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ARTICLE

# Behaviour of the minimum degree throughout the *d*-process*

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**Abstract**

The $d$-process generates a graph at random by starting with an empty graph with $n$ vertices, then adding edges one at a time uniformly at random among all pairs of vertices which have degrees at most $d-1$ and are not mutually joined. We show that, in the evolution of a random graph with $n$ vertices under the $d$-process with $d$ fixed, with high probability, for each $j \in \{0, 1, \ldots, d-2\}$, the minimum degree jumps from $j$ to $j+1$ when the number of steps left is on the order of $\ln(n)^{d-j-1}$. This answers a question of Ruciński and Wormald. More specifically, we show that, when the last vertex of degree $j$ disappears, the number of steps left divided by $\ln(n)^{d-j-1}$ converges in distribution to the exponential random variable of mean $\frac{j!}{2(d-1)!}$; furthermore, these $d-1$ distributions are independent.

**Keywords:** Random graph; d-process

**2020 MSC Codes:** Primary: 05C80

## 1. Introduction

There are numerous models that generate different types of sparse random graphs. Among them is the *d-process*, defined in the following way: start with $n$ vertices and 0 edges, and at each time step, choose a pair $\{u, v\}$ uniformly at random over all pairs consisting of vertices which have degree less than $d$ and are not joined to each other by an edge. $d$ could be allowed to change with $n$, but for the rest of this paper $d$ is always a fixed constant (this is also assumed in all relevant citations). Ruciński and Wormald showed that with high probability, abbreviated "w.h.p." (i.e. with probability converging to 1 as $n \to \infty$) the $d$-process ends with $\lfloor dn/2 \rfloor$ edges [11]. There is still much that is unknown about the $d$-process; for example, it is not known whether the $d$-process is *contiguous* with the $d$-uniform random graph model for any $d \geq 2$; i.e. if any event that happens with high probability in one happens with high probability in the other. A recent paper by Molloy, Surya, and Warnke [8] disproves this relation if there is enough "non-uniformity" of the vertex degrees (with an appropriate modification of the $d$-process for non-regular graphs); it also contains a good history of the $d$-process. See ref. [7, Section 9.6] for more on contiguity.

A couple of notable results have been given for the case where $d = 2$: the expected numbers of cycles of constant sizes were studied by Ruciński and Wormald in ref. [10], and in ref. [13], Telcs, Wormald, and Zhou calculated the probability that the 2-process ends with a Hamiltonian cycle.

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In these works, the authors establish estimates on certain graph parameters, such as the number of vertices of a certain degree, that hold throughout the process. This is done with the so-called "differential equations method" for random graph processes, which uses martingale inequalities to give variable bounds; in ref. [14] Wormald gives a thorough description of this method.

More recently, Ruciński and Wormald announced a new analysis of the $d$-process that hinges on a coupling with a balls-in-bins process. This simple argument gives a precise estimate of the probability that the $d$-process ends with $\lfloor dn/2 \rfloor$ edges (i.e. the probability that the $d$-process reaches saturation). This argument includes estimaes for the number of vertices of each degree near the end of the process. This work was presented by Ruciński at the 2023 *Random Structures and Algorithms* conference. Ruciński's presentation included the following problem (which was open at the time): when do we expect the last vertex of degree $j$ (for any $j$ from 0 to $d-2$) to disappear? The question was also stated earlier for $d = 2$ and $j = 0$ by Ruciński and Wormald [10, Question 3]. In November of 2023, after the first release of the pre-print of this paper, Ruciński and Wormald released a pre-print of their balls-and-bins argument which also included an answer to Ruciński's question [12]. Our main result uses the differential equations method (as described in the previous paragraph) and gives a slightly stronger answer:

**Theorem 1.** *Consider the $d$-process on a vertex set of size $n$, and for each $\ell \in \{0\} \cup [d-2]$, let the random variable $T_\ell$ be the step at which the number of vertices of degree at most $\ell$ becomes 0. Then the sequence (over $n$) of random $d-1$-tuples consisting of the variables*

$$V_n^{(\ell)} = \frac{(d-1)!(dn - 2T_\ell)}{\ell!(\ln(n))^{d-1-\ell}}$$

*converges in distribution to the product of $d-1$ independent exponential random variables of mean 1.*

In this paper we use the differential equations method with increasingly precise estimates of certain random variables; these estimates are known as *self-correcting*. Previous results that use self-correcting estimates include [13], [6], [3], [4], [5], and [9]. There have been various approaches to achieving self-correcting estimates; the approach in this paper uses *critical intervals*, regions of possible values of a random variable in which we expect subsequent variables to increase/decrease over time. Critical intervals used in this fashion first appeared in a result of Bohman and Picollelli [6]. For an introduction to and discussion of the method see Bohman, Frieze, and Lubetzky [3].

The proof of Theorem 1 is divided into four sections. In Section 2, we introduce random variables of the form $S_i^{(j)}$ which count the number of vertices of degree at most $j$ after $i$ steps, define approximating functions $s_j(t)$ with the eventual goal of showing that $S_i^{(j)} \approx ns_j(i/n)$ for most of the process, and derive useful properties of these functions. One such property is that, when there are at most $n^c$ steps left for some constant $c < 1$,

$$\frac{s_j(i/n)}{s_{j-1}(i/n)} = \Theta(\ln(n))$$

for each $j$; this hierarchy of functions helps us to focus on each variable $S_i^{(j)}$ independently of the others when it is near 0, which motivates the form of the limiting exponential random variables in Theorem 1. At the end of Section 2 we introduce two martingale inequalities used by Bohman [2] and make a slight modification to them to use later in the paper. In Section 3, we work with a more 'standard martingale method' (without the use of critical intervals) to show that $S_i^{(j)} \approx ns_j(i/n)$ until there are $n^{1-1/(100d)}$ steps left. Here we allow the error bounds to increase over time. In Section 4, we use a more refined martingale method (including the use of critical intervals) to show that $S_i^{(j)} \approx ns_j(i/n)$ continues to hold until there are $\ln(n)^{d-0.8-j}$ steps left; here the error bounds

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*decrease* over time, and so are self-correcting. In Section 5, we complete the proof of Theorem 1 by using a pairing argument to show that, in the last steps of the $d$-process, the behaviour of the random variable in question can be well-approximated by a certain uniform distribution of time steps. Sections 4 and 5 are both parts of a proof by induction over a series of intervals of time steps, though we give each part its own section as the methods used in each are very different.

## 2. Preliminaries

First, two technical notes: we use the standard notation of symbols $o, O, \Theta, \omega, \Omega, \ll, \gg$, and $\sim$ to compare functions asymptotically (e.g. see pages 9-10 of [7]). We also note that, throughout the paper, we assume $n$ to be arbitrarily large.

