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ARTICLE

Decomposing edge-coloured graphs under colour degree constraints[†]

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Abstract

For an edge-coloured graph G, the minimum colour degree of G means the minimum number of colours on edges which are incident to each vertex of G. We prove that if G is an edge-coloured graph with minimum colour degree at least 5, then $V(G)$ can be partitioned into two parts such that each part induces a subgraph with minimum colour degree at least 2. We show this theorem by proving a much stronger form. Moreover, we point out an important relationship between our theorem and Bermond and Thomassen's conjecture in digraphs.

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1. Introduction

When we try to solve a problem in dense graphs, decomposing a graph into two dense parts sometimes plays an important role in proof arguments. This is because one can apply an induction hypothesis to one of the parts so as to obtain a partial configuration, and then use the other part to obtain a desired configuration. Motivated by this natural strategy, a great deal of work has been done in this direction, and a variety of results about such partitions have been obtained. For example, Stiebitz [11] proved that every graph with minimum degree at least $a+b+1$ can be decomposed into two parts A and B such that A has minimum degree at least a and B has minimum degree at least b. We can see that the bound $a+b+1$ is best possible by considering the complete graph of order $a+b+1$. Thomassen [15, 16] conjectured that, similarly, every $(a+b+1)$-connected graph can be decomposed into two parts A and B in such a way that A is

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a-connected and B is b-connected. Thomassen himself [13] confirmed the conjecture for the case that $b \leqslant 2$. However, rather surprisingly, even for the case $b = 3$ this conjecture is still wide open.

The digraph version of this problem was proposed in the Prague Midsummer Combinatorial Workshop in 1995. For integers s and t, does there exist a smallest value $f(s, t)$ such that each digraph D with minimum out degree $\delta^+(D) \geqslant f(s, t)$ admits a vertex partition (D_1, D_2) satisfying $\delta^+(D_1) \geqslant s$ and $\delta^+(D_2) \geqslant t$? Alon [1, 2] posed the following question: Is there a constant c such that $f(1, 2) \leqslant c$? We only know that $f(1, 1) = 3$ by a result of Thomassen [14]. Very little progress has been made on this problem, with the exception of results on specific classes of digraphs, such as tournaments and multipartite tournaments [3, 17]. Recently, this problem was reiterated by Stiebitz [12] in the context of the colouring number of graphs. As observed from the above known results, it seems that these partition problems are very difficult even if we consider some very specific cases.

In this paper, we would like to consider a similar problem in edge-coloured graphs. To state our results, we introduce some notation and definitions.

Given a finite and simple edge-coloured graph G, and a vertex $v \in V(G)$, let $d^c_G(v)$ be the colour degree of v in G, that is, the number of distinct colours assigned on edges which are incident to v. The minimum colour degree of G is denoted by $\delta^c(G)(:= \min\{d^c_G(v){:}v \in V(G)\})$. Let a and b be integers with $a \geqslant b \geqslant 1$. A pair (A, B) is called (a, b)-*feasible* if A and B are disjoint, non-empty subsets of $V(G)$ such that $\delta^c(G[A]) \geqslant a$ and $\delta^c(G[B]) \geqslant b$; in particular, if G contains an (a, b)-feasible pair (A, B) with $V(G) = A \cup B$, then we say that G *has an* (a, b)-*feasible partition*. A graph is called a *properly coloured* graph (for short, PC graph) if no two adjacent edges have the same colour.

Our main motivation for the theorems stated below is the following conjecture.

Conjecture 1.1. *Let a, b be integers with $a \geqslant b \geqslant 2$, and let G be an edge-coloured graph with $\delta^c(G) \geqslant a + b + 1$. Then G has an (a, b)-feasible partition.*

Observe that this conjecture, if true, would be sharp for a PC K_{a+b+1}. In this paper we will prove this conjecture in various particular cases. First, we will prove it for general graphs in the case $a = b = 2$.

Theorem 1.2. *Conjecture 1.1 is true for $a = b = 2$.*

In fact Theorem 1.2 will be given by proving a much stronger result. To state our result, we first recall the following theorem, which is on the existence of vertex-disjoint directed cycles in digraphs.

Theorem 1.3. (**Thomassen [14]**). *For each natural number k there exists a (smallest) number $f(k)$ such that every digraph D with $\delta^+(D) \geqslant f(k)$ contains k vertex-disjoint directed cycles.*

Bermond and Thomassen [4] conjectured that $f(k) = 2k - 1$, Alon [1] showed that $f(k) \leqslant 64k$ and recently Bucić [5] improved this bound to $24k$.

As above, for $k \geqslant 1$ let $f(k)$ be a function such that every directed graph D satisfying $\delta^+(D) \geqslant f(k)$ contains k disjoint directed cycles. Define a function $g(k)$ as follows:

$$g(k) = \begin{cases} 2 & k = 1, \\ \max\{f(k) + 1, g(k-1) + 3\} & k \geqslant 2. \end{cases}$$

We generalize the concept of (a, b)-feasible partitions as follows. For $k \geqslant 2$, if $V(G)$ can be partitioned into k parts $A_1, A_2, \ldots, A_k$ such that $\delta^c(G[A_i]) \geqslant a_i$ holds for each $1 \leqslant i \leqslant k$, then we say that G *has an* $(a_1, a_2, \ldots, a_k)$-*feasible partition*. In this paper, we will mainly focus on the case where $(a_1, a_2, \ldots, a_k) = (2, 2, \ldots, 2)$. For simplicity, we shall use the term 2^k-*feasible partition* in

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this special case (thus (2, 2)-feasible partitions are equivalent to 2^2-feasible partitions). Our main result is as follows.

Theorem 1.4. *Let G be an edge-coloured graph with $\delta^c(G) \geqslant g(k)$. Then G has a 2^k-feasible partition.*

Thomassen [14] proved that $f(2) = 3$. Thus $g(2) = \max\{f(2) + 1, g(1) + 3\} = 5$. So Theorem 1.2 can be obtained as a corollary of Theorem 1.4. We then focus on the case $b = 2$ in Conjecture 1.1. We obtained the following partial result.

