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MAPS OF STIEFEL MANIFOLDS AND A BORSUK–ULAM THEOREM

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1. Introduction

We are concerned with the following classical version of the Borsuk–Ulam theorem: Let $f: S^n \to R^k$ be a map and let $A_f = \{x \in S^n \mid fx = f(-x)\}$. Then, if $k \leqq n$, $A_f \neq \phi$. In fact, theorems due to Yang [**17**] give an estimation of the size of A_f in terms of the cohomology index. This classical theorem concerns the antipodal action of the group $G = \mathbb{Z}_2$ on S^n. It has been generalized and extended in many ways (see a comprehensive expository article by Steinlein [**16**]). This author ([**9, 10**)] and Nakaoka [**14**] proved "continuous" or "parameterized" versions of the theorem. Analogous theorems for actions of the groups $G = S^1$ or S^3 have been proved in [**11**], and [**12**]; compare also [**4, 5, 6**].

A tool in estimating the size of the set A_f (for $G = \mathbb{Z}_2$) in terms of index is the first Stiefel–Whitney class of a space with a free involution. Similarly, such an estimate for $G = S^1$ makes use of the first Chern class; and for $G = S^3$ the first Pontriagin class is used. A natural question arises of whether there exists a corresponding result using other characteristic classes.

Various extensions of the concept of index were defined and used by Fadell and Husseini (see [**5, 6**]). In a forthcoming paper [**7**], Fadell and Husseini introduce a general notion of index for an arbitrary compact Lie group action as an ideal-valued function. I arrived independently at the concept of an ideal-valued index and presented my results, with an application to a geometric situation, at the NATO Advanced Study Institute on "Variational Methods in Nonlinear Problems" held in Montreal in July 1986, where Fadell presented his joint results with Husseini; this is how I first learned about their recent work. I understand, however, that Fadell and Husseini defined their general notion of index before me and I am pleased to acknowledge their priority in developing the index theory. In fact, a suggestion that the index can be defined as an ideal is mentioned in Remark (3.5) of [**5**]. I shall use the notation Ind^G for the G-index introduced by Fadell and Husseini and prove some properties of Ind^G (Proposition (3.3) and Theorem (3.4)) which will be needed in this paper.

In theorems of the Borsuk–Ulam type for a general compact Lie group G we usually consider a map $f: X \to W$ of a G-space W, for instance, into a representation space for G; and we try to estimate the size of the set A_f where the G-symmetry becomes degenerate under f. The degeneracy set may be defined in various ways depending on the context.

For instance, if $f: X \to W$ is an equivariant map of X into a representation space W, we may want A_f to be the set of zeros of f. More generally, for any invariant subset $\tilde{W}$ of W, we can set $A_f(\tilde{W}) := f^{-1}\tilde{W}$. If we don't want to start necessarily with an equivariant map $f: X \to W$, we can apply the averaging construction, replace f by its average $\operatorname{Av} f: X \to W$ and define $A_f(\tilde{W}) := (\operatorname{Av} f)^{-1}\tilde{W}$. (Compare [11] and [12]). The classical Borsuk–Ulam theorem asserts that for any map $f: S^n \to \mathbb{R}^n$ there is a point in S^n where the average of f (with respect to the antipodal actions on the source space and on the target space) is zero.

1.1. Example. The following example is a direct generalization of the Borsuk–Ulam–Yang situation of $f: S^n \to \mathbb{R}^{k+1}$ from the group $G = \mathbb{Z}_2$ $(O(1))$ to $G = O(m)$:

Let X be the Stiefel manifold $V_m(\mathbb{R}^{m+n})$ of orthonormal m-frames in $\mathbb{R}^{m+n}$ and let $f: X = V_m(\mathbb{R}^{m+n}) \to (\mathbb{R}^{m+k})^m = W$ (be a map. In other words, f assigns to every m-frame in $V_m(\mathbb{R}^{m+n})$ an m-tuple of vectors in $\mathbb{R}^{m+k}$. Let $\tilde{W}$ be the subset of W consisting of the m-tuples which are not linearly independent. We are asking about the size of $A_f = (\operatorname{Av} f)^{-1}\tilde{W}$; i.e., A_f is the degeneracy set in our example. Here the group $G = O(m)$ acts on $V_m(\mathbb{R}^{m+n})$ and on $(\mathbb{R}^{m+k})^m$ in the standard way: Thus if $w \in W$, then w can be thought of as an $(m+k) \times m$ matrix (having $m+k$ rows and m columns). For $g \in O(m)$ we define $gw := w \cdot g$, where the dot is the matrix multiplication. Then the action is free in $W - \tilde{W}$: if $w \in W - \tilde{W}$, i.e., Rank $w = m$, and $w \cdot g = w$, there is an $(m \times m)$-submatrix A of w which is invertible. Then $A \cdot g = A$ and thus g is the identity. The converse is also true: if $w \in W$ and Rank $w < m$, then one can find a matrix $g \in O(m)$ other than the identity such that $w \cdot g = w$. Of course, if $m = 1$, then $X = S^n$, $W = \mathbb{R}^{k+1}$, and we are in the Borsuk–Ulam–Yang situation. We will prove a theorem in which the size of A_f is described in terms of index, in a way similar to the assertion of the Borsuk–Ulam theorem. An estimate of the size of A_f will be given in terms of cohomology, but, as a corollary we will find a lower bound for the covering dimension, $\dim A_f$, of A_f: In the case $m = 2$, we will show that $\dim A_f \geqq 2n - k - 1$; if, in addition, $k = n - 1$ and $n = 2^s - 1$, then $\dim A_f \geqq n + 1$.

There exist also "continuous" versions of the results proved here, for spaces and maps over a base space. They are analogous to those of [9, 10, 12]. We shall deal with them in a future paper.

2. Index

Let G be a compact Lie group. We shall be using the Alexander–Spanier cohomology with coefficients in $\mathbb{Z}_2$ (which will be suppressed from the notation) and the Borel equivariant cohomology. If X is a G-space then $X_G := EG_G \times X$ where EG is a universal space for G, G acts on $EG \times X$ by $g(e, x) = (ge, gx)$ and $EG_G \times X := (EG \times X)/G$. The map $X_G \to (EG)/G = BG$ induced by the first projection $EG \times X \to EG$ is a bundle with fibre X. If G acts trivially on X then $X_G \cong BG \times X$.

The equivariant cohomology of X is $H_G^* X := H^* X_G$. If G acts freely on X then the map $X_G \to X/G$ induced by the second projection $EG \times X \to X$ is a bundle with a contractible fibre EG; hence $H_G^* X \cong H^*(X/G)$.

If $(*)$ denotes a one-point space then the constant map $EG \to (*)$ induces an isomorphism $H_G^*(*) \cong H_G^* EG$. We will be identifying the groups $H_G^*(*)$, $H_G^* EG$ and $H^* BG$ under this isomorphism.

Proposition (compare [**11**, (5.1)]) **2.1.** *Let X be a free G-space, let $\varphi: X \to EG$ be an equivariant map and let $c = c_X: X \to (*)$ be the constant map. Then under the identification $H_G^* EG = H_G^*(*) = H^* BG$ and $H_G^* X = H^*(X/G)$ we have $\varphi^* = c^*: H^* BG \to H^*(X/G)$.*

Definition 2.2. Let G be a compact Lie group and let X be a G-space. Then *the G-index of X* is defined to be the kernel of the G-cohomology homomorphism induced by the constant map $c_X: X \to (*)$;

$$\mathrm{Ind}^G X := \mathrm{Ker}(c^*: H_G^*(*) \to H_G^* X)$$

Thus the index of X is an ideal in the G-cohomology ring of a point.