In this section we set up sequences of random variables, describe how the evolution of the $d$-process depends on these, and deduce properties of certain *approximating functions*; such functions are used to estimate the number of vertices of given degrees throughout the process (much of this is also described in ref. [13] with similar notation; the one major difference is that we use $i$ instead of $t$ for the number of time steps, and $t$ instead of $x$ for the corresponding time variable). Consider a sequence of graphs $G_0, G_1, \dots, G_{\lfloor dn/2 \rfloor}$, where $G_0$ is the empty graph of $n$ vertices, and for $i \in [n]$, let $G_i$ be formed by adding an edge uniformly at random to $G_{i-1}$ so that the maximum degree of $G_i$ is at most $d$ (in the unlikely event that there are no valid edges to add after $s$ steps for some $s < \lfloor dn/2 \rfloor$, let $G_i = G_s$ for all $i > s$). Next, we define several sequences of random variables: For all $i, j, j_1, j_2$ such that $0 \le i \le \lfloor \frac{dn}{2} \rfloor$, $0 \le j \le d$, and $0 \le j_1 \le j_2 \le d-1$, define:

$Y_i^{(j)} :=$ the number of vertices in $G_i$ with degree $j$

$S_i^{(j)} :=$ the number of vertices in $G_i$ with degree *at most* $j$

$Z_i^{(j_1,j_2)} :=$ the number of edges $\{v_1, v_2\}$ in $G_i$ for which

$$\min\{\deg(v_1), \deg(v_2)\} = j_1 \quad \text{and} \quad \max\{\deg(v_1), \deg(v_2)\} = j_2$$

$$Z_i := \sum_{0 \le j_1 \le j_2 \le d-1} Z_i^{(j_1,j_2)}.$$

By definition and by edge-counting we have

$$\sum_{j=0}^{d} Y_i^{(j)} = n \qquad \text{and} \qquad \sum_{j=1}^{d} jY_i^{(j)} = 2i.$$

Combining the equations above gives us

$$\sum_{j=0}^{d-1} S_i^{(j)} = \sum_{j=0}^{d-1} (d-j)Y_i^{(j)} = dn - 2i. \tag{1}$$

One can also quickly verify from (1) and by definition of $S_i^{(d-1)}$, $Z_i^{(j_1,j_2)}$, and $Z_i$, that

$$dS_i^{(d-1)} \ge \max\{2Z_i, dn - 2i\} \qquad \text{and} \qquad jY_i^{(j)} \ge \sum_{k \le j} Z_i^{(k,j)} + \sum_{k \ge j} Z_i^{(j,k)}. \tag{2}$$

Throughout the process we will keep track of the variables $S^{(j)}$ using martingale arguments. This is sufficient for us, as the $Y$ variables can be derived from the $S$ variables, and because none of the $Z$ variables will have any significant effect in any of our calculations, as we will see later.

Our next step is to estimate the expected on-step change of $S_i^{(j)}$, known as the "trend hypothesis" in ref. [14]. Note that $S_i^{(j)} - S_{i+1}^{(j)}$ equals the number of vertices of degree $j$ that are picked at the $i+1$ time step; hence, for all $j \in \{0\} \cup [d-1]$:

$$\begin{aligned}\mathbb{E}\left[S_{i+1}^{(j)} - S_i^{(j)} \mid G_i\right] &= \frac{-Y_i^{(j)}\left(S_i^{(d-1)}-1\right) + \sum_{k\le j} Z_i^{(k,j)} + \sum_{k\ge j} Z_i^{(j,k)}}{\binom{S_i^{(d-1)}}{2} - Z_i} \\ &= \frac{-2Y_i^{(j)}}{S_i^{(d-1)}}\left(1+O\left(\frac{1}{dn-2i}\right)\right) \qquad \text{by (2)} \qquad (3) \\ &= \frac{-2Y_i^{(j)}}{S_i^{(d-1)}} + O\left(\frac{1}{dn-2i}\right). \qquad (4)\end{aligned}$$

For $j \in \{0\} \cup [d-1]$ we define approximating functions $y_j{:}[0, d/2) \to \mathbb{R}$ and $s_j{:}[0, d/2) \to \mathbb{R}$: let $y_j(t)$, $s_j(t)$ be functions such that $s_j = \sum_{k=0}^{j} y_k$, $y_0(0) = 1$ and $y_k(0) = 0$ for all $k \in [d-1]$ (equivalent to $s_j(0) = 1$ for all $j$), and (assuming the "dummy functions" $y_{-1}(t) = s_{-1}(t) = 0$):

$$\frac{ds_j}{dt} = \frac{-2y_j}{s_{d-1}} = \frac{2(s_{j-1}-s_j)}{s_{d-1}} \qquad \frac{dy_j}{dt} = \frac{2(y_{j-1}-y_j)}{s_{d-1}}. \qquad (5)$$

By (5) and the chain rule, for all $j \in [d-1]$:

$$\frac{dy_j}{dy_0} - \frac{y_j}{y_0} = -\frac{y_{j-1}}{y_0}.$$

Since the above equation is first-order linear, we have, for some constant $C_j$:

$$y_j = y_0\left(-\int \frac{y_{j-1}}{y_0^2}dy_0 + C_j\right).$$

Using the above recursively with initial conditions, we have, for all $j \in [d-1]$:

$$y_j = \frac{y_0(-\ln(y_0))^j}{j!}. \qquad (6)$$

To solve for an explicit formula relating $y_0$ and $t$, note that, by (5):

$$\frac{d}{dt}\left(\sum_{j=0}^{d-1}(d-j)y_j\right) = \frac{-2\sum_{j=0}^{d-1} y_j}{s_{d-1}} = -2,$$

hence, using initial conditions (note the resemblance to (1)):

$$\sum_{j=0}^{d-1} s_j = \sum_{j=0}^{d-1}(d-j)y_j = d-2t. \qquad (7)$$

Equations (6) and (7) together give us a complete description of the functions $y_j$ and $s_j$. We will now prove some useful properties of these functions. To start, we can combine (6) and (7) to get

$$\sum_{j=0}^{d-1} \frac{y_0(-\ln(y_0))^j(d-j)}{j!} = d-2t. \qquad (8)$$

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Note that, by continuity of $y_0$ and by (8), $y_0 > 0$ over its domain. Next, by summing up (6) over $j \in \{0\} \cup [d-1]$ ((6) holds for $j = 0$ also) one can see that $s_{d-1}$ is positive if $y_0 \leq 1$. This, combined with $\frac{dy_0}{dt} = \frac{-2y_0}{s_{d-1}}$, tells us that $y_0$ is decreasing and $s_{d-1}$ is positive. It follows from $y_0 \in (0, 1]$ and (6) that each $y_j$ is positive for $t \neq 0$. In turn, this implies that $0 \leq y_j \leq s_j \leq s_{d-1}$ for each $j$. From this it follows that $ds_{d-1}$ is at least the left expression of (7), so $s_{d-1} \geq 1 - \frac{2t}{d}$. We make a special note of the last couple of properties mentioned:

$$0 \leq y_j \leq s_j \leq s_{d-1} \text{ for all } j \quad \text{and} \quad s_{d-1}(i/n) \geq 1 - \frac{2i}{dn}. \tag{9}$$