Theorem 1.5. *Let a be an integer with $a \geqslant 2$, and let K_n be an edge-coloured complete graph of order n with $\delta^c(K_n) \geqslant a + 3$. Then K_n has an $(a, 2)$-feasible partition.*

Also, in [7], it is shown that any edge-coloured complete bipartite graph $K_{m,n}$ with $\delta^c(K_{m,n}) \geqslant 3$ contains a PC cycle of length 4. This immediately yields the following (because when we remove a cycle of length 4 from $K_{m,n}$, the colour degree of each vertex in the remaining graph decreases by at most 2).

Theorem 1.6. *Let a be an integer with $a \geqslant 1$. If an edge-coloured complete bipartite graph $K_{m,n}$ satisfies $\delta^c(K_{m,n}) \geqslant a + 2$, then $K_{m,n}$ admits an $(a, 2)$-feasible partition.*

Regarding Conjecture 1.1 in the general case, by using the probabilistic method, we get the following result.

Theorem 1.7. *Let a, b be integers with $a \geqslant b \geqslant 1$. If G is an edge-coloured graph with $|V(G)| = n$ and $\delta^c(G) \geqslant 2\ln n + 4(a - 1)$, then G has an (a, b)-feasible partition.*

Although our results might look a bit modest, proving Conjecture 1.1 even for the case $b = 2$ seems quite hard. This is because we could give a big improvement on the bound of $f(k)$ if it is true.

Theorem 1.8. *If Conjecture 1.1 is true for $b = 2$, then $f(k) \leqslant 3k - 1$.*

Solving Conjecture 1.1 completely seems a very difficult problem.

This paper is organized as follows. In Section 2 we define and characterize the structure of 'minimally 2-coloured graphs'. In Sections 3 and 4 we give the proofs of Theorems 1.4 and 1.7, respectively. In Section 5 we prove Theorems 1.5 and 1.8. In particular, Theorem 1.8 is obtained by a much stronger result (see Proposition 5.3 in Section 5).

Before delivering the proofs, we shall need more notation. For a subgraph H of G with $E(H) \neq \emptyset$, let $\mathrm{col}_G(H)$ be the set of colours assigned to $E(H)$. Also, for a pair of vertex-disjoint subgraphs M, N in G, let $\mathrm{col}_G(M, N)$ be the set of colours on edges between M and N in G. For a vertex v of G, let $N^c_G(v) = \mathrm{col}_G(v, N_G(v))$. By definition, note that $d^c_G(v) = |N^c_G(v)|$. When there is no ambiguity, we often write $\mathrm{col}(e)$ for $\mathrm{col}_G(e)$, $\mathrm{col}(H)$ for $\mathrm{col}_G(H)$, $\mathrm{col}\,(M, N)$ for $\mathrm{col}_G(M, N)$ and $d^c(v)$ for $d^c_G(v)$. For a colour $i \in \mathrm{col}(G)$, let G^i denote the subgraph of G induced by the edges of colour i. We say a colour i *appears* k times at a vertex v if $d_{G^i}(v) = k$.

2. Minimally 2-coloured graphs

To consider our problem, utilizing the structure of minimal subgraphs H with $\delta^c(H) \geqslant 2$ will be very important. An edge-coloured graph G is 2-*coloured* if $\delta^c(G) \geqslant 2$. Specifically, we say a graph G is *minimally* 2-*coloured* if $\delta^c(G) \geqslant 2$ holds but any proper subgraph H of G has minimum colour

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degree less than 2 in H. By definition, note that every PC cycle is a minimally 2-coloured graph. An edge-coloured graph obtained from two disjoint cycles by joining a path is a *generalized bowtie* (more briefly, call it *g-bowtie*). We allow the case where the path joining two cycles is empty. In that case, the g-bowtie becomes a graph obtained from two disjoint cycles by identifying one vertex in each cycle. Note also that $K_1 + 2K_2$ (*i.e.* a graph obtained from two disjoint triangles by identifying one vertex of each triangle) is a g-bowtie with minimum order.

We have the following characterization of minimally 2-coloured graphs, which will be used to prove our main result.

Theorem 2.1. *If an edge-coloured graph G is minimally 2-coloured, then G is either a PC cycle or a 2-coloured g-bowtie containing no PC cycles.*

In order to prove this theorem, we shall need the following structural theorem characterizing edge-coloured graphs containing no PC cycles.

Theorem 2.2. (Grossman and Häggkvist [9], Yeo [18]). *Let G be an edge-coloured graph containing no PC cycles. Then there is a vertex $z \in V(G)$ such that no component of $G - z$ is joined to z with edges of more than one colour.*

Proof of Theorem 2.1. Let G be a minimally 2-coloured graph. If G contains a subgraph H which is a PC cycle or a 2-coloured g-bowtie containing no PC cycles, then $G = H$ (otherwise, by deleting vertices in $V(G) \setminus V(H)$ or edges in $E(G) \setminus E(H)$, we obtain a smaller 2-coloured graph). Hence, it is sufficient to prove that if G contains no PC cycle, then G contains a 2-coloured g-bowtie. Apply Theorem 2.2 to G. Since G is minimally 2-coloured, we may assume that G is connected and there is a vertex $z \in V(G)$ such that $G - z$ contains two components H_1 and H_2, with all the edges between z and H_i having colour i for $i = 1, 2$.

Let $zx_1x_2 \cdots x_p$ and $zy_1y_2 \cdots y_q$, respectively, be longest PC paths in $G \backslash H_2$ and $G \backslash H_1$ starting from z. Set $x_0 = z$ and $y_0 = z$. Since $d^c_{G\backslash H_2}(x) \geqslant 2$ and $d^c_{G\backslash H_1}(y) \geqslant 2$ for arbitrary vertices $x \in V(H_1)$ and $y \in V(H_2)$, we have $p, q \geqslant 2$, and there exist vertices x_i and y_j for some i, j with $0 \leqslant i \leqslant p-2$ and $0 \leqslant j \leqslant q-2$ such that $\mathrm{col}\,(x_px_i) \neq \mathrm{col}\,(x_{p-1}x_p)$ and $\mathrm{col}\,(y_qy_j) \neq \mathrm{col}\,(y_{q-1}y_q)$. Since G contains no PC cycle, we have $\mathrm{col}\,(x_px_i) = \mathrm{col}\,(x_ix_{i+1})$ and $\mathrm{col}\,(y_qy_j) = \mathrm{col}\,(y_jy_{j+1})$. Together, the path $x_ix_{i-1} \cdots x_1zy_1y_2 \cdots y_j$ and cycles $x_ix_{i+1} \cdots x_px_i$ and $y_jy_{j+1} \cdots y_qy_j$ form a 2-coloured g-bowtie.