In the classical case, when $G = \mathbb{Z}_2$ is acting freely on X, the non-trivial element of $\mathbb{Z}_2$ represents a free involution on X. In the case $BG \cong \mathbb{R}P^\infty$, $H_G^*(*) \cong H^* \mathbb{R}P^\infty$ is a polynomial algebra over $\mathbb{Z}_2$ on one generator in dimension one, the first Stiefel–Whitney class $w_1 \in H_{\mathbb{Z}_2}^1(*) \cong H^1 \mathbb{R}P^\infty$. Its image under c^* in $H_{\mathbb{Z}_2}^1 X = H^1(X/\mathbb{Z}_2)$ is the characteristic class of the involution, $w_1(X) = c^* w_1$. The kernel of c^* is the ideal generated by w_1^{n+1}, for some integer n, and thus $\mathrm{Ind}^{\mathbb{Z}}(X)$ can be identified with that integer; it is the largest integer n such that $w_1^n(X) \neq 0$. This corresponds to the classical definition of index of space with a free involution. In an analogous way, for free actions of $G = S^1$ and $G = S^3$ (and for cohomology with rational coefficients), the index can also be identified with an integer (compare [**4**, **5**, **6**, **11**, **12**]).

That the index is natural is expressed by the following proposition.

Proposition 2.3. *Let X and Y by G-spaces and let $f: X \to Y$ be an equivariant map. Then*

$$\mathrm{Ind}^G Y \subset \mathrm{Ind}^G X.$$

The proof is immediate.

The following theorem is a general principle of which the classical Borsuk–Ulam–Yang theorem is a special case (compare Remark (5.2)).

Theorem 2.4. *Let X and W be G-spaces and assume that X is paracompact. Let $f: X \to W$ be an equivariant map, and let $\tilde{W}$ be a closed invariant subset of W. Then*

$$(\mathrm{Ind}^G f^{-1} \tilde{W}) \cdot (\mathrm{Ind}^G (W - \tilde{W})) \subset \mathrm{Ind}^G X.$$

Remark. The proof is analogous to the proof of part (b) of Proposition (2.8) of [**6**] (additivity property of the integer-valued index). Compare also [**9**, p. 113], [**10**, p. 160], [**11**, p. 161] and [**12**, p. 148]. Thus (2.4) expresses a crucial principle used in these proofs.

Proof. As before, for every space Y, $c=c_Y\colon Y\to(*)$ is the constant map of Y into a one point space. Let $A_f=f^{-1}\tilde{W}$ (thus A_f corresponds to the singularity set in the Borsuk–Ulam–Yang situation) and let $a\in \mathrm{Ind}^G A_f$; that is, $c^*_{A_f}a=0$. Consider $c^*_X a\in H^*_G X$. Thus $(c^*_X a)\,|\,A_f=0$. By the continuity of H^*_G it follows that there exists a neighbourhood N of A_f in X such that $(c^*_X a)\,|\,N=0$. Thus $c^*_X a=j^*a'$, where $a'\in H^*_G(X,N)$ and $j\colon X\to(X,N)$ is the inclusion. Let $b\in \mathrm{Ind}^G(W-\tilde{W})$. Since we have an equivariant map $X-A_f\to W-\tilde{W}$, we have by (2.3) that $\mathrm{Ind}^G(W-\tilde{W})\subset \mathrm{Ind}^G(X-A_f)$. Hence $c^*_X b\in \mathrm{Ind}^G(X-A_f)$, that is, $c^*_X b\,|\,(X-A_f)=0$. Thus $c^*_X b=j^*b'$, where $b'\in H^*_G(X,X-A_f)$ and $j\colon X\to(X,X-A_f)$ is the inclusion. It follows that $c^*_X(ab)=(c^*_X a)(c^*_X b)=(j^*a')(j^*b')=j^*(a'b')=0$. Therefore $ab\in \mathrm{Ind}^G X$. □

3. The cohomology of grassmannians

Let $V_m=V_m(\mathbb{R}^\infty)$ denote the Stiefel manifold of orthonormal m-frames in $\mathbb{R}^\infty$. If $O(m)$ acts on V_m in the standard way (by the right multiplication) then the orbit space of the action is the infinite Grassmann space $G_m=G_m(\mathbb{R}^\infty)$ of m-dimensional subspaces of $\mathbb{R}^\infty$ and the orbit map $V_m\to G_m$ is a classifying bundle for $O(m)$. Thus $H^*_{O(m)}V_m\cong H^*G_m$. The cohomology of G_m (with coefficients in $\mathbb{Z}_2$) is a polynomial algebra $\mathbb{Z}_2[w_1,\dots,w_m]$ freely generated by the Stiefel–Whitney classes $w_i\in H^iG_m$ of the standard m-plane bundle associated to the principal bundle $V_m\to G_m$.

