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A rigidity result for non-local semilinear equations

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We consider a possibly anisotropic integrodifferential semilinear equation, driven by a non-decreasing nonlinearity. We prove that if the solution grows less than the order of the operator at infinity, then it must be affine (possibly constant).

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1. Introduction

The idea that bounded harmonic functions are constant dates back to Liouville and Cauchy in 1844. Several generalizations of this result, also involving nonlinear equations and more general growth of the solution at infinity have since appeared in the literature (see [11] for a detailed review of this topic).

The aim of this paper is to obtain a rigidity result for integrodifferential semilinear equations of fractional order $2s$, with $s \in (0,1)$.

We recall that fractional integrodifferential operators are a classical topic in analysis, whose study arises in may different fields including harmonic analysis [25], partial differential equations [5] and probability [1]. The study of these operators is also relevant to concrete situations, in related real-world applications, such as quantum mechanics [12], water waves [7], meteorology [6], crystallography [16], biology [13], finance [22] and high technology [26].

The type of integrodifferential operators that we consider here are of the form

$$\mathcal{I}u(x) := \int_{S^{n-1}\times\mathbb{R}} (u(x+\vartheta r) + u(x-\vartheta r) - 2u(x)) \frac{\mathrm{d}\mu(\vartheta)\,\mathrm{d}r}{|r|^{1+2s}}. \tag{1.1}$$

In this notation, $r \in \mathbb{R}$ is integrated by the usual Lebesgue measure $\mathrm{d}r$, while $\vartheta \in S^{n-1}$ is integrated by a measure $\mathrm{d}\mu(\vartheta)$, which is called the 'spectral measure' in the literature; these measures satisfy the following non-degeneracy assumptions: there exist $\lambda, \Lambda \in (0, +\infty)$ such that

$$\inf_{\nu \in S^{n-1}} \int_{S^{n-1}} |\nu \cdot \vartheta|^{2s} \, \mathrm{d}\mu(\vartheta) \geqslant \lambda \quad \text{and} \quad \mu(S^{n-1}) \leqslant \Lambda. \tag{1.2}$$

Particularly famous cases of spectral measures are those induced by singular kernels, i.e. when $\mathrm{d}\mu(\vartheta) = \mathcal{K}_0(\vartheta) \, \mathrm{d}\mathcal{H}^{n-1}(\vartheta)$, with $0 < \inf_{S^{n-1}} \mathcal{K}_0 \leqslant \sup_{S^{n-1}} \mathcal{K}_0 < +\infty$. Note that in this particular case the spectral measure is absolutely continuous with respect to the standard Hausdorff measure on S^{n-1}, and the operator in (1.1) comes from integration against the homogeneous kernel

$$\mathcal{K}(y) := |y|^{-n-2s} \mathcal{K}_0\left(\frac{y}{|y|}\right), \tag{1.3}$$

in the sense that

$$\mathcal{I}u(x) = \int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x)) \mathcal{K}(y) \, \mathrm{d}y. \tag{1.4}$$

Of course, the case when $\mathcal{K}$ is equal to constant boils down to the fractional Laplacian, i.e. to the case in which, up to a normalization factor, $\mathcal{I} = -(-\Delta)^s$.

The literature has recently shown increasing efforts to study these types of anisotropic operators (see, for example, [14, 15, 19–21]). Recall that the picture given by the general operator in (1.1) is often quite special when compared with the isotropic case, and sometimes even surprising: for instance, a complete regularity theory in the setting of (1.1), (1.2) does not hold, and explicit counterexamples can be constructed (see [20]).

We shall consider the equation $\mathcal{I}u = f(u)$. This type of equation is often called 'semilinear' since the nonlinearity depends only on the values of the solution itself (for this reason, solutions of semilinear equations may satisfy geometric properties better than the solutions of arbitrary equations).

Our main result states that if f is non-decreasing, then solutions of $\mathcal{I}u = f(u)$ whose growth at infinity is bounded by $|x|^\kappa$, with κ less than the order of operator, must necessarily be affine (and, in fact, constant when the nonlinearity is non-trivial). More precisely, we have the following.

