:  $\exists, \wedge, \neg$ .

The word  $\bar{x}$ , which can be the empty word  $\varepsilon$ , is called *vector of variables*. Note that the formulas  $\exists? \varepsilon \varphi$  and  $\exists! \varepsilon \varphi$  are respectively equivalent to *true* and to  $\varphi$  in any model  $M$ .

*Notation 3.1.1*

Let  $Q$  be a quantifier taken from  $\{\forall, \exists, \exists!, \exists?\}$ . Let  $\bar{x}$  be vector of variables taken from  $V$ . We write:

$$Q\bar{x} \varphi \wedge \phi \text{ for } Q\bar{x}(\varphi \wedge \phi).$$

*Example 3.1.2*

Let  $I = \{1, \dots, n\}$  be a finite set. Let  $\varphi$  and  $\phi_i$  with  $i \in I$  be formulas. Let  $\bar{x}$  and  $\bar{y}_i$  with  $i \in I$  be vectors of variables. We write:

$$\begin{aligned} \exists \bar{x} \varphi \wedge \neg \phi_1 & \quad \text{for } \exists \bar{x}(\varphi \wedge \neg \phi_1), \\ \forall \bar{x} \varphi \wedge \phi_1 & \quad \text{for } \forall \bar{x}(\varphi \wedge \phi_1), \\ \exists! \bar{x} \varphi \wedge \bigwedge_{i \in I} (\exists \bar{y}_i \phi_i) & \quad \text{for } \exists! \bar{x}(\varphi \wedge (\exists \bar{y}_1 \phi_1) \wedge \dots \wedge (\exists \bar{y}_n \phi_n) \wedge \text{true}), \\ \exists? \bar{x} \varphi \wedge \bigwedge_{i \in I} \neg(\exists \bar{y}_i \phi_i) & \quad \text{for } \exists? \bar{x}(\varphi \wedge (\neg(\exists \bar{y}_1 \phi_1)) \wedge \dots \wedge (\neg(\exists \bar{y}_n \phi_n)) \wedge \text{true}). \end{aligned}$$

*Property 3.1.3*

If  $T \models \exists? \bar{x} \varphi$  then

$$T \models (\exists \bar{x} \varphi \wedge \neg \phi) \leftrightarrow ((\exists \bar{x} \varphi) \wedge \neg(\exists \bar{x} \varphi \wedge \phi)). \tag{7}$$

*Proof*

Let  $M$  be a model of  $T$  and let  $\exists \bar{x} \varphi' \wedge \neg \phi'$  be an instantiation of  $\exists \bar{x} \varphi \wedge \neg \phi$  by individuals of  $M$ . Let us denote by  $\varphi'_1$  the  $M$ -formula  $(\exists \bar{x} \varphi' \wedge \neg \phi')$  and by  $\varphi'_2$  the  $M$ -formula  $(\exists \bar{x} \varphi') \wedge \neg(\exists \bar{x} \varphi' \wedge \phi')$ . To show the equivalence (7), it is enough to show that

$$M \models \varphi'_1 \leftrightarrow \varphi'_2. \tag{8}$$

If  $M \models \neg(\exists \bar{x} \varphi')$  then  $M \models \neg \varphi'_1$  and  $M \models \neg \varphi'_2$ , thus the equivalence (8) holds.

If  $M \models \exists \bar{x} \varphi'$ . Since  $T \models \exists? \bar{x} \varphi'$ , there exists a unique vector  $\bar{i}$  of individuals of  $M$  such that  $M \models \varphi'_{\bar{x} \leftarrow \bar{i}}$ . Two cases arise:

If  $M \models \neg(\varphi'_{\bar{x} \leftarrow \bar{i}})$ , then  $M \models (\varphi' \wedge \neg \phi')_{\bar{x} \leftarrow \bar{i}}$ , thus  $M \models \varphi'_1$ . Since  $\bar{i}$  is unique and since  $M \models \neg(\varphi'_{\bar{x} \leftarrow \bar{i}})$ , there exists no vector  $\bar{u}$  of individuals of  $M$  such that  $M \models (\varphi' \wedge \phi')_{\bar{x} \leftarrow \bar{u}}$ . Consequently,  $M \models \neg(\exists \bar{x} \varphi' \wedge \phi')$  and thus  $M \models \varphi'_2$ . We have  $M \models \varphi'_1$  and  $M \models \varphi'_2$ , thus the equivalence (8) holds.

If  $M \models \varphi'_{\bar{x} \leftarrow \bar{i}}$ , then  $M \models (\varphi' \wedge \phi')_{\bar{x} \leftarrow \bar{i}}$  and thus  $M \models \neg \varphi'_2$ . Since  $\bar{i}$  is unique and since  $M \models \varphi'_{\bar{x} \leftarrow \bar{i}}$ , there exists no vector  $\bar{u}$  of individuals of  $M$  such that  $M \models (\varphi' \wedge \neg \phi')_{\bar{x} \leftarrow \bar{u}}$ . Consequently,  $M \models \neg(\exists \bar{x} \varphi' \wedge \neg \phi')$  and thus  $M \models \neg \varphi'_1$ . We have  $M \models \neg \varphi'_1$  and  $M \models \neg \varphi'_2$ , thus the equivalence (8) holds.  $\square$

*Corollary 3.1.4*

If  $T \models \exists? \bar{x} \varphi$  then

$$T \models (\exists \bar{x} \varphi \wedge \bigwedge_{i \in I} \neg \phi_i) \leftrightarrow ((\exists \bar{x} \varphi) \wedge \bigwedge_{i \in I} \neg(\exists \bar{x} \varphi \wedge \phi_i)).$$



*Proof*

Let  $\psi$  be the formula  $\neg(\bigwedge_{i \in I} \neg\phi_i)$ . The formula  $\exists \bar{x} \varphi \wedge \bigwedge_{i \in I} \neg\phi_i$ , is equivalent in  $T$  to  $\exists \bar{x} \varphi \wedge \neg\psi$ . Since  $T \models \exists ?\bar{x} \varphi$ , then according to Property 3.1.3 the preceding formula is equivalent in  $T$  to  $(\exists \bar{x} \varphi) \wedge \neg(\exists \bar{x} \varphi \wedge \psi)$ , which is equivalent in  $T$  to  $(\exists \bar{x} \varphi) \wedge \neg(\exists \bar{x} \varphi \wedge \neg(\bigwedge_{i \in I} \neg\phi_i))$ , thus to  $(\exists \bar{x} \varphi) \wedge \neg(\exists \bar{x} \varphi \wedge (\bigvee_{i \in I} \phi_i))$ , which is equivalent in  $T$  to  $(\exists \bar{x} \varphi) \wedge \neg(\exists \bar{x} (\bigvee_{i \in I} (\varphi \wedge \phi_i)))$ , thus to  $(\exists \bar{x} \varphi) \wedge \neg(\bigvee_{i \in I} (\exists \bar{x} \varphi \wedge \phi_i))$ , which is finally equivalent in  $T$  to  $(\exists \bar{x} \varphi) \wedge \bigwedge_{i \in I} \neg(\exists \bar{x} \varphi \wedge \phi_i)$ .  $\square$

*Property 3.1.5*

If  $T \models \exists !\bar{x} \varphi$  then

$$T \models (\exists \bar{x} \varphi \wedge \neg\phi) \leftrightarrow \neg(\exists \bar{x} \varphi \wedge \phi).$$

*Corollary 3.1.6*

If  $T \models \exists !\bar{x} \varphi$  then

$$T \models (\exists \bar{x} \varphi \wedge \bigwedge_{i \in I} \neg\phi_i) \leftrightarrow \bigwedge_{i \in I} \neg(\exists \bar{x} \varphi \wedge \phi_i).$$

### 3.2 The infinite quantifier

Let  $M$  be a model. Let  $T$  be a theory. Let  $\varphi(x)$  be an  $M$ -formula and let  $\Psi(u)$  be a set of formulas having at most  $u$  as free variable. Let us now present our infinite quantifier  $\exists_{\infty}^{\Psi(u)}$ . The main intuitions behind this quantifier come from an aim to get a full elimination of quantifiers in complex  $M$ -formulas of the form  $\exists x \varphi(x) \wedge \bigwedge_{j \in \{1, \dots, n\}} \neg\psi_j(x)$  using the fact that the domain of  $M$  is infinite.

*Definition 3.2.1*

We write

$$M \models \exists_{\infty}^{\Psi(u)} x \varphi(x), \tag{9}$$

if for every instantiation  $\exists x \varphi'(x)$  of  $\exists x \varphi(x)$  by individuals of  $M$  and for every finite subset  $\{\psi_1(u), \dots, \psi_n(u)\}$  of elements of  $\Psi(u)$ , the set of the individuals  $i$  of  $M$  such that  $M \models \varphi'(i) \wedge \bigwedge_{j \in \{1, \dots, n\}} \neg\psi_j(i)$  is infinite.

We write  $T \models \exists_{\infty}^{\Psi(u)} x \varphi(x)$ , if for each model  $M$  of  $T$  we have (9).

This infinite quantifier holds only for models whose set of individuals is infinite. Note that if  $\Psi(u) = \{false\}$  then (9) simply means that  $M$  contains an infinite set of individuals  $i$  such that  $\varphi(i)$ . Informally, the notation (9) states that there exists a full elimination of quantifiers in formulas of the form  $\exists x \varphi(x) \wedge \bigwedge_{j \in \{1, \dots, n\}} \neg\psi_j(x)$  due to an infinite set of valuations of  $x$  in  $M$  which satisfy this formula.

*Property 3.2.2*

Let  $J$  be a finite (possibly empty) set. Let  $\varphi(x)$  and  $\varphi_j(x)$  with  $j \in J$  be  $M$ -formulas. If  $T \models \exists_{\infty}^{\Psi(u)} x \varphi(x)$  and if for each  $\varphi_j(x)$ , at least one of the following properties holds:

- $T \models \exists ?x \varphi_j(x)$ ,
- there exists  $\psi_j(u) \in \Psi(u)$  such that  $T \models \forall x \varphi_j(x) \rightarrow \psi_j(x)$ ,

then

$$T \models \exists x \varphi(x) \wedge \bigwedge_{j \in J} \neg \varphi_j(x)$$

*Proof*

Let  $M$  be a model of  $T$  and let  $\exists x \varphi'(x) \wedge \bigwedge_{j \in J} \neg \varphi'_j(x)$  be an instantiation of  $\exists x \varphi(x) \wedge \bigwedge_{j \in J} \neg \varphi_j(x)$  by individuals of  $M$ . Suppose that the conditions of Property 3.2.2 hold and let us show that

$$M \models \exists x \varphi'(x) \wedge \bigwedge_{j \in J} \neg \varphi'_j(x). \tag{10}$$

Let  $J'$  be the set of the  $j \in J$  such that  $M \models \exists ?x \varphi'_j(x)$  and let  $m$  be the cardinality of  $J'$ . Since for all  $j \in J'$ ,  $M \models \exists ?x \varphi'_j(x)$ , then for every set  $\mathcal{M}'$  of individuals of  $M$  such that  $Cardinality(\mathcal{M}') > m$ , there exists  $i \in \mathcal{M}'$  such that

$$M \models \bigwedge_{j \in J'} \neg \varphi'_j(i). \tag{11}$$

On the other hand, since  $T \models \exists_{\infty}^{\Psi(u)} x \varphi(x)$  and according to Definition 3.2.1 we know that for every finite subset  $\{\psi_1(u), \dots, \psi_n(u)\}$  of  $\Psi(u)$ , the set of the individuals  $i$  of  $M$  such that  $M \models \varphi'(i) \wedge \bigwedge_{k=1}^n \neg \psi_k(i)$  is infinite. Since for all  $j \in J - J'$  we have  $M \models \forall x \varphi_j(x) \rightarrow \psi_j(x)$ , thus  $M \models \forall x (\neg \psi_j(x)) \rightarrow (\neg \varphi_j(x))$ , then there exists an infinite set  $\xi$  of individuals  $i$  of  $M$  such that  $M \models \varphi'(i) \wedge \bigwedge_{j \in J - J'} \neg \varphi'_j(i)$ . Since  $\xi$  is infinite then  $Cardinality(\xi) > m$ , and thus according to (11) there exists at least an individual  $i \in \xi$  such that  $M \models \varphi'(i) \wedge (\bigwedge_{j \in J - J'} \neg \varphi'_j(i)) \wedge (\bigwedge_{k \in J'} \neg \varphi'_k(i))$ . Thus, we have  $M \models \exists x \varphi'(x) \wedge \bigwedge_{j \in J} \neg \varphi'_j(x)$ .  $\square$

*Property 3.2.3*

If  $T \models \exists_{\infty}^{\Psi(u)} x \varphi(x)$  then  $T \models \exists_{\infty}^{\Psi(u)} x true$ .

*Proof*

Let  $M$  be a model of  $T$ . If  $T \models \exists_{\infty}^{\Psi(u)} x \varphi(x)$  then  $M \models \exists_{\infty}^{\Psi(u)} x \varphi(x)$ . According to Definition 3.2.1 there exists an infinite set of individuals  $i$  such that  $M \models \varphi(i) \wedge \bigwedge_{j \in J} \neg \varphi_j(i)$  with  $\varphi_j(u) \in \Psi(u)$  for all  $j \in J$ . Thus there exists an infinite set of individuals  $i$  such that  $M \models true \wedge \bigwedge_{j \in J} \neg \varphi_j(i)$ , i.e.  $M \models \exists_{\infty}^{\Psi(u)} x true$  and thus  $T \models \exists_{\infty}^{\Psi(u)} x true$ .  $\square$

### 3.3 Decomposable theory

We present in this section a formal definition of the *decomposable theories*. Informally, this definition simply states that in every decomposable theory  $T$  each formula of the form  $\exists \bar{x} \alpha$  with  $\alpha$  a  $T$ -closed set is equivalent in  $T$  to a decomposed formula of the form  $\exists \bar{x}' \alpha' \wedge (\exists \bar{x}'' \alpha'' \wedge (\exists \bar{x}''' \alpha'''))$ , where the formulas  $\exists \bar{x}' \alpha'$ ,  $\exists \bar{x}'' \alpha''$  and  $\exists \bar{x}''' \alpha'''$  have elegant properties which can be expressed using vectorial quantifiers.

*Definition 3.3.1*

A theory  $T$  is called *decomposable* if there exists a set  $\Psi(u)$  of formulas having at most  $u$  as free variable, a  $T$ -closed set  $A$  and three sets  $A'$ ,  $A''$  and  $A'''$  of formulas

of the form  $\exists \bar{x} \alpha$  with  $\alpha \in A$  such that:

1. Every formula of the form  $\exists \bar{x} \alpha \wedge \psi$ , with  $\alpha \in A$  and  $\psi$  any formula, is equivalent in  $T$  to a wnfv decomposed formula of the form

$$\exists \bar{x}' \alpha' \wedge (\exists \bar{x}'' \alpha'' \wedge (\exists \bar{x}''' \alpha''' \wedge \psi)),$$

with  $\exists \bar{x}' \alpha' \in A'$ ,  $\exists \bar{x}'' \alpha'' \in A''$  and  $\exists \bar{x}''' \alpha''' \in A'''$ .

2. If  $\exists \bar{x}' \alpha' \in A'$  then  $T \models \exists ? \bar{x}' \alpha'$  and for each free variable  $y$  in  $\exists \bar{x}' \alpha'$ , at least one of the following properties holds:
  - $T \models \exists ? y \bar{x}' \alpha'$ ,
  - there exists  $\psi(u) \in \Psi(u)$  such that  $T \models \forall y (\exists \bar{x}' \alpha') \rightarrow \psi(y)$ .
3. If  $\exists \bar{x}'' \alpha'' \in A''$  then for each  $x_i''$  of  $\bar{x}''$  we have  $T \models \exists_{\infty}^{\Psi(u)} x_i'' \alpha''$ .
4. If  $\exists \bar{x}''' \alpha''' \in A'''$  then  $T \models \exists ! \bar{x}''' \alpha'''$ .
5. If the formula  $\exists \bar{x}' \alpha'$  belongs to  $A'$  and has no free variables then this formula is either the formula  $\exists \text{true}$  or  $\exists \text{false}$ .

Since  $A$  is  $T$ -closed, then  $A$  is a subset of  $AT$ . While the formulas of  $A''$  and  $A'''$  accept full elimination of quantifiers according to the properties of the quantifiers  $\exists !$  and  $\exists_{\infty}^{\Psi(u)}$ , the formulas of  $A'$  can possibly not accept elimination of quantifiers. This is due to the second point of Definition 3.3.1 which states that  $T \models \exists ? \bar{x}' \alpha'$ . The computation of the sets  $A$ ,  $A'$ ,  $A''$ ,  $A'''$  and  $\Psi(u)$  for a theory  $T$  depends on the axiomatization of  $T$ . Generally, it is enough to know how to solve a formula of the form  $\exists \bar{x} \alpha$  with  $\alpha \in FL$  to get a first intuition on the sets  $A'$ ,  $A''$ ,  $A'''$  and  $\Psi(u)$ . Informally, the sets  $A'$ ,  $A''$  and  $A'''$  can be called according to their linked vectorial quantifier, i.e.  $A'$  is the *at most one solution set* and contains formulas which accept at most one solution in  $T$  and possibly not accept full elimination of quantifiers, the set  $A''$  is the *infinite instantiation set* and contains formulas that accept an infinite set of solutions in  $T$ . The set  $A'''$  is the *unique solution set* and contains formulas which have one and only solution in  $T$ . The set  $\Psi(u)$  contains generally simple formulas of the form  $\exists \bar{x} \alpha$  with at most one free variable and  $\alpha \in A$ . It can also be reduced for example to the set  $\{\text{false}\}$ . Note that the sets  $A'$  and  $A'''$  are generally not empty since for every model  $M$  of any theory  $T$  we have  $M \models \exists ?_{\varepsilon} x = y$  and  $M \models \exists ! x x = y$ .

### Property 3.3.2

Let  $T$  be a decomposable theory. Every formula of the form  $\exists \bar{x} \alpha$ , with  $\alpha \in A$ , is equivalent in  $T$  to a wnfv formula of the form  $\exists \bar{x}' \alpha'$  with  $\exists \bar{x}' \alpha' \in A'$ .

### Proof

Let  $\exists \bar{x} \alpha$  be a formula with  $\alpha \in A$ . According to Definition 3.3.1 this formula is equivalent in  $T$  to a wnfv formula of the form

$$\exists \bar{x}' \alpha' \wedge (\exists \bar{x}'' \alpha'' \wedge (\exists \bar{x}''' \alpha''')),$$

with  $\exists \bar{x}' \alpha' \in A'$ ,  $\exists \bar{x}'' \alpha'' \in A''$  and  $\exists \bar{x}''' \alpha''' \in A'''$ . Since  $\exists \bar{x}''' \alpha''' \in A'''$  then according to Definition 3.3.1 we have  $T \models \exists ! \bar{x}''' \alpha'''$  and thus using Property 3.1.5 (with  $\phi = \text{false}$ ) the preceding formula is equivalent in  $T$  to

$$\exists \bar{x}' \alpha' \wedge (\exists \bar{x}'' \alpha''),$$

which is equivalent in  $T$  to

$$\exists \bar{x}' \alpha' \wedge (\exists x''_1 \dots x''_{n-1} (\exists x''_n \alpha'')).$$

Since  $\exists \bar{x}'' \alpha'' \in A''$  then according to Definition 3.3.1 we have  $T \models \exists_{\infty}^{(u)} x''_n \alpha''$  and thus  $T \models \exists x''_n \alpha''$ . The preceding formula is equivalent in  $T$  to

$$\exists \bar{x}' \alpha' \wedge (\exists x''_1 \dots x''_{n-1} \text{true}),$$

which is finally equivalent in  $T$  to

$$\exists \bar{x}' \alpha'.$$

□

Using Property 3.3.2 and the fifth point of Definition 3.3.1 we get

*Corollary 3.3.3*

Let  $T$  be a decomposable theory. Every formula, without free variables, of the form  $\exists \bar{x} \alpha$ , with  $\alpha \in A$ , is equivalent in  $T$  either to *true* or to *false*.

*Theorem 3.3.4*

If  $T$  is decomposable then  $T$  is complete.