The proof is complete. □

3. Proof of Theorem 1.4

First we prove the following proposition.

Proposition 3.1. *Let G be an edge-coloured graph with $\delta^c(G) \geqslant a + b - 1$. If G contains an (a, b)-feasible pair, then there exists an (a, b)-feasible partition of G.*

Proof. Let (A, B) be an (a, b)-feasible pair such that $A \cup B$ is maximal. If (A, B) is not an (a, b)-feasible partition, then $A \cup B = V(G) \backslash S$ with $S \neq \emptyset$. Since (A, B) is maximal, $(A, B \cup S)$ is not a feasible pair. Hence there exists a vertex x in S such that $d^c_{G[B\cup S]}(x) \leqslant b - 1$. Recall that $d^c_G(x) \geqslant a + b - 1$. So $d^c_{G[A\cup x]}(x) \geqslant a$. Thus $(A \cup x, B)$ is a feasible pair, which is a contradiction to the maximality of (A, B). This proves that (A, B) is an (a, b)-feasible partition of G. □

It is easy to check that the following proposition is also true.

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Proposition 3.2. *Let G be an edge-coloured graph with $\delta^c(G) \geqslant \sum_{i=1}^{k}(a_i - 1) + 1$. If G contains k disjoint subgraphs $H_1, H_2, \ldots, H_k$ such that $\delta^c(H_i) \geqslant a_i$ for $i = 1, 2, \ldots, k$, then G admits an $(a_1, a_2, \ldots, a_k)$-feasible partition.*

In what follows, we will keep the above propositions in mind and use these facts as a matter of course.

Proof of Theorem 1.4. We prove the theorem by contradiction. We say (G, k) is a counterexample if $\delta^c(G) \geqslant g(k)$, but G does not admit a 2^k-feasible partition. Let (G, k) be a counterexample such that (G, k) is chosen according to the following order of preferences.

(i) k is minimum,
(ii) $|G|$ is minimum,
(iii) $|E(G)|$ is minimum,
(iv) $|\operatorname{col}(G)|$ is maximum.

By the choice of (G, k), we know that $\delta^c(G) = g(k)$ and $k \geqslant 2$. Let

$$S_v = \{u : d^c_{G-v}(u) = d^c_G(u) - 1\}.$$

Now we prove the following claims.

Claim 1. *G contains no rainbow triangles.*

Proof. Suppose that G contains a rainbow triangle $xyzx$. Then let G' be the edge-coloured graph obtained by deleting vertices x, y, z from G. Thus $\delta^c(G') \geqslant g(k) - 3 \geqslant g(k-1)$. By the assumption of (G, k), we know that $(G', k-1)$ is not a counterexample. So G' admits a 2^{k-1}-feasible partition. Together with the triangle $xyzx$, we get a 2^k-feasible partition of G, a contradiction. □

Claim 2. *$S_v \neq \emptyset$ for all $v \in V(G)$.*

Proof. Suppose that $S_v = \emptyset$ for some vertex $v \in V(G)$. Then $\delta^c(G - v) \geqslant \delta^c(G) = g(k)$. By the assumption of (G, k), we know that $(G - v, k)$ is not a counterexample. So $G - v$ admits a 2^k-feasible partition. By Proposition 3.2, G also has a 2^k-feasible partition, a contradiction. □

Claim 3. *For each edge $uv \in E(G)$, either $u \in S_v$ or $v \in S_u$.*

Proof. Suppose that there exists an edge $uv \in E(G)$ such that $u \notin S_v$ and $v \notin S_u$. Then let G' be the edge-coloured graph obtained by deleting the edge uv from G. Thus $\delta^c(G') = \delta^c(G) = g(k)$. By the assumption of (G, k), we know that (G', k) is not a counterexample. So G' admits a 2^k-feasible partition, which is also a 2^k-feasible partition of G, a contradiction. □

Claim 4. *For each colour $i \in \operatorname{col}(G)$, the coloured graph G^i is a star.*

Proof. Claim 3 implies that G contains no monochromatic triangle or monochromatic P_4 (P_4 denotes a path on four vertices). Thus for every colour $i \in \operatorname{col}(G)$, each component of G^i is a star. If G^i contains more than one component, then colour one of the components with a colour not in $\operatorname{col}(G)$. Thus, we get a counterexample with more colours than G, which contradicts the choice of G. □

Claim 4 implies that for each monochromatic star with centre v, every leaf of the star belongs to S_v.

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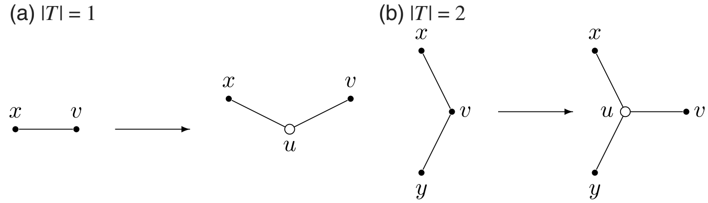

Figure 1. Cases of $|T|$.

Claim 5. *For $u, v \in V(G)$, if $u \in S_v$ and $v \notin S_u$, then $S_u \cap N_G(v) \neq \emptyset$.*

Proof. Suppose to the contrary that there exist vertices $u, v \in V(G)$ satisfying $u \in S_v$, $v \notin S_u$ and $S_u \cap N_G(v) = \emptyset$. Then col (vu) appears only once at u and more than once at v. By Claim 4, the colour col (vu) can only appear at $\{v\} \cup S_v$, and in particular not at S_u. Now we construct an edge-coloured graph G' by deleting the vertex u and adding edges $\{vx{:}x \in S_u\}$ to G with all of them coloured by col (vu) (since $S_u \cap N_G(v) = \emptyset$, this is possible without resulting multi-edges). For each vertex $x \in V(G')\backslash S_u$, we have $d^c_{G'}(x) = d^c_G(x)$. For each vertex $y \in S_u$, we have $N^c_{G'}(y) = (N^c_G(y)\backslash \text{col}\,(uy)) \cup \text{col}\,(vu)$. Since the colour col (vu) does not appear at S_u, we have $d^c_{G'}(y) = |N^c_{G'}(y)| = |N^c_G(y)| = d^c_G(y)$. This implies that $\delta^c(G') \geqslant \delta^c(G) = g(k)$. Note that $|G'| = |G| - 1$. By the assumption of G, we know that G' must admit a 2^k-feasible partition. By Theorem 2.1, G' contains k disjoint subgraphs $H_1, H_2, \ldots, H_k$ such that H_i is either a PC cycle or a minimally 2-coloured g-bowtie containing no PC cycles for $i = 1, 2, \ldots, k$. If $\bigcup_{i=1}^k E(H_i) \subseteq E(G)$, then we can find a 2^k-partition of G as desired, a contradiction. If $\bigcup_{i=1}^k E(H_i) \nsubseteq E(G)$, then all the edges in $T = (\bigcup_{i=1}^k E(H_i)) \setminus E(G)$ form a monochromatic star with the vertex v as a centre. Thus, without loss of generality, assume that $T \subseteq E(H_1)$.