The orbit space of the standard free action (right multiplication) of $O(m)$ on the Stiefel manifold $V_m(\mathbb{R}^{m+n})$ is the real Grassmann manifold $G_m(\mathbb{R}^{m+n})$ of m-dimensional subspaces of $\mathbb{R}^{m+n}$. Thus $H^*_{O(m)}V_m(\mathbb{R}^{m+n})\cong H^*G_m(\mathbb{R}^{m+n})$. There exist two quite different descriptions of the cohomology of $G_m(\mathbb{R}^{m+n})$. On the one hand, Chern [2] gave a description of the cohomology ring $H^*G_m(\mathbb{R}^{m+n})$ by means of a specific cellular decomposition of the Grassmann manifolds constructed by Ehresmann [3] which, in turn was based on the work of Schubert [15]. By letting $n\to\infty$, one obtains a decomposition of the infinite Grassmannian $G_m=G_m(\mathbb{R}^\infty)$. In this decomposition of G_m evey cell represents a free generator of the cohomology group in the respective dimension, a monomial in the Stiefel–Whitney classes. On the other hand, the cohomology of $G_m(\mathbb{R}^{m+n})$ was described in Borel's thesis [1] as a quotient of the polynomial ring on the universal Stiefel–Whitney classes and their duals. A pleasing exposition of the first approach is given in Milnor [13]; compare also Hiller [8].

We can write the total Stiefel–Whitney class as a formal series $w=1+w_1+w_2+\cdots$ and define the total dual class $\bar{w}=1+\bar{w}_1+\bar{w}_2+\cdots$ as the formal inverse of w, i.e., by the relation

$$w\bar{w}=1 \tag{3.1}$$

(compare [13, §4]). Relation (3.1) can be used to express the dual classes $\bar{w}_i$'s in terms of $w_1,\dots,w_m$. It contains a countable number of relations, one in each positive dimension.

Definition 3.2. Let $J(m,n)$ denote the ideal in $\mathbb{Z}_2[w_1,\dots,w_m]$ generated by $\bar{w}_{1+n},\dots,\bar{w}_{m+n}$ expressed as polynomials in $w_1,\dots,w_m$ by using (3.1).

Theorem 3.3. *$J(m,n)$ is the $O(m)$-index of $V_m(\mathbb{R}^{m+n})$.*

Borel [1] showed that the albegra $H^*_{O(m)}V_m \cong H^*_{O(m)}(*) \cong H^*G_m$ is isomorphic to the quotient algebra $\mathbb{Z}_2[w_1,\dots,w_m,\bar{w}_1,\bar{w}_2,\dots]/I(m)$, where $I(m)$ is the ideal in $\mathbb{Z}_2[w_1,\dots,w_m,\bar{w}_1,\bar{w}_2,\dots]$ generated by the homogeneous terms of $w\bar{w}$ of positive dimension. As shown by Borel, it follows that

$$H^*G_m(\mathbb{R}^{m+n}) \cong \mathbb{Z}_2[w_1,\dots,w_m,\bar{w}_1,\dots,\bar{w}_n]/I(m,n), \tag{3.4}$$

where $I(m,n)$ is the ideal in the polynomial algebra $\mathbb{Z}_2[w_1,\dots,w_m,\bar{w}_1,\dots,\bar{w}_m]$ generated by the $m+n$ terms of $(1+w_1+\cdots+w_m)(1+\bar{w}_1+\cdots+\bar{w}_n)$ of positive dimension. The relations corresponding to the first homogeneous terms of the latter product (in dimensions $1,\dots,n$) yield n equations