*Proof*

Let  $T$  be a decomposable theory which satisfies the five conditions of Definition 3.3.1. Let us show that  $T$  is complete using Property 2.3.1 and by taking formulas of the form  $\exists \bar{x} \alpha$ , with  $\alpha \in A$ , as basic formulas. Note that according to Definition 3.3.1, the sets  $A'$ ,  $A''$  and  $A'''$  contain formulas of the form  $\exists \bar{x} \alpha$  with  $\alpha \in A$ .

Let us show that the first condition of Property 2.3.1 holds, i.e. every flat formula is equivalent in  $T$  to a wnfv Boolean combination of basic formulas. According to Definition 3.3.1 the set  $A$  is  $T$ -closed, i.e. (i) every flat formula is equivalent in  $T$  to a wnfv formula which belongs to  $A$ . Let  $\alpha$  be a flat formula. According to (i)  $\alpha$  is equivalent in  $T$  to a wnfv formula  $\beta$  which belongs to  $A$ . Since  $\beta$  is equivalent in  $T$  to  $\exists \varepsilon \beta$  and  $\beta \in A$  then  $\alpha$  is equivalent in  $T$  to a wnfv basic formula<sup>3</sup>. Thus, the first condition of Property 2.3.1 holds.

Let us show that the second condition of Property 2.3.1 holds, i.e. every basic formula without free variables is either equivalent to *true* or to *false* in  $T$ . Let  $\exists \bar{x} \alpha$  with  $\alpha \in A$  be a basic formula without free variables. According to Corollary 3.3.3 either  $T \models \exists \bar{x} \alpha$  or  $T \models \neg(\exists \bar{x} \alpha)$ . Thus, the second condition of Property 2.3.1 holds.

Let us show now that the third condition of Property 2.3.1 holds, i.e. every formula of the form

$$\exists x (\bigwedge_{i \in I} (\exists \bar{x}_i \alpha_i)) \wedge (\bigwedge_{j \in J} \neg(\exists \bar{y}_j \beta_j)), \tag{12}$$

with  $\alpha_i \in A$  for all  $i \in I$  and  $\beta_j \in A$  for all  $j \in J$ , is equivalent in  $T$  to a wnfv Boolean combination of basic formulas, i.e. to a wnfv Boolean combination of formulas of the form  $\exists \bar{x} \alpha$  with  $\alpha \in A$ . By lifting all the quantifications  $\exists \bar{x}_i$  after having possibly

<sup>3</sup> Of course a basic formula is a particular case of a Boolean combination of basic formulas.

renamed the variables<sup>4</sup> which appear in each  $\bar{x}_i$ , the formula (12) is equivalent in  $T$  to a wnfv formula of the form

$$\exists \bar{x} (\bigwedge_{i \in I} \alpha_i) \wedge \bigwedge_{j \in J} \neg(\exists \bar{y}_j \beta_j),$$

with  $\alpha_i \in A$  for all  $i \in I$  and  $\beta_j \in A$  for all  $j \in J$ . According to Definition 3.3.1 the set  $A$  is  $T$ -closed and thus closed under conjunction. The preceding formula is equivalent in  $T$  to a wnfv formula of the form

$$\exists \bar{x} \alpha \wedge \bigwedge_{j \in J} \neg(\exists \bar{y}_j \beta_j),$$

with  $\alpha \in A$  and  $\beta_j \in A$  for all  $j \in J$ . According to the first point of Definition 3.3.1 the preceding formula is equivalent in  $T$  to a wnfv formula of the form

$$\exists \bar{x}' \alpha' \wedge (\exists \bar{x}'' \alpha'' \wedge (\exists \bar{x}''' \alpha''' \wedge \bigwedge_{j \in J} \neg(\exists \bar{y}_j \beta_j))),$$

with  $\exists \bar{x}' \alpha' \in A'$ ,  $\exists \bar{x}'' \alpha'' \in A''$ ,  $\exists \bar{x}''' \alpha''' \in A'''$  and  $\beta_j \in A$  for all  $j \in J$ . Since  $\exists \bar{x}''' \alpha''' \in A'''$  then according to the fourth point of Definition 3.3.1  $T \models \exists ! \bar{x}''' \alpha'''$ . Thus, using Corollary 3.1.6 the preceding formula is equivalent in  $T$  to

$$\exists \bar{x}' \alpha' \wedge (\exists \bar{x}'' \alpha'' \wedge \bigwedge_{j \in J} \neg(\exists \bar{x}''' \alpha''' \wedge (\exists \bar{y}_j \beta_j))).$$

By lifting all the quantifies  $\exists \bar{y}_j$  after having possibly renamed the variables which appear in each  $\bar{y}_j$ , the preceding formula is equivalent in  $T$  to

$$\exists \bar{x}' \alpha' \wedge (\exists \bar{x}'' \alpha'' \wedge \bigwedge_{j \in J} \neg(\exists \bar{x}''' \exists \bar{y}_j \alpha''' \wedge \beta_j)).$$

According to Definition 3.3.1 the sets  $A'$ ,  $A''$  and  $A'''$  contain formulas of the form  $\exists \bar{x} \alpha$  with  $\alpha \in A$ , thus  $\alpha''' \in A$ . Since  $\beta_j \in A$  for all  $j \in J$  and since  $A$  is  $T$ -closed (i.e. closed under conjunction. . . ) then for all  $j \in J$  the formula  $\alpha''' \wedge \beta_j$  belongs to  $A$ . Thus, the preceding formula is equivalent in  $T$  to a wnfv formula of the form

$$\exists \bar{x}' \alpha' \wedge (\exists \bar{x}'' \alpha'' \wedge \bigwedge_{j \in J} \neg(\exists \bar{y}_j \beta_j)),$$

with  $\exists \bar{x}' \alpha' \in A'$ ,  $\exists \bar{x}'' \alpha'' \in A''$ , and  $\beta_j \in A$  for all  $j \in J$ . According to Property 3.3.2 the preceding formula is equivalent in  $T$  to a wnfv formula of the form

$$\exists \bar{x}' \alpha' \wedge (\exists \bar{x}'' \alpha'' \wedge \bigwedge_{j \in J} \neg(\exists \bar{y}'_j \beta'_j)),$$

with  $\exists \bar{x}' \alpha' \in A'$ ,  $\exists \bar{x}'' \alpha'' \in A''$ , and  $\exists \bar{y}'_j \beta'_j \in A'$  for all  $j \in J$ . Let us denote by  $J_1$ , the set of the  $j \in J$  such that  $x''_n$  does not have free occurrences in the formula  $\exists \bar{y}'_j \beta'_j$ . Thus, the preceding formula is equivalent in  $T$  to

$$\exists \bar{x}' \alpha' \wedge (\exists x''_1 \dots \exists x''_{n-1} \left[ (\bigwedge_{j \in J_1} \neg(\exists \bar{y}'_j \beta'_j)) \wedge (\exists \bar{x}''_n \alpha'' \wedge \bigwedge_{j \in J - J_1} \neg(\exists \bar{y}'_j \beta'_j)) \right]). \tag{13}$$

Since  $\exists \bar{x}'' \alpha'' \in A''$  and  $\exists \bar{y}'_j \beta'_j \in A'$  for all  $j \in J$ , then according to Property 3.2.2 and the points 2 and 3 of Definition 3.3.1, the formula (13) is equivalent in  $T$  to

$$\exists \bar{x}' \alpha' \wedge (\exists x''_1 \dots \exists x''_{n-1} (\text{true} \wedge \bigwedge_{j \in J_1} \neg(\exists \bar{y}'_j \beta'_j))).$$

<sup>4</sup> We must rename the variables of  $\bar{x}_i$  only if they have free occurrences in a formula  $\alpha_k$  of (12) with  $k \in I$  and  $i \neq k$ .

By repeating the three preceding steps  $(n - 1)$  times, by denoting by  $J_k$  the set of the  $j \in J_{k-1}$  such that  $x''_{(n-k+1)}$  does not have free occurrences in  $\exists \bar{y}'_j \beta'_j$ , and by using  $(n - 1)$  times Property 3.2.3, the preceding formula is equivalent in  $T$  to

$$\exists \bar{x}' \alpha' \wedge \bigwedge_{j \in J_n} \neg(\exists \bar{y}'_j \beta'_j).$$

Since  $\exists \bar{x}' \alpha' \in A'$  then according to the second point of Definition 3.3.1 we have  $T \models \exists ?\bar{x}' \alpha'$ . Thus, using Corollary 3.1.4 the preceding formula is equivalent in  $T$  to

$$(\exists \bar{x}' \alpha') \wedge \bigwedge_{j \in J_n} \neg(\exists \bar{x}' \alpha' \wedge (\exists \bar{y}'_j \beta'_j)).$$

By lifting all the quantifies  $\exists \bar{y}'_j$  after having possibly renamed the variables which appear in each  $\bar{y}'_j$ , the preceding formula is equivalent in  $T$  to

$$(\exists \bar{x}' \alpha') \wedge \bigwedge_{j \in J_n} \neg(\exists \bar{x}' \exists \bar{y}'_j \alpha' \wedge \beta'_j).$$

According to Definition 3.3.1 the sets  $A'$ ,  $A''$  and  $A'''$  contain formulas of the form  $\exists \bar{x} \alpha$  with  $\alpha \in A$ . Thus, since  $\exists \bar{x}' \alpha' \in A'$  and  $\exists \bar{y}'_j \beta'_j \in A'$  for all  $j \in J_n$ , then  $\alpha' \in A$  and  $\beta_j \in A$  for all  $j \in J_n$ . Since the set  $A$  is  $T$ -closed, it is closed under conjunction, then for all  $j \in J_n$  the formula  $\alpha' \wedge \beta'_j$  belongs to  $A$  and thus, the preceding formula is equivalent in  $T$  a wnfv formula of the form

$$(\exists \bar{x} \alpha) \wedge \bigwedge_{j \in J_n} \neg(\exists \bar{y}_j \beta_j),$$

with  $\alpha \in A$  and  $\beta_j \in A$  for all  $j \in J_n$ . This formula is a Boolean combination of formulas of the form  $\exists \bar{x} \alpha$  with  $\alpha \in A$ , i.e. a Boolean combination of basic formulas. Thus, the third condition of Property 2.3.1 holds.

Since  $T$  satisfies the three conditions of Property 2.3.1, then  $T$  is a complete theory.  $\square$

According to Theorem 3.3.4 and Corollary 2.3.2, we have the following corollary:

*Corollary 3.3.5*

If  $T$  is decomposable and if for all  $\exists \bar{x}' \alpha' \in A'$  we have  $\bar{x}' = \varepsilon$ , then  $T$  accepts full elimination of quantifiers.

*Proof*

Let  $T$  be a decomposable theory such that for all  $\exists \bar{x}' \alpha' \in A'$  we have  $\bar{x}' = \varepsilon$ . Let  $\varphi$  be a formula which can possibly contain free variables. In the proof of Theorem 3.3.4 we have shown that  $T$  satisfies the three conditions of Property 2.3.1 using formulas of the forms  $\exists \bar{x} \alpha$  with  $\alpha \in A$  as basic formulas. Thus, according to Corollary 2.3.2, the formula  $\varphi$  is equivalent in  $T$  to a wnfv Boolean combination of basic formulas, i.e. Boolean combination of formulas of the form  $\exists \bar{x} \alpha$  with  $\alpha \in A$ . According to Property 3.3.2 each of these basic formulas is equivalent in  $T$  to a wnfv formula of the form  $\exists \bar{x}' \alpha'$  which belongs to  $A'$ . Since for all  $\exists \bar{x}' \alpha' \in A'$  we have  $\bar{x}' = \varepsilon$  and since  $\alpha' \in A$  (according to the structure of the set  $A'$  defined in Definition 3.3.1) then the formula  $\varphi$  is equivalent in  $T$  to a boolean combination of elements of  $A$ . Since  $T$  is decomposable then  $A$  is a  $T$ -closed set and thus  $A \subseteq AT$ . Then, the formula  $\varphi$  is equivalent in  $T$  to a boolean combination  $\phi$  of conjunctions of atomic formulas. According to the syntax of the atomic formulas defined in Section 2, it is clear that  $\phi$  does not contain quantifiers.  $\square$

This corollary makes the connection between the set  $A'$  and the notion of full elimination of quantifiers. In fact, if  $T$  is decomposable and does not accept full elimination of quantifiers then it is enough to add axioms to  $T$  which enable the elimination of all the quantifiers of the formulas of  $A'$  to get a theory which accepts a full elimination of quantifiers. The sets  $A''$  and  $A'''$  are not concerned by this notion since in any decomposable theory  $T$  the formulas of  $A''$  and  $A'''$  accept full elimination of quantifiers due to their associated vectorial quantifiers:  $\exists!$  and  $\exists_{\infty}^{\Psi(u)}$ . On the other hand, if  $T$  is a decomposable theory which satisfies Corollary 3.3.5 then we can interest ourselves in getting the smallest subset  $T^*$  of axioms of  $T$ , such that  $T^*$  still accepts full elimination of quantifiers. For that it is enough to remove axiom by axiom from  $T$  and check each time if the theory still satisfies Corollary 3.3.5. This corollary shows also the fact that a decomposable theory  $T$  does not mean that  $T$  accepts full elimination of quantifiers. In fact, the theories of infinite trees, finite trees and finite or infinite trees as defined by Maher Maher (1988) do not accept full elimination of quantifiers but are decomposable and thus complete (Djelloul 2006a).

### 3.4 Simple decomposable theories

We present in this sub-section two examples of simple decomposable theories. The first one is a simple axiomatization of an infinite set of distinct individuals with an empty set of function and relation symbols. This theory denoted by  $Eq$  can be seen as a small extension of the Clark equational theory CET (Clark 1978), even if according to our syntax the equality symbol is considered as a primitive logical symbol together with its usual properties (commutativity, transitivity...). The second theory is the theory of additive rational or real numbers with addition and subtraction. The goal of these examples is to show the decomposability of simple theories whose properties are well known and do not need addition of proofs. An other example of a non-simple decomposable theory (finite or infinite trees) is given in Section 5 with a detailed study of the properties of this theory.

Let us assume for all this sub-section that the variables of  $V$  are ordered by a strict linear dense order relation without endpoints denoted by  $\succ$ .

#### Equality theory

Let  $Eq$  be a theory together with an empty set of function and relation symbols and whose axioms is the infinite set of propositions of the following form

$$(1_n) \quad \forall x_1 \dots \forall x_n \exists y \neg(x_1 = y) \wedge \dots \wedge \neg(x_n = y), \quad (14)$$

where all the variables  $x_1 \dots x_n$  are distinct and ( $n \neq 0$ ). The form (14) is called *diagram of axiom* and for each value of  $n$  there exists an *axiom* of  $Eq$ . For example the following property is true in  $Eq$ :

$$Eq \models \exists x \neg(x = y) \wedge \neg(x = z).$$

The theory  $Eq$  has as model an infinite set of distinct individuals.

Note that since  $Eq$  has an empty set of function and relation symbols, then  $AT = FL$  and thus all the equations of  $Eq$  are flat equations. Let  $x$  and  $y$  be two distinct variables. We call *leader* of the equation  $x = y$  the variable  $x$ . A conjunction  $\alpha$  of flat formulas is called ( $>$ )-solved in  $Eq$  if: (1) *false* is not a sub-formula of  $\alpha$ , (2) if  $x = y$  is a sub-formula of  $\alpha$  then<sup>5</sup>  $x > y$ , (3) each equation of  $\alpha$  has a distinct leader which does not occur in the other equations of  $\alpha$ .

#### Property 3.4.1

Every conjunction of flat formulas is equivalent in  $Eq$  either to *false* or to a ( $>$ )-solved conjunction of equations.

Let  $x, y$  and  $z$  be variables such that  $x > y > z$ . The conjunction  $x = x \wedge y = z$  is not ( $>$ )-solved because in the equation  $x = x$  we have  $x \not> x$ . By the same way, the conjunction  $x = y \wedge y = z$  is not ( $>$ )-solved because  $y$  is leader in  $y = z$  and occurs also in  $x = y$ . The conjunctions *true* and  $x = z \wedge y = z$  are ( $>$ )-solved. The computation of a possibly ( $>$ )-solved conjunction of equations from a conjunction of flat formulas in  $Eq$  is evident<sup>6</sup> and proceeds using the usual properties of the equality (commutativity, substitution, transitivity...) and by replacing each formula of the form  $x = x$  respectively  $\alpha \wedge \textit{false}$  by *true* respectively by *false*.

#### Property 3.4.2

Let  $\alpha$  be a ( $>$ )-solved conjunction of equations. Let  $\bar{x}$  be the vector of the leaders of the equations of  $\alpha$ . We have:

1.  $Eq \models \exists! \bar{x} \alpha$ .
2. For all  $x \in V$  we have  $Eq \models \exists_{\infty}^{\{\textit{false}\}} x \textit{true}$ .
3. For all  $x \in \textit{var}(\alpha)$  we have  $Eq \models \exists ? x \alpha$ .

The first point holds because all the leaders of the equations of  $\alpha$  are distinct and have one and only occurrence in  $\alpha$ . Thus, for every instantiation of the right hand sides of each equation, there exists one and only one value for the left hand sides and thus for the leaders. The second point is a consequence of the diagram of axiom (14) which states that for every finite set of distinct variables  $x_1 \dots x_n$  there exists a variable  $y$  which is different from all the  $x_i$ . Thus, in each model of  $Eq$  there exists an infinite set of individuals. Thus according to Definition 3.2.1 we have  $Eq \models \exists_{\infty}^{\{\textit{false}\}} x \textit{true}$ . The third point holds since in a ( $>$ )-solved conjunction of equations we have no formulas of the form  $x = x$  (because  $x \not> x$ ). Thus, using the properties of the equality for every model of  $Eq$  and for every instantiation of the variables of  $\textit{var}(\alpha) - \{x\}$  either there exists a unique solution of  $x$  or there exists a contradiction in the instantiations and thus there exists no values for  $x$ .

<sup>5</sup> Recall that  $>$  is a strict linear dense order relation and thus  $x \not> x$ . In other terms  $x = x$  is not ( $>$ )-solved.

<sup>6</sup>

(1)  $y = x \implies x = y$ . (2)  $x = y \wedge x = z \implies x = y \wedge z = y$ . (3)  $x = y \wedge z = x \implies x = y \wedge z = y$ .  
 (4)  $\textit{false} \wedge \alpha \implies \textit{false}$ . (5)  $x = x \implies \textit{true}$ .

The rules (1), (2) and (3) are applied only if  $x > y$ .



## Property 3.4.3

The theory  $Eq$  is decomposable.

## Proof

We show that  $Eq$  satisfies the conditions of Definition 3.3.1. The sets  $A$ ,  $A'$ ,  $A''$ ,  $A'''$  and  $\Psi(u)$  are chosen as follows:

- $A$  is the set  $FL$ .
- $A'$  is the set of formulas of the form  $\exists \varepsilon \alpha'$  where  $\alpha'$  is either a ( $>$ )-solved conjunction of equations or the formula *false*.
- $A''$  is the set of formulas of the form  $\exists \bar{x}'' \alpha''$  *true*.
- $A'''$  is the set of formulas of the form  $\exists \bar{x}''' \alpha'''$  with  $\alpha'''$  a ( $>$ )-solved conjunction of equations and  $\bar{x}'''$  the vector of the leaders of the equations of  $\alpha'''$ .
- $\Psi(u) = \{false\}$ .

It is obvious that  $FL$  is  $Eq$ -closed and  $A'$ ,  $A''$  and  $A'''$  contain formulas of the form  $\exists \bar{x} \alpha$  with  $\alpha \in FL$ .