Next, we want to understand the behaviour of each function when $t$ is close to $\frac{d}{2}$, as this is the most critical point of the process. Consider (8) again. As $t \to \frac{d}{2}$, $y_0 \to 0$, so $\frac{y_0(-\ln(y_0))^{d-1}}{(d-1)!}$ will be the most dominant term on the left; hence,

$$t \to \frac{d}{2} \implies y_0 \sim \frac{(d-1)!(d-2t)}{(-\ln(d-2t))^{d-1}}.$$

This, combined with (6) gives us, for all $j \in \{0\} \cup [d-1]$:

$$t \to \frac{d}{2} \implies y_j(t) \sim s_j(t) \sim \frac{(d-1)!(d-2t)}{j!(-\ln(d-2t))^{d-1-j}}. \tag{10}$$

For large enough $t$ (and hence, for a large enough step $i$), we can approximate the above expression:

$$i \geq \frac{dn}{2} - n^{1-1/(100d)} \implies ny_j(i/n) \sim ns_j(i/n) = \Theta\left(\ln(n)^{-d+1+j}\left(\frac{dn}{2} - i\right)\right). \tag{11}$$

One can now see that, near the end of the process, $s_j/s_{j-1} = \Theta(\ln(n))$, as mentioned in the introduction.

Finally, we introduce two martingale inequalities from a result of Bohman [2] which will be used in Section 4 in a slightly modified form. The original inequalities are as follows:

**Lemma 2** (Lemma 6 from [2]). *Suppose $a, \eta$, and $N$ are positive, $\eta \leq N/2$, and $a < \eta m$. If $0 = A_0, A_1, \ldots, A_m$ is a submartingale such that $-\eta \leq A_{i+1} - A_i \leq N$ for all $i$, then*

$$\mathbb{P}[A_m \leq -a] \leq e^{-\frac{a^2}{3\eta N m}}.$$

**Lemma 3** (Lemma 7 from [2]). *Suppose $a, \eta$, and $N$ are positive, $\eta \leq N/10$, and $a < \eta m$. If $0 = A_0, A_1, \ldots, A_m$ is a supermartingale such that $-\eta \leq A_{i+1} - A_i \leq N$ for all $i$, then*

$$\mathbb{P}[A_m \geq a] \leq e^{-\frac{a^2}{3\eta N m}}.$$

We present the following modification, which removes the requirement $a < \eta m$ and modifies one of the inequalities slightly:

**Corollary 4.** *Suppose $a, \eta$, and $N$ are positive, and $\eta \leq N/2$. If $0 = A_0, A_1, \ldots, A_m$ is a submartingale such that $-\eta \leq A_{i+1} - A_i \leq N$ for all $i$, then*

$$\mathbb{P}[A_m \leq -a] \leq e^{-\frac{a^2}{3\eta N m}}.$$

**Corollary 5.** *Suppose $a, \eta$, and $N$ are positive, and $\eta \leq N/10$. If $0 = A_0, A_1, \ldots, A_m$ is a supermartingale such that $-\eta \leq A_{i+1} - A_i \leq N$ for all $i$, then*

$$\mathbb{P}[A_m \geq a] \leq e^{-\frac{a^2}{3\eta N m}} + e^{-\frac{a}{6N}}.$$

Corollary 4 is nearly immediate from Lemma 2: first, one can extend the result to include $a = \eta m$ by using left-continuity (with respect to $a$) of both sides of the inequality; we hence assume $a > \eta m$. Since $A_{i+1} - A_i \geq -\eta > -a/m$, then $A_m = A_m - A_0 > -a$. We now derive Corollary 5 from Lemma 3: assume $a \geq \eta m$, and let $m' \in \mathbb{Z}^+$ such that $a < \eta m' \leq 2a$. Extend the martingale by adding variables $A_{m+1}, \ldots, A_{m'}$ which are all equal to $A_m$. Apply Lemma 3 with $m$ replaced with $m'$, and use $\eta m' \leq 2a$ to get

$$\mathbb{P}[A_m \geq a] = \mathbb{P}[A_{m'} \geq a] \leq e^{-\frac{a}{6N}}.$$

Combining the case $a < \eta m$ from Lemma 3 and the case $a \geq \eta m$ above gives Corollary 5.

## 3. First phase

Let $i_{trans} = \lfloor \frac{dn}{2} - n^{1-1/(100d)} \rfloor$. The objective of this section is to prove the following Theorem:

**Theorem 6.** *Define $E_{first}(i) := n^{0.6} \left(\frac{dn}{dn-2i}\right)^{4d}$. With high probability, for all $i \leq i_{trans}$ and all $j \in \{0\} \cup [d-1]$:*

$$\left| S_i^{(j)} - ns_j\left(\frac{i}{n}\right) \right| \leq E_{first}(i). \tag{12}$$

Now we define two new random variables for each $j$:

$$\begin{aligned} S_i^{(j)+} &:= S_i^{(j)} - ns_j(i/n) - E_{first}(i) \\ S_i^{(j)-} &:= S_i^{(j)} - ns_j(i/n) + E_{first}(i). \end{aligned}$$

Next, we introduce a stopping time $T$, defined as the first step $i \leq i_{trans}$ for which (12) is *not* satisfied for some $j$; if (12) always holds, then let $T = \infty$. Although this stopping time is not necessarily needed to prove Theorem 6, it does make some calculations easier, and moreover, a similar stopping time *will* be necessary for the following section; hence, this serves as a good warm-up. Let variable name $W$ be introduced to equip this stopping time to variable $S$, i.e.

$$W_i^{(j)+} := \begin{cases} S_i^{(j)+}, i < T \\ S_T^{(j)+}, i \geq T \end{cases} \qquad W_i^{(j)-} := \begin{cases} S_i^{(j)-}, i < T \\ S_T^{(j)-}, i \geq T. \end{cases}$$

Note that $W_i^{(j)+}$ corresponds to the upper boundary and $W_i^{(j)-}$ to the lower one in the sense that crossing the corresponding boundary will make the corresponding variable change signs; furthermore, the inequality of Theorem 6 holds if and only if $W_{i_{trans}}^{(j)+} \leq 0$ and $W_{i_{trans}}^{(j)-} \geq 0$ for each $j$. We now state our martingale Lemma:

**Lemma 7.** *Restricted to $i \leq i_{trans}$, for all $j$, $\left(W_i^{(j)-}\right)_i$ is a submartingale and $\left(W_i^{(j)+}\right)_i$ is a supermartingale.*

**Proof.** Here we just prove the first part of the Lemma; the second part follows from nearly identical calculations. Fix some arbitrary $i \leq i_{trans}$; we need to show that

$$\mathbb{E}\left[W_{i+1}^{(j)-} - W_i^{(j)-} \mid G_i\right] \geq 0.$$

Also assume that $T \geq i+1$, else $W_{i+1}^{(j)-} - W_i^{(j)-} = 0$ and we are done; it follows that $W_i^{(j)-} = S_i^{(j)-}$, $W_{i+1}^{(j)-} = S_{i+1}^{(j)-}$, and (12) holds for the fixed $i$. By (4) and (5) and using Taylor's Theorem, we have, for some $\psi \in [i, i+1]$:

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$$\mathbb{E}\left[S_{i+1}^{(j)-} - S_i^{(j)-} \mid G_i\right] = \frac{-2Y_i^{(j)}}{S_i^{(d-1)}} + O\left(\frac{1}{dn-2i}\right) + \frac{2y_j(i/n)}{s_{d-1}(i/n)} - \frac{d^2}{d\mu^2}\left(\frac{ns_j(\mu/n)}{2}\right)\Big|_{\mu=\psi}$$
$$+\left(E_{first}(i+1) - E_{first}(i)\right).$$

We split the above expression $\left(\text{excluding } O\left(\frac{1}{dn-2i}\right)\right)$ into three summands.

1. Here we make use of the fact that $Y_i^{(j)} = S_i^{(j)} - S_i^{(j-1)}$ and $y_j(t) = s_j(t) - s_{j-1}(t)$. We have (putting $s_{j-1}$ and $S_i^{(-1)} = 0$):

$$\begin{aligned}
\frac{-2Y_i^{(j)}}{S_i^{(d-1)}} + \frac{2y_j(i/n)}{s_{d-1}(i/n)} &= \frac{-2S_i^{(j)} + 2ns_j(i/n) + 2S_i^{(j-1)} - 2ns_{j-1}(i/n)}{ns_{d-1}(i/n)} \\
&\quad + 2Y_i^{(j)}\left(\frac{1}{ns_{d-1}(i/n)} - \frac{1}{S_i^{(d-1)}}\right) \\
&\geq \frac{-4E_{first}(i)}{ns_{d-1}(i/n)} + 2Y_i^{(j)}\left(\frac{1}{ns_{d-1}(i/n)} - \frac{1}{S_i^{(d-1)}}\right) \qquad \text{by (12) and } i < T \\
&\geq \frac{-4E_{first}(i)}{ns_{d-1}(i/n)} - \frac{2Y_i^{(j)}E_{first}(i)}{S_i^{(d-1)}(ns_{d-1}(i/n))} \qquad \text{by (12) and } i < T \\
&\geq \frac{-6E_{first}(i)}{ns_{d-1}(i/n)}.
\end{aligned}$$

2. 

$$\begin{aligned}
-\frac{d^2}{d\mu^2}\left(\frac{ns_j(\mu/n)}{2}\right)\Big|_{\mu=\psi} &= \frac{2}{n}\left(\frac{s_{d-1}(\psi/n)(y_{j-1}(\psi/n) - y_j(\psi/n)) + y_j(\psi/n)y_{d-1}(\psi/n)}{(s_{d-1}(\psi/n))^3}\right) \\
&= O\left(\frac{1}{dn-2i}\right) \qquad \text{by (9).}
\end{aligned}$$

3. For some $\phi \in [i, i+1]$:

$$\begin{aligned}
E_{first}(i+1) - E_{first}(i) &= \frac{dE_{first}(\mu)}{d\mu}\Big|_{\mu=\phi} \\
&= 8d^{4d+1}n^{4d+0.6}\left(dn - 2\phi\right)^{-4d-1} \\
&= (1+o(1))\frac{8dE_{first}(i)}{dn-2i}. \qquad (13)
\end{aligned}$$

Now we put the three bounds together:

$$\begin{aligned}
\mathbb{E}\left[Y_{i+1}^- - Y_i^- \mid G_i\right] &\geq \frac{7dE_{first}(i)}{dn-2i} - \frac{6E_{first}(i)}{ns_{d-1}(i/n)} + O\left(\frac{1}{dn-2i}\right) \\
&\geq \frac{dE_{first}(i) + O(1)}{dn-2i} \qquad \text{by (9)} \\
&\geq 0.
\end{aligned}$$

□

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Next, we need a Lipschitz condition on each of our variables. Note that $S^{(j)}_{i+1} - S^{(j)}_i$ is either $-2, -1$, or 0; also, one can quickly verify that $|s_j((i+1)/n) - s_j(i/n)| \le \frac{2}{n}$ by (5) and (9), and $\left|E_{first}(i+1) - E_{first}(i)\right| = o(1)$ by (13). Hence, we have, for all $i \le i_{trans}$ and all $j$:

$$\max\left\{\left|W^{(j)+}_{i+1} - W^{(j)+}_i\right|, \left|W^{(j)-}_{i+1} - W^{(j)-}_i\right|\right\} \le 5. \tag{14}$$

We conclude the proof of Theorem 6 by noting that, by Lemma 7 and (14), we can use the standard Hoeffding-Azuma inequality for martingales (e.g. Theorem 7.2.1 in [1]) to show that $\mathbb{P}\left[W^{(j)+}_{i_{trans}} > 0\right]$ and $\mathbb{P}\left[W^{(j)-}_{i_{trans}} < 0\right]$ are both $o(1)$. For example, for the variable $W^{(j)+}_i$ one would get

$$\mathbb{P}\left[W^{(j)+}_{i_{trans}} > 0\right] \le \exp\left\{-\frac{n^{1.2}}{50 i_{trans}}\right\} = o(1).$$

## 4. Second phase

The second phase is where the more sophisticated tools will be used, including the use of critical intervals, self-correcting estimates, and a more general martingale inequality. Furthermore, this phase is broken up into $d-1$ sub-phases, in relation to when each of the $d-1$ sequences $S^{(j)}$ (for $j \le d-2$) terminate at 0. First, a few definitions: for all $k \in \{0\} \cup [d-2]$, define

$$i_{after}(k) := \left\lfloor \frac{dn}{2} - \ln(n)^{d-1.01-k} \right\rfloor.$$

These step values will govern the endpoints of the sub-phases: define for all $k \in \{0\} \cup [d-2]$:

$$I_k := \begin{cases} \left[i_{trans} + 1, i_{after}(0)\right], & k = 0 \\ \left[i_{after}(k-1) + 1, i_{after}(k)\right], & k > 0. \end{cases}$$