Since H_1 is either a PC cycle or a minimally 2-coloured g-bowtie containing no PC cycles, for each vertex $a \in H_1$ and each colour $j \in \text{col}\,(H_1)$, the colour j appears at most twice at a in H_1. Thus we have $1 \leqslant |T| \leqslant 2$.

If $|T| = 1$, then let xv be the unique edge in T. Replace xv in H_1 with the path xuv (see Figure 1(a)). We obtain an edge-coloured graph H'_1 in G with $\delta^c(H'_1) \geqslant 2$ (since $\delta^c(H_1) \geqslant 2$, $x \in S_u$, $u \in S_v$ and $\text{col}_G\,(uv) = \text{col}_{G'}\,(xv)$). Thus $H'_1, H_2, \ldots, H_k$ imply a 2^k-feasible partition of G, a contradiction.

If $|T| = 2$, then let $T = \{vx, vy\}$. Since col $(vx) =$ col (vy), we know that H_1 is a minimally 2-coloured g-bowtie with v being an end-vertex of the connecting path in H_1. Delete the edges vx, vy and add vertex u and edges uv, ux, uy in H_1 (see Figure 1(b)). We obtain a g-bowtie H'_1 in G with $\delta^c(H'_1) \geqslant 2$ (since $\delta^c(H_1) \geqslant 2$, $x, y \in S_u$, $u \in S_v$ and $\text{col}_G\,(uv) = \text{col}_{G'}\,(vx) = \text{col}_{G'}\,(vy)$). Thus $H'_1, H_2, \ldots, H_k$ imply a 2^k-feasible partition of G, a contradiction. □

Claim 6. *There exists an edge $xy \in E(G)$ such that $x \in S_y$ and $y \in S_x$.*

Proof. Suppose not. Then by Claim 3, we can construct an oriented graph D by orienting each edge $e = uv \in E(G)$ from u to v if and only if $v \in S_u$. Then $d^+_D(v) \geqslant 2$ for each vertex $v \in V(D)$. Let $T_i(v) = \{u{:}\ \text{col}\,(uv) = i\}$.

Subclaim 6.1. *For each vertex $v \in V(G)$ and colours $i, j \in \text{col}\,(G)$ with $i \neq j$, if $|T_i(v)| \geqslant 2$ and $|T_j(v)| \geqslant 2$, then the following statements hold:*

(a) *$T_i(v) \cap T_j(v) = \emptyset$ and $E(T_i(v), T_j(v)) = \emptyset$,*
(b) *$G[T_i(v)]$ contains at least one edge.*

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Proof. (a) By definition, we know that $T_i(v) \cap T_j(v) = \emptyset$. Since $|T_i(v)| \geqslant 2$ and $|T_j(v)| \geqslant 2$, we know that $T_i(v) \cup T_j(v) \subseteq S_v$. Let $u_i \in T_i(v)$ and $u_j \in T_j(v)$. Then colours i and j appear only once at u_i and u_j, respectively. If $u_iu_j \in E(G)$, then vu_iu_jv is a rainbow triangle, a contradiction. So we have $E(T_i(v), T_j(v)) = \emptyset$.

(b) Suppose that $G[T_i(v)]$ is empty for some colour i with $|T_i(v)| \geqslant 2$. Then choose $u \in T_i(v)$. We have $u \in S_v$ and $v \notin S_u$. Applying Claim 5 to u and v, we obtain $S_u \cap N_G(v) \neq \emptyset$. Let $z \in S_u \cap N_G(v)$. We have $z \in N_G(v)$, and as $G[T_i(v)]$ is empty, z does not belong to $T_i(v)$. So $\mathrm{col}\,(zv) \neq i$. Recall that $\mathrm{col}\,(vu) = i$, $u \in S_v$ and $z \in S_u$. It is easy to check that $C = zuvz$ is a rainbow triangle in G, a contradiction. □

Subclaim 6.2. *For each vertex $v \in V(G)$, there is exactly one colour $i \in \mathrm{col}\,(G)$ such that $|T_i(v)| \geqslant 2$.*

Proof. Given a vertex v, by Claim 2, we can find a vertex $u \in S_v$. By the assumption on G, we have $v \notin S_u$. Let $i = \mathrm{col}\,(uv)$. Then $|T_i(v)| \geqslant 2$. This implies that for each vertex $v \in V(G)$, there is at least one colour $i \in \mathrm{col}\,(G)$ such that $|T_i(v)| \geqslant 2$. Now, suppose to the contrary that there exists a vertex $v \in V(G)$ and colours $i, j \in \mathrm{col}\,(G)$ with $i \neq j$ satisfying $|T_i(v)| \geqslant 2$ and $|T_j(v)| \geqslant 2$. By Subclaim 6.1, we can choose edges u_iw_i from $G[T_i(v)]$ and u_jw_j from $G[T_j(v)]$. Let $F = G[v, u_i, w_i, u_j, w_j]$. Then $\delta^c(F) \geqslant 2$. Now we will discuss on the minimum colour degree of $G - F$.