Let us show that  $Eq$  satisfies the first condition of Definition 3.3.1. Let  $\psi$  be any formula and  $\alpha \in FL$ . Let  $\bar{x}$  be a vector of variables. Let us choose an order  $>$  such that the variables of  $\bar{x}$  are greater than the free variables of  $\exists \bar{x} \alpha$ . According to Property 3.4.1 two cases arise:

- If the formula  $\alpha$  is equivalent to *false* in  $Eq$ , then the formula  $\exists \bar{x} \alpha \wedge \psi$  is equivalent in  $Eq$  to a decomposed formula of the form

$$\exists \varepsilon false \wedge (\exists \varepsilon true \wedge (\exists \varepsilon true \wedge \psi)).$$

- If the formula  $\alpha$  is equivalent in  $Eq$  to a ( $>$ )-solved conjunction  $\beta$  of equations, then let  $X_l$  be the set of the variables of  $\bar{x}$  which are leader in the equations of  $\beta$  and let  $X_n$  be the set of the variables of  $\bar{x}$  which are not leader in the equations of  $\beta$ . The formula  $\exists \bar{x} \alpha \wedge \psi$  is equivalent in  $Eq$  to a decomposed formula of the form

$$\exists \bar{x}' \alpha' \wedge (\exists \bar{x}'' \alpha'' \wedge (\exists \bar{x}''' \alpha''' \wedge \psi)), \quad (15)$$

with  $\bar{x}' = \varepsilon$ . The formula  $\alpha'$  contains the conjunction of the equations of  $\beta$  whose leaders do not belong to  $X_l$ . The vector  $\bar{x}''$  contains the variables of  $X_n$ . The formula  $\alpha''$  is the formula *true*. The vector  $\bar{x}'''$  contains the variables of  $X_l$ . The formula  $\alpha'''$  is the conjunction of the equations of  $\beta$  whose leaders belong to  $X_l$ . According to our construction it is clear that  $\exists \bar{x}' \alpha' \in A'$ ,  $\exists \bar{x}'' \alpha'' \in A''$  and  $\exists \bar{x}''' \alpha''' \in A'''$ . Let us show that (15) and  $\exists \bar{x} \alpha \wedge \psi$  are equivalent in  $Eq$ . Let  $X$ ,  $X'$ ,  $X''$  and  $X'''$  be the sets of the variables of the vectors<sup>7</sup>  $\bar{x}$ ,  $\bar{x}'$ ,  $\bar{x}''$  and  $\bar{x}'''$ . If  $\alpha$  is equivalent to *false* in  $Eq$  then the equivalence of the decomposition is evident. Else  $\beta$  is a ( $>$ )-solved conjunction of equations and thus according to our construction we have:  $X = X' \cup X'' \cup X'''$ ,  $X' \cap X'' = \emptyset$ ,  $X' \cap X''' = \emptyset$ ,  $X'' \cap X''' = \emptyset$ ,  $X' = \emptyset$ , for all  $x_i'' \in X''$  we have  $x_i'' \notin var(\alpha')$  and for all  $x_i''' \in X'''$  we have  $x_i''' \notin var(\alpha' \wedge \alpha''')$ . This is due to the definition of the ( $>$ )-solved conjunction of flat formulas and the order  $>$  which has been chosen such that the quantified variables of  $\exists \bar{x} \alpha$  are greater than the free variables of  $\exists \bar{x} \alpha$ .

<sup>7</sup> Of course if  $\bar{x} = \varepsilon$  then  $X = \emptyset$

On the other hand, each equation in  $\beta$  occurs in  $\alpha' \wedge \alpha'' \wedge \alpha'''$  and each equation in  $\alpha' \wedge \alpha'' \wedge \alpha'''$  occurs in  $\beta$  and thus  $Eq \models \beta \leftrightarrow (\alpha' \wedge \alpha'' \wedge \alpha''')$ . We have shown that the vectorial quantifications are coherent and the equivalence  $Eq \models \beta \leftrightarrow \alpha' \wedge \alpha'' \wedge \alpha'''$  holds. According to Property 3.4.1 we have  $Eq \models \alpha \leftrightarrow \beta$  and thus, the decomposition keeps the equivalence in  $Eq$ . Let us decompose for example

$$\exists xyz v = w \wedge z = z \wedge z = x \wedge v = y.$$

Let us choose the order  $>$  such that  $x > y > z > v > w$ . Note that the quantified variables are greater than the free variables. Let us now ( $>$ )-solve the conjunction  $v = w \wedge z = z \wedge z = x \wedge v = y$ . Thus the preceding formula is equivalent in  $Eq$  to

$$\exists xyz v = w \wedge x = z \wedge y = w.$$

We have  $X_l = \{x, y\}$  and  $X_n = \{z\}$ . Thus, the preceding formula is equivalent in  $Eq$  to the following decomposed formula:

$$\exists \varepsilon v = w \wedge (\exists z \text{ true} \wedge (\exists xy x = z \wedge y = w)).$$

The theory  $Eq$  satisfies the second condition of Definition 3.3.1 according to the third point of Property 3.4.2 and using the fact that  $\bar{x}' = \varepsilon$ . The theory  $Eq$  satisfies the third condition of Definition 3.3.1 according to the second point of Property 3.4.2. The theory  $Eq$  satisfies the fourth condition of Definition 3.3.1 according to the first point of Property 3.4.2. The theory  $Eq$  satisfies the last condition of Definition 3.3.1 because  $A'$  is of the form  $\exists \varepsilon \alpha'$  where  $\alpha'$  is either the formula *false* or a ( $>$ )-solved conjunction of equations. Thus, if  $\exists \varepsilon \alpha'$  has no free variables, then either  $\alpha' = \text{true}$  or  $\alpha' = \text{false}$ .  $\square$

Note that  $Eq$  accepts full elimination of quantifiers. In fact Corollary 3.3.5 illustrates this result since for all  $\exists \bar{x}' \alpha' \in A'$  we have  $\bar{x}' = \varepsilon$ .

### Additive rational or real numbers theory

Let  $F = \{+, -, 0, 1\}$  be a set of function symbols of respective arities 2, 1, 0, 0. Let  $R = \emptyset$  be an empty set of relation symbols. Let  $Ra$  be the theory of additive rational or real numbers together with addition and subtraction.

#### Notation 3.4.4

Let  $a$  be a positive integer and  $t_1, \dots, t_n$  terms. We denote by:

- $Z$  the set of the integers.
- $t_1 + t_2$ , the term  $+t_1t_2$ .
- $t_1 + t_2 + t_3$ , the term  $+t_1(+t_2t_3)$ .
- $0.t_1$ , the term 0.
- $-a.t_1$ , the term  $\underbrace{(-t_1) + \dots + (-t_1)}_a$ .
- $a.t_1$ , the term  $\underbrace{t_1 + \dots + t_1}_a$ .
- $\sum_{i=1}^n t_i$ , the term  $\overline{t_1 + t_2 + \dots + t_n} + 0$ , where  $\overline{t_1 + t_2 + \dots + t_n}$  is the term  $t_1 + t_2 + \dots + t_n$  in which we have removed all the  $t_i$ 's which are equal to 0. For  $n = 0$  the term  $\sum_{i=1}^n t_i$  is reduced to the term 0.

The axiomatization of  $Ra$  is the set of propositions of one of the 8 following forms:

- 1  $\forall x \forall y x + y = y + x,$
- 2  $\forall x \forall y \forall z x + (y + z) = (x + y) + z,$
- 3  $\forall x x + 0 = x,$
- 4  $\forall x x + (-x) = 0,$
- 5 <sub>$n$</sub>   $\forall x n.x = 0 \rightarrow x = 0,$
- 6 <sub>$n$</sub>   $\forall x \exists! y n.y = x,$
- 7  $\forall x \forall y \forall z (x = y) \leftrightarrow (x + z = y + z),$
- 8  $\neg(0 = 1).$

with  $n$  an non-null integer. This theory has two usual models: rational numbers  $Q$  with addition and subtraction in  $Q$  and real numbers  $R$  with addition and subtraction in  $R$ .

We call *block* every conjunction  $\alpha$  of formulas of the form: *true*, *false*,  $\sum_{i=1}^n a_i.x_i = a_0.1$  with  $x_1, \dots, x_n$  distinct variables and  $a_i \in Z$  for all  $i \in \{0, 1, \dots, n\}$ . We call leader of an equation of the form  $\sum_{i=1}^n a_i.x_i = a_0.1$  the greatest variables  $x_k$  ( $k \in \{1, \dots, n\}$ ) according to the order  $\succ$  such that  $a_k \neq 0$ . A block  $\alpha$  is called ( $\succ$ )-solved in  $Ra$  if (1) each equation of  $\alpha$  has a distinct leader which does not occur in the other equations of  $\alpha$  and (2)  $\alpha$  does not contain sub-formulas of the form  $0 = a_0.1$  or *false* with  $a_0 \in Z$ . According to the axiomatization of  $Ra$  we show easily that:

*Property 3.4.5*

For all  $k \in \{1, \dots, n\}$  we have:

$$Ra \models \sum_{i=1}^n a_i.x_i = a_0.1 \leftrightarrow a_k.x_k = \sum_{i=1, i \neq k}^n (-a_i).x_i + a_0.1$$

*Property 3.4.6*

Every block is equivalent in  $Ra$  either to *false* or to a ( $\succ$ )-solved block.

Let  $x, y$  and  $z$  be variables such that  $x \succ y \succ z$ . The block  $2.x + y = (-1).1 \wedge 2.z + y = 2.1$  is not ( $\succ$ )-solved because  $y$  is leader in the second equation and occurs also in the first one. By the same way, the block  $x + y = 3.1 \wedge 0 = 0.1$  is not ( $\succ$ )-solved because  $0 = 0.1$  occurs in it. The blocks *true* and  $x + 2.z = 4.1 \wedge 3.y + 2.z = 3.1$  are ( $\succ$ )-solved. The computation of a possibly ( $\succ$ )-solved block is evident<sup>8</sup> and proceeds using Property 3.4.5 and a usual technique of substitution and simplification by replacing each equation of the form  $0 = a_0.1$  by *false* if  $a_0 \neq 0$  and by *true* otherwise and each formula of the form *false*  $\wedge \alpha$  by *false*.

<sup>8</sup>

$$(1) 0 = 0.1 \implies \textit{true}. \quad (2) 0 = a_0.1 \implies \textit{false}. \quad (3) \textit{false} \wedge \alpha \implies \textit{false}.$$

$$(4) \left[ \sum_{i=1}^n a_i.x_i = a_0.1 \wedge \sum_{i=1}^n b_i.x_i = b_0.1 \right] \implies \left[ \sum_{i=1}^n a_i.x_i = a_0.1 \wedge \sum_{i=1}^n (b_k a_i - a_k b_i).x_i = (b_k a_0 - a_k b_0).1 \right].$$

In the rule (2)  $a_0 \neq 0$ . In the rule (4)  $x_k$  is the leader of the block  $\sum_{i=1}^n a_i.x_i = a_0.1$  and  $b_k \neq 0$ .

*Property 3.4.7*

Let  $\alpha$  be a ( $\succ$ )-solved block and  $\bar{x}$  be the vector of the leaders of the equations of  $\alpha$ . We have:

1.  $Ra \models \exists! \bar{x} \alpha$ .
2. For all  $x \in V$  we have  $Ra \models \exists_{\infty}^{\{false\}} x true$ .
3. For all  $x \in var(\alpha)$  we have  $Ra \models \exists? x \alpha$ .

The first point holds because all the leaders are distinct and do not occur in the other equations. Thus, if we transform each equation of the form  $\sum_{i=1}^n a_i.x_i = a_0.1$  using Property 3.4.5 into a formula of the form  $a_k.x_k = \sum_{i=1, i \neq k}^n (-a_i).x_i + a_0.1$  with  $x_k$  the leader of this equation, then we get a conjunction of equations whose left hand sides are distinct and do not occur in the right hand sides. Thus, for each instantiation of the right hand sides of these equations there exists one and only value for the left hand sides and thus for the leaders according to axiom 6 of  $Ra$ . The second point holds because according to axiom 8 we have  $Ra \models \neg(0 = 1)$  thus using axiom 7 we have  $Ra \models \neg(0 + 1 = 1 + 1)$ . Then using axiom 3 we get  $Ra \models \neg(1 = 1 + 1)$ . Thus using the transitivity of the equality we have  $Ra \models \neg(0 = 1 + 1)$ . If we repeat the preceding steps  $n$  times we get  $n$  different individuals in all models of  $Ra$ . Thus for every model of  $Ra$  there exists an infinite set of individuals. Thus according to Definition 3.2.1 we have  $Ra \models \exists_{\infty}^{\{false\}} x true$ . The third point is evident according to the form of the blocks and the definition of the ( $\succ$ )-solved block.

*Property 3.4.8*

The theory  $Ra$  is decomposable.

*Proof*

We show that  $Ra$  satisfies the conditions of Definition 3.3.1. The sets  $A, A', A'', A'''$  and  $\Psi(u)$  are chosen as follows:

- $A$  is the set of blocks.
- $A'$  is the set of formulas of the form  $\exists \varepsilon \alpha'$  where  $\alpha'$  is either a ( $\succ$ )-solved block or the formula *false*.
- $A''$  is the set of formulas of the form  $\exists \bar{x}'' true$ .
- $A'''$  is the set of formulas of the form  $\exists \bar{x}''' \alpha'''$  with  $\alpha'''$  a ( $\succ$ )-solved block and  $\bar{x}'''$  the vector of the leaders of the equations of  $\alpha'''$ .
- $\Psi(u) = \{false\}$ .

Let us denote by  $BL$  the set of the blocks. It is clear that  $A', A''$  and  $A'''$  contain formulas of the form  $\exists \bar{x} \alpha$  with  $\alpha \in BL$ . Let us show that  $BL$  is  $Ra$ -closed: (i) According to the definition of  $BL$  we have  $BL \subseteq AT$ . (ii)  $BL$  is closed under conjunction. (iii) Let  $\alpha$  be a flat formula. If  $\alpha$  is the formula *true*, *false*,  $x = 0$  or  $x = 1$  then it is a block<sup>9</sup>. Else the following transformations transform  $\alpha$  to a block

$$\begin{aligned} x = y & \implies x + (-1).y = 0.1 \\ x = -y & \implies x + y = 0.1 \\ x = y + z & \implies x + (-1).y + (-1).z = 0.1 \end{aligned}$$

<sup>9</sup> The formulas  $x = 0$  and  $x = 1$  are blocks because the notations  $1.x, 0.1$  and  $1.1$  denote the terms  $x, 0$  and  $1$  according to Notation 3.4.4

From (i), (ii) and (iii)  $BL$  is  $Ra$ -closed. Let us show that  $Ra$  satisfies the first condition of Definition 3.3.1. Let  $\psi$  be any formula and  $\alpha \in BL$ . Let  $\bar{x}$  be a vector of variables. Let us choose an order  $\succ$  such that the variables of  $\bar{x}$  are greater than the free variables of  $\exists \bar{x} \alpha$ . According to Property 3.4.6 two cases arise:

- If  $\alpha$  is equivalent to *false* in  $Ra$ , then the formula  $\exists \bar{x} \alpha \wedge \psi$  is equivalent in  $Ra$  to a decomposed formula of the form

$$\exists \varepsilon \text{false} \wedge (\exists \varepsilon \text{true} \wedge (\exists \varepsilon \text{true} \wedge \psi)).$$

- If  $\alpha$  is equivalent in  $T$  to a  $(\succ)$ -solved block  $\beta$ , then let  $X_l$  be the set of the variables of  $\bar{x}$  which are leader in the equations of  $\beta$  and let  $X_n$  be the set of the variables of  $\bar{x}$  which are not leader in the equations of  $\beta$ . The formula  $\exists \bar{x} \alpha \wedge \psi$  is equivalent in  $T$  to a decomposed formula of the form

$$\exists \bar{x}' \alpha' \wedge (\exists \bar{x}'' \alpha'' \wedge (\exists \bar{x}''' \alpha''' \wedge \psi)), \tag{16}$$

with  $\bar{x}' = \varepsilon$ . The formula  $\alpha'$  contains the conjunction of the equations of  $\beta$  whose leaders do not belong to  $X_l$ . The vector  $\bar{x}''$  contains the variables of  $X_n$ . The formula  $\alpha''$  is the formula *true*. The vector  $\bar{x}'''$  contains the variables of  $X_l$ . The formula  $\alpha'''$  is the conjunction of the equations of  $\beta$  whose leaders belong to  $X_l$ . According to our construction it is clear that  $\exists \bar{x}' \alpha' \in A'$ ,  $\exists \bar{x}'' \alpha'' \in A''$  and  $\exists \bar{x}''' \alpha''' \in A'''$ . Let us show that (16) and  $\exists \bar{x} \alpha \wedge \psi$  are equivalent in  $Ra$ . Let  $X, X', X''$  and  $X'''$  be the sets of the variables of the vectors  $\bar{x}, \bar{x}', \bar{x}''$  and  $\bar{x}'''$ . If  $\alpha$  is equivalent to *false* in  $Ra$  then the equivalence of the decomposition is evident. Else  $\beta$  is a  $(\succ)$ -solved block and thus according to our construction we have:  $X = X' \cup X'' \cup X'''$ ,  $X' \cap X'' = \emptyset$ ,  $X' \cap X''' = \emptyset$ ,  $X'' \cap X''' = \emptyset$ ,  $X' = \emptyset$ , for all  $x'_i \in X''$  we have  $x'_i \notin \text{var}(\alpha')$  and for all  $x'''_i \in X'''$  we have  $x'''_i \notin \text{var}(\alpha' \wedge \alpha'')$ . This is due to the definition of  $(\succ)$ -solved blocks and the order  $\succ$  which has been chosen such that the quantified variables of  $\exists \bar{x} \alpha$  are greater than the free variables of  $\exists \bar{x} \alpha$ . On the other hand, each equation of  $\beta$  occurs in  $\alpha' \wedge \alpha'' \wedge \alpha'''$  and each equation in  $\alpha' \wedge \alpha'' \wedge \alpha'''$  occurs in  $\beta$  and thus  $Ra \models \beta \leftrightarrow (\alpha' \wedge \alpha'' \wedge \alpha''')$ . We have shown that the vectorial quantifications are coherent and the equivalence  $Ra \models \beta \leftrightarrow \alpha' \wedge \alpha'' \wedge \alpha'''$  holds. According to Property 3.4.6 we have  $Ra \models \alpha \leftrightarrow \beta$  and thus, the decomposition keeps the equivalence in  $Ra$ . Let us decompose for example

$$\exists xyz \ 2.v + w = 3.1 \wedge v + x = 2.1 \wedge v + x + 2.z = 4.1$$

Let us choose the order  $\succ$  such that  $x \succ y \succ z \succ v \succ w$ . Note that the quantified variables are greater than the free variables. Let us now  $(\succ)$ -solve the block  $2.v + w = 3.1 \wedge v + x = 2.1 \wedge v + x + 2.z = 4.1$ . Thus the preceding formula is equivalent in  $Ra$  to

$$\exists xyz \ 2.v + w = 3.1 \wedge 2.x + (-1).w = 1 \wedge z = 1$$

We have  $X_l = \{x, z\}$  and  $X_n = \{y\}$  thus the preceding formula is equivalent in  $Ra$  to the following decomposed formula

$$\exists \varepsilon \ 2.v + w = 3.1 \wedge (\exists y \text{true} \wedge (\exists xz \ 2.x + (-1).w = 1 \wedge z = 1)).$$

The theory  $Ra$  satisfies the second condition of Definition 3.3.1 according to the third point of Property 3.4.7 and using the fact that  $\bar{x}' = \varepsilon$ . The theory  $Ra$  satisfies the

third condition of Definition 3.3.1 according to the second point of Property 3.4.7. The theory  $Ra$  satisfies the fourth condition of Definition 3.3.1 according to the first point of Property 3.4.7. The theory  $Ra$  satisfies the last condition of Definition 3.3.1 because  $A'$  is of the form  $\exists \varepsilon \alpha'$  where  $\alpha'$  is either a ( $\succ$ )-solved block or the formula *false*. Thus, if  $\alpha'$  does not contain free variables then according to the definition of the ( $\succ$ )-solved blocks  $\alpha'$  does not contain formulas of the form  $0 = a_0 1$  and thus  $\alpha'$  is either the formula *true* or the formula *false*.  $\square$

Note that  $Ra$  accepts full elimination of quantifiers. In fact Corollary 3.3.5 illustrates this result since for all  $\exists \bar{x}' \alpha' \in A'$  we have  $\bar{x}' = \varepsilon$ .

#### 4 A general algorithm for solving first-order formulas in a decomposable theory $T$

Let  $T$  be a decomposable theory together with its set of function symbols  $F$  and its set of relation symbols  $R$ . The sets  $\Psi(u)$ ,  $A$ ,  $A'$ ,  $A''$  and  $A'''$  are now known and fixed.

##### 4.1 Normalized formula

*Definition 4.1.1*

A normalized formula  $\varphi$  of depth  $d \geq 1$  is a formula of the form

$$\neg(\exists \bar{x} \alpha \wedge \bigwedge_{i \in I} \varphi_i), \tag{17}$$

with  $I$  a finite (possibly empty) set,  $\alpha \in FL$  and the  $\varphi_i$ 's are normalized formulas of depth  $d_i$  with  $d = 1 + \max\{0, d_1, \dots, d_n\}$  and all the quantified variables of  $\varphi$  have distinct names and different from the names of the free variables.