Next, for all $i, j, k$ such that $0 \le j < d$, $0 \le k < d-1$, and $i \in I_k$, define error functions

$$E_{j,k}(i) = E_j(i) := 2^k \ln(n)^{0.05} (n s_j(i/n))^{0.7}.$$

Note that, by (11), we have

$$E_j(i) = \Theta\left(\ln(n)^{-0.7d+0.75+0.7j}\left(\frac{dn}{2} - i\right)^{0.7}\right). \tag{15}$$

Finally, for any $r \in \mathbb{R}_+$ and $\ell \in [d-2]$, define

$$i(r, \ell) = \frac{dn}{2} - \left(\frac{\ell!}{2(d-1)!}\right) r(\ln n)^{d-1-\ell}.$$

The following Theorem will be proved by induction over the $d-1$ sub-phases governed by the index $k$:

**Theorem 8.** *For each $k \in \{0\} \cup [d-2]$:*

1. *With high probability, for all integers $j \in [0, d-1]$ and $i \in I_k$:*

$$\left|S^{(j)}_i - n s_j\left(\frac{i}{n}\right)\right| \le 4E_j(i). \tag{16}$$

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2. $S^{(k)}_{i_{after}(k)} = 0$ *with high probability. Furthermore, for any* $k+1$*-tuple* $\{r_0, r_1, \ldots, r_k\} \in (\mathbb{R}_+ \cup \{0\})^{k+1}$*:*

$$\mathbb{P}\left(\bigcap_{\ell=0}^{k}\left(S^{(\ell)}_{\lfloor i(r_\ell,\ell)\rfloor}=0\right)\right) \to \exp\left\{-\sum_{\ell=0}^{k} r_\ell\right\}.$$

In the end, it is only the second statement with $k = d-2$ that matters for proving Theorem 1. We make the connection here:

**Proof of Theorem** 1 from Theorem 8. First, note that $S^{(\ell)}_{\lfloor i(r_\ell,\ell)\rfloor} = 0$ is the same as $T_\ell \le i(r_\ell, \ell)$, hence by Theorem 8:

$$\mathbb{P}\left(\bigcap_{\ell=0}^{d-2}(T_\ell \le i(r_\ell,\ell))\right) \to \exp\left\{-\sum_{\ell=0}^{d-2} r_\ell\right\}.$$

Using the Principle of Inclusion-Exclusion plus a simple limiting argument, one can derive

$$\mathbb{P}\left(\bigcap_{\ell=0}^{d-2}\left(\frac{(d-1)!(dn-2T_\ell)}{\ell!(\ln(n))^{d-1-\ell}} \le r_\ell\right)\right) = \mathbb{P}\left(\bigcap_{\ell=0}^{d-2}(T_\ell \ge i(r_\ell,\ell))\right) \to \prod_{\ell=0}^{d-2}\left(1-e^{-r_\ell}\right),$$

hence the $d-1$-dimensional random vector with entries $V_n^{(\ell)} = \frac{(d-1)!(dn-2T_\ell)}{\ell!(\ln(n))^{d-1-\ell}}$ converges in distribution to the product of $d-1$ independent exponential variables of mean 1. □

The rest of this section is for proving the first statement of Theorem 8 (for some fixed $k$ using induction), and Section 5 will be for proving the second statement (again, for some fixed $k$ using induction, assuming the first statement holds for the same $k$). Hence, for the rest of the paper we will fix some $k \in \{0\} \cup [d-2]$.

First, we note that (16) holds w.h.p. for all $j < k$ by a simple argument: by induction on the second statement of Theorem 8, w.h.p. if $i \in I_k$ then $S_i^{(j)} = 0$. By (11) and by definition of $E_j(i)$, if $i \in I_k$ then $ns_j(i/n) \ll E_j(i)$, completing the argument.

Next, we prove that (16) holds for $j = d-1$ if it holds for all other values of $j$: by combining (1) and (7), we have

$$\begin{aligned}
\left|S_i^{(d-1)} - ns_{d-1}\left(\frac{i}{n}\right)\right| &= \left|\sum_{j=0}^{d-2}\left(S_i^{(j)} - ns_j\left(\frac{i}{n}\right)\right)\right| \\
&\le \sum_{j=0}^{d-2} 4E_j(i) \qquad \text{by (16) for } j \le d-2 \\
&< 4E_{d-1}(i) \qquad \text{by (15).}
\end{aligned}$$

Hence, for the rest of this section, we need to show the first statement of Theorem 8 for $j \in [k, d-2]$. From now on we always assume $j$ to be in this range. We will *also* assume that, for all $\lambda < k$, $S_i^{(\lambda)} = 0$ if $i \in I_k$ (which holds w.h.p. from above).

In this section we will make use of so-called *critical intervals*, ranges of possible values for $S_i^{(j)}$ in which we apply a martingale argument. The lower critical interval will be

$$[ns_j(i/n) - 4E_j(i), ns_j(i/n) - 3E_j(i)],$$

and the upper critical interval will be

$$[ns_j(i/n) + 3E_j(i), ns_j(i/n) + 4E_j(i)].$$

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Our goal is to show that w.h.p. $S_i^{(j)}$ does not cross either critical interval; however, we first need to show that $S_i^{(j)}$ sits between the critical intervals at the beginning of the phase (this is the reason why $E_j(i)$ has the $2^k$ factor; it makes a sudden jump in size between phases to accommodate a new martingale process), which is the statement of our first Lemma of this section:

**Lemma 9.** *W.h.p., for all $j \in [k, d-2]$ (putting $i_{after}(-1) = i_{trans}$ for convenience of notation):*

$$\left| S^{(j)}_{i_{after}(k-1)+1} - ns_j\left(\frac{i_{after}(k-1)+1}{n}\right)\right| < 3E_j(i_{after}(k-1)+1).$$

**Proof.** First, recall that $S^{(j)}_{i+1} - S^{(j)}_i \in \{-2, -1, 0\}$ and $|ns_j((i+1)/n) - ns_j(i/n)| \leq 2$ for any $i$ and $j$ (see paragraph above (14)). Second, consider the change in the bound itself between $i_{after}(k-1)$ and $i_{after}(k-1)+1$: by definitions of $i_{trans}, E_{first}, E_j$, and by (15), we have $1 \ll E_{first}(i_{trans}) = \Theta\left(n^{0.64}\right), E_j(i_{trans}+1) = \omega\left(n^{0.69}\right)$, and $1 \ll E_j(i_{after}(k-1)) \approx \frac{1}{2}(E_j(i_{after}(k-1)+1)$ for $k > 0$. Hence, by induction on the first statement of Theorem 8 and by Theorem 6, the statement of the Lemma follows. $\square$

