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A model of a spring-mass-damper system with temperature-dependent friction†

S. MIGORSKI[1], A. OCHAL[1], M. SHILLOR[2] and M. SOFONEA[3]

[1]*Faculty of Mathematics and Computer Science, Institute of Computer Science, Jagiellonian University, 30348 Krakow, Poland*
emails: migorski@ii.uj.edu.pl; ochal@ii.uj.edu.pl
[2]*Department of Mathématics and Statistics, Oakland University, Rochester, MI 48309, USA*
email: shillor@oakland.edu
[3]*Laboratoire de Mathematicques et Physique, University of Perpignan Via Domitia, 66860 Perpignan, France*
email: sofonea@univ-perp.fr

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This work models and analyses the dynamics of a general spring-mass-damper system that is in frictional contact with its support, taking into account frictional heat generation and a reactive obstacle. Friction, heat generation and contact are modelled with subdifferentials of, possibly non-convex, potential functions. The model consists of a non-linear system of first-order differential inclusions for the position, velocity and temperature of the mass. The existence of a global solution is established and additional assumptions yield its uniqueness. Nine examples of conditions arising in applications, for which the analysis results are valid, are presented.

Key words: Spring-mass-damper system; Contact; Temperature-dependent friction; Normal compliance; Heat generation; Differential inclusion; Clarke subdifferential; Existence and uniqueness

1 Introduction

Friction is a complex physical phenomenon that includes a number of different processes that take place on contact surfaces, some of which depend heavily on temperature. This dependence is not well understood, [19]. Therefore, there is a need to construct sufficiently inclusive models that take into account the temperature dependence of processes, which is partially addressed here.

The general setting in this work is that of a mass moving on a support, say a rail, when friction and frictional heat generation are taken into account and with a possible obstacle in its way. Moreover, we take into account the possible full compression of spring's coils. This paper is a considerable extension of [5], where a similar setting, but without friction or thermal effects, was studied. Related isothermal results can be found in [14].

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The aim of this work is two-fold. First, we show how one can apply the rapidly developing theory of differential inclusions to describe contact processes that involve frictional heat generation and temperature dependence of the friction coefficient. We do it in a 'simple' setting that avoids many of the mathematical complications related to higher dimensions, making the mathematical approach much more transparent. Moreover, these simple settings are of importance since they allow for easier experimental measurements and identification of the system parameters and their dependence on the temperature. These parameter functions then may be used in more realistic applications. The second aim is to extend the differential inclusions theory to include temperature-like effects.

We describe nine examples of conditions of contact, friction and frictional heat generation that fit the theory. In practical applications one can use various combinations of these conditions to construct appropriate models for the processes at hand. Then we extend the theory of Hemivariational Inequalities (HVIs) by allowing the coefficients in various subdifferential conditions to be temperature-dependent. We establish the solvability of a general model by using our extended arguments of non-linear differential inclusions.

Models of frictional contact of a thermoviscoelastic body with a rigid foundation were investigated in [10]; however, the friction coefficient was assumed to be independent of the temperature. Derivation of various models with thermal effects and wear, from thermodynamic considerations, can be found in [22]. A simplified model for a braking system in which frictional heat generation was taken into account was studied in [2, 13], where the friction coefficient was assumed to be slip rate and temperature-dependent.

First-order differential inclusions in Hilbert spaces were considered in [1], where these inclusions were governed by Lipschitz perturbations of maximal monotone operators. Here we extend the theory by adding a result for a coupled system of inclusions.

The rest of the paper is structured as follows. Section 2 describes a general temperature-dependent frictional contact model for a spring-mass-damper system. It consists of three first-order non-linear differential inclusions involving the subdifferentials of various, possibly non-convex, potential or superpotential functions. In Section 3 we provide nine examples of conditions in spring-mass-damper systems without or with friction and thermal effects that fall within the theoretical results of Section 4. These results guarantee the existence of solutions for models that contain these conditions. In Section 4 we state and prove our main theoretical results, Theorems 4.1 and 4.2, which also include the unique solvability of the general model. The proofs are based on various results from the theory of first-order differential inclusions, extending some known results. We conclude the paper in Section 5 with comments, and mention some unresolved issues. For the convenience of the reader, we provide in the Appendix preliminary mathematical materials related to subdifferentials that are used in the analysis of our model.

2 Physical setting and mathematical model

We study the thermo-mechanics of a spring-mass-damper system that is moving on a rail, may come in contact with an obstacle and take into account friction and the accompanied frictional heat generation. The model consists of a force balance equation, which describes the system's mechanical behaviour, and an equation for the system's thermal behaviour. Including frictional generation is very important in many applications, and it is well

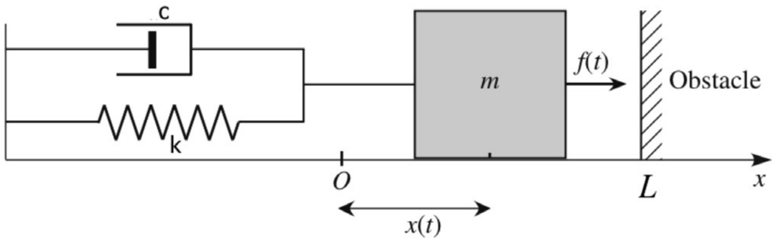

FIGURE 1. The spring-mass-damper system in frictional contact.

known that frictional contact can be associated with considerable heat generation, such as the heating of tyres when breaking abruptly, or the rubbing of hands on a cold day. Moreover, we study the case where the friction coefficient depends on temperature, which often is the case (e.g., [19] and the references therein).

We use this setting to construct a rather general model for the combined processes of friction, contact and heat generation. Then we show nine examples of processes that can be cast in this mathematical form.

The system consists of a rigid moving body, the 'mass', that is attached to a non-linear spring-damper device at one end and constrained by wall at the other. The mass is in frictional contact with a rigid support along the x-axis and its motion is restricted by the presence of a reactive obstacle situated at distance $L > 0$. The mass of the body is denoted by m, and the setting is depicted in Figure 1.

A related isothermal frictional problem with material damage of the spring has been studied in [5], where it was also assumed that the compression of a spring-damper device was restricted. An isothermal model with frictional contact described by a subdifferential of non-convex functions was analysed in [14].

We denote by $[0, T]$ the time interval of interest ($T > 0$) and, unless specified otherwise, $0 \leqslant t \leqslant T$. An external force $f = f(t)$, which may be periodic, is acting on the mass. Let $x = x(t)$ denote the horizontal displacement of the mass from its equilibrium point, and let $v = v(t) = \dot{x}(t)$ be its velocity. We let dots over a variable represent time derivatives, i.e. $\dot{x}(t) = dx/dt$, and $\ddot{x}(t) = d^2x/dt^2$. Let θ_{amb} represent the ambient temperature, Θ_{abs} be the absolute temperature, and denote $\theta = \Theta_{\text{abs}} - \theta_{\text{amb}}$.