*Example 4.1.2*

Let  $f$  and  $g$  be two 1-ary function symbols which belong to  $F$ . The formula

$$\neg \left[ \exists \varepsilon true \wedge \left[ \begin{array}{l} \neg(\exists x y = fx \wedge x = y \wedge \neg(\exists \varepsilon y = gx)) \wedge \\ \neg(\exists \varepsilon x = z) \end{array} \right] \right]$$

is a normalized formula of depth equals to three. The formulas  $\neg(\exists \varepsilon true)$  and  $\neg(\exists \varepsilon false)$  are two normalized formulas of depth 1. The smallest value of a depth of a normalized formula is 1. Normalized formulas of depth 0 are not defined and do not exist.

*Property 4.1.3*

Every formula  $\varphi$  is equivalent in  $T$  to a wnfv normalized formula of depth  $d \geq 1$ .

*Proof*

It is easy to transform any formula to a wnfv normalized formula, it is enough for example to follow the followings steps:

1. Introduce a supplement of equations and existentially quantified variables to transform the conjunctions of atomic formulas into conjunctions of flat formulas.

2. Express all the quantifiers, constants and logical connectors using only the logical symbols  $\neg$ ,  $\wedge$  and  $\exists$ . This can be done using the following transformations<sup>10</sup> of sub-formulas:

$$\begin{aligned}(\varphi \vee \phi) &\implies \neg(\neg\varphi \wedge \neg\phi), \\(\varphi \rightarrow \phi) &\implies \neg(\varphi \wedge \neg\phi), \\(\varphi \leftrightarrow \phi) &\implies (\neg(\varphi \wedge \neg\phi) \wedge \neg(\phi \wedge \neg\varphi)), \\(\forall x \varphi) &\implies \neg(\exists x \neg\varphi).\end{aligned}$$

3. If the formula  $\varphi$  obtained does not start with the logical symbol  $\neg$ , then replace it by  $\neg(\exists \varepsilon \text{ true} \wedge \neg\varphi)$ .
4. Name the quantified variables by distinct names and different from the names of the free variables.
5. Lift the quantifier before the conjunction, i.e.  $\varphi \wedge (\exists \bar{x} \psi)$  or  $(\exists \bar{x} \psi) \wedge \varphi$ , becomes  $\exists \bar{x} \varphi \wedge \psi$  because the free variables of  $\varphi$  are distinct from those of  $\bar{x}$ .
6. Group the quantified variables into a vectorial quantifier, i.e.  $\exists \bar{x}(\exists \bar{y} \varphi)$  or  $\exists \bar{x}\exists \bar{y} \varphi$  becomes  $\exists \bar{x}\bar{y} \varphi$ .
7. Insert empty vectors and formulas of the form *true* to get the normalized form using the following transformations of sub-formulas:

$$\neg\left(\bigwedge_{i \in I} \neg\varphi_i\right) \implies \neg(\exists \varepsilon \text{ true} \wedge \bigwedge_{i \in I} \neg\varphi_i), \quad (18)$$

$$\neg(\alpha \wedge \bigwedge_{i \in I} \neg\varphi_i) \implies \neg(\exists \varepsilon \alpha \wedge \bigwedge_{i \in I} \neg\varphi_i), \quad (19)$$

$$\neg(\exists \bar{x} \bigwedge_{j \in J} \neg\varphi_j) \implies \neg(\exists \bar{x} \text{ true} \wedge \bigwedge_{j \in J} \neg\varphi_j). \quad (20)$$

with  $\alpha \in FL$ ,  $I$  a finite (possibly empty) set and  $J$  a finite non-empty set.

If the starting formula does not contain the logical symbol  $\leftrightarrow$  then this transformation will be linear, i.e. there exists a constant  $k$  such that  $n_2 \leq kn_1$ , where  $n_1$  is the size of the starting formula and  $n_2$  the size of the normalized formula. We show easily by contradiction that the final formula obtained after application of these steps is normalized.  $\square$

#### Example 4.1.4

Let  $f$  be a 2-ary function symbol which belongs to  $F$ . Let us apply the preceding steps to transform the following formula into a normalized formula which is equivalent in  $T$ :

$$(fuv = fwu \wedge (\exists x u = x)) \vee (\exists u \forall w u = fvw).$$

Note that the formula does not start with  $\neg$  and the variables  $u$  and  $w$  are free in  $fuv = fwu \wedge (\exists x u = x)$  and bound in  $\exists u \forall w u = fvw$ .

Step 1: Let us first transform the equations into flat equations. The preceding formula is equivalent in  $T$  to

$$(\exists u_1 u_1 = fuv \wedge u_1 = fwu \wedge (\exists x u = x)) \vee (\exists u \forall w u = fvw). \quad (21)$$

<sup>10</sup> These equivalences are true in the empty theory and thus in any theory  $T$ .

Step 2: Let us now express the quantifier  $\forall$  using  $\neg, \wedge$  and  $\exists$ . Thus, the formula (21) is equivalent in  $T$  to

$$(\exists u_1 u_1 = fuv \wedge u_1 = fwu \wedge (\exists x u = x)) \vee (\exists u \neg(\exists w \neg(u = fvw))).$$

Let us also express the logical symbol  $\vee$  using  $\neg, \wedge$  and  $\exists$ . Thus, the preceding formula is equivalent in  $T$  to

$$\neg(\neg(\exists u_1 u_1 = fuv \wedge u_1 = fwu \wedge (\exists x u = x)) \wedge \neg(\exists u \neg(\exists w \neg(u = fvw)))). \tag{22}$$

Step 3: The formula starts with  $\neg$ , then we move to Step 4.

Step 4: The occurrences of the quantified variables  $u$  and  $w$  in  $(\exists u \neg(\exists w \neg(u = fvw)))$  must be renamed. Thus, the formula (22) is equivalent in  $T$  to

$$\neg(\neg(\exists u_1 u_1 = fuv \wedge u_1 = fwu \wedge (\exists x u = x)) \wedge \neg(\exists u_2 \neg(\exists w_1 \neg(u_2 = fvw_1)))).$$

Step 5: By lifting the existential quantifier  $\exists x$ , the preceding formula is equivalent in  $T$  to

$$\neg(\neg(\exists u_1 \exists x u_1 = fuv \wedge u_1 = fwu \wedge u = x) \wedge \neg(\exists u_2 \neg(\exists w_1 \neg(u_2 = fvw_1)))).$$

Step 6: Let us group the two quantified variables  $x$  and  $u_1$  into a vectorial quantifier. Thus, the preceding formula is equivalent in  $T$  to

$$\neg(\neg(\exists u_1 x u_1 = fuv \wedge u_1 = fwu \wedge u = x) \wedge \neg(\exists u_2 \neg(\exists w_1 \neg(u_2 = fvw_1)))).$$

Step 7: Let us introduce empty vectors of variables and formulas of the form *true* to get the normalized formula. According to the rule (18), the preceding formula is equivalent in  $T$  to

$$\neg \left[ \exists \varepsilon \text{true} \wedge \left[ \begin{array}{l} \neg(\exists u_1 x u_1 = fuv \wedge u_1 = fwu \wedge u = x) \wedge \\ \neg(\exists u_2 \neg(\exists w_1 \neg(u_2 = fvw_1))) \end{array} \right] \right],$$

which using the rule (19) with  $I = \emptyset$  is equivalent in  $T$  to

$$\neg \left[ \exists \varepsilon \text{true} \wedge \left[ \begin{array}{l} \neg(\exists u_1 x u_1 = fuv \wedge u_1 = fwu \wedge u = x) \wedge \\ \neg(\exists u_2 \neg(\exists w_1 \neg(\exists \varepsilon u_2 = fvw_1))) \end{array} \right] \right],$$

which using the rule (20) is equivalent in  $T$  to

$$\neg \left[ \exists \varepsilon \text{true} \wedge \left[ \begin{array}{l} \neg(\exists u_1 x u_1 = fuv \wedge u_1 = fwu \wedge u = x) \wedge \\ \neg(\exists u_2 \text{true} \wedge \neg(\exists w_1 \text{true} \wedge \neg(\exists \varepsilon u_2 = fvw_1))) \end{array} \right] \right].$$

This is a normalized formula of depth 4.

### 4.2 Working formula

#### Definition 4.2.1

A working formula  $\varphi$  of depth  $d \geq 1$  is a formula of the form

$$\neg(\exists \bar{x} \alpha \wedge \bigwedge_{i \in I} \varphi_i), \tag{23}$$

with  $I$  a finite (possibly empty) set,  $\alpha \in A$  and the  $\varphi_i$ 's are working formulas of depth  $d_i$  with  $d = 1 + \max\{0, d_1, \dots, d_n\}$  and all the quantified variables of  $\varphi$  have



distinct names and different from the names of the free variables. Working formulas of depth 0 are not defined and do not exist.

*Property 4.2.2*

Every formula is equivalent in  $T$  to a wnfv working formula.

*Proof*

Let  $\varphi$  be a formula. According to Property 4.1.3,  $\varphi$  is equivalent in  $T$  to a wnfv normalized formula  $\phi$  of the form

$$\neg(\exists \bar{x} \alpha \wedge \bigwedge_{i \in I} \varphi_i), \quad (24)$$

with  $\alpha \in FL$ ,  $I$  a finite possibly empty set and all the  $\varphi_i$  are normalized formulas. Let us show by recurrence on the depth  $d$  of (24) that the formula (24) is equivalent in  $T$  to a working formula.

(1) Let us show first that the recurrence is true for  $d = 1$ , i.e. every normalized formula of the form  $\neg(\exists \bar{x} \alpha)$  with  $\alpha \in FL$  is equivalent in  $T$  to a working formula. Since  $T$  is decomposable then according to Definition 3.3.1 the set  $A$  is  $T$ -closed, i.e. (i)  $A \subseteq AT$ , (ii)  $A$  is closed under conjunction and (iii) every flat formula is equivalent in  $T$  to a formula which belongs to  $A$ . Since  $\alpha \in FL$ , then according to (iii)  $\alpha$  is equivalent in  $T$  to a conjunction  $\beta$  of elements of  $A$ . According to (ii)  $\beta$  belongs to  $A$ . Thus, the formula  $\neg(\exists \bar{x} \alpha)$  is equivalent in  $T$  to  $\neg(\exists \bar{x} \beta)$  with  $\beta \in A$  which is a working formula of depth 1.

(2) Let us suppose now that the recurrence is true for  $d \leq n$  and let us show that it is true for  $d = n + 1$ . Let

$$\neg(\exists \bar{x} \alpha \wedge \bigwedge_{i \in I} \varphi_i), \quad (25)$$

be a normalized formula of depth  $n + 1$  with  $\alpha \in FL$  and all the  $\varphi_i$  are normalized formulas of depth  $d_i \leq n$ . According to the hypothesis of recurrence the preceding formula is equivalent in  $T$  to a formula of the form

$$\neg(\exists \bar{x} \alpha \wedge \bigwedge_{i \in I} \varphi_i), \quad (26)$$

with  $\alpha \in FL$  and all the  $\varphi_i$  are working formulas. Since  $T$  is decomposable then according to Definition 3.3.1 the set  $A$  is  $T$ -closed, i.e. (i)  $A \subseteq AT$ , (ii)  $A$  is closed under conjunction and (iii) every flat formula is equivalent in  $T$  to a formula which belongs to  $A$ . Since  $\alpha \in FL$ , then according to (iii)  $\alpha$  is equivalent in  $T$  to a conjunction  $\beta$  of elements of  $A$ . According to (ii)  $\beta$  belongs to  $A$ . Thus, the formula (26) is equivalent in  $T$  to

$$\neg(\exists \bar{x} \beta \wedge \bigwedge_{i \in I} \varphi_i),$$

with  $\beta \in A$  and all the  $\varphi_i$  are working formulas. The preceding formula is a working formula. From (1) and (2) our recurrence is true.  $\square$

Example 4.2.3

In the theory  $Ra$  of additive rational numbers, the formula

$$\neg \left[ \exists \varepsilon \text{ true} \wedge \left[ \begin{array}{l} \neg(\exists x y = -z \wedge z = x + y) \wedge \\ \neg(\exists \text{ true} \wedge \neg(\exists w \text{ true} \wedge \neg(\exists \varepsilon z = w))) \end{array} \right] \right],$$

is a normalized formula of depth 4 which is equivalent in  $Ra$  to the following working formula

$$\neg \left[ \exists \varepsilon \text{ true} \wedge \left[ \begin{array}{l} \neg(\exists x y + z = 0.1 \wedge z + (-1).x + (-1).y = 0.1) \wedge \\ \neg(\exists \text{ true} \wedge \neg(\exists w \text{ true} \wedge \neg(\exists \varepsilon z + (-1).w = 0.1))) \end{array} \right] \right].$$

Definition 4.2.4

A *solved* formula is a working formula of the form

$$\neg(\exists \bar{x}' \alpha' \wedge \bigwedge_{i \in I} \neg(\exists \bar{y}'_i \beta'_i)), \tag{27}$$

where  $I$  is a finite (possibly empty) set,  $\exists \bar{x}' \alpha' \in A'$ ,  $\exists \bar{y}'_i \beta'_i \in A'$  for all  $i \in I$ ,  $\alpha'$  is different from the formula *false* and all the  $\beta'_i$  are different from the formulas *true* and *false*.

Property 4.2.5

Let  $\varphi$  be a conjunction of solved formulas without free variables. The conjunction  $\varphi$  is either the formula  $\neg \text{true}$  or the formula *true*.

Proof

Recall first that we write  $\bigwedge_{i \in I} \varphi_i$ , and call *conjunction* each formula of the form  $\varphi_{i_1} \wedge \varphi_{i_2} \wedge \dots \wedge \varphi_{i_n} \wedge \text{true}$ . Let  $\varphi$  be a conjunction of solved formulas without free variables. According to Definition 4.2.4,  $\varphi$  is of the form

$$\left( \bigwedge_{i \in I} \neg(\exists \bar{x}'_i \alpha'_i \wedge \bigwedge_{j \in J_i} \neg(\exists \bar{y}'_{ij} \beta'_{ij})) \right) \wedge \text{true} \tag{28}$$

with

1.  $I$  a finite (possibly empty) set,
2.  $(\exists \bar{x}'_i \alpha'_i) \in A'$  for all  $i \in I$ ,
3.  $(\exists \bar{y}'_{ij} \beta'_{ij}) \in A'$  for all  $i \in I$  and  $j \in J_i$ ,
4.  $\alpha'_i$  different from *false* for all  $i \in I$ ,
5.  $\beta'_{ij}$  different from *true* and *false* for all  $i \in I$  and  $j \in J_i$ .

Since these solved formulas don't have free variables and since  $T$  is a decomposable theory then according to the fifth point of Definition 3.3.1 of a decomposable theory and the conditions 2 and 3 of (28) we have:

(\*) each formula  $\exists \bar{x}'_i \alpha'_i$  and each formula  $\exists \bar{y}'_{ij} \beta'_{ij}$  is either the formula  $\exists \varepsilon \text{true}$  or  $\exists \varepsilon \text{false}$ . According to (\*) and the condition 5 of (28), all the sets  $J_i$  must be empty, thus  $\varphi$  is of the form

$$\left( \bigwedge_{i \in I} \neg(\exists \bar{x}'_i \alpha'_i) \right) \wedge \text{true} \tag{29}$$

According to (\*) and (29), the formula  $\varphi$  is of the form

$$\left( \bigwedge_{i \in I'} \neg(\exists \varepsilon \text{false}) \right) \wedge \left( \bigwedge_{j \in I-I'} \neg(\exists \varepsilon \text{true}) \right) \wedge \text{true}$$

According to the condition 4 of (28), the set  $I'$  must be empty and thus  $\varphi$  is of the form

$$\left(\bigwedge_{i \in I} \neg(\exists \text{true})\right) \wedge \text{true}$$

If  $I = \emptyset$  then  $\varphi$  is the formula *true*. Else, according to our assumptions, we do not distinguish two formulas which can be made equal using the following transformations of sub-formulas:

$$\begin{aligned} \varphi \wedge \varphi &\Longrightarrow \varphi, & \varphi \wedge \psi &\Longrightarrow \psi \wedge \varphi, & (\varphi \wedge \psi) \wedge \phi &\Longrightarrow \varphi \wedge (\psi \wedge \phi), \\ \varphi \wedge \text{true} &\Longrightarrow \varphi, & \varphi \vee \text{false} &\Longrightarrow \varphi. \end{aligned}$$

Thus  $\varphi$  is the formula

$$\neg \text{true}$$

□

#### Property 4.2.6

Every solved formula is equivalent in  $T$  to a wnfv Boolean combination of elements of  $A'$ .

#### Proof

Let  $\varphi$  be a solved formula. According to Definition 4.2.4, the formula  $\varphi$  is of the form

$$\neg(\exists \bar{x}' \alpha' \wedge \bigwedge_{i \in I} \neg(\exists \bar{y}'_i \beta'_i)),$$

with  $\exists \bar{x}' \alpha' \in A'$  and  $\exists \bar{y}'_i \beta'_i \in A'$  for all  $i \in I$ . Since  $\exists \bar{x}' \alpha' \in A'$  then according to Definition 3.3.1 we have  $T \models \exists ?\bar{x}' \alpha'$  and thus according to Corollary 3.1.4, the preceding formula is equivalent in  $T$  to the following wnfv formula

$$\neg((\exists \bar{x}' \alpha') \wedge \bigwedge_{i \in I} \neg(\exists \bar{x}' \alpha' \wedge (\exists \bar{y}'_i \beta'_i))).$$

According to the definition of working formula, all the quantified variables have distinct names and different from the names of the free variables, thus the preceding formula is equivalent in  $T$  to the wnfv formula

$$\neg((\exists \bar{x}' \alpha') \wedge \bigwedge_{i \in I} \neg(\exists \bar{x}' \bar{y}'_i \alpha' \wedge \beta'_i)).$$

Since  $\exists \bar{x}' \alpha' \in A'$  and  $\exists \bar{y}'_i \beta'_i \in A'$  for all  $i \in I$ , then  $\alpha' \in A$  and  $\beta'_i \in A$ . Since  $A$  is  $T$ -closed then it is closed under conjunction and thus  $\alpha' \wedge \beta'_i \in A$  for all  $i \in I$ . According to Property 3.3.2 the preceding formula is equivalent in  $T$  to a wnfv formula of the form

$$\neg((\exists \bar{x}' \alpha') \wedge \bigwedge_{i \in I} \neg(\exists \bar{z}'_i \delta'_i)),$$

with  $\exists \bar{x}' \alpha' \in A'$  and  $\exists \bar{z}'_i \delta'_i \in A'$  for all  $i \in I$ . Which is finally equivalent in  $T$  to

$$(\neg(\exists \bar{x}' \alpha')) \vee \bigvee_{i \in I} (\exists \bar{z}'_i \delta'_i).$$

□

4.3 The rewriting rules

We present now the rewriting rules which transform a working formula  $\varphi$  of any depth  $d$  into a wnfv conjunction  $\phi$  of solved formulas which is equivalent to  $\varphi$  in  $T$ . To apply the rule  $p_1 \implies p_2$  to the working formula  $p$  means to replace in  $p$ , a sub-formula  $p_1$  by the formula  $p_2$ , by considering that the connector  $\wedge$  is associative and commutative.