THEOREM 1.1. *Let $f \in C(\mathbb{R})$ be non-decreasing. Let $u \in C^2(\mathbb{R}^N)$ be a solution of*

$$\mathcal{I}u(x) = f(u(x)) \quad \text{for any } x \in \mathbb{R}^n. \tag{1.5}$$

Assume that

$$|u(x)| \leqslant K(1 + |x|^\kappa), \tag{1.6}$$

for some $K \geqslant 0$ and $\kappa \in [0, 2s)$.

Then the following classifications hold:

(i) *if f is not identically zero, then u is constant, say $u(x) = c$ for any $x \in \mathbb{R}^n$, and $f(c) = 0$;*

(ii) *if f is identically zero, then u is an affine function, say $u(x) = \varpi \cdot x + c$, for some $\varpi \in \mathbb{R}^n$ and $c \in \mathbb{R}$; in this case, if, in addition, $\kappa < 1$, then $\varpi = 0$ and u is constant.*

Remark 1.2. Theorem 1.1 holds here for the very general integrodifferential operator in (1.1), (1.2). Moreover, to the best of our knowledge, theorem 1.1 is new, even in the case of the regular spectral measure in (1.3), (1.4), and, perhaps quite surprisingly, even in the isotropic case of the fractional Laplacian.

On the other hand, when $\mathcal{I}$ is replaced by the Laplacian (which is formally the above case with $s = 1$) theorem 1.1 is a well-known result in the framework of classical Liouville-type theorems (see, for example, [11,23]). As a matter of fact, the counterpart of (1.5) in the classical case is the semilinear equation $\Delta u = f(u)$, set in the whole of $\mathbb{R}^n$. This equation has been studied extensively, not least in connection with phase transition models (see, for example, [2, 17] and the references therein). Its fractional analogue also has physical relevance, since it appears, for instance, in the study of phase transitions arising from long-range interactions and in the dynamics of atom dislocations in crystals (see, for example, [3, 4, 8, 9, 18, 24]).

Our strategy for proving theorem 1.1 is to show that $f(u(x))$ must be identically equal to zero in any case. Therefore, u is a solution of $\mathcal{I}u = 0$, and this will allow us to use a Liouville-type theorem in order to obtain the desired classification. Towards this goal, one uses the subcritical growth of the solution u to compare the solution with suitable barriers. Of course, the construction of appropriate barriers is the main novelty with respect to the classical case, since the non-locality of the operator mostly comes into play in this framework.

Remark 1.3. We also stress that theorem 1.1 has a natural generalization *when only the second condition in (1.2) is satisfied*: in this case, the thesis of theorem 1.1 becomes that

$$f(u(x)) = 0 \quad \text{for all } x \in \mathbb{R}^n. \tag{1.7}$$

In particular, by (1.7), in this case we still have that $\mathcal{I}u(x) = 0$ for all $x \in \mathbb{R}^n$, and if a solution exists, then the nonlinearity f must have at least one zero.

Another consequence of (1.7) is that *if f is not identically zero*, then u must be either bounded from above or bounded from below.

In addition, *if the zeros of f are isolated*, then one obtains for free that u is also constant in this case.

On the other hand, to obtain the complete thesis of theorem 1.1, we need also to assume that *the first assumption in (1.2) is satisfied*, in order to use a result in [19].

Note that theorem 1.1 has a natural, and simple, generalization that deals with the case in which (1.6) is replaced by a one-sided inequality. In this spirit, we present the following result.

Theorem 1.4. *Let $f \in C(\mathbb{R})$ be non-decreasing. Let $u \in C^2(\mathbb{R}^N)$ be a solution of*

$$\mathcal{I}u(x) = f(u(x)) \quad \textit{for any } x \in \mathbb{R}^n.$$

Then, if

$$u(x) \leqslant K(1 + |x|^\kappa),$$

for some $K \geqslant 0$ *and* $\kappa \in [0, 2s)$*, we have that*

$$\mathcal{I}u(x) \leqslant 0 \quad \text{for any } x \in \mathbb{R}^n.$$

Similarly, if

$$u(x) \geqslant -K(1 + |x|^{\kappa}),$$

for some $K \geqslant 0$ *and* $\kappa \in [0, 2s)$*, we have that*

$$\mathcal{I}u(x) \geqslant 0 \quad \text{for any } x \in \mathbb{R}^n.$$

We also point out that the regularity assumptions of u in theorems 1.1 and 1.4 were taken for simplicity, and in concrete cases one does not need u to be initially smooth (for instance, in the setting of [15] one can deal with viscosity solutions, and in the setting of [19] one can deal with weak solutions).