Next, like in Section 3, we define two new random variables for each $j$ and $i \in I_k$:

$$\begin{aligned} S_i^{(j)+} &:= S_i^{(j)} - ns_j(i/n) - 4E_j(i) \\ S_i^{(j)-} &:= S_i^{(j)} - ns_j(i/n) + 4E_j(i). \end{aligned}$$

We also re-introduce the stopping time $T$, now defined as the first step $i \in I_k$ for which (16) is *not* satisfied for some $j$; if (16) always holds, then let $T = \infty$. Let variable name $W$ be introduced to equip this stopping time to variable $S$, i.e.

$$W_i^{(j)+} := \begin{cases} S_i^{(j)+}, i < T \\ S_T^{(j)+}, i \geq T \end{cases} \qquad W_i^{(j)-} := \begin{cases} S_i^{(j)-}, i < T \\ S_T^{(j)-}, i \geq T. \end{cases}$$

Note that $W_i^{(j)+}$ corresponds to the upper critical interval, and $W_i^{(j)-}$ to the lower one. Furthermore, the inequality of Theorem 8 holds if and only if $W^{(j)+}_{i_{after}(k)} \leq 0$ and $W^{(j)-}_{i_{after}(k)} \geq 0$ for each $j$ (here we must make use of our assumption that $S_i^{(\lambda)} = 0$ for all $\lambda < k$). The next Lemma states that, within their respective critical intervals, they are a supermartingale and submartingale respectively:

**Lemma 10.** *For all $i \in I_k$ and for all $j \in [k, d-2]$, $\mathbb{E}\left[W^{(j)-}_{i+1} - W^{(j)-}_i \mid G_i\right] \geq 0$ whenever $W_i^{(j)-} \leq E_j(i)$, and $\mathbb{E}\left[W^{(j)+}_{i+1} - W^{(j)+}_i \mid G_i\right] \leq 0$ whenever $W_i^{(j)+} \geq -E_j(i)$.*

**Proof.** Here we just prove the first part of the Lemma; the second part follows from nearly identical calculations. By the same logic as in the proof of Lemma 7 we work with $S^{(j)-}$ instead of $W^{(j)-}$ and assume that (16) holds for all $j$. We also have the same expected change as in Lemma 7, except with $E_{first}(i)$ replaced with $4E_j(i)$:

$$\begin{aligned} \mathbb{E}\left[S^{(j)-}_{i+1} - S^{(j)-}_i \mid G_i\right] = \frac{-2Y_i^{(j)}}{S_i^{(d-1)}} + O\left(\frac{1}{dn-2i}\right) + \frac{2y_j(i/n)}{s_{d-1}(i/n)} - \frac{d^2}{d\mu^2}\left(\frac{ns_j(\mu/n)}{2}\right)\Big|_{\mu=\psi} \\ + 4(E_j(i+1) - E_j(i)). \end{aligned}$$

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We split the above expression $\left(\text{excluding } O\left(\frac{1}{dn-2i}\right)\right)$ into three summands, assuming $S_i^{(j)-} \leq E_j(i) \iff S_i^{(j)} - ns_j(i/n) \leq -3E_j(i)$ (for convenience, for the case $j = 0$, we put $S_i^{(j-1)}$, $s_{j-1}$, and $E_{j-1}$ all equal to 0):

1.

$$\begin{aligned}
\frac{-2Y_i^{(j)}}{S_i^{(d-1)}} + \frac{2y_j(i/n)}{s_{d-1}(i/n)} &= \frac{-2S_i^{(j)} + 2ns_j(i/n) + 2S_i^{(j-1)} - 2ns_{j-1}(i/n)}{ns_{d-1}(i/n)} \\
&\quad + 2Y_i^{(j)}\left(\frac{1}{ns_{d-1}(i/n)} - \frac{1}{S_i^{(d-1)}}\right) \\
&\geq \frac{6E_j(i) - 8E_{j-1}(i)}{ns_{d-1}(i/n)} - \frac{8S_i^{(j)}E_{d-1}(i)}{S_i^{(d-1)}(ns_{d-1}(i/n))} && \text{by (16)} \\
&\geq \frac{5.9E_j(i)}{ns_{d-1}(i/n)} - \frac{9S_i^{(j)}E_{d-1}(i)}{(ns_{d-1}(i/n))^2} && \text{by (15), (16), (11), and } i \leq i_{after}(k) \\
&\geq \frac{5.9E_j(i)}{ns_{d-1}(i/n)} - \frac{9(ns_j(i/n))E_{d-1}(i)}{(ns_{d-1}(i/n))^2} - \frac{36E_j(i)E_{d-1}(i)}{(ns_{d-1}(i/n))^2} && \text{by (16)} \\
&= \left(\frac{E_j(i)}{ns_{d-1}(i/n)}\right)\left(5.9 - 9\left(\frac{s_j(i/n)}{s_{d-1}(i/n)}\right)^{0.3} - \frac{36 * 2^k \ln(n)^{0.05}}{(ns_{d-1}(i/n))^{0.3}}\right) \\
&\geq \frac{5.8E_j(i)}{ns_{d-1}(i/n)} && \text{by } i \leq i_{after}(k) \text{ and (11).}
\end{aligned}$$

2. Just as in the proof of Lemma 7:

$$-\frac{d^2}{d\mu^2}\left(\frac{ns_j(\mu/n)}{2}\right)\Big|_{\mu=\psi} = O\left(\frac{1}{dn-2i}\right).$$

3.

$$\begin{aligned}
4(E_j(i+1) - E_j(i)) &= 4\frac{dE_j(\mu)}{d\mu}\Big|_{\mu=\phi} && \text{for some } \phi \in [i, i+1] \\
&= (4)(2^k)\ln(n)^{0.05}\left(\frac{0.7}{(ns_j(\phi/n))^{0.3}}\right)\left(\frac{-2y_j(\phi/n)}{s_{d-1}(\phi/n)}\right) && \text{by (5)} \\
&= (4 + o(1))(2^k)\ln(n)^{0.05}\left(\frac{0.7}{(ns_j(\phi/n))^{0.3}}\right)\left(\frac{-2s_j(\phi/n)}{s_{d-1}(\phi/n)}\right) && \text{by (10)} \\
&= \frac{-(5.6 + o(1))E_j(\phi)}{ns_{d-1}(\phi/n)} = \frac{-(5.6 + o(1))E_j(i)}{ns_{d-1}(i/n)}.
\end{aligned} \tag{17}$$