The horizontal motion is effected by the applied force $f = f(t)$, the tangential spring-damper restitution force $F_d(t)$, the frictional resistance force $F_{\text{fr}}(t)$ and the reaction force of the obstacle $F_{\text{ob}}(t)$. Therefore, the total horizontal force acting on the body is

$$F_{\text{tot}}(t) = f(t) + F_d(t) + F_{\text{fr}}(t) + F_{\text{ob}}(t). \tag{2.1}$$

The restitution force of the device opposes the motion and satisfies

$$-F_d(t) = k(t, x(t), v(t)). \tag{2.2}$$

Examples of spring-damper devices for which the restitution force F_d satisfies (2.2) are presented in Section 3.

We generalize Coulomb's law of dry friction, as explained in Examples 3.4–3.7, by allowing for a general temperature-dependent superpotential of dissipation and assume that the rail's frictional resistance satisfies a subdifferential inclusion of the form

$$-F_{\mathrm{fr}}(t) \in \mu(\theta(t))\partial j(v(t)). \tag{2.3}$$

Here ∂j is a subdifferential (in the sense of Clarke) of the friction superpotential function j (which may be non-convex) and $\mu = \mu(\theta)$ is the temperature-dependent friction coefficient. We note that, as can be seen in [19], the dependence of the friction coefficient on the temperature can be pronounced but is very complicated, and having j as a general function allows us to take into account such types of behaviour.

We turn to the reaction force F_{ob} of the obstacle. We assume that

$$-F_{\mathrm{ob}}(t) \in p(\theta(t))\partial\varphi(x(t)), \tag{2.4}$$

where $p = p(\theta)$ is the temperature-dependent stiffness coefficient of the obstacle and $\partial\varphi$ represents the (Clarke) subdifferential of the contact potential function φ (possibly non-convex). Contact conditions of the form (2.4) can be found in the examples in Section 3, where an obstacle is situated at L (Figure 1). We note that in the examples we have

$$\begin{cases} x(t) < L & \Longrightarrow \quad F_{\mathrm{ob}}(t) = 0, \\ x(t) \geqslant L & \Longrightarrow \quad F_{\mathrm{ob}}(t) \leqslant 0. \end{cases} \tag{2.5}$$

The condition means that when there is no contact (i.e. $x(t) < L$), there is no reaction from the obstacle, and when there is contact ($x(t) \geqslant L$), the obstacle reaction opposes the motion. In this case the difference $x(t)-L$ represents the interpenetration of the mass and obstacle surface asperities. Note that the contact condition (2.4) is temperature-dependent, which is one of the main novelties in the models we consider in this paper.

The system's mechanical behaviour is governed by

$$m\dot{v}(t) = F_{\mathrm{tot}}(t). \tag{2.6}$$

Collecting now the expressions in (2.1)–(2.6) we obtain

$$m\dot{v}(t) \in f(t) - k(t, x(t), v(t)) - p(\theta(t))\,\partial\varphi(x(t)) - \mu(\theta(t))\,\partial j(v(t)), \tag{2.7}$$

for $t \in [0, T]$. We also prescribe the initial position $x(0) = x_0$ and velocity $v(0) = v_0$.

Finally, we describe the system's thermal behaviour. We denote by c_{th} the averaged specific heat of the body and so the heat capacity of the body is mc_{th}. The heat generated by friction is denoted by $P = P(t)$. This energy dissipates in part to the environment at the rate $h = h(t) \geqslant 0$, and the rest causes a rise in body's temperature. Therefore, the energy rate balance equation is

$$mc_{th}\dot{\theta}(t) = P(t) - h(t). \tag{2.8}$$

We assume that

$$P(t) \in \mu(\theta(t))\,\partial l(v(t)), \tag{2.9}$$

where l is the frictional heat generation superpotential and, again, ∂l denotes its

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subdifferential. Since friction generates heat only when there is motion, we suppose that

$$\partial l(0) = \{0\}. \tag{2.10}$$

The examples in Section 3 satisfy this condition. Also, we assume that the rate of energy dissipated to the environment depends only on the temperature, i.e.

$$h(t) = h(\theta(t)). \tag{2.11}$$

Indeed, it is customary to assume that

$$h(\theta(t)) = h_{\mathrm{ex}}(\theta(t) - \theta_{\mathrm{amb}}),$$

where h_{ex} is the coefficient of heat exchange. Combining (2.8), (2.9) and (2.11) yields

$$mc_{th}\dot{\theta}(t) \in \mu(\theta(t))\,\partial l(v(t)) - h(\theta(t)), \tag{2.12}$$

for $t \in [0, T]$. System's initial temperature is $\theta(0) = \theta_0$.

Next, we rescale the inclusions (2.7) and (2.12), assume that $m = 1$ and $c_{th} = 1$ and combine the resulting inclusions with the initial conditions. In this manner we obtain the following:

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Problem 2.1 Find the displacement, velocity and temperature functions $x, v, \theta : [0, T] \to \mathbb{R}$, respectively, such that, for all $t \in [0, T]$,

$$\begin{cases} \dot{x}(t) = v(t), \\ \dot{v}(t) \in f(t) - k(t, x(t), v(t)) - p(\theta(t))\,\partial\varphi(x(t)) - \mu(\theta(t))\,\partial j(v(t)), \\ \dot{\theta}(t) \in \mu(\theta(t))\,\partial l(v(t)) - h(\theta(t)), \\ x(0) = x_0, \quad v(0) = v_0, \quad \theta(0) = \theta_0. \end{cases} \tag{2.13}$$

In Section 4 we prove the existence of solutions to this model.

We also consider the case where the functions p and μ do not depend on the temperature, $p(\theta) = \widetilde{p}$ and $\mu(\theta) = \widetilde{\mu}$, and $\partial\varphi$, ∂l are single-valued functions, i.e. $\partial\varphi(r) = \psi(r)$ and $\partial l(r) = w(r)$. Then Problem 2.1 reduces to the following simplified model, for which we prove the uniqueness of the solution too.

Problem 2.2 Find the displacement, velocity and temperature functions $x, v, \theta : [0,T] \to \mathbb{R}$, respectively, such that, for all $t \in [0,T]$,

$$\begin{cases} \dot{x}(t) = v(t), \\ \dot{v}(t) \in f(t) - k(t, x(t), v(t)) - \widetilde{p}\, \psi(x(t)) - \widetilde{\mu}\, \partial j(v(t)), \\ \dot{\theta}(t) = \widetilde{\mu}\, w(v(t)) - h(\theta(t)), \\ x(0) = x_0, \quad v(0) = v_0, \quad \theta(0) = \theta_0. \end{cases}$$

The analysis of Problems 2.1 and 2.2 can be found in Section 4, but first we describe examples of settings that fit these two problems.