$$\begin{aligned}
 (1) \quad & \neg \left[ \begin{array}{l} \exists \bar{x} \alpha \wedge \varphi \wedge \\ \neg(\exists \bar{y} \text{ true}) \end{array} \right] \implies \text{true} \\
 (2) \quad & \neg \left[ \exists \bar{x} \text{ false} \wedge \varphi \right] \implies \text{true} \\
 (3) \quad & \neg \left[ \begin{array}{l} \exists \bar{x} \alpha \wedge \\ \bigwedge_{i \in I} \neg(\exists \bar{y}_i \beta_i) \end{array} \right] \implies \neg \left[ \begin{array}{l} \exists \bar{x}' \bar{x}'' \alpha' \wedge \alpha'' \wedge \\ \bigwedge_{i \in I} \neg(\exists \bar{x}''' \bar{y}_i \alpha''' \wedge \beta_i)^* \end{array} \right] \\
 (4) \quad & \neg \left[ \begin{array}{l} \exists \bar{x} \alpha \wedge \\ \bigwedge_{i \in I} \neg(\exists \bar{y}'_i \beta'_i) \end{array} \right] \implies \neg \left[ \begin{array}{l} \exists \bar{x}' \alpha' \wedge \\ \bigwedge_{i \in I'} \neg(\exists \bar{y}'_i \beta'_i) \end{array} \right] \\
 (5) \quad & \neg \left[ \begin{array}{l} \exists \bar{x} \alpha \wedge \varphi \wedge \\ \neg \left[ \begin{array}{l} \exists \bar{y}' \beta' \wedge \\ \bigwedge_{i \in I} \neg(\exists \bar{z}'_i \delta'_i) \end{array} \right] \end{array} \right] \implies \left[ \begin{array}{l} \neg(\exists \bar{x} \alpha \wedge \varphi \wedge \neg(\exists \bar{y}' \beta')) \wedge \\ \bigwedge_{i \in I} \neg(\exists \bar{x} \bar{y}' \bar{z}'_i \alpha \wedge \beta' \wedge \delta'_i \wedge \varphi)^* \end{array} \right]
 \end{aligned}$$

with  $\alpha \in A$ ,  $\varphi$  a conjunction of working formulas and  $I$  a finite (possibly empty) set. In the rule (3), the formula  $\exists \bar{x} \alpha$  is equivalent in  $T$  to a decomposed formula of the form  $\exists \bar{x}' \alpha' \wedge (\exists \bar{x}'' \alpha'' \wedge (\exists \bar{x}''' \alpha'''))$

with  $\exists \bar{x}' \alpha' \in A'$ ,  $\exists \bar{x}'' \alpha'' \in A''$ ,  $\exists \bar{x}''' \alpha''' \in A'''$  and  $\exists \bar{x}''' \alpha'''$  different from  $\exists \varepsilon \text{ true}$ . All the  $\beta_i$ 's belong to  $A$ . The formula  $(\exists \bar{x}''' \bar{y}_i \alpha''' \wedge \beta_i)^*$  is the formula  $(\exists \bar{x}''' \bar{y}_i \alpha''' \wedge \beta_i)$  in which we have renamed the variables of  $\bar{x}'''$  by distinct names and different from the names of the free variables. In the rule (4), the formula  $\exists \bar{x} \alpha$  is not an element of  $A'$  and is equivalent in  $T$  to a decomposed formula of the form  $\exists \bar{x}' \alpha' \wedge (\exists \bar{x}'' \alpha'' \wedge (\exists \varepsilon \text{ true}))$

with  $\exists \bar{x}' \alpha' \in A'$  and  $\exists \bar{x}'' \alpha'' \in A''$ . Each formula  $\exists \bar{y}'_i \beta'_i$  is an element of  $A'$ .  $I'$  is the set of the  $i \in I$  such that  $\exists \bar{y}'_i \beta'_i$  does not have free occurrences of any variable of  $\bar{x}''$ . In the rule (5),  $I \neq \emptyset$ ,  $\exists \bar{y}' \beta' \in A'$  and  $\exists \bar{z}'_i \delta'_i \in A'$  for all  $i \in I$ . The formula  $(\exists \bar{x} \bar{y}' \bar{z}'_i \alpha \wedge \beta' \wedge \delta'_i \wedge \varphi)^*$  is the formula  $(\exists \bar{x} \bar{y}' \bar{z}'_i \alpha \wedge \beta' \wedge \delta'_i \wedge \varphi)$  in which we have renamed the variables of  $\bar{x}$  and  $\bar{y}'$  by distinct names and different from the names of the free variables.

Property 4.3.1

Every repeated application of the preceding rewriting rules on any working formula  $\varphi$ , terminates and produces a wnfv conjunction  $\phi$  of solved formulas which is equivalent to  $\varphi$  in  $T$ .

*Proof, first part:* The application of the rewriting rules terminates. Let us consider the 3-tuple  $(n_1, n_2, n_3)$  where the  $n_i$ 's are the following positive integers:

- $n_1 = \alpha(p)$ , where the function  $\alpha$  is defined as follows:

$$\begin{aligned} & - \alpha(true) = 0, \\ & - \alpha(\neg(\exists \bar{x} a \wedge \varphi)) = 2^{\alpha(\varphi)}, \\ & - \alpha(\bigwedge_{i \in I} \varphi_i) = \sum_{i \in I} \alpha(\varphi_i), \end{aligned}$$

with  $a \in A$ ,  $\varphi$  a conjunction of working formulas and the  $\varphi_i$ 's working formulas. Note that if  $\alpha(p_2) < \alpha(p_1)$  then  $\alpha(p[p_2]) < \alpha(p)$  where  $p[p_2]$  is the formula obtained from  $p$  when we replace the occurrence of the formula  $p_1$  in  $p$  by  $p_2$ . This function has been introduced in (Vorobyov 1996) and (Colmerauer 2003) to show the non-elementary complexity of all algorithms solving propositions in the theory of finite or infinite trees. It has also the property to decrease if the depth of the working formula decreases after application of distributions as it is done in our rule (5).

- $n_2 = \beta(p)$ , where the function  $\beta$  is defined as follows:

$$\begin{aligned} & - \beta(true) = 0, \\ & - \beta(\neg(\exists \bar{x} a \wedge \bigwedge_{i \in I} \varphi_i)) = \left\{ \begin{array}{l} 4^{1 + \sum_{i \in I} \beta(\varphi_i)} \text{ if } \exists \bar{x}''' \alpha''' \neq \exists \varepsilon true, \\ 1 + \sum_{i \in I} \beta(\varphi_i) \text{ if } \exists \bar{x}''' \alpha''' = \exists \varepsilon true \end{array} \right\} \end{aligned}$$

with the  $\varphi_i$ 's working formulas and  $T \models (\exists \bar{x} \alpha) \leftrightarrow (\exists \bar{x}' \alpha' \wedge (\exists \bar{x}'' \alpha'' \wedge (\exists \bar{x}''' \alpha''')))$ .

We show that:

$$\beta(\neg(\exists \bar{x} \alpha \wedge \bigwedge_{i \in I} \neg(\exists \bar{y}_i \lambda_i))) > \beta(\neg(\exists \bar{z} \delta \wedge \bigwedge_{i \in I} \neg(\exists \bar{w}_i \gamma_i)))$$

where  $I$  is a finite possibly empty set, the formula  $\exists \bar{x} \alpha$  is equivalent in  $T$  to a decomposed formula of the form  $\exists \bar{x}' \alpha' \wedge (\exists \bar{x}'' \alpha'' \wedge (\exists \bar{x}''' \alpha'''))$  with  $\exists \bar{x}''' \alpha''' \neq \exists \varepsilon true$ , the formula  $\exists \bar{z} \delta$  is equivalent in  $T$  to a decomposed formula of the form  $\exists \bar{z}' \delta' \wedge (\exists \bar{z}'' \delta'' \wedge (\exists \varepsilon true))$  and all the  $\lambda_i$  and  $\gamma_i$  belong to  $A$  and have no particular conditions.

- $n_3$  is the number of sub-formulas of the form  $\neg(\exists \bar{x} \alpha \wedge \varphi)$  with  $\exists \bar{x} \alpha \notin A'$  and  $\varphi$  a conjunction of working formulas.

For each rule, there exists an integer  $i$  such that the application of this rule decreases or does not change the values of the  $n_j$ 's, with  $1 \leq j < i$ , and decreases the value of  $n_i$ . This integer  $i$  is equal to: 1 for the rules (1), (2) and (5), 2 for the rule (3) and 3 for the rule (4). To each sequence of formulas obtained by a finite application of the preceding rewriting rules, we can associate a series of 3-tuples  $(n_1, n_2, n_3)$  which is strictly decreasing in the lexicographic order. Since the  $n_i$ 's are positive integers, they cannot be negative, thus this series of 3-tuples is a finite series and the application of the rewriting rules terminates.

*Proof, second part:* Let us show now that for each rule of the form  $p \implies p'$  we have  $T \models p \leftrightarrow p'$  and the formula  $p'$  remains a conjunction of working formulas. It is clear that the rules (1) and (2) are correct.

Correctness of the rule (3):

$$\neg \left[ \begin{array}{l} \exists \bar{x} \alpha \wedge \\ \bigwedge_{i \in I} \neg(\exists \bar{y}_i \beta_i) \end{array} \right] \implies \neg \left[ \begin{array}{l} \exists \bar{x}' \bar{x}'' \alpha' \wedge \alpha'' \wedge \\ \bigwedge_{i \in I} \neg(\exists \bar{x}''' \bar{y}_i \alpha''' \wedge \beta_i) \end{array} \right]$$

where the formula  $\exists \bar{x} \alpha$  is equivalent in  $T$  to a decomposed formula of the form  $\exists \bar{x}' \alpha' \wedge (\exists \bar{x}'' \alpha'' \wedge (\exists \bar{x}''' \alpha'''))$

with  $\exists \bar{x}' \alpha' \in A'$ ,  $\exists \bar{x}'' \alpha'' \in A''$ ,  $\exists \bar{x}''' \alpha''' \in A'''$  and  $\exists \bar{x}''' \alpha'''$  different from  $\exists \varepsilon true$ .

Let us show the correctness of this rule. According to the conditions of application of this rule, the formula  $\exists \bar{x} \alpha$  is equivalent in  $T$  to a decomposed formula of the form  $\exists \bar{x}' \alpha' \wedge (\exists \bar{x}'' \alpha'' \wedge (\exists \bar{x}''' \alpha'''))$

with  $\exists \bar{x}' \alpha' \in A'$ ,  $\exists \bar{x}'' \alpha'' \in A''$ ,  $\exists \bar{x}''' \alpha''' \in A'''$  and  $\exists \bar{x}''' \alpha'''$  different from  $\exists \varepsilon true$ . Thus, the left formula of this rewriting rule is equivalent in  $T$  to the formula

$$\neg(\exists \bar{x}' \alpha' \wedge (\exists \bar{x}'' \alpha'' \wedge (\exists \bar{x}''' \alpha''' \wedge \bigwedge_{i \in I} \neg(\exists \bar{y}_i \beta_i))))$$

Since  $\exists \bar{x}''' \alpha''' \in A'''$ , then according to the fourth point of Definition 3.3.1 we have  $T \models \exists ! \bar{x}''' \alpha'''$ , thus using Corollary 3.1.6 the preceding formula is equivalent in  $T$  to

$$\neg(\exists \bar{x}' \alpha' \wedge (\exists \bar{x}'' \alpha'' \wedge \bigwedge_{i \in I} \neg(\exists \bar{x}''' \alpha''' \wedge (\exists \bar{y}_i \beta_i))))$$

According to the definition of the working formula the quantified variables have distinct names and different from the names of the free variables, thus we can lift the quantifications and then the preceding formula is equivalent in  $T$  to

$$\neg(\exists \bar{x}' \alpha' \wedge (\exists \bar{x}'' \alpha'' \wedge \bigwedge_{i \in I} \neg(\exists \bar{x}''' \bar{y}_i \alpha''' \wedge \beta_i)))$$

i.e. to

$$\neg(\exists \bar{x}' \bar{x}'' \alpha' \wedge \alpha'' \wedge \bigwedge_{i \in I} \neg(\exists \bar{x}''' \bar{y}_i \alpha''' \wedge \beta_i)^*),$$

where the formula  $(\exists \bar{x}''' \bar{y}_i \alpha''' \wedge \beta_i)^*$  is the formula  $(\exists \bar{x}''' \bar{y}_i \alpha''' \wedge \beta_i)$  in which we have renamed the variables of  $\bar{x}'''$  by distinct names and different from the names of the free variables. Thus, the rewriting rule (3) is correct in  $T$ .

Correctness of the rule (4):

$$\neg \left[ \begin{array}{l} \exists \bar{x} \alpha \wedge \\ \bigwedge_{i \in I} \neg(\exists \bar{y}'_i \beta'_i) \end{array} \right] \implies \neg \left[ \begin{array}{l} \exists \bar{x}' \alpha' \wedge \\ \bigwedge_{i \in I'} \neg(\exists \bar{y}'_i \beta'_i) \end{array} \right]$$

where the formula  $\exists \bar{x} \alpha$  is not an element of  $A'$  and is equivalent in  $T$  to a decomposed formula of the form  $\exists \bar{x}' \alpha' \wedge (\exists \bar{x}'' \alpha'' \wedge (\exists \varepsilon true))$

with  $\exists \bar{x}' \alpha' \in A'$  and  $\exists \bar{x}'' \alpha'' \in A''$ . Each formula  $\exists \bar{y}'_i \beta'_i$  is an element of  $A'$ .  $I'$  is the set of the  $i \in I$  such that  $\exists \bar{y}'_i \beta'_i$  does not have free occurrences of any variable of  $\bar{x}''$ .

Let us show the correctness of this rule. According to the conditions of application of this rule, the formula  $\exists \bar{x} \alpha$  is equivalent in  $T$  to a decomposed formula of the form  $\exists \bar{x}' \alpha' \wedge (\exists \bar{x}'' \alpha'' \wedge (\exists \varepsilon true))$

with  $\exists \bar{x}' \alpha' \in A'$  and  $\exists \bar{x}'' \alpha'' \in A''$ . Moreover, each formula  $\exists \bar{y}'_i \beta'_i$  belongs to  $A'$ . Thus, the left formula of this rewriting rule is equivalent in  $T$  to the formula

$$\neg(\exists \bar{x}' \alpha' \wedge (\exists \bar{x}'' \alpha'' \wedge \bigwedge_{i \in I} \neg(\exists \bar{y}'_i \beta'_i)))$$

Let us denote by  $I_1$ , the set of the  $i \in I$  such that  $x''_n$  does not have free occurrences in the formula  $\exists \bar{y}'_i \beta'_i$ , thus the preceding formula is equivalent in  $T$  to

$$\neg(\exists \bar{x}' \alpha' \wedge (\exists x''_1 \dots \exists x''_{n-1} [(\bigwedge_{i \in I_1} \neg(\exists \bar{y}'_i \beta'_i)) \wedge (\exists \bar{x}''_n \alpha'' \wedge \bigwedge_{i \in I - I_1} \neg(\exists \bar{y}'_i \beta'_i))])). \tag{30}$$

Since  $\exists \bar{x}'' \alpha'' \in A''$  and  $\exists \bar{y}'_i \beta'_i \in A'$  for every  $i \in I - I_1$ , then according to Property 3.2.2 and the conditions 2 and 3 of Definition 3.3.1, the formula (30) is equivalent in  $T$  to

$$\neg(\exists \bar{x}' \alpha' \wedge (\exists x''_1 \dots \exists x''_{n-1} (true \wedge \bigwedge_{i \in I_1} \neg(\exists \bar{y}'_i \beta'_i)))). \tag{31}$$

By repeating the three preceding steps  $(n - 1)$  times, by denoting by  $I_k$  the set of the  $i \in I_{k-1}$  such that  $x''_{(n-k+1)}$  does not have free occurrences in  $\exists \bar{y}'_i \beta'_i$ , and by using  $(n - 1)$  times Property 3.2.3, the preceding formula is equivalent in  $T$  to

$$\neg(\exists \bar{x}' \alpha' \wedge \bigwedge_{i \in I_n} \neg(\exists \bar{y}'_i \beta'_i)),$$

Thus, the rule (4) is correct in  $T$ .

*Correctness of the rule (5):*

$$\neg \left[ \begin{array}{l} \exists \bar{x} \alpha \wedge \varphi \wedge \\ \neg \left[ \begin{array}{l} \exists \bar{y}' \beta' \wedge \\ \bigwedge_{i \in I} \neg(\exists \bar{z}'_i \delta'_i) \end{array} \right] \end{array} \right] \implies \left[ \begin{array}{l} \neg(\exists \bar{x} \alpha \wedge \varphi \wedge \neg(\exists \bar{y}' \beta')) \wedge \\ \bigwedge_{i \in I} \neg(\exists \bar{x} \bar{y}' \bar{z}'_i \alpha \wedge \beta' \wedge \delta'_i \wedge \varphi)^* \end{array} \right]$$

where  $I \neq \emptyset$  and the formulas  $\exists \bar{y}' \beta'$  and  $\exists \bar{z}'_i \delta'_i$  are elements of  $A'$  for all  $i \in I$ .

Let us show the correctness of this rule. Since  $\exists \bar{y}' \beta' \in A'$  then according to the second point of Definition 3.3.1 we have  $T \models \exists ?\bar{y}' \beta'$ , thus using Corollary 3.1.4 the preceding formula is equivalent in  $T$  to

$$\neg \left[ \begin{array}{l} \exists \bar{x} \alpha \wedge \varphi \wedge \\ \neg \left[ (\exists \bar{y}' \beta') \wedge \bigwedge_{i \in I} \neg(\exists \bar{y}' \beta' \wedge (\exists \bar{z}'_i \delta'_i)) \right] \end{array} \right]$$

According to the definition of working formula the quantified variables have distinct names and different from the names of the free variables, thus we can lift the quantifications and then the preceding formula is equivalent in  $T$  to

$$\neg \left[ \begin{array}{l} \exists \bar{x} \alpha \wedge \varphi \wedge \\ \neg \left[ (\exists \bar{y}' \beta') \wedge \bigwedge_{i \in I} \neg(\exists \bar{y}' \bar{z}'_i \beta' \wedge \delta'_i) \right] \end{array} \right]$$

thus to

$$\neg \left[ \begin{array}{l} \exists \bar{x} \alpha \wedge \varphi \wedge \\ \left[ \neg(\exists \bar{y}' \beta') \vee \bigvee_{i \in I} (\exists \bar{y}' \bar{z}'_i \beta' \wedge \delta'_i) \right] \end{array} \right]$$

After having distributed the  $\wedge$  on the  $\vee$  and lifted the quantification  $\exists \bar{y}'z'_i$  we get

$$\neg \left[ \begin{array}{l} (\exists \bar{x} \alpha \wedge \varphi \wedge \neg(\exists \bar{y}' \beta')) \vee \\ \bigvee_{i \in I} (\exists \bar{x} \bar{y}' z'_i \alpha \wedge \varphi \wedge \beta' \wedge \delta'_i) \end{array} \right]$$

which is equivalent in  $T$  to

$$\left[ \begin{array}{l} \neg(\exists \bar{x} \alpha \wedge \varphi \wedge \neg(\exists \bar{y}' \beta')) \wedge \\ \bigwedge_{i \in I} \neg(\exists \bar{x} \bar{y}' z'_i \alpha \wedge \varphi \wedge \beta' \wedge \delta'_i) \end{array} \right] \tag{32}$$

In order to satisfy the definition of the working formulas we must rename the variables of  $\bar{x}$  and  $\bar{y}'$  by distinct names and different from the names of the free variables. Let us denote by  $(\exists \bar{x} \bar{y}' z'_i \alpha \wedge \varphi \wedge \beta' \wedge \delta'_i)^*$  the formula  $(\exists \bar{x} \bar{y}' z'_i \alpha \wedge \varphi \wedge \beta' \wedge \delta'_i)$  in which we have renamed the variables of  $\bar{x}$  and  $\bar{y}'$  by distinct names and different from the names of the free variables. Thus, the formula (32) is equivalent in  $T$  to

$$\left[ \begin{array}{l} \neg(\exists \bar{x} \alpha \wedge \varphi \wedge \neg(\exists \bar{y}' \beta')) \wedge \\ \bigwedge_{i \in I} \neg(\exists \bar{x} \bar{y}' z'_i \alpha \wedge \varphi \wedge \beta' \wedge \delta'_i)^* \end{array} \right]$$

Thus, the rule (5) is correct in  $T$ .

*Proof, third part:* Every finite application of the rewriting rules on a working formula produces a wnfv conjunction of solved formulas.

Recall that we write  $\bigwedge_{i \in I} \varphi_i$ , and call *conjunction* each formula of the form  $\varphi_{i_1} \wedge \varphi_{i_2} \wedge \dots \wedge \varphi_{i_n} \wedge true$ . In particular, for  $I = \emptyset$ , the conjunction  $\bigwedge_{i \in I} \varphi_i$  is reduced to *true*. Moreover, we do not distinguish two formulas which can be made equal using the following transformations of sub-formulas:

$$\begin{aligned} \varphi \wedge \varphi &\implies \varphi, & \varphi \wedge \psi &\implies \psi \wedge \varphi, & (\varphi \wedge \psi) \wedge \phi &\implies \varphi \wedge (\psi \wedge \phi), \\ \varphi \wedge true &\implies \varphi, & \varphi \vee false &\implies \varphi. \end{aligned}$$

Let us show first that every substitution of a sub-working formula of a conjunction of working formulas by a conjunction of working formulas produces a conjunction of working formulas. Let  $\bigwedge_{i \in I} \varphi_i$  be a conjunction of working formulas. Let  $\varphi_k$  with  $k \in I$  be an element of this conjunction of depth  $d_k$ . Two cases arise:

1. We replace  $\varphi_k$  by a conjunction of working formulas. Thus, let  $\bigwedge_{j \in J_k} \phi_j$  be a conjunction of working formulas which is equivalent to  $\varphi_k$  in  $T$ . The conjunction of working formulas  $\bigwedge_{i \in I} \varphi_i$  is equivalent in  $T$  to

$$\left( \bigwedge_{i \in I - \{k\}} \varphi_i \right) \wedge \left( \bigwedge_{j \in J_k} \phi_j \right)$$

which is clearly a conjunction of working formulas.