If $\delta^c(G - F) \geqslant g(k-1)$, then by the assumption of (G, k), $G - F$ has a 2^{k-1}-feasible partition. Together with F, we obtain a 2^k-feasible partition of G, a contradiction. So we have $\delta^c(G - F) < g(k-1)$. Let $x \in V(G - F)$ be a vertex satisfying $d^c_{G-F}(x) = \delta^c(G - F)$. Since $\delta^c(G) \geqslant g(k) \geqslant g(k-1) + 3$ and $|F| = 5$, we have

$$4 \leqslant |\,\mathrm{col}\,(x, F)| \leqslant 5.$$

For vertices $a \in \{u_i, w_i\}$ and $b \in \{u_j, w_j\}$, if $|\,\mathrm{col}\,(x, \{a, b, v\})| \geqslant 3$, then it is easy to check that either $xavx$ or $xbvx$ is a rainbow triangle, a contradiction (note that $\mathrm{col}\,(xv) \in \{i, j\}$ is possible). So we have $|\,\mathrm{col}\,(x, \{a, b, v\})| \leqslant 2$. Note that $|\,\mathrm{col}\,(x, F)| \geqslant 4$. This ensures that $vx \notin E(G)$ and $|\,\mathrm{col}\,(x, \{u_i, w_i, u_j, w_j\})| = 4$. Thus $C = xu_ivu_jx$ is a rainbow cycle of length 4. Suppose that there exists a vertex $y \in V(G - C)$ such that $d^c_{G-C}(y) < g(k-1)$. Then $|\,\mathrm{col}\,(y, C)| \geqslant 4$. Note that $u_i, u_j \in S_v$. Thus either yu_ivy or yu_jvy is a rainbow triangle, a contradiction. Hence we have $\delta^c(G - C) \geqslant g(k-1)$. By the assumption of G, the graph $G - C$ has a 2^{k-1}-feasible partition. Together with $G[V(C)]$, we get a 2^k-feasible partition of G, a contradiction. □

Subclaim 6.2 implies that at least $g(k) - 1$ colours appear only once at v for each vertex $v \in V(G)$. Thus, we have $\delta^-(D) \geqslant g(k) - 1 \geqslant f(k)$. So D contains k disjoint directed cycles, which correspond to k disjoint PC cycles in G, a contradiction. □

Claim 7. *For each edge $xy \in E(G)$ satisfying $x \in S_y$ and $y \in S_x$, we have*

(a) $|N^c_G(x) \cup N^c_G(y) - \mathrm{col}\,(xy)| \leqslant g(k) - 1$, *and*
(b) $N_G(x) - y = N_G(y) - x = \{v_i : 1 \leqslant i \leqslant g(k) - 1\}$, *where* $\mathrm{col}\,(xv_i) = \mathrm{col}\,(yv_i)$ *and* $\mathrm{col}\,(xv_i) \neq \mathrm{col}\,(xv_j)$ *for* $i, j \in [1, g(k) - 1]$ *with* $i \neq j$.

Proof. (a) Since G contains no rainbow triangles and $\mathrm{col}\,(xy)$ appears only once at x and y, respectively, we have $\mathrm{col}\,(xu) = \mathrm{col}\,(yu)$ for all $u \in N_G(x) \cap N_G(y)$. Now let $G' = G/xy$. Then G' is well-defined and $d^c_{G'}(v) = d^c_G(v)$ for all vertices in $V(G)\backslash\{x, y\}$. Let z be the new vertex resulting from contracting the edge xy.

Suppose that $|N^c_G(x) \cup N^c_G(y) - \mathrm{col}\,(xy)| \geqslant g(k)$. Then $d^c_{G'}(z) \geqslant g(k)$. Thus we have $\delta^c(G') \geqslant g(k)$. By the choice of G, we know that G' must admit a 2^k-feasible partition. By Theorem 2.1,

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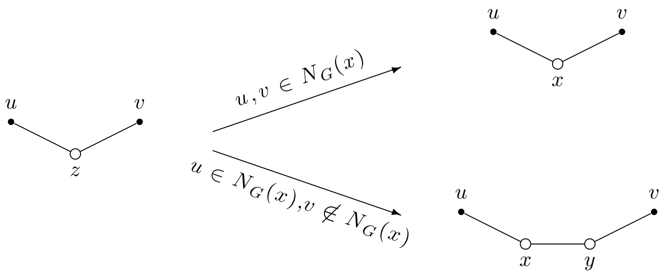

Figure 2. $d_{H_1}(z) = 2$.

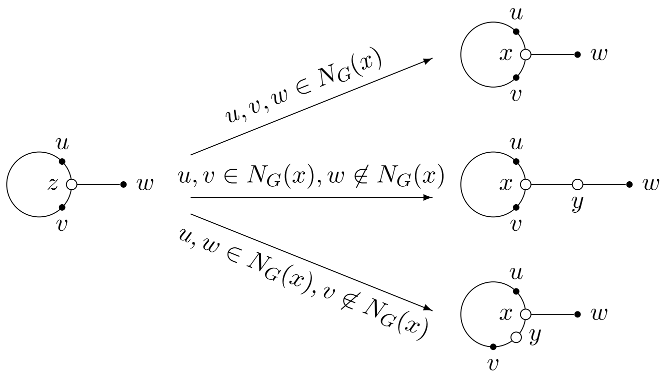

Figure 3. $d_{H_1}(z) = 3$.

G' contains k disjoint subgraphs $H_1, H_2, \ldots, H_k$ such that H_i $(i = 1, 2, \ldots, k)$ is either a PC cycle or a minimally 2-coloured g-bowtie containing no PC cycles.

If $z \notin \bigcup_{i=1}^{k} V(H_i)$, then $H_1, H_2, \ldots, H_k$ are k-disjoint subgraphs of G. This implies a 2^k-feasible partition of G, a contradiction. So we can assume that $z \in V(H_1)$. Evidently we have $2 \leqslant d_{H_1}(z) \leqslant 4$.

If $d_{H_1}(z) = 2$, then let $N_{H_1}(z) = \{u, v\}$ (see Figure 2). By the symmetry between x and y, and that between u and v, it suffices to discuss the following two cases:

(i) $u, v \in N_G(x)$,
(ii) $u \in N_G(x)$ and $v \notin N_G(x)$.

If $u, v \in N_G(x)$, then replace z with x. If $u \in N_G(x)$ and $v \notin N_G(x)$, then replace the path uzv with $uxyv$. In all cases, we can transform H_1 into a graph $H_1' \subseteq G$ such that $\delta^c(H_1') \geqslant 2$ and $V(H_1') \cap V(H_i) = \emptyset$ for $i = 2, 3 \ldots, k$. Thus $H_1', H_2, \ldots, H_k$ imply the existence of a 2^k-feasible partition of G, a contradiction.