3 Examples

We now present nine examples of elements of spring-mass-damper systems that can be incorporated into the general model of Section 2. These include non-linear restitution forces, models for friction, contact conditions and a heat generation condition. By using various combinations of these conditions, one can construct models for a large number of systems and devices. These all lead to inclusions of the forms (2.2), (2.3), (2.4) and (2.9), respectively. In all the examples, the relevant assumptions from (H_1)–(H_4) of Section 4 hold, and so we deduce by Theorem 4.1 the existence of solutions to models that include these conditions.

We start with the examples of spring-mass-damper systems that satisfy (2.2). We use the usual rules to obtain differential equations corresponding to springs and dampers connected in series or in parallel, see, for instance, [8, 11, 21].

Example 3.1 Non-linear spring. *We follow [5, 14] and consider a mass-spring device that consists of a spring that becomes non-linear when all the coils are compressed, and so can be described as follows. Let $-L_0$ be the position of the mass when all the coils are fully compressed, then when the spring is in tension, $0 \leqslant x$, or in compression but $-L_0 \leqslant x < 0$, the restitution force is proportional and opposite to the position,*

$$-F_d = k_1 x, \qquad -L_0 \leqslant x.$$

Here k_1 is the stiffness coefficient of the spring (a positive constant). When the coils are fully compressed, an additional resistance force $F_{add} = -k_2(-x - L_0)$ arises, since the force needed to further compress of the spring is much bigger. The coils' stiffness coefficient k_2 is a large positive constant, and therefore

$$-F_d = k_1 x - k_2(-x - L_0), \qquad x \leqslant -L_0.$$

Next, we denote by $r_+ = \max\{0, r\}$ the positive part of r, and gather the above conditions, thus

$$-F_d(t) = k_1 x(t) - k_2(-x(t) - L_0)_+ \tag{3.1}$$

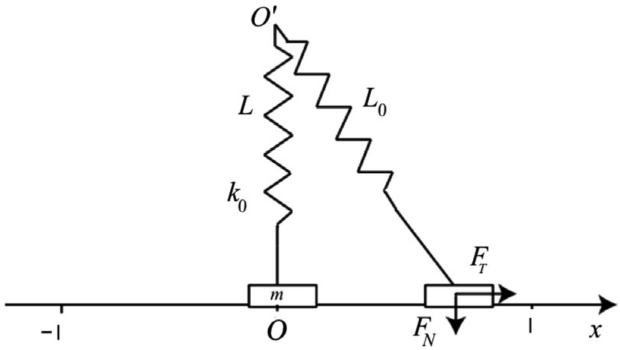

FIGURE 2. The setting of VeCHSS.

for $t \in [0, T]$. The restitution force (3.1) satisfies equation (2.2) with $k(t,r,s) = k_1 r - k_2(-r - L_0)_+$. Clearly, this function satisfies assumption (H_4) below.

Example 3.2 *We consider the Vertically Compressed Horizontally Sliding Spring (VeCHSS – pronounced 'vex') as an example of a device with a non-linear spring stiffness, and a range of displacements where the stiffness is negative. The details of the model can be found in [3,9]. Here we show that F_d satisfies an equality of the form (2.2).*

The system consists of a mass m that is attached to a vertically positioned compressed spring which is pinned at the pivot O', as depicted in Figure 2. The natural length of the spring is L_0 and the compressed length is L, with $L < L_0$, (the length $O'O$ in the figure), and we denote by $k_0 > 0$ the spring constant.

When the mass is perturbed from the equilibrium position $x = 0$ (O in Figure 2), the spring expands up to its natural length L_0 causing the mass to move along the x-axis.

It is seen that $L_0^2 L^2 + 1^2$, and a straightforward decomposition of the forces, based on the geometry in Figure 2, shows that the force the spring exerts on the mass has horizontal and vertical components F_τ and F_N, given respectively by

$$\begin{aligned} F_\tau(t) &= -k_0 x(t)\left(1 - \frac{L_0}{\sqrt{x^2(t) + L^2}}\right), \\ F_N(t) &= -k_0 L(t)\left(1 - \frac{L_0}{\sqrt{x^2(t) + L^2}}\right). \end{aligned} \tag{3.2}$$

The vertical component F_N is balanced by the normal reaction of the rail and is not involved in the force balance equation. However, it is involved in friction, which was studied in [9] but not considered here.

We conclude from (3.2) that the restitution force F_d satisfies (2.2) with

$$k(t,r,s) = k_0 r\left(1 - \frac{L_0}{\sqrt{r^2 + L^2}}\right), \tag{3.3}$$

where we wrote r instead of x to conform to the notation in (H_4) in Section 4. The function k satisfies assumption (H_4), and therefore the existence results hold true for a problem with

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this restitution force. We also remark that if the point O' is moving on the vertical axis then L depends on t and therefore the function k in (3.3) depends on time.

Example 3.3 Spring and damper connected in parallel. *In this example the system consists of a linear damper with viscosity coefficient $c(>0)$ connected in parallel with a linear spring of stiffness k (>0), as shown in Figure 1. Then the restitution force is*

$$-F_d(t) = cv(t) + kx(t),$$

which is of the form (2.2) with $k(t,r,s) = c\,s + k\,r$, $(v = \dot{x})$ and satisfies assumption (H_4). A generalization of this model can be obtained by connecting in parallel a non-linear damper with a non-linear spring, described by the functions k_1 and k_2, respectively, assumed to be Lipschitz continuous and $k_1(0) = k_2(0) = 0$. In this case (2.2) holds with $k(t,r,s) = k_1(s) + k_2(r)$. It is straightforward to see that assumption (H_4) holds in this case.

We turn to examples of friction laws of the form (2.3).

Example 3.4 Coulomb's law of dry friction. *In many publications friction is modelled by Coulomb's law, where the friction force F_{fr} satisfies*

$$|F_{fr}(t)| \leqslant F_b, \qquad F_{fr}(t) = -F_b \frac{v(t)}{|v(t)|} \quad \text{if} \quad v(t) \neq 0, \tag{3.4}$$

where F_b is a positive constant, the so-called friction bound. According to condition (3.4) the maximal frictional resistance is F_b, and when slip takes place $(v \neq 0)$, the frictional resistance force has modulus F_b and acts in the opposite direction of the motion. Since we are interested in thermal effects of friction, we also assume that the friction bound depends on the temperature and denote $F_b = \mu(\theta)$, where μ satisfies assumption (H_5). It is straightforward to show that (3.4) is equivalent to the inclusion of the form (2.3) with

$$j(r) = |r|, \qquad r \in \mathbb{R}. \tag{3.5}$$

Indeed, the friction potential $j : \mathbb{R} \to \mathbb{R}$ is convex and Lipschitz continuous and its subdifferential is given by

$$\partial j(r) = \begin{cases} -1 & \textit{if } r < 0, \\ [-1, 1] & \textit{if } r = 0, \\ 1 & \textit{if } r > 0. \end{cases}$$

Then (3.4) may be written as

$$-F_{fr}(t) \in \mu(\theta(t))\partial|v(t)|. \tag{3.6}$$

The multifunction $\mu(\theta)\partial|v|$ is depicted in Figure 3.