The rest of the paper is organized as follows. In § 2 we collect some preliminary integral computations that will be used in § 3 to construct a useful barrier. Roughly speaking, this barrier replaces the classical paraboloid in our non-local framework (of course, checking the properties of the paraboloid in the classical case is much simpler than constructing barriers in non-local cases). The proofs of theorems 1.1 and 1.4 are stated in § 4.

2. Toolbox

Below we give some preliminary integral computations that are needed in § 3 to construct a suitable barrier. For convenience, we use the following notation:

$$\mathcal{I}_1 v(x) := \int_{S^{n-1}\times(-1,1)} (v(x+\vartheta r) + v(x-\vartheta r) - 2v(x)) \frac{\mathrm{d}\mu(\vartheta)\,\mathrm{d}r}{|r|^{1+2s}}$$

and

$$\mathcal{I}_2 v(x) := \int_{S^{n-1}\times(\mathbb{R}\setminus(-1,1))} (v(x+\vartheta r) + v(x-\vartheta r) - 2v(x)) \frac{\mathrm{d}\mu(\vartheta)\,\mathrm{d}r}{|r|^{1+2s}},$$

in order to distinguish the integration performed when $|r| < 1$ from that when $|r| \geqslant 1$.

2.1. Estimates near the origin

We estimate $\mathcal{I}_1 v$ and $\mathcal{I}_2 v$ near the origin, according to the following lemmas.

LEMMA 2.1. *Let* $v \in C^2(B_3)$*. Then, for any* $x \in B_1$*,*

$$\mathcal{I}_1 v(x) \leqslant C$$

for some $C > 0$ *possibly depending on* n*,* s*,* Λ *and* $\|v\|_{C^2(B_2)}$*.*

Proof. If $x, y \in B_1$, we obtain from a Taylor expansion that

$$|v(x+y) + v(x-y) - 2v(x)| \leqslant \|D^2 v\|_{L^\infty(B_2)} |y|^2.$$

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Hence, by integration, we get

$$\begin{aligned}\mathcal{I}_1 v(x) &\leqslant \int_{S^{n-1}\times(-1,1)} \|D^2 v\|_{L^\infty(B_2)} |r|^2 \frac{\mathrm{d}\mu(\vartheta)\,\mathrm{d}r}{|r|^{1+2s}}\\ &\leqslant \Lambda \|D^2 v\|_{L^\infty(B_2)} \int_0^1 |r|^{1-2s}\,\mathrm{d}r\end{aligned}$$

due to (1.2), which gives the desired result. □

Lemma 2.2. *Let*

$$\gamma \in (0, 2s). \tag{2.1}$$

Let $v\colon \mathbb{R}^n \to [0,+\infty)$ *be a measurable function such that* $v(x) \leqslant |x|^\gamma$ *for any* $x \in \mathbb{R}^n$. *Then, for any* $x \in B_1$,

$$\mathcal{I}_2 v(x) \leqslant C,$$

for some $C > 0$ *possibly depending on* n, s, Λ *and* γ.

Proof. Let $x \in B_1$ and $y \in \mathbb{R}^n \setminus B_1$. Then $|x| \leqslant 1 \leqslant |y|$ and so

$$\begin{aligned}|v(x+y)+v(x-y)-2v(x)| &\leqslant |v(x+y)| + |v(x-y)| + 2|v(x)|\\ &\leqslant |x+y|^\gamma + |x-y|^\gamma + 2|x|^\gamma\\ &\leqslant 2(|x|+|y|)^\gamma + 2|x|^\gamma\\ &\leqslant (2^{\gamma+1}+2)|y|^\gamma.\end{aligned}$$