Now we put the above bounds together (using (11), (15), and $i \leq i_{after}(k) \leq i_{after}(j)$):

$$\begin{aligned}
\mathbb{E}\left[S_{i+1}^{(j)-} - S_i^{(j)-} \mid G_i\right] &\geq \frac{0.01E_j(i)}{ns_{d-1}(i/n)} + O\left(\frac{1}{dn-2i}\right) \\
&\geq 0.
\end{aligned}$$

□

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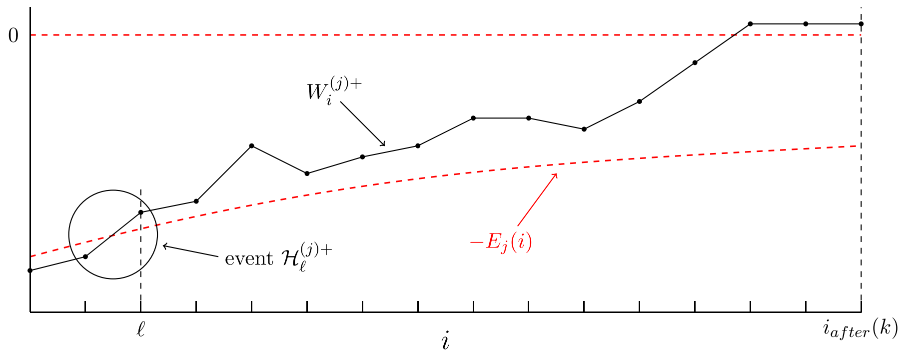


**Figure 1.** Visual representation of event $\mathcal{E}_\ell^{(j)+}$.

We introduce the next Lemma to get sufficiently small bounds on the one-step changes in each time step (this is known as the "bounded hypothesis" from [14]):

**Lemma 11.** *For all $i \in I_k$ and all $j \in [k, d-2]$,*

$$-3 < W_{i+1}^{(j)\xi} - W_i^{(j)\xi} < \ln(n)^{-d+1.06+j}$$

*where "$\xi$" can be either "+" or "−".*

**Proof.** Like in the proofs of Lemma 7 and 10, we assume that $W^\xi = S^{(j)\xi}$ ($\xi$ is + or −), else $W_{i+1}^\xi - W_i^\xi = 0$. Again, we have $-2 \le S_{i+1}^{(j)} - S_i^{(j)} \le 0$. Secondly, we have

$$\begin{aligned}
&|-ns_j((i+1)/n) + ns_j(i/n) - CE_j(i+1) + CE_j(i)| \\
&\le |-ns_j((i+1)/n) + ns_j(i/n)| + |-CE_j(i+1) + CE_j(i)| \\
&= O\left(\frac{y_j(i/n)}{s_{d-1}(i/n)} + \frac{E_j(i)}{ns_{d-1}(i/n)}\right) \qquad \text{by (5), (11), and (17)} \\
&= o\left(\ln(n)^{-d+1.06+j}\right) \qquad \text{by (11), (15), and } i \le i_{after}(k) \le i_{after}(j).
\end{aligned}$$

Combining the inequalities completes the proof. □

To put this all together to prove the first part of Theorem 8, we introduce a series of events: first, let $\mathcal{E}^{(j)+}$ denote the event that $W_{i_{after}(k)}^{(j)+} > 0$ and $\mathcal{E}^{(j)-}$ denote the event that $W_{i_{after}(k)}^{(j)-} < 0$. Let $\mathcal{E} = \left(\bigcup_{j\ge k} \mathcal{E}^{(j)+}\right) \cup \left(\bigcup_{j\ge k} \mathcal{E}^{(j)-}\right)$; we seek to bound $\mathbb{P}[\mathcal{E}]$, since $\mathcal{E}$ is the event that (16) *doesn't* hold for some $i \in I_k$. Next, for all $\ell \in I_k$, let $\mathcal{H}_\ell^{(j)+}$ be the event that $W_{\ell-1}^{(j)+} < -E_j(\ell-1)$ and $W_\ell^{(j)+} \ge -E_j(\ell)$, and let

$$\mathcal{E}_\ell^{(j)+} := \mathcal{H}_\ell^{(j)+} \cap \left\{W_i^{(j)+} \ge -E_j(i) \text{ for all } i \ge \ell\right\} \cap \left\{W_{i_{after}(k)}^{(j)+} > 0\right\}.$$

$\left(\text{see Fig. 1 for a visual representation of event } \mathcal{E}_\ell^{(j)+}\right)$

Similarly, for all $\ell \in I_k$, let $\mathcal{H}_\ell^{(j)-}$ be the event that $W_{\ell-1}^{(j)-} > E_j(\ell-1)$ and $W_\ell^{(j)-} \le E_j(\ell)$, and let

$$\mathcal{E}_\ell^{(j)-} := \mathcal{H}_\ell^{(j)-} \cap \left\{W_i^{(j)-} \le E_j(i) \text{ for all } i \ge \ell\right\} \cap \left\{W_{i_{after}(k)}^{(j)-} < 0\right\}.$$

Finally, note that, by Lemma 9, with high probability we must have

$$W_{i_{after}(k-1)+1}^{(j)+} < -E_j(i_{after}(k-1)+1) \;\text{ and }\; W_{i_{after}(k-1)+1}^{(j)-} > E_j(i_{after}(k-1)+1).$$

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Furthermore, assuming these two inequalities hold (and, once again, assuming that $S_i^\lambda = 0$ if $\lambda < k$), then if $W^{(j)+}_{i_{after}(k)} > 0$ for some $j$, one of the events $\mathcal{E}^{(j)+}_\ell$ must happen; likewise, if $W^{(j)-}_{i_{after}(k)} < 0$ for some $j$, one of the events $\mathcal{E}^{(j)-}_\ell$ must happen; hence, $\mathcal{E}^{(j)+} = \bigcup_\ell \mathcal{E}^{(j)+}_\ell$ and $\mathcal{E}^{(j)-} = \bigcup_\ell \mathcal{E}^{(j)-}_\ell$.

We are now ready to prove the first statement of Theorem 8 in full.