We note that the function (3.5) satisfies assumption (H_2) and since j is convex, its subdifferential is monotone, thus the friction potential (3.5) satisfies condition (H_2^*) with $m_j = 0$.

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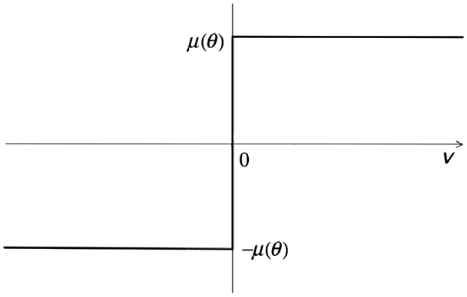

FIGURE 3. Coulomb's law (3.6) – the stick state takes place when the force lies in the vertical solid segment at $v = 0$.

Example 3.5 Regularized friction law. *Since function $j = |r|$ in (3.5) is non-differentiable at $r = 0$, to deal with it in analysis and even more so in numerical computations, we have several ways to regularize it. A simple way is to replace (3.6) with a better behaved condition,*

$$-F_{fr}(t) = \mu(\theta(t))\,\frac{v(t)}{\sqrt{v(t)^2 + \rho^2}}, \tag{3.7}$$

where ρ is a small positive regularization parameter and μ satisfies condition (H_5). Formally, (3.6) is obtained from (3.7) in the limit $\rho \to 0$. Condition (3.7) is obtained from the potential function $j : \mathbb{R} \to \mathbb{R}$ defined by

$$j(r) = \sqrt{r^2 + \rho^2} - \rho, \qquad r \in \mathbb{R}. \tag{3.8}$$

Function j is convex and continuously differentiable and therefore locally Lipschitz and so at each $r \in \mathbb{R}$, the subdifferential has only one value that is the derivative at r, i.e. $\partial j(r) = \{j'(r)\}$. Thus, it is straightforward to see that the friction condition (3.7) is of the form (2.3) with the choice (3.8) for j. Moreover, we have $|j'(r)| \leqslant 1$ for all $r \in \mathbb{R}$, which implies that assumption (H_2) is satisfied, and again, using the convexity of j, we deduce that condition (H_2^*) is satisfied with $m_j = 0$.

Example 3.6 Non-monotone friction law. *An examination of the regularized friction condition (3.7) shows that it can be recovered from a general relation of the form*

$$-F_{fr}(t) = \mu(\theta(t))q(v(t)), \tag{3.9}$$

with an appropriate choice of the function $q : \mathbb{R} \to \mathbb{R}$. For this reason we consider in what follows the friction law (3.9) in which μ satisfies assumption (H_5) and q is a prescribed continuous function, which is positive for a positive argument and negative for a negative argument. We need the last condition since frictional resistance opposes the slip rate. Since q is not assumed to be monotone, we refer to the friction law (3.9) as a non-monotone friction law, see Figure 4. Condition (3.9) represents one of the various friction laws that arise in geomechanics, rock interface analysis and in the modelling of reinforced concrete in concrete structures, see e.g. [17, p. 53] and references therein. Let $j : \mathbb{R} \to \mathbb{R}$ be the function defined

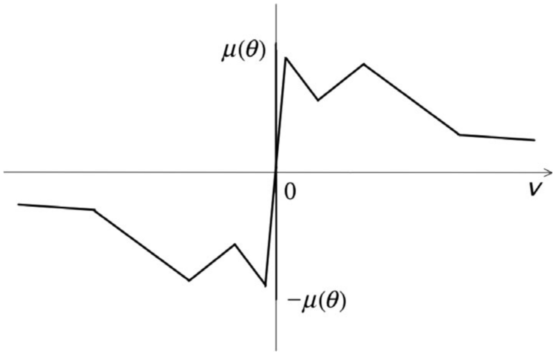

FIGURE 4. Non-monotone friction law (3.9).

by

$$j(r) = \int_0^r q(s)\,ds. \tag{3.10}$$

Then, using the arguments in [17, p. 17], it is seen that the friction condition (3.9) is equivalent to the multi-valued condition (2.3). Note that, unlike the situation in Examples 3.4 and 3.5, the function q is not assumed to be increasing, so the potential function (3.10) is not necessarily convex. However, j is locally Lipschitz and $\partial j(r) = \{q(r)\}$ *for all* $r \in \mathbb{R}$. *Moreover, if q satisfies the growth condition*

$$|q(r)| \leqslant c_2\,(1 + |r|), \quad r \in \mathbb{R}, \tag{3.11}$$

with $c_2 > 0$, *then j satisfies condition* (H_2).

Assume now that $q : \mathbb{R} \to \mathbb{R}$ *is a Lipschitz continuous function. Then q satisfies the inequalities*

$$|q(r)| \leqslant q_0\,(1 + |r|), \quad (q(r_1) - q(r_2))(r_1 - r_2) \geqslant -\widetilde{q}_0\,|r_1 - r_2|^2 \tag{3.12}$$

for all $r,\ r_1, r_2 \in \mathbb{R}$, *with some positive constants* q_0 *and* $\widetilde{q}_0$. *By definition (3.10) of j and the equality* $\partial j(r) = \{q(r)\}$, *it follows that (3.12) implies that j satisfies assumption* (H_2^*).

Example 3.7 *Multi-valued friction law. Equality (3.9) describes a single-valued relation between the friction force and the velocity of the mass. However, the study of partial cracking and crushing of adhesive bonding of materials leads to multi-valued relations between the friction force and the tangential velocity on the contact surface, see e.g. [17, Section 2.4] for details. We refer to such a law as multi-valued friction law, see Figure 5. It can be obtained in the following way: Consider a function* $q \in L^\infty_{loc}(\mathbb{R})$ *such that the limits exist and satisfy*

$$\lim_{\xi \to r^-} q(\xi) = q(r-) \qquad \text{and} \qquad \lim_{\xi \to r^+} q(\xi) = q(r+),$$

for all $r \in \mathbb{R}$. *Let* $\widehat{q} : \mathbb{R} \to 2^{\mathbb{R}}$ *be the multi-valued function that results from q by 'filling in*

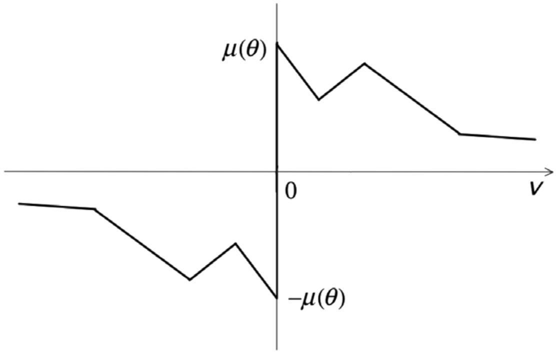

FIGURE 5. Non-monotone multi-valued friction law (3.13).

the gap procedure',

$$\widehat{q}(r) = \begin{cases} [q(r-), q(r+)] & \text{if} \quad q(r-) \leqslant q(r+) \\ [q(r+), q(r-)] & \text{if} \quad q(r+) \leqslant q(r-) \end{cases} \quad \textit{for all } r \in \mathbb{R},$$

where $[\cdot, \cdot]$ *denotes a vertical interval. Let* $j : \mathbb{R} \to \mathbb{R}$ *be defined by (3.10). It was proved in [4] that in this case* $\partial j(r) = \widehat{q}(r)$ *for all* $r \in \mathbb{R}$.