$$w_k + w_{k-1}\bar{w}_1 + \cdots + w_1\bar{w}_{k-1} + \bar{w}_k = 0, \; k = 1,\dots,n$$

which can be solved recursively for $\bar{w}_1,\dots,\bar{w}_n$ (see [**13**, p. 40]). Substituting the resulting formulas to the remaining m homogeneous terms of the product (in dimensions $1+n,\dots,m+n$) we obtain the ideal $J(m,n)$. Thus $H^*G_m(\mathbb{R}^{m+n}) \cong \mathbb{Z}_2[w_1,\dots,w_m]/J(m,n)$. The $O(m)$-index of $V_m(\mathbb{R}^{m+n})$ is $\mathrm{Ind}^{O(m)}V_m(\mathbb{R}^{m+n}) = (\mathrm{Ker}(c^*\colon H^*_{O(m)}(*) \to H^*_{O(m)}V_m(\mathbb{R}^{m+n}))$. Since the action of $O(m)$ on $V_m(\mathbb{R}^{m+n})$ is free, $H^*_{O(m)}V_m(\mathbb{R}^{m+n}) \cong H^*G_m(\mathbb{R}^{m+n})$ and, under this isomorphism, the kernel of c^* coincides with the kernel of $\varphi^*\colon H^*G_m \to H^*G_m(\mathbb{R}^{m+n})$, where φ is a classifying map for $V_m(\mathbb{R}^{m+n})$ (compare [**11**, (5.1)]); for instance, φ can be in the inclusion $G_m(\mathbb{R}^{m+n}) \to G_m$. Thus φ^* corresponds to the quotient map $\mathbb{Z}_2[w_1,\dots,w_m] \to \mathbb{Z}_2[w_1,\dots,w_m]/J(m,n)$ whose kernel is $J(m,n)$. □

Let $J(m,n)_r$ denote the r-dimensional component of the ideal $J(m,n)$. Consider the map

$$\gamma(r,n)\colon H^{r-n-1}G_m \oplus \cdots \oplus H^{r-n-m}G_m \to H^rG_m$$

defined by $(x_1,\dots,x_m) \to \bar{w}_{1+n}x_1 + \cdots + \bar{w}_{m+n}x_m$.

Lemma 3.5. *$J(m,n)_r = \mathrm{Im}\,\gamma(r,n)$.*

For the proof it is enough to observe that every element in $J(m,n)_r$ can be written as an element of $\mathrm{Im}\,\gamma(r,n)$ by grouping similar terms with respect to $\bar{w}_{1+n},\dots,\bar{w}_{m+n}$. □

If $r \leqq n$ then $J(m,n)_r = 0$ and the inclusion $G_m(\mathbb{R}^{m+n}) \subset G_m$ induces an isomorphism $H^rG_m \cong H^rG_m(\mathbb{R}^{m+n})$. Thus for $r \leqq n$, $H^rG_m(\mathbb{R}^{m+n})$ is additively generated by all monomials $w_1^{q_1},\dots,w_m^{q_m}$ of a total degree $q_1 + 2q_2 + \cdots + mq_m = r$. In this range of r, the rank of $H^rG_m(\mathbb{R}^{m+n})$ is equal to the number $p_m(r)$ of all partitions of r into at most m integers (see [**13**, p. 85]).

Let $p_m^n(r)$ denote the number of partitions of r into at most m positive integers each of which is $\leqq n$.

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Proposition 3.6. *Rank* $H^rG_2(\mathbb{R}^{2+n})=p_2^n(r)$.

Proof. If $r\leqq n$ then this rank is $p_2(r)$ which is equal to $p_2^n(r)$. Suppose $n\leqq r\leqq 2n$. Since $G_2(\mathbb{R}^{2+n})$ is a $2n$-manifold, by the Poincaré Duality, Rank $H^rG_2(\mathbb{R}^{2+n})=p_2(2n-r)=p_2^n(r)$. □

Remark 3.7. For $m=2$, $p_2(r)=[r/2]+1$.