So, we integrate, recalling (1.2), and we see that

$$\begin{aligned}\mathcal{I}_2 v(x) &\leqslant \int_{S^{n-1}\times(\mathbb{R}^n\setminus(-1,1))} (2^{\gamma+1}+2)|r|^\gamma \frac{\mathrm{d}\mu(\vartheta)\,\mathrm{d}r}{|r|^{1+2s}}\\ &\leqslant 2(2^{\gamma+1}+2)\Lambda \int_1^{+\infty} r^{\gamma-1-2s}\,\mathrm{d}r.\end{aligned} \tag{2.2}$$

Then we use (2.1) to obtain the desired result. □

2.2. Estimates far from the origin

Now, we estimate $\mathcal{I}v = \mathcal{I}_1 v + \mathcal{I}_2 v$ at infinity.

Lemma 2.3. *Let* γ *be as in (2.1) and let* $v\colon \mathbb{R}^n \to \mathbb{R}$ *be a measurable function such that* $v(x) \leqslant |x|^\gamma$ *for any* $x \in \mathbb{R}^n$.

Assume also that $v(x) = |x|^\gamma$ *for any* $x \in \mathbb{R}^n \setminus B_1$. *Then, for any* $x \in \mathbb{R}^n \setminus B_1$,

$$\mathcal{I}v(x) \leqslant C$$

for some $C > 0$ *possibly depending on* n, s, Λ *and* γ.

Proof. Fix $x \in \mathbb{R}^n \setminus B_1$. Then $v(x) = |x|^\gamma$. Moreover, $v(x \pm y) \leqslant |x \pm y|^\gamma$, and so

$$v(x+y)+v(x-y)-2v(x) \leqslant |x+y|^\gamma + |x-y|^\gamma - 2|x|^\gamma.$$

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Therefore, setting $\omega := x/|x|$ and changing variable $r := |x|\varrho$, we have that

$$\begin{aligned}
\mathcal{I}v(x) &= \int_{S^{n-1}\times\mathbb{R}} (v(x+\vartheta r) + v(x-\vartheta r) - 2v(x)) \frac{\mathrm{d}\mu(\vartheta)\,\mathrm{d}r}{|r|^{1+2s}} \\
&\leqslant \int_{S^{n-1}\times\mathbb{R}} (|x+\vartheta r|^\gamma + |x-\vartheta r|^\gamma - 2|x|^\gamma) \frac{\mathrm{d}\mu(\vartheta)\,\mathrm{d}r}{|r|^{1+2s}} \\
&= |x|^{\gamma-2s} \int_{S^{n-1}\times\mathbb{R}} (|\omega+\vartheta\varrho|^\gamma + |\omega-\vartheta\varrho|^\gamma - 2) \frac{\mathrm{d}\mu(\vartheta)\,\mathrm{d}\varrho}{|\varrho|^{1+2s}} \\
&= |x|^{\gamma-2s} \int_{S^{n-1}\times\mathbb{R}} \frac{g(\vartheta\varrho) + g(-\vartheta\varrho) - 2g(0)}{|\varrho|^{1+2s}}\,\mathrm{d}\mu(\vartheta)\,\mathrm{d}\varrho, \qquad (2.3)
\end{aligned}$$

where, for any $\eta \in \mathbb{R}^n$, we set $g(\eta) := |\omega + \eta|^\gamma$. Note that

$$|g(\eta)| \leqslant (|\omega| + |\eta|)^\gamma = (1 + |\eta|)^\gamma. \qquad (2.4)$$

Moreover, $g \in C^\infty(B_{1/2})$ and, for any $\eta \in B_{1/2}$, we have that

$$\begin{aligned}
\partial_i g(\eta) &= \gamma|\omega+\eta|^{\gamma-2}(\omega_i + \eta_i), \\
\partial^2_{ij} g(\eta) &= \gamma(\gamma-2)|\omega+\eta|^{\gamma-4}(\omega_i+\eta_i)(\omega_j+\eta_j) + \gamma|\omega+\eta|^{\gamma-2}\delta_{ij}.
\end{aligned}$$