Now assume that the friction force satisfies the multi-valued relation

$$-F_{fr}(t) \in \mu(\theta(t))\widehat{q}(v(t)) \quad \text{for all} \quad t \in [0, T], \tag{3.13}$$

where μ *satisfies condition* (H_5). *We conclude that the friction law (3.13) is of the form (2.3). Moreover, if* q *satisfies (3.11), then the function* j *defined by (3.10) satisfies* (H_2).

As a concrete example, consider the function $q : \mathbb{R} \to \mathbb{R}$ *given by*

$$q(r) = \begin{cases} -1 & \textit{if} \quad r < 0, \\ 1 & \textit{if} \quad r \geqslant 0. \end{cases}$$

This is a non-decreasing function such that $q \in L^{\infty}_{loc}(\mathbb{R})$, $|q(r)| \leqslant 1$ *for all* $r \in \mathbb{R}$. *In this case, using the filling in the gap procedure, we obtain*

$$\widehat{q}(r) = \begin{cases} -1 & \textit{if} \quad r < 0, \\ [-1, 1] & \textit{if} \quad r = 0, \\ 1 & \textit{if} \quad r > 0, \end{cases} \tag{3.14}$$

which shows that the multi-valued friction laws (3.13) and (3.14) reduce to Coulomb's friction law (3.4) with the friction bound $F_b = \mu(\theta)$.

We next present an example of a contact condition of the form (2.4).

Example 3.8 Non-monotone contact condition. *In many publications the contact between a body and a deformable obstacle is modelled by the normal compliance condition. This* *condition was introduced in [12, 16] for isothermal contact problems with elastic and viscoelastic bodies and was used in a large number of publications (see e.g. [11, 20] and the references therein). This condition assumes that the reaction of the foundation depends on the interpenetration of asperities on body's surface and those on the foundation. In our setting, the generalized normal compliance condition is*

$$-F_{ob}(t) = pq(x(t) - L), \tag{3.15}$$

where p is a stiffness coefficient and the function $q : \mathbb{R} \to \mathbb{R}$ *describes the obstacle reaction. It vanishes for negative arguments when there is no contact (*$x(t) < L$*), and is positive when contact takes place (*$x(t) > L$*), resisting the contact force. Since q is not assumed to be monotone, we refer to (3.15) as a* non-monotone contact condition.

Now, since we are interested in the thermal aspects of the process, we assume that the stiffness coefficient depends on the temperature, i.e. $p = p(\theta(t))$*. Then (3.15) reads as*

$$-F_{ob}(t) = p(\theta(t))q(x(t) - L). \tag{3.16}$$

Assume that q is a continuous function, and let $\varphi : \mathbb{R} \to \mathbb{R}$ *be given by*

$$\varphi(r) = \int_0^{r-L} q(s)\,ds \quad \text{for all} \quad r \in \mathbb{R}.$$

Then, using the arguments in [17, p. 17], it follows that the contact condition (3.16) is equivalent to the multi-valued condition (2.4). It is clear that φ *is locally Lipschitz and* $\partial\varphi(r) = \{\, q(r - L) \,\}$ *for all* $r \in \mathbb{R}$*. Moreover, if* $|q(r)| \leqslant c_1(1 + |r|)$ *for all* $r \in \mathbb{R}$ *with* $c_1 > 0$*, then* φ *satisfies* (H_1)*.*

Now we turn to one example of a condition of the form (2.9).

Example 3.9 Non-monotone heat generation condition. *We assume that the heat generated by the friction satisfies an equation of the form*

$$P(t) = \mu(\theta(t))q(v(t)), \tag{3.17}$$

where $\mu = \mu(\theta)$ *is the friction coefficient (actually, the friction bound!) and* $q : \mathbb{R} \to \mathbb{R}$ *represents the contribution of the speed to heat generation. We assume it to be a continuous function such that* $q(0) = 0$*, since friction generates heat only when there is motion. Let* $l : \mathbb{R} \to \mathbb{R}$ *be the function defined by*

$$l(r) = \int_0^{r} q(s)\,ds.$$

Then the arguments in [17, p. 17] imply that the heat generation condition (3.17) is equivalent to the multi-valued condition (2.9). The function l is locally Lipschitz and $\partial l(r) = \{\, q(r) \,\}$ *for all* $r \in \mathbb{R}$*. Moreover, if* $|q(r)| \leqslant c_3(1 + |r|)$ *for all* $r \in \mathbb{R}$ *with* $c_3 > 0$*, we deduce that l satisfies* (H_3)*, and since* $q(0) = 0$*, we note that condition (2.10) holds.*

We end this section with the remark that various other examples of multi-valued contact and heat generation conditions of the forms (2.4) and (2.9) can be constructed by using the 'filling in the gap procedure' presented in Example 3.7. Since the arguments are similar, we just note that there are many additional examples to which our results in Theorem 4.1 can be applied.

4 Existence and uniqueness results

In this section we consider the existence and possible uniqueness of solutions of Problems 2.1 and 2.2. We begin with the study of Problem 2.1 and to that end we make the following assumptions on the problem data.

The *obstacle contact potential function* φ satisfies:

$\underline{(H_1)}$: $\varphi : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function such that

$$|\partial\varphi(r)| \leqslant c_1(1 + |r|) \quad \text{for all} \quad r \in \mathbb{R} \text{ with } c_1 \geqslant 0.$$

The *friction superpotential function* j satisfies:

$\underline{(H_2)}$: $j : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function such that

$$|\partial j(r)| \leqslant c_2(1 + |r|) \quad \text{for all} \quad r \in \mathbb{R} \text{ with } c_2 \geqslant 0.$$

The *friction heat generation potential function* l satisfies:

$\underline{(H_3)}$: $l : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function such that

$$|\partial l(r)| \leqslant c_3(1 + |r|) \quad \text{for all} \quad r \in \mathbb{R} \text{ with } c_3 \geqslant 0.$$

The system's *spring stiffness function* k satisfies:

$\underline{(H_4)}$: $k : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function such that
$k(\cdot, r, s)$ is measurable for all $r,\ s \in \mathbb{R}$,
$k(t, \cdot, \cdot)$ is continuous for all $t \in [0, T]$ and

$$|k(t, r, s)| \leqslant c_4(t) + c_5(|r| + |s|)$$

for all $t \in [0, T]$, $r, s \in \mathbb{R}$ with $c_4 \in L^1(0, T)$, $c_4, c_5 > 0$.