4. Maps of Stiefel manifolds

We return now to our example to Section 1. Thus $X=V_m(\mathbb{R}^{m+n})$ and $W=(\mathbb{R}^{m+k})^m$ have the standard (right) action of $0(m)$, $f:X\to W$ is a map, $\tilde{W}$ is the subset of W consisting of m-tuples of vectors in $\mathbb{R}^{m+n}$ which are not linearly independent, $W_0=W-\tilde{W}$, and $A_f=(\mathrm{Av}\,f)^{-1}\tilde{W}$. Then the Gram–Schmidt orthogonalization process provides a homotopy equivalence $W_0\cong V_m(\mathbb{R}^{m+k})$. Thus $\mathrm{Ind}^{0(m)}X=J(m,n)$, $\mathrm{Ind}^{0(m)}W_0=J(m,k)$ and by (2.4) we have the following inclusion.

Theorem 4.1. $(\mathrm{Ind}^{0(m)}A_f)\cdot J(m,k)\subset J(m,n)$.

Remark 4.2. If $m=1$ then $X=S^n$, $W=\mathbb{R}^{k+1}$, $J(m,n)$ is the ideal in $H^*_{\mathbb{Z}_2}(X)\cong H^*\mathbb{R}P^\infty\cong \mathbb{Z}_2[w_1]$ generated by w_1^{n+1}, and $J(m,k)$ is the ideal generated by w_1^{k+1}. The index $\mathrm{Ind}^{0(1)}A_f=\mathrm{Ind}^{\mathbb{Z}_2}A_f$ is generated by w_1^{j+1}, for some integer j, the classical $\mathbb{Z}_2$-index of A_f. In this case, the inclusion of (4.1) is equivalent to the inequality $j\geqq n-(k+1)$, which is the assertion of the classical Borsuk–Ulam–Yang theorem.

Thus Theorem 4.1 contains information about the size of A_f: it asserts that the cohomology ring of A_f cannot be too small: its index is bounded above. However, deciding in a particular case which universal cohomology class of $H^*_{0(m)}(*)$ survive by not finding themselves in the index of A_f, can be a non-trivial task. In an effort to extract a more specific information about the size of A_f, we shall attempt to determine a highest possible dimension where the cohomology of $A_f/0(m)$, or the $0(m)$-cohomology of A_f, is non-zero. This will be done in the next section in the case $m=2$. In Section 6 we will be able to obtain a better result for the special case when $m=2$, $n=2^s-1$ and $k=n-1$.

5. The case $m=2$

We shall keep the notation of Section 1. Thus for a map $f:V_2(\mathbb{R}^{n+2})\to(\mathbb{R}^{k+2})^2$ we have

$$(\mathrm{Ind}^{0(2)}A_f)\cdot J(2,k)\subset J(2,n). \tag{5.1}$$

Theorem 5.2. *If* $k<n$ *and* $f:V_2(\mathbb{R}^{n+2})\to(\mathbb{R}^{k+2})^2$ *is a map then* $H^*(A_f/0(2))$ *is non-zero in a dimension at least* $2n-k-2$.

Corollary 5.3. *The covering dimension of* $A_f/O(2)$, $\dim A_f/O(2)$, *is at least* $2n-k-2$. *Furthermore, since the orbit map* $A_f \to A_f/O(2)$ *is a bundle with fibre* $O(2)$, $\dim A_f \geqq 2n-k-1$.

Proof of (5.2). According to (3.6), Rank $H^{2n}G_2(\mathbb{R}^{n+2}) = p_2^n(2n) = 1$; this also follows from the fact that $G_2(\mathbb{R}^{n+2})$ is a $2n$-dimensional manifold. Let $v_{2n} \in H^{2n}G_2(\mathbb{R}^{n+2}) \cong \mathbb{Z}_2$ be the non-zero class (in fact, v_{2n} is the image under the natural map $H^{2n}G_2 \to H^{2n}G_2(\mathbb{R}^{n+2})$ of $w_2^n \in H^{2n}G_2$; see [**8**, Lemma (1.2)]. Thus the $2n$-component $J(2,n)_{2n}$ of the index does not contain the entire $H^{2n}G_2$. This means that there is a class (in this case, it is w_2^n) in $H^{2n}G_2$ which is not in $J(2,n)_{2n}$.