Consequently, for any $\eta \in B_{1/2}$,

$$|D^2 g(\eta)| \leqslant C_0|\omega+\eta|^{\gamma-2}$$

for some $C_0 > 0$ depending on γ and n, and $|\omega+\eta| \geqslant |\omega| - |\eta| \geqslant \frac{1}{2}$. Therefore,

$$\|D^2 g\|_{L^\infty(B_{1/2})} \leqslant 2^{2-\gamma} C_0.$$

This, together with a Taylor expansion, implies that, for any $\eta \in B_{1/2}$,

$$|g(\eta) + g(-\eta) - 2g(0)| \leqslant \|D^2 g\|_{L^\infty(B_{1/2})}|\eta|^2 \leqslant 2^{2-\gamma}C_0|\eta|^2.$$

Hence, recalling (2.4) and (1.2), we obtain that

$$\begin{aligned}
&\int_{S^{n-1}\times\mathbb{R}} \frac{g(\vartheta\varrho) + g(-\vartheta\varrho) - 2g(0)}{|\varrho|^{1+2s}}\,\mathrm{d}\mu(\vartheta)\,\mathrm{d}\varrho \\
&\qquad \leqslant C\bigg(\int_{S^{n-1}\times(-1/2,1/2)} \frac{|\varrho|^2}{|\varrho|^{1+2s}}\,\mathrm{d}\mu(\vartheta)\,\mathrm{d}\varrho \\
&\qquad\qquad + \int_{S^{n-1}\times(\mathbb{R}\setminus(-1/2,1/2))} \frac{(1+|\varrho|)^\gamma}{|\varrho|^{1+2s}}\,\mathrm{d}\mu(\vartheta)\,\mathrm{d}\varrho\bigg) \\
&\qquad \leqslant C\Lambda\bigg(\int_{(-1/2,1/2)} \frac{|\varrho|^2}{|\varrho|^{1+2s}}\,\mathrm{d}\varrho + \int_{\mathbb{R}\setminus(-1/2,1/2)} \frac{(1+|\varrho|)^\gamma}{|\varrho|^{1+2s}}\,\mathrm{d}\varrho\bigg) \\
&\qquad \leqslant C'\Lambda \qquad (2.5)
\end{aligned}$$

for some $C, C' > 0$, due to (2.1). We insert this into (2.3) to obtain the desired estimate (by possibly renaming the constants). □

We observe that condition (2.1) has been used to make some integrals converge (e.g. those in (2.2) and (2.5)). In addition, when $\gamma \geqslant 2s$, the fractional Laplacian of functions growing with $|x|^\gamma$ is not well defined.

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3. Construction of an auxiliary barrier

Here we use the estimate in § 2, and borrow some ideas from [10] to construct a useful auxiliary function.

LEMMA 3.1. *Let $\gamma \in (0, 2s)$. There exists a function $v \in C^\infty(\mathbb{R}^n)$ such that, for some $C > 0$,*

$$v(0) = 0, \tag{3.1}$$

$$0 \leqslant v(x) \leqslant |x|^\gamma \quad \textit{for any } x \in \mathbb{R}^n, \tag{3.2}$$

$$v(x) = |x|^\gamma \quad \textit{if } |x| \geqslant 1, \tag{3.3}$$

$$\sup_{x \in \mathbb{R}^n} \mathcal{I}v(x) \leqslant C. \tag{3.4}$$

Proof. Let $\tau \in C^\infty(\mathbb{R}^n)$ be such that $0 \leqslant \tau \leqslant 1$ in the whole of $\mathbb{R}^n$, and let $\tau = 1$ in $B_{1/2}$ and $\tau = 0$ in $\mathbb{R}^n \setminus B_1$. We define $v(x) := (1 - \tau(x))|x|^\gamma$. In this way, conditions (3.1)–(3.3) are satisfied.