The *friction coefficient function* μ*, which is the friction bound,* satisfies:

$\underline{(H_5)}$: $\mu : \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$0 \leqslant \mu(r) \leqslant \mu_0 \qquad \text{for all } r \in \mathbb{R} \text{ with } \mu_0 > 0.$$

The *obstacle stiffness coefficient function* p satisfies:

$\underline{(H_6)}$: $p : \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$0 \leqslant p(r) \leqslant p_0 \quad \text{for all } r \in \mathbb{R} \text{ with } p_0 > 0.$$

The *heat exchange function* h satisfies:

$\underline{(H_7)}$: $h : \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$0 < h(r) \leqslant h_0 \quad \text{for all } r \in \mathbb{R} \quad \text{with } h_0 > 0.$$

The *applied force* f and the *initial conditions* satisfy:
$\underline{(H_8)}$: $f \in L^1(0,T)$, $x_0, v_0, \theta_0 \in \mathbb{R}$.
Following is our main existence result.

Theorem 4.1 *Assume that* (H_1)–(H_8) *hold. Then Problem* 2.1 *has at least one solution that satisfies* $(x, v, \theta) \in W^{1,1}(0, T; \mathbb{R}^3)$, *hence*

$$(x, \theta) \in W^{2,1}(0, T) \times W^{1,1}(0, T).$$

Proof. We formulate Problem 2.1 as the following Cauchy problem

$$\begin{cases} \dot{u}(t) \in F(t, u(t)) & \text{a.e. } t \in [0, T], \\ u(0) = u_0 \end{cases} \tag{4.1}$$

and then we apply Theorem A1. Let $u(t) = (x(t), v(t), \theta(t))$, $u_0 = (x_0, v_0, \theta_0)$ and let us introduce the multifunction $F : [0, T] \times \mathbb{R}^3 \to 2^{\mathbb{R}^3} \setminus \{\emptyset\}$ given by

$$F(t, q, r, s) = \begin{pmatrix} r \\ f(t) - k(t, q, r) - p(s)\, \partial\varphi(q) - \mu(s)\, \partial j(r) \\ \mu(s)\, \partial l(r) - h(s) \end{pmatrix} \tag{4.2}$$

for all $(t, q, r, s) \in [0, T] \times \mathbb{R}^3$.

We now check that F satisfies condition $H(F)$ below. The fact that the multifunction F has non-empty, convex and compact values follows from the analogous properties of the Clarke subdifferential (e.g. [7, Proposition 5.6.9]). Since $f(\cdot)$ and $k(\cdot, q, r)$ are measurable for all $q, r \in \mathbb{R}$, it follows that the multifunction F satisfies condition $H(F)$(i).

To check the validity of condition $H(F)$(ii), let $t \in (0, T)$, $\{(q_n, r_n, s_n)\}$, $\{(\alpha_n, \beta_n, \gamma_n)\} \subset \mathbb{R}^3$, $(\alpha_n, \beta_n, \gamma_n) \in F(t, q_n, r_n, s_n)$ with $\alpha_n \to \alpha$, $\beta_n \to \beta$, $\gamma_n \to \gamma$, $q_n \to q$, $r_n \to r$, $s_n \to s$ in $\mathbb{R}$. Hence, there exist $\zeta_n \in \partial\varphi(q_n)$, $\eta_n \in \partial j(r_n)$ and $\vartheta_n \in \partial l(r_n)$ such that for a.e. $t \in [0, T]$, we have

$$\begin{cases} \alpha_n = r_n, \\ \beta_n = f(t) - k(t, q_n, r_n) - p(s_n)\, \zeta_n - \mu(s_n)\, \eta_n, \\ \gamma_n = \mu(s_n)\, \vartheta_n - h(s_n). \end{cases}$$

Using the growth conditions on φ, j and l, we may suppose that $\zeta_n \to \zeta$, $\eta_n \to \eta$ and $\vartheta_n \to \vartheta$ in $\mathbb{R}$. Then it follows from the hypotheses (H_2) and (H_3) that

$$\begin{cases} \alpha = r, \\ \beta = f(t) - k(t, q, r) - p(s)\, \zeta - \mu(s)\, \eta, & \text{a.e. } t \in [0, T], \\ \gamma = \mu(s)\, \vartheta - h(s). \end{cases} \tag{4.3}$$

Since the graphs of $\partial\varphi$, ∂j and ∂l are closed in $\mathbb{R} \times \mathbb{R}$ (cf. [7, Proposition 5.6.10]), we infer $\zeta \in \partial\varphi(q)$, $\eta \in \partial j(r)$ and $\vartheta \in \partial l(r)$. Thus, we have

$$\beta \in f(t) - k(t,q,r) - p(s)\,\partial\varphi(q) - \mu(s)\,\partial j(r) \quad \text{for a.e. } t \in [0,T],$$
$$\gamma \in \mu(s)\,\partial l(r) - h(s),$$

which means that $(\alpha, \beta, \gamma) \in F(t,q,r,s)$ and so $H(F)$(ii) holds.

Next, by the hypotheses (H_1)–(H_7), we obtain

$$\|F(t,q,r,s)\|_{\mathbb{R}^3} \leqslant |r| + |f(t)| + |k(t,q,r))| + |p(s)\,\partial\varphi(q)| + |\mu(s)\,\partial j(r)|$$
$$+ |\mu(s)\,\partial l(r)| + |h(s)| \leqslant |r| + |f(t)| + c_4(t) + c_5(|q| + |r|) + p_0 c_1(1 + |q|)$$
$$+ \mu_0\, c_2(1 + |r|) + \mu_0\, c_3(1 + |r|) + h_0 \leqslant \widetilde{c}(t) + \overline{c}\,\|(q,r,s)\|_{\mathbb{R}^3}$$

for $(t,q,r,s) \in [0,T] \times \mathbb{R}^3$, where $\widetilde{c}(t) = |f(t)| + c_4(t) + p_0\, c_1 + \mu_0(c_2 + c_3) + h_0$ and $\overline{c} = \max\{1 + c_5 + \mu_0\,(c_2 + c_3), c_5 + p_0\, c_1\}$. We conclude that condition $H(F)$(iii) is satisfied.