If $d_{H_1}(z) = 3$, then H_1 must be a minimally 2-coloured g-bowtie with z being an end-vertex of the connecting path. Let $N_{H_1}(z) = \{u, v, w\}$ with u, v on a same cycle in H_1 (see Figure 3). By the symmetry between x and y, and that between u and v, it suffices to discuss the following three cases:

(i) $\{u, v, w\} \subseteq N_G(x)$,
(ii) $\{u, v\} \subseteq N_G(x)$ and $w \notin N_G(x)$,
(iii) $\{u, w\} \subseteq N_G(x)$ and $v \notin N_G(x)$.

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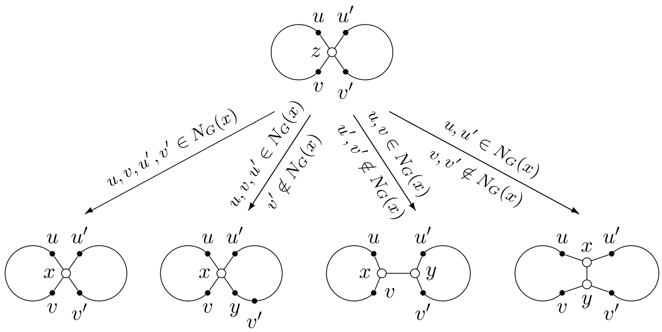

Figure 4. $d_{H_1}(z) = 4$.

If $\{u, v, w\} \subseteq N_G(x)$, then replace z with x. If $\{u, v\} \subseteq N_G(x)$ and $w \notin N_G(x)$, then replace zw with xyw. If $\{u, w\} \subseteq N_G(x)$ and $v \notin N_G(x)$, then replace zv with xyv. Finally, in all cases, we can transform H_1 into a graph $H_1' \subseteq G$ such that $\delta^c(H_1') \geqslant 2$ and $V(H_1') \cap V(H_i) = \emptyset$ for $i = 2, 3 \ldots, k$. Thus $H_1', H_2, \ldots, H_k$ imply a 2^k-feasible partition of G, a contradiction.

If $d_{H_1}(z) = 4$, then H_1 is a minimally 2-coloured g-bowtie with two cycles overlapped on the vertex z. Let $N_{H_1}(z) = \{u, v, u', v'\}$ with u, v on one cycle and u', v' on the other cycle (see Figure 4). By the symmetry between x and y and between u and v and that between u' and v', it suffices to discuss the following four cases:

(i) $\{u, v, u', v'\} \subseteq N_G(x)$,
(ii) $\{u, v, u'\} \subseteq N_G(x)$ and $v' \notin N_G(x)$,
(iii) $\{u, v\} \subseteq N_G(x)$ and $\{u', v'\} \cap N_G(x) = \emptyset$,
(iv) $\{u, u'\} \subseteq N_G(x)$ and $\{v, v'\} \cap N_G(x) = \emptyset$.

If $\{u, v, u', v'\} \subseteq N_G(x)$, then replace z with x. If $\{u, v, u'\} \subseteq N_G(x)$ and $v' \notin N_G(x)$, then replace the path zv' with xyv'. If $\{u, v\} \subseteq N_G(x)$ and $\{u', v'\} \cap N_G(x) = \emptyset$, then split z into the edge xy such that the resulting graph is still a g-bowtie. If $\{u, u'\} \subseteq N_G(x)$ and $\{v, v'\} \cap N_G(x) = \emptyset$, then split z into the edge xy in an orthogonal direction such that the resulting graph is a cycle with one chord xy. Finally, in all cases, we can transform H_1 into a graph $H_1' \subseteq G$ such that $\delta^c(H_1') \geqslant 2$ and $V(H_1') \cap V(H_i) = \emptyset$ for $i = 2, 3 \ldots, k$. Thus $H_1', H_2, \ldots, H_k$ imply a 2^k-feasible partition of G, a contradiction.

(b) By Claim 7(a) and the fact that $d^c_G(x), d^c_G(y) \geqslant g(k)$, we have $N^c_G(x) = N^c_G(y)$ and $d^c_G(x) = d^c_G(y) = g(k)$. For each colour $j \in N^c_G(x)$ and $j \neq \text{col}\,(xy)$, since G^j is a star and the colour j appears at x and y, we know that x, y must be leaf vertices of G^j. Let v_j be the centre of G^j. The proof is complete. □

Now let $xy \in E(G)$ with $x \in S_y$, $y \in S_x$ and $\{v_i{:}1 \leqslant i \leqslant g(k) - 1\}$ be the set of vertices described in Claim 7. Without loss of generality, let $\text{col}\,(xv_i) = i$ for $i \in [1, g(k) - 1]$. Let H be the subgraph of G induced by $\{x, y\} \cup \{v_i : 1 \leqslant i \leqslant g(k) - 1\}$ and $R = G - H$.

Claim 8. *For* $1 \leqslant i \leqslant g(k) - 1$, $\text{col}\,(v_i, S_{v_i}) = \{i\}$.

Proof. Suppose to the contrary that there exists a vertex $u \in S_{v_i}$ such that $\text{col}\,(uv_i) \neq i$. If $u = v_j$ for some j with $1 \leqslant j \leqslant g(k) - 1$ and $j \neq i$, then $\text{col}\,(uv_i) = j$ (since xv_iv_jx is not a rainbow triangle). Since the colour j appears at least twice at $v_j(= u)$, we know that $u \notin S_{v_i}$, a contradiction. Now

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the vertex u must belong to $V(R)$. Since each G^j $(1 \leqslant j \leqslant g(k)-1)$ is a star and $\mathrm{col}\,(uv_i) \neq i$, we have $\mathrm{col}\,(uv_i) \notin [1, g(k)-1]$. If $v_i \in S_u$, then by applying Claim 7 to the edge uv_i, we have $N_G(u) - v_i = N_G(v_i) - u$. Since $x \in N_G(v_i)$, we have $x \in N_G(u)$, namely, $u \in N_G(x)$, a contradiction. So we have $v_i \notin S_u$. Applying Claim 5 to uv_i, we obtain a vertex $v \in S_u \cap N_G(v_i)$. Note that $\mathrm{col}\,(uv_i) \notin [1, g(k)-1]$ and G contains no rainbow triangle, we have $v \in R - u$ and $\mathrm{col}\,(vv_i) = \mathrm{col}\,(uv_i)$. Let $F = G[x, y, v_i, u, v]$. It is easy to check that $\delta^c(F) \geqslant 2$.