Furthermore, v satisfies all the assumptions of lemmas 2.1–2.3. Thus, using such results, we obtain condition (3.4). □

4. Proof of the main results

Proof of theorem 1.1. The proof relies on a modification of a classical argument (see, for example, [11, 23]). In our setting, the barrier constructed in lemma 3.1 will replace (at least from one side) the classical paraboloid. The details of the argument are as follows. Let f, u, K and κ be as in theorem 1.1. Let $\gamma := \frac{1}{2}(2s + \kappa)$. By construction,

$$\gamma \in (\kappa, 2s), \tag{4.1}$$

so we can use the barrier v constructed in lemma 3.1. We fix $\varepsilon > 0$ and an arbitrary point $x_0 \in \mathbb{R}^n$, and we define

$$\left.\begin{aligned} w_1(x) &:= u(x) - u(x_0) + 2\varepsilon - \varepsilon v(x - x_0), \\ w_2(x) &:= u(x) - u(x_0) - 2\varepsilon + \varepsilon v(x - x_0). \end{aligned}\right\} \tag{4.2}$$

Note that

$$\begin{aligned} \limsup_{|x| \to +\infty} w_1(x) &\leqslant \limsup_{|x| \to +\infty} [u(x) + |u(x_0)| + 2\varepsilon - \varepsilon v(x - x_0)] \\ &\leqslant \limsup_{|x| \to +\infty} [K(1 + |x|^\kappa) + |u(x_0)| + 2\varepsilon - \varepsilon |x - x_0|^\gamma] \\ &= -\infty \end{aligned}$$

and

$$\begin{aligned} \liminf_{|x| \to +\infty} w_2(x) &\geqslant \liminf_{|x| \to +\infty} [u(x) - |u(x_0)| - 2\varepsilon + \varepsilon v(x - x_0)] \\ &\geqslant \liminf_{|x| \to +\infty} [-K(1 + |x|^\kappa) - |u(x_0)| - 2\varepsilon + \varepsilon |x - x_0|^\gamma] \\ &= +\infty, \end{aligned}$$

where we have used (1.6), (3.3) and (4.1). Consequently, the maxima of w_1 and w_2 are attained, i.e. there exist $y_1, y_2 \in \mathbb{R}^n$ such that

$$w_1(y) \leqslant w_1(y_1) \quad \text{and} \quad w_2(y) \geqslant w_2(y_2) \quad \text{for any } y \in \mathbb{R}^n. \tag{4.3}$$

Accordingly, for any $y \in \mathbb{R}^n$, we have

$$\left.\begin{aligned} w_1(y_1+y) + w_1(y_1-y) - 2w_1(y_1) \leqslant 0, \\ w_2(y_1+y) + w_2(y_1-y) - 2w_2(y_2) \geqslant 0. \end{aligned}\right\} \tag{4.4}$$

On the other hand,

$$\left.\begin{aligned} &w_1(y_1+y) + w_1(y_1-y) - 2w_1(y_1) \\ &\qquad = u(y_1+y) + u(y_1-y) - 2u(y_1) \\ &\qquad\qquad - \varepsilon(v(y_1+y-x_0) + v(y_1-y-x_0) - 2v(y_1-x_0)), \\ &w_2(y_2+y) + w_2(y_2-y) - 2w_2(y_2) \\ &\qquad = u(y_2+y) + u(y_2-y) - 2u(y_2) \\ &\qquad\qquad + \varepsilon(v(y_2+y-x_0) + v(y_2-y-x_0) - 2v(y_2-x_0)). \end{aligned}\right\} \tag{4.5}$$

By comparing (4.4) and (4.5), we obtain that

$$\left.\begin{aligned} 0 &\geqslant \int_{S^{n-1}\times\mathbb{R}} (w_1(y_1+\vartheta r) + w_1(y_1-\vartheta r) - 2w_1(y_1)) \frac{\mathrm{d}\mu(\vartheta)\,\mathrm{d}r}{|r|^{1+2s}} \\ &= \mathcal{I}u(y_1) - \varepsilon \mathcal{I}v(y_1-x_0), \\ 0 &\leqslant \int_{S^{n-1}\times\mathbb{R}} (w_2(y_2+\vartheta r) + w_2(y_2-\vartheta r) - 2w_2(y_2)) \frac{\mathrm{d}\mu(\vartheta)\,\mathrm{d}r}{|r|^{1+2s}} \\ &= \mathcal{I}u(y_2) + \varepsilon \mathcal{I}v(y_2-x_0). \end{aligned}\right\} \tag{4.6}$$