We are now in a position to apply Theorem A1, which guarantees the existence of a solution $(x, v, \theta) \in W^{1,1}(0,T;\mathbb{R}^3)$ of the Cauchy problem (4.1) with the multifunction F (4.2). Since this Cauchy problem is equivalent to Problem 2.1, we conclude that Problem 2.1 has at least one solution that satisfies $x, v, \theta \in W^{1,1}(0,T)$, which implies that $x \in W^{2,1}(0,T)$. □

Now we turn to Problem 2.2, which is a particular case of Problem 2.1. We consider the following hypotheses on the data of Problem 2.2.

$\underline{(H_1^*)}$: $\psi : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function.
$\underline{(H_2^*)}$: $j : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function such that

$$|\partial j(r)| \leqslant c_2(1 + |r|) \quad \text{for all } r \in \mathbb{R} \text{ with } c_2 \geqslant 0,$$

and moreover it satisfies the following relaxed monotonicity condition

$$(\partial j(r_1) - \partial j(r_2))(r_1 - r_2) \geqslant -m_j|r_1 - r_2|^2 \quad \text{for all } r_1, r_2 \in \mathbb{R} \text{ with } m_j \geqslant 0.$$

$\underline{(H_3^*)}$: $w : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function.
$\underline{(H_4^*)}$: $k : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function such that $k(\cdot, r, s)$ is measurable for all $r, s \in \mathbb{R}$, $k(t, \cdot, \cdot)$ satisfies the Lipschitz condition

$$|k(t, r_1, s_1) - k(t, r_2, s_2)| \leqslant L_k(|r_1 - r_2| + |s_1 - s_2|)$$

for all $r_i, s_i \in \mathbb{R}, i = 1, 2$, uniformly for all $t \in [0,T]$, with $L_k > 0$.
$\underline{(H_5^*)}$: $\widetilde{\mu}$ is a non-negative constant.
$\underline{(H_6^*)}$: $\widetilde{p}$ is a non-negative constant.
$\underline{(H_7^*)}$: $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function.

We have the following existence and uniqueness result for Problem 2.2.

Theorem 4.2 *Assume that* (H_1^*)–(H_7^*) *and* (H_8) *hold. Then Problem* 2.2 *admits a unique solution that satisfies* $(x, v, \theta) \in W^{1,1}(0,T;\mathbb{R}^3)$, *i.e.* $(x, \theta) \in W^{2,1}(0,T) \times W^{1,1}(0,T)$.

Proof. Let $u(t) = (x(t), v(t), \theta(t))$, $u_0 = (x_0, v_0, \theta_0)$ and consider the function $g\colon [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ and the multifunction $G\colon \mathbb{R}^3 \to 2^{\mathbb{R}^3} \setminus \{\emptyset\}$, defined by

$$g(t,q,r,s) = \begin{pmatrix} r \\ f(t) - k(t,q,r) - \widetilde{p}\,\psi(q) \\ \widetilde{\mu}\,w(r) - h(s) \end{pmatrix}, \tag{4.4}$$

$$G(q,r,s) = \widetilde{\mu}\,\partial j(r) \tag{4.5}$$

for all $t \in [0,T]$ and $(q,r,s) \in \mathbb{R}^3$. Then it is easy to see that $(x,v,\theta)\colon [0,T] \to \mathbb{R}^3$ is a solution of Problem 2.2 if and only if $u\colon [0,T] \to \mathbb{R}^3$ is a solution of the Cauchy problem

$$\begin{cases} \dot{u}(t) \in g(t,u(t)) - G(u(t)) \quad \text{a.e. } t \in [0,T], \\ u(0) = u_0, \end{cases} \tag{4.6}$$

with g and G given by (4.4) and (4.5), respectively. We note that assumptions (H_1^*), (H_3^*)–(H_7^*) and (H_8) imply that the function g satisfies condition $H(g)$. Moreover, assumptions (H_2^*) and (H_5^*) imply that the multi-valued function G satisfies condition $H(G)$. We conclude from Theorem A2 that the Cauchy problem (4.6) with (4.4), (4.5) has a unique solution $u \in W^{1,1}(0,T;\mathbb{R}^3)$. Therefore, since this Cauchy problem is equivalent to Problem 2.2, it follows that Problem 2.2 has a unique solution $(x,v,\theta) \in W^{1,1}(0,T;\mathbb{R}^3)$. This concludes the proof. □

5 Conclusions

The paper presents a ‘simple’ model for the process of temperature-dependent frictional contact in a setting where a mass that is attached to a spring-damper system moves under the influence of an applied force. This ‘simple’ setting is transparent in that it avoids various complications arising in higher dimensions and allows for simpler notation while containing most of the essential mathematical ideas about the evolution of the system. The mathematical analysis is more transparent, easier and still retains the flavour of more complicated 3D settings.

This ‘simple’ model allows for easy numerical simulations and may be used as a benchmark for more complicated problems. Also, the setting is relatively easy to be set experimentally, which may be used for parameter identification, such as the dependence of friction coefficient on temperature, to be used in more realistic applications. This is, indeed, one of the issues that is open and needs to be addressed for accurate real-world simulations.

The existence of the solutions for the model was established by using arguments for non-linear differential inclusions. The uniqueness of the solution was also established under rather restrictive additional assumptions. The list of examples presented in Section 3 shows that these results are applicable to a large variety of models of specific spring-mass-damper systems with different contact conditions, friction laws and heat generation processes.

Subsequent stages of this research will deal with numerical simulations of the process. We also intend to investigate control strategies for such systems by controlling the stiffness and the viscous damping coefficients so as to optimize the system response.

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Appendix

We present in the appendix preliminary material that is used in the paper and refer the reader for further details to [6, 7, 15, 18].

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function. The generalized directional derivative of φ at $x \in \mathbb{R}$ in the direction $v \in \mathbb{R}$, denoted by $\varphi^0(x; v)$, is defined by

$$\varphi^0(x; v) = \limsup_{y \to x,\ \lambda \downarrow 0} \frac{\varphi(y + \lambda v) - \varphi(y)}{\lambda}.$$

The generalized subdifferential of φ at x, denoted by $\partial\varphi(x)$, is the set

$$\partial\varphi(x) = \{\, \zeta \in \mathbb{R} \mid \varphi^0(x; v) \geqslant \zeta\, v \text{ for all } v \in \mathbb{R}\,\}.$$

Let n be a positive integer and denote by $\|\cdot\|_{\mathbb{R}^n}$ the the norm on $\mathbb{R}^n$. Everywhere below, we use the standard notations $L^p(0, T; \mathbb{R}^n)$ and $W^{1,p}(0, T; \mathbb{R}^n)$ for the Bochner–Lebesgue and the Bochner–Sobolev function spaces, respectively, defined on $[0, T]$ with values in $\mathbb{R}^n$. In the particular case where $n = 1$, we use the notations $L^p(0, T)$ and $W^{1,p}(0, T)$ for the spaces $L^p(0, T; \mathbb{R})$ and $W^{1,p}(0, T; \mathbb{R})$, respectively.