2. We replace a strict sub-working formula of  $\varphi_k$  by a conjunction of working formulas. Thus, let  $\phi$  be a sub-working formula of  $\varphi_k$  of depth  $d_\phi < d_k$  (thus



$\phi$  is different from  $\varphi_k$ ). Thus,  $\varphi_k$  has a sub-working formula<sup>11</sup> of the form

$$\neg(\exists \bar{x} \alpha \wedge (\bigwedge_{l \in L} \psi_l) \wedge \phi),$$

where  $L$  is a finite (possibly empty) set and all the  $\psi_l$  are working formulas.

Let  $\bigwedge_{j \in J} \phi_j$  be a conjunction of working formulas which is equivalent to  $\phi$  in  $T$ . Thus the preceding sub-working formula of  $\varphi_k$  is equivalent in  $T$  to

$$\neg(\exists \bar{x} \alpha \wedge (\bigwedge_{l \in L} \psi_l) \wedge (\bigwedge_{j \in J} \phi_j)),$$

which is clearly a sub-working formula and thus  $\varphi_k$  is equivalent to a working formula and thus  $\bigwedge_{i \in I} \varphi_i$  is equivalent to a conjunction of working formulas.

From 1 and 2 we deduce that (i) every substitution of a sub-working formula of a conjunction of working formulas by a conjunction of working formulas produces a conjunction of working formulas.

Since each rule transforms a working formula into a conjunction of working formulas, then according to (i) every finite application of the rewriting rules on a working formula produces a conjunction of working formulas. Let us show now that each of these final working formulas is solved.

Let  $\varphi$  be a working formula. According to all what we have shown, every finite application of our rules on  $\varphi$  produces a conjunction  $\phi$  of working formulas. Suppose that the rules terminate and one of the working formulas of  $\phi$  is not solved. Let  $\psi$  be this formula, two cases arise:

**Case 1:**  $\psi$  is a working formula of depth greater than 2. Thus,  $\psi$  has a sub-formula of the form

$$\neg \left[ \begin{array}{l} \exists \bar{x} \alpha \wedge \psi_1 \wedge \\ \neg [ \exists \bar{y} \beta \wedge \bigwedge_{i \in I} \neg(\exists \bar{z}_i \delta_i) ] \end{array} \right]$$

where  $\psi_1$  is a conjunction of working formulas,  $I$  is a nonempty set and  $\alpha, \beta$  and  $\delta_i$  are elements of  $A$  for all  $i \in I$ . Let  $(\exists \bar{y}' \beta' \wedge (\exists \bar{x}'' \beta'' \wedge (\exists \bar{y}''' \beta''')))$

be the decomposed formula in  $T$  of  $\exists \bar{y} \beta$  and let  $(\exists \bar{z}'_i \delta'_i \wedge (\exists \bar{z}''_i \delta''_i \wedge (\exists \bar{z}'''_i \delta'''_i)))$

be the decomposed formula in  $T$  of  $\exists \bar{z}_i \delta_i$ . If  $\exists \bar{y}''' \beta'''$  is not the formula  $\exists \text{true}$  then the rule (3) can still be applied which contradicts our supposition. Thus, suppose that

$$\exists \bar{y}''' \beta''' = \exists \text{true} \tag{33}$$

If there exists  $k \in I$  such that  $\exists \bar{z}'''_k \delta'''_k$  is not the formula  $\exists \text{true}$  then the rule (3) can be still applied (with  $I = \emptyset$ ) which contradicts our supposition. Thus, suppose that

$$\exists \bar{z}'''_i \delta'''_i = \exists \text{true} \tag{34}$$

for all  $i \in I$ . If there exists  $k \in I$  such that  $\exists \bar{z}_k \delta_k$  is not an element of  $A'$  then since we have (34), the rule (4) can still be applied (with  $I = \emptyset$ ) which contradicts our supposition. Thus, suppose that

$$\exists \bar{z}_i \delta_i \in A' \tag{35}$$

<sup>11</sup> By considering that the set of the sub-formulas of any formula  $\varphi$  contains also the whole formula  $\varphi$ .

for all  $i \in I$ . If  $\exists \bar{y}\beta$  is not an element of  $A'$  then since we have (33) and (35), the rule (4) can still be applied which contradicts our supposition. Thus, suppose that

$$\exists \bar{y}\beta \in A' \quad (36)$$

Since we have (35) and (36) then the rule (5) can still be applied which contradicts all our suppositions.

**Case 2:**  $\psi$  is a working formula of the form

$$\neg(\exists \bar{x} \alpha \wedge \bigwedge_{i \in I} \neg(\exists \bar{y}_i \beta_i))$$

where at least one of the following conditions holds:

1.  $\alpha$  is the formula *false*,
2. there exists  $k \in I$  such that  $\beta_k$  is the formula *true* or *false*,
3. there exists  $k \in I$  such that  $\exists \bar{y}_k \beta_k \notin A'$ ,
4.  $\exists \bar{x} \alpha \notin A'$ .

If the condition (1) holds then the rule (2) can still be applied which contradicts our suppositions. If the condition (2) holds then the rules (1) and (2) can still be applied which contradicts our suppositions. If the condition (3) holds then the rule (3) or (4) (with  $I = \emptyset$ ) can still be applied which contradicts our suppositions. If the condition (4) holds then according to the preceding point  $\exists \bar{y}_i \beta_i \in A'$  for all  $i \in I$  and thus the rule (3) or (4) can still be applied which contradicts our suppositions.

From Case 1 and Case 2, our suppositions are always false thus  $\psi$  is a solved formula and thus  $\phi$  is a conjunction of solved formulas.

#### 4.4 The algorithm of resolution

Having any formula  $\psi$ , the resolution of  $\psi$  proceeds as follows:

1. Transform the formula  $\psi$  into a normalized formula and then into a working formula  $\varphi$  which is wnfv and equivalent to  $\psi$  in  $T$ .
2. Apply the preceding rewriting rules on  $\varphi$  as many time as possible. At the end we obtain a conjunction  $\phi$  of solved formulas.

According to Property 4.3.1, the application of the rewriting rules on a formula  $\psi$  without free variables produces a conjunction  $\phi$  of solved formulas which is equivalent to  $\psi$  in  $T$  and does not contain free variables. According to Property 4.2.5,  $\phi$  is either the formula *true* or *¬true*, thus either  $T \models \psi$  or  $T \models \neg\psi$  and thus  $T$  is a complete theory. We can now present our main result:

##### Corollary 4.4.1

If  $T$  is a decomposable theory then every formula is equivalent in  $T$  either to *true* or to *false* or to a Boolean combination of elements of  $A'$  which has at least one free variable.

*Remark 4.4.2*

There exists another way to solve the first-order formulas in  $T$  specially in the case where there exists at least one free variable in the initial formula  $\psi$  and when the goal of the resolution is to have explicit and understanding solutions of these free variables in  $\psi$ . In this case it is better to run the preceding algorithm on  $\neg\psi$ . Let then

$$\bigwedge_{i \in I} \neg(\exists \bar{x}'_i \alpha'_i \wedge \bigwedge_{j \in J_i} \neg(\exists \bar{y}'_{ij} \beta'_{ij}))$$

be the conjunction of solved formulas obtained by application of the preceding rules on  $\neg\psi$ . The formula

$$\bigvee_{i \in I} (\exists \bar{x}'_i \alpha'_i \wedge \bigwedge_{j \in J_i} \neg(\exists \bar{y}'_{ij} \beta'_{ij}))$$

is a wnfv disjunction of formulas which is equivalent to  $\psi$  in  $T$ . It is more easy to understand the solutions of the free variables of this disjunction of solved formulas than those of a conjunction of solved formulas.

### 5 The theory $\mathcal{T}$ of finite or infinite trees

#### 5.1 The axioms

The theory  $\mathcal{T}$  of finite or infinite trees built on an **infinite** set  $F$  of distinct function symbols has as axioms the infinite set of propositions of one of the three following forms:

$$\begin{aligned} \forall \bar{x} \forall \bar{y} \quad \neg f \bar{x} = g \bar{y} & \quad [1] \\ \forall \bar{x} \forall \bar{y} \quad f \bar{x} = f \bar{y} \rightarrow \bigwedge_i x_i = y_i & \quad [2] \\ \forall \bar{x} \exists ! \bar{z} \quad \bigwedge_i z_i = t_i[\bar{x}\bar{z}] & \quad [3] \end{aligned}$$

where  $f$  and  $g$  are distinct function symbols taken from  $F$ ,  $\bar{x}$  is a vector of possibly non-distinct variables  $x_i$ ,  $\bar{y}$  is a vector of possibly non-distinct variables  $y_i$ ,  $\bar{z}$  is a vector of distinct variables  $z_i$  and  $t_i[\bar{x}\bar{z}]$  is a term which begins with an element of  $F$  followed by variables taken from  $\bar{x}$  or  $\bar{z}$ . Note that this theory does not accept full elimination of quantifiers. In fact, in the formula  $\exists x y = f(x)$  we can not remove or eliminate the quantifier  $\exists x$ .

#### 5.2 Properties of $\mathcal{T}$

Suppose that the variables of  $V$  are ordered by a strict linear dense order relation without endpoints denoted by  $\succ$ .

*Definition 5.2.1*

A conjunction  $\alpha$  of flat equations is called ( $\succ$ )-solved if all its left-hand sides are distinct and  $\alpha$  does not contain equations of the form  $x = x$  or  $y = x$ , where  $x$  and  $y$  are variables such that  $x \succ y$ .

*Property 5.2.2*

Every conjunction  $\alpha$  of flat formulas is equivalent in  $\mathcal{T}$  either to false or to a ( $\succ$ )-solved conjunction of flat equations.

*Proof*

To prove this property we introduce the following rewriting rules:

- (1)  $false \wedge \alpha \implies false,$
- (2)  $x = fy_1 \dots y_m \wedge x = gz_1 \dots z_n \implies false,$
- (3)  $x = fy_1 \dots y_n \wedge x = fz_1 \dots z_n \implies x = fy_1 \dots y_n \wedge \bigwedge_{i \in \{1, \dots, n\}} y_i = z_i,$
- (4)  $x = x \implies true$
- (5)  $y = x \implies x = y$
- (6)  $x = y \wedge x = fz_1 \dots z_n \implies x = y \wedge y = fz_1 \dots z_n$
- (7)  $x = y \wedge x = z \implies x = y \wedge y = z$

with  $\alpha$  any formula and  $f$  and  $g$  two distinct function symbols taken from  $F$ . The rules (5), (6) and (7) are applied only if  $x > y$ . This condition prevents infinite loops.

Let us prove now that every repeated application of the preceding rewriting rules on any conjunction  $\alpha$  of flat formulas, is terminating and producing either the formula *false* or a ( $>$ )-solved conjunction of flat equations which is equivalent to  $\alpha$  in  $\mathcal{T}$ .

*Proof, first part:* The application of the rewriting rules terminates. Since the variables which occur in our formulas are ordered by the strict linear order relation without endpoints " $>$ ", we can number them by positive integers such that

$$x > y \leftrightarrow no(x) > no(y),$$

where  $no(x)$  is the number associated to the variable  $x$ . Let us consider the 4-tuple  $(n_1, n_2, n_3, n_4)$  where the  $n_i$ 's are the following positive integers:

- $n_1$  is the number of occurrences of sub-formulas of the form  $x = fy_1 \dots y_n$ , with  $f \in F$ ,
- $n_2$  is the number of occurrences of atomic formulas,
- $n_3$  is the sum of the  $no(x)$ 's for all occurrences of a variable  $x$ ,
- $n_4$  is the number of occurrences of formulas of the form  $y = x$ , with  $x > y$ .

For each rule, there exists an integer  $i$  such that the application of this rule decreases or does not change the values of the  $n_j$ 's, with  $1 \leq j < i$ , and decreases the value of  $n_i$ . This integer  $i$  is equal to: 2 for the rule (1), 1 for the rules (2) and (3), 3 for the rules (4), (6) and (7), 4 for the rule (5). To each sequence of formulas obtained by a finite application of the preceding rewriting rules, we can associate a series of 4-tuples  $(n_1, n_2, n_3, n_4)$  which is strictly decreasing in the lexicographic order. Since the  $n_i$ 's are positive integers, they cannot be negative, thus this series of 4-tuples is a finite series and the application of the rewriting rules terminates.

*Proof, second part:* The rules preserve equivalence in  $\mathcal{T}$ . The rule (1) is evident in  $\mathcal{T}$ . The rules (2) preserves the equivalence in  $\mathcal{T}$  according to the axiom 1. The rule (3) preserves the equivalence in  $\mathcal{T}$  according to the axiom 2. The rules (4), (5), (6) and (7) are evident in  $\mathcal{T}$ .

*Proof, third part:* The application of the rewriting rules terminates either by *false* or by a ( $>$ )-solved conjunction of flat equations. Suppose that the application of the

rewriting rules on a conjunction  $\alpha$  of flat formulas terminates by a formula  $\beta$  and at least one of the following conditions holds:

1.  $\beta$  is not the formula *false* and has at least a sub-formula of the form *false*,
2.  $\beta$  has two equations with the same left-hand side,
3.  $\beta$  contains equations of the form  $x = x$  or  $y = x$  with  $x \succ y$ .

If the condition 1 holds then the rule (1) can still be applied which contradicts our supposition. If the condition 2 holds then the rules (2), (3), (6) and (7) can still be applied which contradicts our supposition. If the condition 3 holds then the rules (4) and (5) can still be applied which contradicts our supposition. Thus, the formula  $\beta$  according to Definition 5.2.1 is either the formula *false* or a ( $\succ$ )-solved conjunction of flat equations.  $\square$

Let us introduce now the notion of *reachable variable* and *reachable equation*.

### Definition 5.2.3

The equations and variables reachable from the variable  $u$  in the formula

$$\exists \bar{x} \bigwedge_{i=1}^n v_i = t_i$$

are those who occur in at least one of its sub-formulas of the form  $\bigwedge_{j=1}^m v_{k_j} = t_{k_j}$ , where  $v_{k_1}$  is the variable  $u$  and  $v_{k_{j+1}}$  occurs in the term  $t_{k_j}$  for all  $j \in \{1, \dots, m\}$ . The equations and variables reachable of this formula are those who are reachable from a variables which does not occur in  $\bar{x}$ .

### Example 5.2.4

In the formula

$$\exists uvwz = fw \wedge v = gvu \wedge w = fw,$$

the equations  $z = fw$  and  $v = gvu$  and the variables  $u$  and  $v$  are reachable. On the other hand the equation  $w = fw$  and the variable  $w$  are not reachable.

According to the axioms [1] and [2] of  $\mathcal{T}$  we have the following property

### Property 5.2.5

Let  $\alpha$  be a conjunction of flat equations. If all the variables of  $\bar{x}$  are reachable in  $\exists \bar{x} \alpha$  then  $\mathcal{T} \models \exists ?\bar{x} \alpha$ .

According to the axiom 3 we have:

### Property 5.2.6

Let  $\alpha$  be a ( $\succ$ )-solved conjunction of flat equations and let  $\bar{x}$  be the vector of its left-hand sides. We have  $\mathcal{T} \models \exists !\bar{x} \alpha$ .

## 5.3 $\mathcal{T}$ is decomposable

### Property 5.3.1

$\mathcal{T}$  is a decomposable theory.

Let us show that  $\mathcal{T}$  satisfies the conditions of Definition 3.3.1.

5.3.2 Choice of the sets  $\Psi(u)$ ,  $A$ ,  $A'$ ,  $A''$  and  $A'''$

Let  $F_0$  be the set of the 0-ary function symbols of  $F$ . The sets  $\Psi(u)$ ,  $A$ ,  $A'$ ,  $A''$  and  $A'''$  are chosen as follows:

- $\Psi(u)$  is the set  $\{false\}$  if  $F - F_0 = \emptyset$ , else it contains formulas of the form  $\exists \bar{y} u = f \bar{y}$  with  $f \in F - F_0$ ,
- $A$  is the set  $FL$ ,
- $A'$  is the set of formulas of the form  $\exists \bar{x}' \alpha'$  such that
  - $\alpha'$  is either the formula *false* or a ( $\succ$ )-solved conjunction of flat equations where the order  $\succ$  is such that all the variables of  $\bar{x}'$  are greater than the free variables of  $\exists \bar{x}' \alpha'$ ,
  - all the variables of  $\bar{x}'$  and all the equations of  $\alpha'$  are reachable in  $\exists \bar{x}' \alpha'$ ,
- $A''$  is the set of formulas of the form  $\exists \bar{x}'' true$ ,
- $A'''$  is the set of formulas of the form  $\exists \bar{x}''' \alpha'''$  such that  $\alpha'''$  is a ( $\succ$ )-solved conjunction of flat equations and  $\bar{x}'''$  is the vector of the left-hand sides of the equations of  $\alpha'''$ .

It is clear that  $FL$  is  $\mathcal{T}$ -closed and  $A'$ ,  $A''$  and  $A'''$  contain formulas of the form  $\exists \bar{x} \alpha$  with  $\alpha \in FL$ . Let us now show that  $\mathcal{T}$  satisfies the five condition of Definition 3.3.1

5.3.3  $\mathcal{T}$  satisfies the first condition

Let us show that every formula of the form  $\exists \bar{x} \alpha \wedge \psi$ , with  $\alpha \in FL$  and  $\psi$  any formula, is equivalent in  $\mathcal{T}$  to a wnfv formula of the form

$$\exists \bar{x}' \alpha' \wedge (\exists \bar{x}'' \alpha'' \wedge (\exists \bar{x}''' \alpha''' \wedge \psi)), \tag{37}$$

with  $\exists \bar{x}' \alpha' \in A'$ ,  $\exists \bar{x}'' \alpha'' \in A''$  and  $\exists \bar{x}''' \alpha''' \in A'''$ .

Let us choose the order  $\succ$  such that all the variables of  $\bar{x}$  are greater than the free variables of  $\exists \bar{x} \alpha$ . According to Property 5.2.2 two cases arise:

Either  $\alpha$  is equivalent to *false* in  $\mathcal{T}$ . Thus,  $\bar{x}' = \bar{x}'' = \bar{x}''' = \varepsilon$ ,  $\alpha' = false$  and  $\alpha'' = \alpha''' = true$ .

Or,  $\alpha$  is equivalent to a ( $\succ$ )-solved conjunction  $\beta$  of flat equations. Let  $X$  be the set of the variables of the vector  $\bar{x}$ . Let  $Y_{rea}$  be the set of the reachable variables of  $\exists \bar{x} \beta$ . Let  $Lhs$  be the set of the variables which occur in a left-hand side of an equation of  $\beta$ . We have:

- $\bar{x}'$  contains the variables of  $X \cap Y_{rea}$ .
- $\bar{x}''$  contains the variables of  $(X - Y_{rea}) - Lhs$ .
- $\bar{x}'''$  contains the variables of  $(X - Y_{rea}) \cap Lhs$ .
- $\alpha'$  is the conjunction of the reachable equations of  $\exists \bar{x} \beta$ .
- $\alpha''$  is the formula *true*.
- $\alpha'''$  is the conjunction of the unreachable equations of  $\exists \bar{x} \beta$ .