On the other hand, also because $G_2(\mathbb{R}^{k+2})$ is a $2k$-dimensional manifold, $H^{2n}G_2(\mathbb{R}^{k+2}) = 0$ for $k<n$. Therefore the $2n$-component $J(2,k)_{2n} = \operatorname{Im}\gamma(2n,k)$ of the index (cf. (3.5)) contains the entire cohomology module $H^{2n}G_2$. This means that the map

$$\gamma(2n,k)\colon H^{2n-k-1}G_2 \oplus H^{2n-k-2}G_2 \to H^{2n}G_2$$

is surjective. It follows that there exists a pair $(x,y) \in H^{2n-k-1}G_2 \oplus H^{2n-k-2}G_2$ such that $\gamma(2n,k)(x,y) \notin J(2,n)$. By the definition of γ, $\bar{w}_{k+1}x + \bar{w}_{k+2}y \notin J(2,n)$. But $\bar{w}_{k+1}$ and $\bar{w}_{k+2}$ are in $J(2,k) = \operatorname{Ind}^{O(2)}V_2(\mathbb{R}^{k+2})$, hence it follows (5.1) implies that either $x \notin \operatorname{Ind}^{O(2)}A_f$ or $y \notin \operatorname{Ind}^{O(2)}A_f$. This means that either the image of x or the image of y is a non-zero class in $H^*_{O(2)}A_f \cong H^*(A_f/O(2))$. □

6. The case $m=2$ and $n=2^s-1$

In the case $m=2$, $n=2^s-1$ and $k=n-1$ the result of (5.2) can be improved.

Theorem 6.1. *If* $n=2^s-1$ *and* $f\colon V_2(\mathbb{R}^{n+2}) \to (\mathbb{R}^{n+1})^2$ *is a map then* $H^n(A_f/O(2)) \neq 0$.

Just as in (5.3) we have

Corollary 6.2. *The covering dimension* $\dim A_f/O(2) \geqq n$; *hence* $\dim A_f \geqq n+1$.

Lemma 6.3.

$$\frac{(2^s-i-1)}{(2^s-2i-1)!\,i!} \equiv 0 \bmod 2, \ \textit{for } i>0.$$

Proof.

$$\frac{(2^s-i-1)!}{(2^s-2i-1)!\,i!} = \frac{(2^s-2i)(2^s-2i+1)\dots(2^s-i-1)}{i!}$$

$$\equiv \frac{(2i)(2i-1)\dots(i+1)}{i!} \bmod 2$$

$$= \frac{(2i)!}{i!\,i!} = \frac{(2i-1)!}{(i-1)!\,i!} + \frac{(2i-1)!}{i!\,(i-1)!} \equiv 0 \bmod 2,$$

because the binomial coefficients $(a+b)!/a!b!$ satisfy the relation

$$\frac{(a+b)!}{a!b!}=\frac{(a+b-1)!}{(a-1)!b!}+\frac{(a+b-1)!}{a!(b-1)!}. \qquad \square$$

Proof of 6.1. Relation (6.1) for $k=n-1$ now reads

$$(\mathrm{Ind}^{O(2)} A_f)\cdot J(2,n-1)\subset J(2,n). \tag{6.4}$$

The ideal $J(2,n-1)$ is generated by $\bar{w}_n$ and $\bar{w}_{n+1}$, and $J(2,n)$ is generated by $\bar{w}_{n+1}$ and $\bar{w}_{n+2}$ in $\mathbb{Z}_2[w_1,w_2]$. The dual class $\bar{w}_n$ is a polynomial consisting of the terms of

$$\bar{w}=(1+(w_1+w_2))^{-1}=1+(w_1+w_2)+(w_1+w_2)^2+\cdots$$

of total degree $n=2^s-1$. Thus

$$\bar{w}_n=\sum_{i=0}^{2^{s-1}-1}\frac{(2^s-i-1)!}{(2^s-2i-1)!i!}w_1^{2^s-2i-1}w_2^i.$$