We will show that for each vertex $z \in G - F$, $|\mathrm{col}\,(z, F)| \leqslant 3$. For $z \in R \setminus V(F)$, the assertion holds since z has no neighbour to x or y. Thus we may assume that $z = v_j$ for some j with $1 \leqslant j \leqslant g(k)-1$ and $j \neq i$. If $zv_i \notin E(G)$ or $\mathrm{col}\,(zv_i) = j$, then we have the desired conclusion. So we may assume that z is adjacent to v_i and $\mathrm{col}\,(zv_i) = i$ (otherwise, zxv_iz is a rainbow triangle). Since there is no rainbow triangle and G^i is a star, we can easily check that $zu \notin E(G)$. So z satisfies the desired property.

Now, $\delta^c(G-F) \geqslant g(k) - 3 \geqslant g(k-1)$. So $G-F$ admits a 2^{k-1}-feasible partition. Together with $G[V(F)]$, we obtain a 2^k-feasible partition of G, a contradiction. □

Claim 9. *There exists a vertex v_i with $1 \leqslant i \leqslant g(k)-1$ such that $S_{v_i} = \{x, y\}$.*

Proof. Suppose not. Then there exists a vertex $u_i \in S_{v_i} \backslash \{x, y\}$ for all i with $1 \leqslant i \leqslant g(k)-1$. By Claim 8, $\mathrm{col}\,(u_iv_i) = i$ for $1 \leqslant i \leqslant g(k)-1$. Let $G' = G - \{x, y\}$. Then $\delta^c(G') \geqslant \delta^c(G) \geqslant g(k)$. By the choice of G, the graph G' must admit a 2^k-feasible partition, which implies that G has a 2^k-feasible partition, a contradiction. □

We are now in a position to prove the theorem. Let v_i be the vertex in Claim 9. Since $d^c_H(v_i) \leqslant g(k)-1$ and $d^c_G(v_i) \geqslant g(k)$, there is a vertex $u \in R \cap N_G(v_i)$. Note that $u \notin S_{v_i}$. By Claim 3, we have $v_i \in S_u$. Now apply Claim 5 to the edge uv_i, we have $S_{v_i} \cap N_G(u) \neq \emptyset$. This implies that either $x \in N_G(u)$ or $y \in N_G(u)$, a contradiction.

This completes the proof of Theorem 1.4. □

4. Proof of Theorem 1.7

Proof of Theorem 1.7. Assume $V(G) = \{u_1, u_2, \ldots, u_n\}$. We divide $V(G)$ into two disjoint parts A, B randomly with $\Pr\,(u \in A) = \Pr\,(u \in B) = 1/2$ for each vertex $u \in V(G)$. For each $u \in V(G)$, the bad event A_u means that $u \in A$ and $\{d^c_{G[A]}(u) \leqslant a-1\}$. Let B_u be the bad event that $u \in B$ and $\{d^c_{G[B]}(u) \leqslant b-1\}$. We have

$$\Pr\,(A_u) \leqslant \frac{1}{2}\sum_{j=0}^{a-1}\binom{d^c_G(u)}{j}\left(\frac{1}{2}\right)^{d^c_G(u)} \quad \text{and} \quad \Pr\,(B_u) \leqslant \frac{1}{2}\sum_{j=0}^{b-1}\binom{d^c_G(u)}{j}\left(\frac{1}{2}\right)^{d^c_G(u)}.$$

Thus

$$\begin{aligned}\Pr\,(A_u \cup B_u) = \Pr\,(A_u) + \Pr\,(B_u) &\leqslant \sum_{j=0}^{a-1}\binom{d^c_G(u)}{j}\left(\frac{1}{2}\right)^{d^c_G(u)} \\ &= \sum_{j=d^c_G(u)-a+1}^{d^c_G(u)}\binom{d^c_G(u)}{j}\left(\frac{1}{2}\right)^{d^c_G(u)} = \Pr\,(X \geqslant d^c_G(u) - a + 1),\end{aligned}$$

where $X \sim B(d^c_G(u), 1/2)$.

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Recall that Chernoff's bound: $\Pr[X - E(X) \geqslant n\epsilon] < e^{-2n\epsilon^2}$, where $X \sim B(n, 1/2)$. We get

$$\begin{aligned}\Pr(A_u \cup B_u) &\leqslant \Pr(X \geqslant d^c_G(u) - a + 1)\\ &= \Pr\left(X - \frac{d^c_G(u)}{2} \geqslant \frac{d^c_G(u)}{2} - a + 1\right)\\ &< \exp(-2((d^c_G(u))/2 - a + 1)^2/d^c_G(u)).\end{aligned}$$

Since $d^c_G(u) \geqslant 2\ln n + 4(a-1)$, we have $\exp(-2((d^c_G(u))/2 - a + 1)^2/d^c_G(u)) \leqslant 1/n$. Thus

$$\Pr\left[\bigcup_{u\in V(G)} (A_u \cup B_u)\right] \leqslant \sum_{u\in V(G)} \Pr(A_u \cup B_u) < 1.$$

Hence there exists a partition such that neither event A_u nor B_v happens. So we have an (a, b)-feasible partition. □

5. From (*a*,2)-feasible partitions to Bermond and Thomassen's conjecture

First, we give the proof of Theorem 1.5. In order to prove the theorem, we use the following fact.

Lemma 5.1. [8]. *In any rainbow triangle-free colouring of a complete graph, there exists a vertex partition $(V_1, V_2 \dots, V_t)$ of $V(K_n)$ with $t \geqslant 2$ such that between the parts there are a total of at most two colours, and between every pair of parts V_i, V_j with $i \neq j$ there is only one colour on the edges.*

Proof of Theorem 1.5. If K_n contains a rainbow triangle C, then let $A = C$ and $B = K_n - C$. It follows that $\delta^c(A) \geqslant 2$ and $\delta^c(B) \geqslant a$. So (A, B) is an $(a, 2)$-feasible partition. Now we assume that K_n contains no rainbow triangle. Utilizing Lemma 5.1, we can easily see that $(V_1, \cup_{i=2}^t V_i)$ is an $(a+1, a+1)$-feasible partition, which is also an $(a, 2)$-feasible partition. Thus Theorem 1.5 holds. □

In this section, we will point out a relationship between $(a, 2)$-feasible partitions in edge-coloured graphs and Bermond and Thomassen's conjecture in digraphs. In fact, Bermond and Thomassen's conjecture has not even been confirmed in multipartite tournaments. Recently, Li, Broersma and Zhang [10] revealed a relationship between PC cycles in edge-coloured complete graphs and Bermond and Thomassen's conjecture on multipartite tournaments.