Therefore, using and (1.5) and (3.4), we obtain that

$$0 \geqslant f(u(y_1)) - C\varepsilon \quad \text{and} \quad 0 \leqslant f(u(y_2)) + C\varepsilon. \tag{4.7}$$

Now, we observe that $w_1(x_0) = 2\varepsilon \geqslant 0$ and $w_2(x_0) = -2\varepsilon \leqslant 0$ due to (4.2) and (3.1). So, if we evaluate (4.3) at the point $y := x_0$, we obtain that

$$0 \leqslant w_1(x_0) \leqslant w_1(y_1) \quad \text{and} \quad 0 \geqslant w_2(x_0) \geqslant w_2(y_2). \tag{4.8}$$

Furthermore, using the fact that $v \geqslant 0$ (recall (3.2)), we see from (4.2) that

$$w_1(y_1) \leqslant u(y_1) - u(x_0) + 2\varepsilon \quad \text{and} \quad w_2(y_2) \geqslant u(y_2) - u(x_0) - 2\varepsilon.$$

By comparing this with (4.8), we conclude that

$$u(y_1) \geqslant u(x_0) - 2\varepsilon \quad \text{and} \quad u(y_2) \leqslant u(x_0) + 2\varepsilon.$$

Therefore, since f is non-decreasing, we deduce that

$$f(u(y_1)) \geqslant f(u(x_0) - 2\varepsilon) \quad \text{and} \quad f(u(y_2)) \leqslant f(u(x_0) + 2\varepsilon).$$

We plug this information into (4.7) to obtain

$$0 \geqslant f(u(x_0) - 2\varepsilon) - C\varepsilon \quad \text{and} \quad 0 \leqslant f(u(x_0) + 2\varepsilon) + C\varepsilon. \tag{4.9}$$

Note that x_0 was initially fixed at the beginning and so it is independent of ε (conversely, the points y_1 and y_2 in general may depend on ε). This means that we can pass to the limit as $\varepsilon \to 0^+$ in (4.9) and use the continuity of f to obtain

$$0 \geqslant f(u(x_0)) \quad \text{and} \quad 0 \leqslant f(u(x_0)),$$

i.e. $f(u(x_0)) = 0$. Since x_0 is an arbitrary point of $\mathbb{R}^n$, we have proved that

$$f(u(x)) = 0 \quad \text{for any } x \in \mathbb{R}^n. \tag{4.10}$$

Thus, again using (1.5), we obtain that

$$\mathcal{I}u = 0 \quad \text{in } \mathbb{R}^n.$$

From this and [19, theorem 2.1], we obtain that u is a polynomial of degree $d \in \mathbb{N}$, with d less than or equal to the integer part of κ. In particular, $d \leqslant \kappa < 2s < 2$. Hence, $d \in \{0, 1\}$, and thus u is an affine function. So we can write $u(x) = \varpi \cdot x + c$, for some $\varpi \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

Incidentally, note that, since $d \leqslant \kappa$, when the additional assumption $\kappa < 1$ holds we have that $d = 0$, and consequently $\varpi = 0$ and u is constant. These considerations establish theorem 1.1(ii).

Now we prove that

$$\text{if } f \text{ does not vanish identically, then } \varpi = 0. \tag{4.11}$$

Indeed, if, by contradiction, we have $\varpi \neq 0$, then, given any $r \in \mathbb{R}$, we can take $x_\star := (r - c)|\varpi|^{-2}\varpi$. Then

$$u(x_\star) = \varpi \cdot x_\star + c = r.$$

Thus, by (4.10), we get that $f(r) = f(u(x_\star)) = 0$. Since r was arbitrary, this means that f vanishes identically, contradicting our assumptions. This proves (4.11).

By (4.11) and (4.10) we obtain theorem 1.1(i). This completes the proof of theorem 1.1. □

Proof of theorem 1.4. The proof of theorem 1.4 follows in this case by considering only the function w_1 (to obtain the first statement of theorem 1.4), or only the function w_2 (to obtain the second statement). □

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