We consider the differential inclusion

$$\begin{cases} \dot{u}(t) \in F(t, u(t)) \quad \text{a.e. } t \in [0, T], \\ u(0) = u_0, \end{cases} \tag{A1}$$

in which $u : [0, T] \to \mathbb{R}^n$ is an unknown function, F is a given multi-valued function, $u_0 \in \mathbb{R}^n$ and $T > 0$. By a solution of Cauchy problem (A1), we mean a function $u \in W^{1,1}(0, T; \mathbb{R}^n)$ that satisfies $\dot{u}(t) \in F(t, u(t))$ for almost all $t \in [0, T]$ and $u(0) = u_0$.

To solve Cauchy problem (A1), we consider the following hypothesis on the multifunction F.

$\underline{H(F)}$: $F : [0, T] \times \mathbb{R}^n \to 2^{\mathbb{R}^n} \setminus \{\emptyset\}$ is a multifunction with convex and compact values such that

(i) for every $\xi \in \mathbb{R}^n$, the graph $GrF(\cdot, \xi) = \{\, (t, z) \in [0, T] \times \mathbb{R}^n \mid z \in F(t, \xi)\,\}$ is Borel measurable in $[0, T] \times \mathbb{R}^n$;

(ii) for a.e. $t \in [0, T]$, the graph $GrF(t, \cdot) = \{\, (\xi, z) \in \mathbb{R}^n \times \mathbb{R}^n \mid z \in F(t, \xi)\,\}$ is closed in $\mathbb{R}^n \times \mathbb{R}^n$;

(iii) for a.e. $t \in [0,T]$, all $\xi \in \mathbb{R}^n$ and all $z \in F(t,\xi)$,

$$\|z\|_{\mathbb{R}^n} \leqslant \widetilde{c}(t) + \overline{c}(t)\|\xi\|_{\mathbb{R}^n}$$

with $\widetilde{c}, \overline{c} \in L^1(0,T)$, $\widetilde{c}, \overline{c} \geqslant 0$.

In the study of (A1) we have the following existence result whose proof can be found in [18, p. 114].

Theorem A1 *Assume that the hypothesis $H(F)$ holds and let $u_0 \in \mathbb{R}^n$. Then the differential inclusion (A1) has at least one solution.*

We also consider the following particular form of problem (A1).

$$\begin{cases} \dot{u}(t) \in g(t,u(t)) - G(u(t)) & \text{a.e. } t \in [0,T], \\ u(0) = u_0. \end{cases} \tag{A2}$$

Here the unknown is the function $u\colon [0,T] \to \mathbb{R}^n$, g is a given single-valued function, G is a given multi-valued function and $u_0 \in \mathbb{R}^n$. We need the following hypotheses on the right-hand side of (A2).

$\underline{H(g)}$: $g : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ is a single-valued function such that

(i) $g(\cdot,\xi)$ is measurable on $[0,T]$ for all $\xi \in \mathbb{R}^n$;

(ii) $g(t,\cdot)$ satisfies the Lipschitz condition

$$\|g(t,\xi_1) - g(t,\xi_2)\|_{\mathbb{R}^n} \leqslant L\,\|\xi_1 - \xi_2\|_{\mathbb{R}^n} \quad \text{for all} \quad \xi_1, \xi_2 \in \mathbb{R}^n$$

uniformly for all $t \in [0,T]$, with $L > 0$.

$\underline{H(G)}$: $G\colon \mathbb{R}^n \to 2^{\mathbb{R}^n} \setminus \{\emptyset\}$ is a multifunction with convex and compact values such that

(i) the graph $Gr\,G(\cdot) = \{(\xi,z) \in \mathbb{R}^n \times \mathbb{R}^n \mid z \in G(\xi)\}$ is closed in $\mathbb{R}^n \times \mathbb{R}^n$;

(ii) for all $\xi \in \mathbb{R}^n$ and all $z \in G(\xi)$, we have $\|z\|_{\mathbb{R}^n} \leqslant c_0(1 + \|\xi\|_{\mathbb{R}^n})$ with $c_0 \geqslant 0$;

(iii) for all $\xi_1, \xi_2 \in \mathbb{R}^n$,

$$(G(\xi_1) - G(\xi_2)) \cdot (\xi_1 - \xi_2) \geqslant -m_0\|\xi_1 - \xi_2\|^2_{\mathbb{R}^n}, \quad \text{with } m_0 \geqslant 0.$$

In the study of (A2) we have the following existence and uniqueness result.

Theorem A2 *Assume that the hypotheses $H(g)$ and $H(G)$ hold, and let $u_0 \in \mathbb{R}^n$. Then the differential inclusion (A2) has a unique solution.*

Proof. For the existence part, it is enough to observe that the hypotheses $H(g)$ and $H(G)$ imply the condition $H(F)$ for the function $F\colon [0,T] \times \mathbb{R}^n \to 2^{\mathbb{R}^n} \setminus \{\emptyset\}$ given by

$$F(t,\xi) = g(t,\xi) - G(\xi) \quad \text{for all } t \in [0,T],\ \xi \in \mathbb{R}^n.$$

Therefore, the existence of solution follows from Theorem A1. In order to prove the uniqueness of the solution, let $u_1, u_2 \in W^{1,1}(0,T;\mathbb{R}^n)$ be two solutions of (A2). We have

$$\dot{u}_1(s) - \dot{u}_2(s) = g(s,u_1(s)) - g(s,u_2(s)) + \eta_2(s) - \eta_1(s), \tag{A3}$$

with $\eta_i(s) \in G(u_i(s))$ for a.e. $s \in [0,T]$, $i = 1, 2$, $u_1(0) - u_2(0) = 0$. We multiply (A3) by $u_1(s) - u_2(s)$ and integrate by parts on $(0,t)$ with $t \in [0,T]$. We obtain

$$\begin{aligned}\frac{1}{2}\|u_1(t) - u_2(t)\|^2_{\mathbb{R}^n} &= \int_0^t (g(s,u_1(s)) - g(s,u_2(s))) \cdot (u_1(s) - u_2(s))\,ds \\ &\quad + \int_0^t (\eta_2(s) - \eta_1(s)) \cdot (u_1(s) - u_2(s))\,ds\end{aligned}$$

for all $t \in [0,T]$. Using the hypotheses $H(g)$(ii) and $H(G)$(iii), we obtain

$$\frac{1}{2}\|u_1(t) - u_2(t)\|^2_{\mathbb{R}^n} \leqslant (L + m_0)\int_0^t \|u_1(s) - u_2(s)\|^2_{\mathbb{R}^n}\,ds, \quad \text{for all } t \in [0,T].$$

Now using the Gronwall lemma yields that $u_1(t) - u_2(t) = 0$ for all $t \in [0,T]$. We conclude that $u_1 = u_2$, which completes the proof of the theorem. □

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