By (6.3), all the coefficients in this polynomial for $i>0$ are zero, and for $i=0$, the coefficient of $w_1^{2^s-1}$, is 1. Thus $\bar{w}_n=w_1^{2^s-1}=w_1^n$. Therefore $w_1^n\in J(2,n-1)$. We claim, however, that $w_1^n\notin \mathrm{Ind}^{O(2)} A_f$. For, if w_1^n were in $\mathrm{Ind}^{O(2)} A_f$, relation (7.4) would imply that $w_1^n\cdot w_1^n=w_1^{2n}$ would belong to $J(2,n)$. However, it was proved by Hiller [**8**] that $w_1^{2n}=w_1^{2^{s+1}-2}$ is not zero in $H^*G_2(\mathbb{R}^{n+2})$; that is, $w_1^{2n}\notin J(2,n)$.

This completes the proof. $\square$

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REFERENCES

1. A. Borel, La cohomologie mod 2 de certain espaces homogènes, *Comm. Math. Helv.* **27** 1953), 165–197.

2. S. Chern, On the multiplication in the characteristic ring of a sphere bundle, *Ann. of Math.* **49** (1948), 362–372.

3. C. Ehresmann, Sur la topologie de certains espaces homogènes, *Ann. of Math.* **35** (1934), 396–443.

4. E. R. Fadell, S. Y. Husseini and P. H. Rabinowitz, Borsuk–Ulam theorems for arbitrary S^1 actions and applications, *Trans. Amer. Math. Soc.* **274** (1982), 345–360.

5. E. R. Fadell, S. Y. Husseini, Index theory for G-bundle pairs with applications to Borsuk–Ulam type theorems for G-sphere bundles, preprint.

6. E. R. Fadell and S. Husseini, Relative cohomological index theories, preprint.

7. E. R. Fadell and S. Y. Husseini; *Cohomological Index Theory with Applications to Critical Point Theory and Borsuk–Ulam Theorems.* (A series of lectures, NATO Advanced Study Institute on "Variational Methods in Nonlinear Problems," Montreal, July 1986.)

8. H. L. Hiller, On the cohomology of real Grassmannians, *Trans. Amer. Math. Soc.* **257** (1980), 521–533.

9. J. Jaworowski, A continuous version of the Borsuk–Ulam theory, *Proc. Amer. Math. Soc.* **82** (1981), 112–114.

10. J. Jaworowski, *Fibre Preserving Maps of Sphere Bundles in Vector Space Bundles* (Proc. of the Fixed Point Theory Workshop: Sherbrooke, 1980; Lecture Notes in Mathematics, **886,** Springer–Verlag, 1981), 154–162.

11. J. Jaworowski, The set of balanced orbits of maps of S^1 and S^3 actions, *Proc. Amer. Math. Soc.* **98** (1986), 158–162.

12. J. Jaworowski, *Balanced Orbits for Fibre Preserving Maps of S^1 and S^3 Actions,* (Proc. of the Transformation Groups Symposium, Poznan 1985; Lecture Notes in Mathematics, **1217,** Springer–Verlag, 143–150.

13. J. W. Milnor and J. D. Stasheff, Characteristic Classes, *Ann. of Math. Stud.* **76** (1974).

14. M. Nakaoka, Equivalent point theorems for fibre-preserving maps, *Osaka J. Math.* **21** (1984), 209–815.

15. H. Schubert, *Kalkül der abzählenden Geometrie* (Teubner, Leipzig, 1879).

16. H. Steinlein, Borsuk's theorem and its generalizations and applications: A survey, *Sem. Math. Sup.* **95** (1985), 166–235.

17. C. T. Yang, On theorems of Borsuk–Ulam, Kakutani–Yamabe–Yujobo and Dyson. II, *Ann. of Math.* (2) (1955), 166–283.

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