According to our construction it is clear that  $\exists \bar{x}' \alpha' \in A'$ ,  $\exists \bar{x}'' \alpha'' \in A''$  and  $\exists \bar{x}''' \alpha''' \in A'''$ . Let us show that (37) and  $\exists \bar{x} \alpha \wedge \psi$  are equivalent in  $\mathcal{T}$ . Let  $X'$ ,  $X''$  and  $X'''$  be

the sets of the variables of the vectors  $\bar{x}'$ ,  $\bar{x}''$  and  $\bar{x}'''$ . If  $\alpha$  is equivalent to *false* in  $\mathcal{T}$  then the equivalence of the decomposition is evident. Else  $\beta$  is a conjunction of flat equations and thus according to our construction we have:  $X = X' \cup X'' \cup X'''$ ,  $X' \cap X'' = \emptyset$ ,  $X' \cap X''' = \emptyset$ ,  $X'' \cap X''' = \emptyset$ , for all  $x'_i \in X''$  we have  $x'_i \notin \text{var}(\alpha')$  and for all  $x'''_i \in X'''$  we have  $x'''_i \notin \text{var}(\alpha' \wedge \alpha'')$ . Moreover each equation of  $\beta$  occurs in  $\alpha' \wedge \alpha'' \wedge \alpha'''$  and each equation in  $\alpha' \wedge \alpha'' \wedge \alpha'''$  occurs in  $\beta$  and thus  $\mathcal{T} \models \beta \leftrightarrow (\alpha' \wedge \alpha'' \wedge \alpha''')$ . We have shown that the vectorial quantifications are coherent and the equivalence  $\mathcal{T} \models \beta \leftrightarrow \alpha' \wedge \alpha'' \wedge \alpha'''$  holds. According to Property 5.2.2 we have  $\mathcal{T} \models \alpha \leftrightarrow \beta$  and thus, the decomposition keeps the equivalence in  $\mathcal{T}$ .

#### Example 5.3.4

Let us decompose the following formula  $\varphi$

$$\exists xyvz = fxy \wedge z = fxw \wedge v = fz.$$

First, since  $w$  and  $z$  are free in  $\varphi$  then the order  $>$  will be chosen as follows:

$$x > y > v > w > z.$$

Note that the quantified variables are greater than the free variables. Then, using the rewriting rules of Property 5.2.2 we transform the conjunction of equations to a ( $>$ )-solved formula. Thus, the formula  $\varphi$  is equivalent in  $\mathcal{T}$  to the following formula  $\psi$

$$\exists xyvz = fxy \wedge y = w \wedge v = fz.$$

Since the variables  $x, y, w$  and the equations  $z = fxy, y = w$  are reachable in  $\psi$  then  $\psi$  is equivalent in  $\mathcal{T}$  to the following decomposed formula

$$\exists xyz = fxy \wedge y = w \wedge (\exists \varepsilon \text{ true} \wedge (\exists v v = fz)).$$

It is clear that  $(\exists xy z = fxy \wedge y = w) \in A'$ ,  $(\exists \varepsilon \text{ true}) \in A''$  and  $(\exists v v = fz) \in A'''$ .

#### 5.3.5 $\mathcal{T}$ satisfies the second condition

Let us show that if  $\exists \bar{x}' \alpha' \in A'$  then  $\mathcal{T} \models \exists ? \bar{x}' \alpha'$ . Since  $\exists \bar{x}' \alpha' \in A'$  and according to the choice of the set  $A'$ , either  $\alpha'$  is the formula *false* and thus we have immediately  $\mathcal{T} \models \exists ? \bar{x}' \alpha'$  or  $\alpha'$  is a ( $>$ )-solved conjunction of flat equations and the variables of  $\bar{x}'$  are reachable in  $\exists \bar{x}' \alpha'$ . Thus, using Property 5.2.5 we get  $\mathcal{T} \models \exists ? \bar{x}' \alpha'$ .

Let us show now that if  $y$  is a free variable of  $\exists \bar{x}' \alpha'$  then  $\mathcal{T} \models \exists ? y \bar{x}' \alpha'$  or there exists  $\varphi(u) \in \Psi(u)$  such that  $\mathcal{T} \models \forall y (\exists \bar{x}' \alpha') \rightarrow \varphi(y)$ . Let  $y$  be a free variable of  $\exists \bar{x}' \alpha'$ . It is clear that  $\alpha'$  can not be in this case the formula *false*. Thus, four cases arise:

If  $y$  occurs in a sub-formula of  $\alpha'$  of the form  $y = t(\bar{x}', \bar{z}', y)$ , where  $\bar{z}'$  is the set of the free variables of  $\exists \bar{x}' \alpha'$  which are different from  $y$  and where  $t(\bar{x}', \bar{z}', y)$  is a term which begins by an element of  $F - F_0$  followed by variables taken from  $\bar{x}'$  or  $\bar{z}'$  or  $\{y\}$ , then the formula  $\exists \bar{x}' \alpha'$  implies in  $\mathcal{T}$  the formula  $\exists \bar{x}' y = t(\bar{x}', \bar{z}', y)$ , which implies in  $\mathcal{T}$  the formula  $\exists \bar{x}' \bar{z}' w y = t(\bar{x}', \bar{z}', w)$ , where  $y = t(\bar{x}', \bar{z}', w)$  is the formula  $y = t(\bar{x}', \bar{z}', y)$  in which we have replaced every free occurrence of  $y$  in the term

$t(\bar{x}', \bar{z}', y)$  by the variable  $w$ . According to the choice of the set  $\Psi(u)$ , the formula  $\exists \bar{x}' \bar{z}' w u = t(\bar{x}', \bar{z}', w)$  belongs to  $\Psi(u)$ .

If  $y$  occurs in a sub-formula of  $\alpha'$  of the form  $y = f_0$  with  $f_0 \in F_0$  then according to the third axiom of  $\mathcal{T}$  we have  $\mathcal{T} \models \exists ! y y = f_0$ . Thus (i)  $\mathcal{T} \models \exists ? y \alpha'$ . On the other hand, since  $\alpha'$  is  $(>)$ -solved,  $y$  has no occurrences in an other left-hand side of an equation of  $\alpha'$ , thus since the variables of  $\bar{x}$  are reachable in  $\exists \bar{x}' \alpha'$  (according to the choice of the set  $A'$ ), all the variables of  $\bar{x}'$  keep reachable in  $\exists \bar{x}' y \alpha'$  and thus using (i) and Property 5.2.5 we get  $\mathcal{T} \models \exists ? \bar{x}' y \alpha'$ .

If  $y$  occurs in a sub-formula of  $\alpha'$  of the form  $y = z$  then:

1. According to the choice of the set  $A'$ , the order  $>$  is such that all the variables of  $\bar{x}'$  are greater than the free variables of  $\exists \bar{x}' \alpha'$ .
2. According to Definition 5.2.2 of the  $(>)$ -solved formula, we have  $y > z$ .

From (1) and (2), we deduce that  $z$  is a free variable in  $\exists \bar{x}' \alpha'$ . Since  $\alpha'$  is  $(>)$ -solved,  $y$  has no occurrences in an other left-hand side of an equation of  $\alpha'$ , thus since the variables of  $\bar{x}$  are reachable in  $\exists \bar{x}' \alpha'$  (according to the choice of the set  $A'$ ), all the variables of  $\bar{x}'$  keep reachable in  $\exists \bar{x}' y \alpha'$ . More over, for each value of  $z$  there exists at most a value for  $y$ . Thus, using Property 5.2.5 we get  $\mathcal{T} \models \exists ? \bar{x}' y \alpha'$ .

If  $y$  occurs only in the right-hand sides of the equations of  $\alpha'$  then according to the choice of the set  $A'$ , all the variables of  $\bar{x}'$  and all the equations of  $\alpha'$  are reachable in  $\exists \bar{x}' \alpha'$ . Thus, since  $y$  does not occur in a left-hand side of an equation of  $\alpha'$ , the variable  $y$  and the variables of  $\bar{x}'$  are reachable in  $\exists \bar{x}' y \alpha'$  and thus using Property 5.2.5 we get  $\mathcal{T} \models \exists ? \bar{x}' y \alpha'$ . In all cases  $\mathcal{T}$  satisfies the second condition of Definition 3.3.1.

### 5.3.6 $\mathcal{T}$ satisfies the third condition

First, we present a property which hold in any model  $M$  of  $\mathcal{T}$ . This property results from the axiomatization of  $\mathcal{T}$  (more exactly from axioms 1 and 2) and the infinite set  $F$  of function symbols.

#### Property 5.3.7

Let  $M$  be a model of  $\mathcal{T}$  and let  $f$  be a function symbol taken from  $F - F_0$ . The set of the individuals  $i$  of  $M$ , such that  $M \models \exists \bar{x} i = f \bar{x}$ , is infinite.

Let  $\exists \bar{x}'' \alpha''$  be a formula which belongs to  $A''$ . According to the choice of  $A''$ , this formula is of the form  $\exists \bar{x}'' true$ . Let us show that, for every variable  $x''_j$  of  $\bar{x}''$  we have  $\mathcal{T} \models \exists_{\infty}^{\Psi(u)} x''_j true$ . Two cases arise:

If  $F - F_0 = \emptyset$  then  $\Psi(u) = \{false\}$  and  $F_0$  is infinite since the theory is defined on an infinite set of function symbols. According to axiom 1 of  $\mathcal{T}$ , for all distinct constants  $f$  and  $g$  correspond two distinct individuals in all models of  $\mathcal{T}$ . Thus, since  $F_0$  is infinite there exists an infinite set of individuals in all models of  $\mathcal{T}$  and thus according to Definition 3.2.1 we have:  $\mathcal{T} \models \exists_{\infty}^{\{false\}} x''_j true$ .

If  $F - F_0 \neq \emptyset$  then  $\Psi(u)$  contains formulas of the form  $\exists \bar{z} u = f \bar{z}$  with  $f \in F - F_0$ . Let  $M$  be a model of  $\mathcal{T}$ . Since the formula  $\exists x''_j true$  does not have free variables, it is



already instantiated, and thus according to Definition 3.2.1 it is enough to show that there exists an infinity of individuals  $i$  of  $M$  which satisfy the following condition:

$$M \models \neg\psi_1(i) \wedge \cdots \wedge \neg\psi_n(i), \quad (38)$$

with  $\psi_j(u) \in \Psi(u)$ , i.e. of the form  $\exists \bar{z} u = f\bar{z}$  with  $f \in F - F_0$ . Two cases arise:

- If  $F - F_0$  is a finite set then  $F_0$  is infinite because the theory is defined on infinite set of function symbols. Thus, there exists an infinity of constants  $f_k$  which are different from all the function symbols of all the  $\psi_j(u)$  of (38) and thus using axiom 1 of  $\mathcal{T}$  there exists an infinity of distinct individuals  $i$  such that (38).
- If  $F - F_0$  is infinite then there exists a formula  $\psi(u)^* \in \Psi(u)$  which is different from all the  $\psi_j(u)$  of (38), i.e. which has a function symbol which is different from the function symbols of all the  $\psi_1(u) \cdots \psi_n(u)$ . According to Property 5.3.7 there exists an infinity of individuals  $i$  such that  $M \models \psi(i)^*$ . Since this  $\psi(u)^*$  is different from all the  $\psi_j(u)$ , then according to axiom 1 of  $\mathcal{T}$  there exists an infinite set of individuals  $i$  such that  $M \models \psi(i)^* \wedge \neg\psi_1(i) \wedge \cdots \wedge \neg\psi_n(i)$  and thus such that (38).

### 5.3.8 $\mathcal{T}$ satisfies the fourth condition

Let us show that if  $\exists \bar{x}''' \alpha''' \in A'''$  then  $\mathcal{T} \models \exists ! \bar{x}''' \alpha'''$ . Let  $\exists \bar{x}''' \alpha'''$  be an element of  $A'''$ . According to the choice of the set  $A'''$  and Property 5.2.6 we get immediately  $\mathcal{T} \models \exists ! \bar{x}''' \alpha'''$ .

### 5.3.9 $\mathcal{T}$ satisfies the fifth condition

Let us show that if the formula  $\exists \bar{x}' \alpha'$  belongs to  $A'$  and has no free variables then this formula is either the formula  $\exists \text{true}$  or  $\exists \text{false}$ . Let  $\exists \bar{x}' \alpha'$  be a formula, without free variables, which belongs to  $A'$ . We have

1. According to the choice of the set  $A'$ , all the variables and equations of  $\exists \bar{x}' \alpha'$  are reachable in  $\exists \bar{x}' \alpha'$  and  $\alpha'$  is either the formula *false* or a ( $\succ$ )-solved conjunction of flat equations.
2. Since the formula  $\exists \bar{x}' \alpha'$  has no free variables and according to Definition 5.2.3 there exists in this case neither variables nor equations reachable in  $\exists \bar{x}' \alpha'$ ,

Thus, From (1) and (2),  $\bar{x}'$  is the empty vector, i.e.  $\varepsilon$  and  $\alpha'$  is either the formula *true* or *false*.

## 5.4 Solving first-order formulas in $\mathcal{T}$

Since  $\mathcal{T}$  is decomposable we can apply our general algorithm and solve any first-order formula. Let us first recall the related works about the resolution of tree constraints: the unification of finite terms, i.e. the resolution of conjunctions of equations in the theory of finite trees has first been studied by A. Robinson (Robinson 1965). Some better algorithms with better complexities has been proposed after by

M.S. Paterson and M.N.Wegman (Paterson 1978) and A. Martelli and U. Montanari (Matelli 1982). The resolution of conjunctions of equations in the theory of infinite trees has been studied by G. Huet (Huet 1976), by A. Colmerauer (Colmerauer 1982; Colmerauer 1984) and by J. Jaffar (Jaffar 1984). The resolution of conjunctions of equations and disequations in the theory of finite or infinite trees has been studied by A. Colmerauer (Colmerauer 1984) and H.J. Brckert (Burkert 1988). An incremental algorithm for solving conjunctions of equations and disequations on rational trees has been proposed after by V.Ramachandran and P. Van Hentenryck (Ramachandran 1993). The resolution of universally quantified disequations on finite trees has been also developed by A. Smith (Smith 1991). We will find a general synthesis on this subject in the work of H. Comon (Comon 1991). M. Maher has also shown that every formula is equivalent in  $\mathcal{T}$  to a Boolean combination of existentially quantified solved conjunctions of elementary equations (Maher 1988). Note that we get the same result using Corollary 4.4.1.

In what follows, we first show how to solve some simple formulas without free variables in order to understand the application of the rewriting rules and the role of each rule in  $\mathcal{T}$ , then we give some benchmarks representing real situations on two partner games by full first-order formulas with free variables.

**Simple examples**

*Example 5.4.1*

Let us solve the following formula  $\varphi_1$  in  $\mathcal{T}$  :

$$\exists x \forall y ((\exists z w v y = fz \wedge y = fx \wedge w = gzv) \vee (x = fy \wedge x = fx))$$

Using Property 4.1.3 we first transform the preceding formula into the following normalized formula

$$\neg(\exists \varepsilon true \wedge \neg(\exists x true \wedge \neg \left[ \begin{array}{l} \exists y true \wedge \\ \neg(\exists z w v y = fz \wedge y = fx \wedge w = gzv) \wedge \\ \neg(\exists \varepsilon x = fy \wedge x = fx) \end{array} \right] )) \tag{39}$$

Since  $A = FL$  then the preceding normalized formula is a working formula. Let us decompose the sub-formula

$$\exists z w v y = fz \wedge y = fx \wedge w = gzv. \tag{40}$$

According to Section 5.3.3, the order  $\succ$  is chosen such that  $z \succ w \succ v \succ y \succ x$ . Using the rewriting rules of Property 5.2.2, the sub-formula  $y = fz \wedge y = fx \wedge w = gzv$  is equivalent in  $\mathcal{T}$  to the  $(\succ)$ -solved formula  $y = fz \wedge z = x \wedge w = gzv$ , and thus according to Section 5.3.3 the decomposed formula of (40) is

$$\exists z y = fz \wedge z = x \wedge (\exists v true \wedge (\exists w w = gzv))$$

Since  $(\exists w w = gzv) \neq (\exists \varepsilon true)$  we can apply the rule (3) with  $I = \emptyset$ , thus the formula (39) is equivalent in  $\mathcal{T}$  to

$$\neg(\exists \varepsilon true \wedge \neg(\exists x true \wedge \neg \left[ \begin{array}{l} \exists y true \wedge \\ \neg(\exists v y = fz \wedge z = x) \wedge \\ \neg(\exists \varepsilon x = fy \wedge x = fx) \end{array} \right] )) \tag{41}$$

The sub-formula  $\exists v y = fz \wedge z = x$  is not an element of  $A'$  and is equivalent in  $\mathcal{T}$  to the decomposed formula  $\exists z y = fz \wedge z = x \wedge (\exists v \text{true} \wedge (\exists \varepsilon \text{true}))$ , thus we can apply the rule (4) with  $I = \emptyset$  and the formula (41) is equivalent in  $\mathcal{T}$  to

$$\neg(\exists \varepsilon \text{true} \wedge \neg(\exists x \text{true} \wedge \neg \left[ \begin{array}{l} \exists y \text{true} \wedge \\ \neg(\exists z y = fz \wedge z = x) \wedge \\ \neg(\exists \varepsilon x = fy \wedge x = fx) \end{array} \right] )) \quad (42)$$

Let us decompose now the sub-formula

$$\exists \varepsilon x = fy \wedge x = fx \quad (43)$$

Using the rewriting rules of Property 5.2.2, the sub-formula  $x = fy \wedge x = fx$  is equivalent in  $\mathcal{T}$  to the ( $\succ$ )-solved formula  $x = fy \wedge y = x$  and thus according to Section 5.3.3 the decomposed formula of (43) is

$$\exists \varepsilon x = fy \wedge y = x \wedge (\exists \varepsilon \text{true} \wedge (\exists \varepsilon \text{true}))$$

Since  $(\exists \varepsilon x = fy \wedge x = fx) \notin A'$  then we can apply the rule (4) with  $I = \emptyset$  and thus the formula (42) is equivalent in  $\mathcal{T}$  to

$$\neg(\exists \varepsilon \text{true} \wedge \neg(\exists x \text{true} \wedge \neg \left[ \begin{array}{l} \exists y \text{true} \wedge \\ \neg(\exists z y = fz \wedge z = x) \wedge \\ \neg(\exists \varepsilon x = fy \wedge y = x) \end{array} \right] )) \quad (44)$$

According to Section 5.3.3 the formula  $\exists \varepsilon \text{true} \wedge (\exists y \text{true} \wedge (\exists \varepsilon \text{true}))$  is the decomposed formula of  $\exists y \text{true}$ . Since  $\exists y \text{true} \notin A'$ ,  $(\exists z y = fz \wedge z = x) \in A'$  and  $(\exists \varepsilon x = fy \wedge y = x) \in A'$  then we can apply the rule (4) and thus the formula (44) is equivalent in  $\mathcal{T}$  to

$$\neg(\exists \varepsilon \text{true} \wedge \neg(\exists \varepsilon \text{true} \wedge \neg(\exists \varepsilon \text{true}))) \quad (45)$$

Finally, we can apply the rule (1) thus the formula (45) is equivalent in  $\mathcal{T}$  to  $\neg(\exists \varepsilon \text{true})$ . Thus  $\varphi_1$  is false in  $\mathcal{T}$ .