We prove the following proposition.

Proposition 5.2. *For $k \geqslant 1$ let $d_1, \dots, d_k$ be positive integers, and let*

$$f(d_1, d_2, \dots, d_k), \quad g(d_1, d_2, \dots, d_k) \quad \text{and} \quad h(d_1, d_2, \dots, d_k)$$

be the minimum values which make the following three statements true.

(i) *Every oriented graph D with $\delta^+(D) \geqslant f(d_1, d_2, \dots, d_k)$ has a vertex-partition $(V_1, V_2, \dots, V_k)$ with $\delta^+(D[V_i]) \geqslant d_i$ for $i = 1, 2, \dots, k$.*
(ii) *Every edge-coloured graph G with $\delta^c(G) \geqslant g(d_1, d_2, \dots, d_k)$ has a $(d_1, d_2, \dots, d_k)$-feasible partition.*
(iii) *Every edge-coloured complete graph K with $\delta^c(K) \geqslant h(d_1, d_2, \dots, d_k)$ has a $(d_1, d_2, \dots, d_k)$-feasible partition.*

Then we have

$$f(d_1 - 1, d_2 - 1, \dots, d_k - 1) \leqslant g(d_1, d_2, \dots, d_k) \leqslant h(d_1 + 1, d_2 + 1, \dots, d_k + 1).$$

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Proof. Given an oriented graph D, we construct an edge-coloured graph G with $V(G) = V(D)$, $E(G) = \{uv : uv \in A(D) \text{ or } vu \in A(D)\}$ and $\text{col}_G(uv) = v$ if and only if $uv \in A(D)$. If $\delta^+(D) \geqslant g(d_1, d_2, \ldots, d_k)$, then by the construction, we know that $\delta^c(G) \geqslant g(d_1, d_2, \ldots, d_k)$. Thus, G admits a partition $V_1, V_2, \ldots, V_k$ such that $\delta^c(G[V_i]) \geqslant d_i$ for $i = 1, 2, \ldots, k$. In turn, by the construction, we have $\delta^+(D[V_i]) \geqslant d_i - 1$ for $i = 1, 2, \ldots, k$. So we obtain

$$f(d_1 - 1, d_2 - 1, \ldots, d_k - 1) \leqslant g(d_1, d_2, \ldots, d_k).$$

Given an edge-coloured graph G, we construct an edge-coloured complete graph K with $V(K) = V(G)$, $\text{col}_K(e) = \text{col}_G(e)$ for all $e \in E(G)$, $\text{col}_K(e) = c_0$ for all $e \in E(K) \setminus E(G)$ and $c_0 \notin \text{col}(G)$. If $\delta^c(G) \geqslant h(d_1 + 1, d_2 + 1, \ldots, d_k + 1)$, then $\delta^c(K) \geqslant h(d_1 + 1, d_2 + 1, \ldots, d_k + 1)$. By the definition of h, we know that there exists a partition $V_1, V_2, \ldots, V_k$ of K such that $\delta^c(K[V_i]) \geqslant d_i + 1$ for $i = 1, 2, \ldots, k$. By the construction of K, we have $\delta^c(G[V_i]) \geqslant d_i$ for $i = 1, 2, \ldots, k$. Recall the definition of g. We know that

$$g(d_1, d_2, \ldots, d_k) \leqslant h(d_1 + 1, d_2 + 1, \ldots, d_k + 1).$$

□

Remark. The existence of $f(d_1, d_2, \ldots, d_k)$ for $d_i \geqslant 2$ $(i = 1, 2, \ldots, k)$ and $k \geqslant 2$ is still unknown according to [1]. Proposition 5.2 implies that we could show the existence of $f(d_1, d_2, \ldots, d_k)$ by proving the existence of $g(d_1 + 1, d_2 + 1, \ldots, d_k + 1)$ or $h(d_1 + 2, d_2 + 2, \ldots, d_k + 2)$.

When $d_1 = d_2 = \cdots = d_k = d$, for simplicity we write $f(d, d, \ldots, d)_k$ instead of $f(d_1, d_2, \ldots, d_k)$. This also applies to functions g and h.

The following result provides us the direct consequence of Theorem 1.8.

Proposition 5.3. *If $g(a, 2) \leqslant a + t$ for an integer t and all $a \in \mathbb{N}$, then*

$$f(1, 1, \ldots, 1)_k \leqslant g(2, 2, \ldots, 2)_k \leqslant tk - t + 2.$$

Proof. According to Proposition 5.2, we only need to prove that $g(2, 2, \ldots, 2)_k \leqslant tk - t + 2$. For this we use induction on k. Since $g(a, 2) \leqslant a + t$ for all $a \in \mathbb{N}$. We have $g(2, 2) \leqslant t + 2$. Assume that $g(2, 2, \ldots, 2)_{k-1} \leqslant (k - 2)t + 2$. and let $x = g(2, 2, \ldots, 2)_{k-1}$. Then

$$g(2, 2, \ldots, 2)_k \leqslant g(x, 2) \leqslant x + t \leqslant (k - 1)t + 2 = tk - t + 2.$$

So $g(2, 2, \ldots, 2)_k \leqslant tk - t + 2$ for all $k \geqslant 2$.

The proof is complete. □

Remark. Bermond and Thomassen [4] conjectured that $f(1, 1, \ldots, 1)_k = 2k - 1$ (the conjecture is proposed for simple directed graphs and it is sufficient to prove it in oriented graphs). In 1997, Alon [1] showed that $f(1, 1, \ldots, 1)_k \leqslant 64k$. Recently, this bound has been improved to $24k$ by Bucić [5]. In view of Proposition 5.3, we believe that considering $(a, 2)$-feasible partitions in edge-coloured graphs could be a reasonable approach for improving Bucić's result concerning Bermond and Thomassen's conjecture in digraphs.

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