#### Example 5.4.2

Let us solve the following formula  $\varphi_2$  in  $\mathcal{T}$ :

$$\exists x \forall y ((\exists z y = fz \wedge z = x) \vee (\exists \varepsilon x = fy \wedge y = x) \vee \neg(x = fy)) \quad (46)$$

Using Property 4.1.3 we first transform the preceding formula into the following normalized formula

$$\neg(\exists \varepsilon \text{true} \wedge \neg(\exists x \text{true} \wedge \neg \left[ \begin{array}{l} \exists y x = fy \wedge \\ \neg(\exists z y = fz \wedge z = x) \wedge \\ \neg(\exists \varepsilon x = fy \wedge y = x) \end{array} \right] )) \quad (47)$$

Since  $A = FL$  then the preceding normalized formula is a working formula in  $\mathcal{T}$ . Since  $(\exists y x = fy) \in A'$ ,  $(\exists z y = fz \wedge z = x) \in A'$  and  $(\exists \varepsilon x = fy \wedge y = x) \in A'$  then

we can apply the rule (5), thus the formula (47) is equivalent in  $\mathcal{T}$  to

$$\neg \left[ \begin{array}{l} \exists \varepsilon \text{ true} \wedge \\ \neg(\exists x \text{ true} \wedge \neg(\exists y x = fy)) \wedge \\ \neg(\exists x_1 y_1 z x_1 = fy_1 \wedge y_1 = fz \wedge z = x_1) \wedge \\ \neg(\exists x_2 y_2 x_2 = fy_2 \wedge x_2 = fy_2 \wedge y_2 = x_2) \end{array} \right] \tag{48}$$

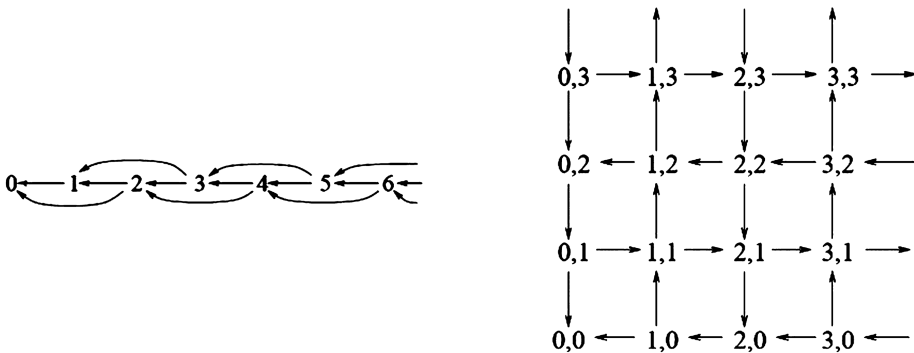
According to Section 5.3.3 the formula  $\exists \varepsilon \text{ true} \wedge (\exists x \text{ true} \wedge (\exists \varepsilon \text{ true}))$  is the decomposed formula of  $\exists x \text{ true}$ . Since  $(\exists x \text{ true}) \notin A'$  and  $(\exists y x = fy) \in A'$  then we can apply the rule (4) and thus the formula (48) is equivalent in  $\mathcal{T}$  to

$$\neg \left[ \begin{array}{l} \exists \varepsilon \text{ true} \wedge \\ \neg(\exists \varepsilon \text{ true}) \wedge \\ \neg(\exists x_1 y_1 z x_1 = fy_1 \wedge y_1 = fz \wedge z = x_1) \wedge \\ \neg(\exists x_2 y_2 x_2 = fy_2 \wedge x_2 = fy_2 \wedge y_2 = x_2) \end{array} \right] \tag{49}$$

Finally we can apply the rule (1), thus the formula (49) is equivalent in  $\mathcal{T}$  to *true*. Thus  $\varphi_2$  is true in  $\mathcal{T}$ .

**Benchmarks: Two partner games**

Let  $(V, E)$  be a directed graph, with  $V$  a set of vertices and  $E \subseteq V \times V$  a set of edges. The sets  $V$  and  $E$  may be empty and the elements of  $E$  are also called positions. We consider a two-partner game which, given an initial position  $x_0$ , consists, one after another, in choosing a position  $x_1$  such that  $(x_0, x_1) \in E$ , then a position  $x_2$  such that  $(x_1, x_2) \in E$  and so on. The first one who cannot play any more has lost and the other one has won. For example the two following infinite graphs correspond to the two following games:



**Game 1** A non-negative integer  $i$  is given and, one after another, each partner subtracts 1 or 2 from  $i$ , but keeping  $i$  non-negative. The first person who cannot play any more has lost.

**Game 2** An ordered pair  $(i, j)$  of non-negative integers is given and, one after another, each partner chooses one of the integers  $i, j$ . Depending on the fact that the chosen integer  $u$  is odd or even, he then increases or decreases the other integer  $v$  by 1, but keeping  $v$  non-negative. The first person who cannot play any more has lost.

Let  $x$  be a position in a game and suppose that it is the turn of person A to play. The position  $x$  is said to be  $k$ -winning if, no matter the way the other person B plays, it is always possible for A to win in making at most  $k$  moves. It is easy to show that

$$winning_k(x) = \left[ \begin{array}{l} \exists y \text{ move}(x, y) \wedge \neg( \\ \exists x \text{ move}(y, x) \wedge \neg( \\ \dots \\ \exists y \text{ move}(x, y) \wedge \neg( \\ \exists x \text{ move}(y, x) \wedge \neg( \\ \text{false} \quad \underbrace{\dots}_{2k} \end{array} \right]$$

where  $\text{move}(x, y)$  means : “ starting from the position  $x$  we play one time and reach the position  $y$ ”. By moving down the negations, we get an embedding of  $2k$  alternated quantifiers. We represent this two games in the algebra of finite or infinite trees  $(A, F)$ , where each position is represented by a tree.

If we take as input of our solver the formula  $winning_k(x)$  we will get as output a formula which represents all the  $k$ -winning positions.

**Game 1:** Suppose that  $F$  contains the 0-ary functional symbol  $0$  and the 1-ary functional symbol  $s$ . We code the vertices  $i$  of the game graph by the trees  $s^i(0)$ <sup>12</sup> The relation  $\text{move}(x, y)$  is defined as follows:

$$\text{move}(x, y) \stackrel{\text{def}}{\leftrightarrow} x = s(y) \vee x = s(s(y)) \vee (\neg(x = 0) \wedge \neg(\exists u x = s(u)) \wedge x = y)$$

For  $winning_1(x)$  our algorithm give the following solved formula:

$$\neg \left[ \exists \varepsilon \text{ true} \wedge \left[ \begin{array}{l} \neg(\exists u x = s(u) \wedge u = 0) \wedge \\ \neg(\exists u_1 u_2 x = s(u_1) \wedge u_1 = s(u_2) \wedge u_2 = 0) \end{array} \right] \right]$$

which corresponds to the solution  $x = s(0) \vee x = s(s(0))$ .

**Game 2:** Suppose that  $F$  contains the functional symbols  $0, f, g, c$  of respective arities 0, 1, 1, 2. We code the vertices  $(i, j)$  of the game graph by the trees  $c(\bar{i}, \bar{j})$  with  $\bar{i} = (fg)^{i/2}(0)$  if  $i$  is even, and  $\bar{i} = g(i-1)$  if  $i$  is odd.<sup>13</sup> The relation  $\text{move}(x, y)$  is defined as follows:

$$\text{move}(x, y) \stackrel{\text{def}}{\leftrightarrow} \text{transition}(x, y) \vee (\neg(\exists uv x = c(u, v)) \wedge x = y)$$

with

$$\text{transition}(x, y) \stackrel{\text{def}}{\leftrightarrow} \left[ \begin{array}{l} \exists uvw \\ \left[ \begin{array}{l} (x = c(u, v) \wedge y = c(u, w)) \vee \\ (x = c(v, u) \wedge y = c(w, u)) \end{array} \right] \\ \wedge \\ \left[ \begin{array}{l} (\exists i u = g(i) \wedge \text{succ}(v, w)) \vee \\ (\neg(\exists i u = g(i)) \wedge \text{pred}(v, w)) \end{array} \right] \end{array} \right]$$

$$\text{succ}(v, w) \stackrel{\text{def}}{\leftrightarrow} \left[ \begin{array}{l} (\exists j v = g(j) \wedge w = f(v)) \vee \\ (\neg(\exists j v = g(j)) \wedge w = g(v)) \end{array} \right]$$

<sup>12</sup> Of course  $s^0(x) = x$  and  $s^{i+1}(x) = s(s^i(x))$ .

<sup>13</sup>  $(fg)^0(x) = x$  and  $(fg)^{i+1}(x) = f(g((fg)^i(x)))$ .

$$pred(v, w) \stackrel{\text{def}}{\leftrightarrow} \left[ \begin{array}{l} (\exists j v = f(j) \wedge \left[ \begin{array}{l} (\exists k j = g(k) \wedge w = j) \vee \\ (\neg(\exists k j = g(k)) \wedge w = v) \end{array} \right] ) \vee \\ (\exists j v = g(j) \wedge \left[ \begin{array}{l} (\exists k j = g(k) \wedge w = v) \vee \\ (\neg(\exists k j = g(k)) \wedge w = j) \end{array} \right] ) \vee \\ (\neg(\exists j v = f(j)) \wedge \neg(\exists j v = g(j)) \wedge \neg(v = 0) \wedge w = v) \end{array} \right]$$

For  $winning_1(x)$  our algorithm give the following solved formula:

$$\neg \left[ \exists \varepsilon true \wedge \left[ \begin{array}{l} \neg(\exists u_1 u_2 u_3 x = c(u_1, u_2) \wedge u_1 = g(u_3) \wedge u_2 = 0 \wedge u_3 = 0) \wedge \\ \neg(\exists u_1 u_2 u_3 x = c(u_1, u_2) \wedge u_2 = g(u_3) \wedge u_1 = 0 \wedge u_3 = 0) \end{array} \right] \right]$$

which corresponds to the solution  $x = c(g(0), 0) \vee x = c(0, g(0))$ .

The times of execution (CPU time in milliseconds) of the formulas  $winning_k(x)$  are given in the following table as well as a comparison with those of Djelloul (2006a). The algorithm was programmed in C++ and the benchmarks are performed on a 2.5Ghz Pentium IV processor, with 1024Mb of RAM.

k (Game 1)	0	1	2	4	10	20	40	80
Our alg	0	0	5	11	178	2630	59430	2553746
Djelloul (2006a)	0	0	5	10	150	2130	45430	1920110
k (Game 2)	0	1	2	4	10	20	40	80
Our alg	0	79	209	508	3830	162393	–	–
Djelloul (2006a)	0	75	180	420	3040	123025	–	–

These benchmarks were first introduced by A. Colmerauer and T. Dao. in (Colmerauer 2003) where the first results of the algorithm of T. Dao (Dao 2000) were presented. We used the same benchmarks in a joint work with T. Dao (Djelloul 2006a) where we gave a more efficient algorithm for solving first-order formulas in finite or infinite trees with better performances. The algorithm (Djelloul 2006a) uses two strategies: (1) a top-down propagation of constraints: where all the super-formulas are propagated to the sub-formulas, then locally solved and finally restored and so on. (2) A bottom-up distribution of sub-formulas to decrease the depth of the formulas. The restorations of constraints defined in the first point uses a particular property which holds only for the theory of finite or infinite trees. This algorithm (Djelloul 2006a) gives good performances and the first step enables us to obtain quickly the solved formulas without losing time with solving sub-formulas which contradict their super-formulas. On the other hand our general algorithm defined in this paper can not use these strategies since it handles general decomposable theories. The main idea is to decompose at each level a quantified conjunction of atomic formulas and to propagate only the third section  $A'''$  into the sub-formulas (rule 3). Then, the rule (4) decreases the size of the conjunction of sub-formulas and eliminates some quantifiers. Finally, the rule (5) decreases the depth of the working

formulas using distribution. This algorithm computes the  $k$ -winning positions with the same bounds of performances for the values of  $k$  as those of (Djelloul 2006a) but takes 5%-30% more time to compute them. This is due to the specific treatments used in (Djelloul 2006a). Unfortunately, this rate (5%-30%) grows with the size of  $k$  and thus with the size of the initial working formula. Anyway, it must be noted that we were able to compute the  $k$ -winning positions of game 1 with  $k = 80$ , which corresponds to a formula involving an alternated embedding of more than 160 quantifiers with a non-specific algorithm for finite or infinite trees.

## 6 Discussion and conclusion

We defined in this paper a new class of theories that we call *decomposable theories* and showed their completeness using a sufficient condition for the completeness of first-order theories. Informally, a decomposable theory is a theory where each quantified conjunction of atomic formulas can be decomposed into three embedded sequences of quantifications having particular properties, which can be expressed with the help of  $\exists?$ ,  $\exists_{\infty}^{\Psi(u)}$  and  $\exists!$ . We deduced from this definition a sufficient condition so that a theory accepts full elimination of quantifiers and showed that there is a strong relation between the set  $A'$  and the notion of full elimination of quantifiers. We have also given a general algorithm for solving first-order formulas in any decomposable theory  $T$ . This algorithm is given in the form of a set of five rewriting which transform a working formula  $\varphi$  to a wnfv conjunction  $\phi$  of solved formulas. In particular if  $\varphi$  is a proposition, then  $\phi$  is either the formula *true* or  $\neg$ *true*.

On the other hand S. Vorobyov (Vorobyov 1996) has shown that the problem of deciding if a proposition is true or not in the theory of finite or infinite trees is non-elementary, i.e. the complexity of all algorithms solving propositions is not bounded by a tower of powers of 2's (top down evaluation) with a fixed height. A. Colmerauer and T. Dao (Colmerauer 2003) have also given a proof of non-elementary complexity of solving constraints in this theory. As a consequence, the complexity of our algorithm and the size of our solved formulas are of this order. We can easily show that the size of our solved formulas is bounded above by a top down tower of powers of 2's, whose height is the maximal depth of nested negations in the initial formula. The function  $\alpha(\varphi)$  used to show the termination of our rules illustrates this result. However, despite this high complexity, we have implemented our algorithm and solved some benchmarks in  $\mathcal{T}$  with formulas having long nested alternated quantifiers (up to 160). This algorithm has given competitive results in term of maximal depth of formulas that can be solved, compared with those of (Djelloul 2006a) but took more time to compute the solved formulas. As a consequence, we are planning with Thom Fruehwirth (Fruehwirth 2002) to add to CHR a general mechanism to treat our normalized formulas. This will enable us to implement quickly and easily other versions of our algorithms in order to get better performances.

Currently, we are trying to find a more abstract characterization and/or a model theoretical characterization of the decomposable theories. The current definition

gives only an algorithmic insight into what it means for a theory to be complete. We expect to add new vectorial quantifiers in the decomposition such as  $\exists^n$  which means *there exists  $n$*  and  $\exists_{0,\infty}^{\Psi(u)}$  which means *there exists zero or infinite*, in order to increase the size of the set of decomposable theories and may be get a much more simple definition than the one defined in this paper. Another interesting challenge is to find which special quantifiers must be added to the decomposable theories to get an equivalence between complete theory and decomposable theory. A first attempt on this subject is actually in progress using the quantifiers  $\exists^n$  and  $\exists_{0,\infty}^{\Psi(u)}$ . It would be also interesting to show if these new quantifiers are enough to prove that every theory which accepts elimination of quantifiers is decomposable.

We have also established a long list of decomposable theories. We can cite for example: the theory of finite trees, of infinite trees, of finite or infinite trees (Djelloul 2006a), of additive rational or real numbers with addition and subtraction, of linear dense order without endpoints, of ordered additive rational or real numbers with addition, subtraction and a linear dense order relation without endpoints, of the combination of trees and ordered additive rational numbers (Djelloul 2005b), of the construction of trees on an ordered set (Djelloul 2005a), of the extension into trees of first-order theories (Djelloul 2006b). It would also be interesting to build some theories that can be decomposed using two completely different sets of  $A$ ,  $A'$ ,  $A''$ ,  $A'''$  and  $\Psi(u)$  and find syntactic or semantic relations between these sets.

Currently, we are showing the decomposability of other fundamental theories such as: theory of lists using a combination of particular trees, theory of queues as done in (Rybina 2001), and the combination of trees and real numbers together with addition, subtraction, multiplication and a linear dense order relation without endpoints. We are also trying to find some formal methods to get easily the sets  $\psi(u)$ ,  $A$ ,  $A'$ ,  $A''$  and  $A'''$  for any decomposable theory  $T$ .

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### References

- BENHAMOU, F., COLMERAUER, A., GARETTA, H., PASERO, R. AND VAN-CANEGHEM, M. 1996. Le manuel de Prolog IV. PrologIA, Marseille, France.
- BURCKERT, H. 1988. Solving disequations in equational theories. In Proceeding of the 9th Conference on Automated Deduction, LNCS 310, pp. 517–526, Springer-Verlag.
- CLARK, K. L. 1978. Negation as failure. In Logic and Data bases. Ed Gallaire, H. and Minker, Plenum.



- COLMERAUER, A. 1982. Prolog and infinite trees. In: K. L. Clark and S.-A. Tarnlund, editors, *Logic Programming*. Academic Press. pp. 231–251.
- COLMERAUER, A. 1984. Equations and inequations on finite and infinite trees. *Proceeding of the International Conference on the Fifth Generation of Computer Systems*, pp. 85–99.
- COLMERAUER, A. 1990. An introduction to Prolog III. *Comm. ACM*, 33(7): 68–90.
- COLMERAUER, A. AND DAO, T. 2003. Expressiveness of full first-order formulas in the algebra of finite or infinite trees, *Constraints*, 8(3): 283–302.
- COMON, H. 1988. Unification et disunification : Theorie et applications. PhD thesis, Institut National Polytechnique de Grenoble.
- COMON, H. AND LESCANNE, P. 1989. Equational problems and disunification. *J. Symbolic Computation*, 7: 371–425.
- COMON, H. 1991. Disunification: a survey. In: J. L. Lassez and G. Plotkin, editors, *Computational Logic: Essays in Honor of Alan Robinson*. MIT Press.
- COMON, H. 1991. Resolution de contraintes dans des algebres de termes. Rapport d'Habilitation, Universite de Paris Sud.
- COURCELLE, B. 1983. Fundamental Properties of Infinite Trees. *Theoretical Computer Science*, 25(2): 95–169.
- COURCELLE, B. 1986. Equivalences and Transformations of Regular Systems applications to Program Schemes and Grammars. *Theoretical Computer Science*, 42: 100–122.
- DAO, T. 2000. Resolution de contraintes du premier ordre dans la theorie des arbres finis ou infinis. These d'informatique, Universite de la mediterrane, France.
- DJELLOUL, K. 2005a. Complete first-order axiomatization of the construction of trees on an ordered set. *Proceedings of the 2005 International Conference on Foundations of Computer Science (FCS'05)*, CSREA Press, pp. 87–93.
- DJELLOUL, K. 2005b. About the combination of trees and rational numbers in a complete first-order theory. *Proceeding of the 5th International conference on frontiers of combining systems FroCoS 2005*, Springer Lecture Notes in Artificial Intelligence, vol 3717, pp. 106–122.
- DJELLOUL, K. AND DAO, T. 2006a. Solving First-Order formulas in the Theory of Finite or Infinite Trees : Introduction to the Decomposable Theories. *Proceeding of the 21st ACM Symposium on Applied Computing (SAC)*. ACM press (to appear).
- DJELLOUL, K. AND DAO, T. 2006b. Complete first-order axiomatization of the M-extended trees. *Proceeding of the 20th Workshop on (constraint) Logic Programming (WLP06)*. INFSYS Research Report 1843-06-02, pp. 111–119.
- FRUEHWIRTH T. AND ABDELNADHER S. 2002. *Essentials of Constraints Programming*. Springer Cognitive technologies.
- HUET, G. 1976. Resolution d'equations dans les langages d'ordre 1, 2, ...  $\omega$ . These d'Etat, Universite Paris 7. France.
- JAFFAR, J. 1984. Efficient unification over infinite terms. *New Generation Computing*, 2(3): 207–219.
- JOHN, E. AND ULLMAN, D. 1979. *Introduction to Automata Theory, Languages and Computation*. Addison-Wesley.
- KUNEN, K. 1987. Negation in logic programming. *J. Logic Program.* 4: 289–308.
- LYNDON, R. C. 1964. *Notes on Logic*. Van Nostrand Mathematical studies.
- MAHER, M. 1988. Complete axiomatization of the algebra of finite, rational and infinite trees. Technical report, IBM T.J.Watson Research Center.
- MALCEV, A. 1971. Axiomatizable classes of locally free algebras of various types. In: B. Wells III, editor, *The Metamathematics of Algebraic Systems*. Anatolii Ivanovic Malcev. Collected Papers: 1936–1967, volume 66, chapter 23, pp. 262–281.

- MATELLI, A. AND MONTANARI, U. 1982. An efficient unification algorithm. *ACM Trans. Lang. Syst.* 4(2): 258–282.
- PATERSON, M. AND WEGMAN, N. 1978. Linear unification. *J. Computer and Systems Science*, 16: 158–167.
- RAMACHANDRAN, V. AND VAN HENTENRYCK, P. 1993. Incremental algorithms for formula solving and entailment over rational trees. *Proceeding of the 13th Conference Foundations of Software Technology and Theoretical Computer Science*, LNCS volume 761, pp. 205–217.
- ROBINSON, J. A. 1965. A machine-oriented logic based on the resolution principle. *J. ACM*, 12(1): 23–41.
- RYBINA, T. AND VORONKOV, A. 2001. A decision procedure for term algebras with queues. *ACM Trans. Computational Logic*, 2(2): 155–181.
- SMITH, A. 1991. Constraint operations for CLP. *Logic Programming: Proceedings of the 8th International Conference*, Paris. pp. 760–774.
- VOROBYOV, S. 1996. An Improved Lower Bound for the Elementary Theories of Trees. *Proceeding of the 13th International Conference on Automated Deduction (CADE'96)*. Springer Lecture Notes in Artificial Intelligence, vol 1104, pp. 275–287.