**Proof of the first part of Theorem 8** *with fixed k*. First, we fix an arbitrary $j$ (in $[k, d-2]$). We prove that $\mathbb{P}\big[\mathcal{E}^{(j)-}\big] = \exp\big\{-\Omega\big(\ln(n)^{0.036}\big)\big\}$; the proof for bounding $\mathbb{P}\big[\mathcal{E}^{(j)+}\big]$ is nearly identical. We will use Corollary 5 to bound $\mathbb{P}\big[\mathcal{E}^{(j)-}_\ell\big]$ for each fixed $\ell$. Given a fixed $\ell$, we define a modified stopping time

$$T_{mod} := \min_{i\in[\ell, i_{after}(k)]} \left\{W_i^{(j)-} > E_j(i) \text{ or } i = T\right\}$$

(letting $T_{mod} = \infty$ if the condition doesn't hold for any $i$ in the range). Let variable $W_i^\ell$ be the variable $W_i^{(j)-}$ defined just on $i \in [\ell, i_{after}(k)]$ equipped with this stopping time (we drop the "$(j)-$" here for convenience); i.e.

$$W_i^\ell := \begin{cases} W_i^{(j)-}, i < T_{mod} \\ W_{T_{mod}}^{(j)-}, i \geq T_{mod}. \end{cases}$$

Note that $\big(W_i^\ell\big)_i$ (over $i \in [\ell, i_{after}(k)]$) is a submartingale by Lemma 10, since our new stopping time negates the need for the condition $W_i^{(j)-} \leq E_j(i)$; also, $\big(W_i^\ell\big)_i$ satisfies Lemma 11. Since we want an upper bound for $\mathbb{P}\big[\mathcal{E}^{(j)-}_\ell\big]$, we can condition on event $\mathcal{H}^{(j)-}_\ell$, as $\mathcal{H}^{(j)-}_\ell \supseteq \mathcal{E}^{(j)-}_\ell$. Now let

$$\begin{aligned} A_i &= -W^\ell_{\ell+i} + W^\ell_\ell, \\ \eta &= \ln(n)^{-d+1.06+j}, \\ N &= 3, \\ m &= i_{after}(k) - \ell, \\ a &= 0.9E_j(\ell). \end{aligned}$$

Note that the conditions of Corollary 5 are satisfied: $0 = A_0$ and $\eta < N/10$ are obvious, Lemma 11 gives us $-\eta \leq A_{i+1} - A_i \leq N$, and $(A_i)_i$ is a supermartingale since $\big(W_i^\ell\big)_i$ is a submartingale. We therefore implement Corollary 5, using $m \leq \frac{dn}{2} - \ell \leq dns_{d-1}(\ell/n)$ (by (9)), (11), and (15):

$$\mathbb{P}[A_m \geq a] \leq e^{-\frac{a^2}{3\eta N m}} + e^{-\frac{a}{6N}} = e^{-\Omega\left(\ln(n)^{0.04}(ns_j(\ell/n))^{0.4}\right)} + e^{-\Omega\left(\ln(n)^{0.05}(ns_j(\ell/n))^{0.7}\right)}. \tag{18}$$

To bound $\mathbb{P}\big[\mathcal{E}^{(j)-}_\ell\big]$, we show that $\mathcal{E}^{(j)-}_\ell \subseteq \{A_m \geq a\}$ and apply (18) while conditioning on $\mathcal{H}^{(j)-}_\ell$. Given $\mathcal{H}^{(j)-}_\ell$ happens, we have $W^\ell_\ell = W^{(j)-}_\ell > 0.9E_j(\ell) = a$ by (15), Lemma 11, and $i \leq i_{after}(j)$. Therefore $\mathcal{E}^{(j)-}_\ell = \mathcal{H}^{(j)-}_\ell \cap \left\{W^\ell_{i_{after}(k)} < 0\right\} \subseteq \{A_m \geq a\}$, hence

$$\mathbb{P}\left[\mathcal{E}^{(j)-}_\ell\right] = e^{-\Omega\left(\ln(n)^{0.04}(ns_j(\ell/n))^{0.4}\right)} + e^{-\Omega\left(\ln(n)^{0.05}(ns_j(\ell/n))^{0.7}\right)}.$$

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We now take a union bound to bound $\mathbb{P}[\mathcal{E}^{(j)-}]$ (using (11) where appropriate):

$$
\begin{aligned}
\mathbb{P}[\mathcal{E}^{(j)-}] &\le \sum_{\ell=i_{after}(k-1)+1}^{i_{after}(k)} \mathbb{P}\left[\mathcal{E}_\ell^{(j)-}\right] \\
&= \sum_{\ell=i_{trans}}^{i_{after}(k)} \left(\exp\left\{-\Omega\left(\ln(n)^{0.04}(ns_j(\ell/n))^{0.4}\right)\right\} + \exp\left\{-\Omega\left(\ln(n)^{0.05}(ns_j(\ell/n))^{0.7}\right)\right\}\right) \\
&= \sum_{\ell=i_{trans}}^{i_{after}(j)} \left(\exp\left\{-\Omega\left(\frac{(dn-2\ell)^{0.4}}{\ln(n)^{0.4d-0.44-0.4j}}\right)\right\} + \exp\left\{-\Omega\left(\frac{(dn-2\ell)^{0.7}}{\ln(n)^{0.7d-0.75-0.7j}}\right)\right\}\right) \\
&= \sum_{p=\lfloor \ln(n)^{d-1.01-j}\rfloor}^{\lceil n^{1-1/(100d)}\rceil} \left(\exp\left\{-\Omega\left(\frac{p^{0.4}}{\ln(n)^{0.4d-0.44-0.4j}}\right)\right\} + \exp\left\{-\Omega\left(\frac{p^{0.7}}{\ln(n)^{0.7d-0.75-0.7j}}\right)\right\}\right) \\
&= \ln(n)^{d-1.01-j} \sum_{q=1}^{\infty} \left(\exp\left\{-\Omega\left(q^{0.4}\ln(n)^{0.036}\right)\right\} + \exp\left\{-\Omega\left(q^{0.7}\ln(n)^{0.043}\right)\right\}\right) \\
&= \exp\left\{-\Omega\left(\ln(n)^{0.036}\right)\right\}.
\end{aligned}
$$

We give a note for the aspects of the proof of bounding $\mathbb{P}[\mathcal{E}^{(j)+}]$ that are different from the above: use the variable $W_i^{(j)+}$ instead of $W_i^{(j)-}$, events $\mathcal{E}_\ell^{(j)+}$ instead of $\mathcal{E}_\ell^{(j)-}$, and $\mathcal{H}_\ell^{(j)+}$ instead of $\mathcal{H}_\ell^{(j)-}$. Define $T_{mod}$ instead as

$$T_{mod} := \min_{i\in[\ell, i_{after}(k)]} \left\{W_i^{(j)+} < -E_j(i) \text{ or } i = T\right\}.$$

Finally, use Corollary 4 instead of Corollary 5 (which will be slightly easier to implement). □

## 5. Final phase

We continue our proof by induction of Theorem 8 with our fixed index $k$; now we prove the second part. We assume the first part of Theorem 8 to hold, as well as the second part of the Theorem for lesser $k$; for example, we have $S^{(k-1)}_{i_{after}(k-1)} = 0$ w.h.p. In this section we focus on the $d$-process for a narrow domain of $i$. Let

$$i_{before}(k) := \left\lfloor \frac{dn}{2} - \ln(n)^{d-0.8-k}\right\rfloor.$$

We will consider the $d$-process starting at step $i_{before}(k)$ assuming that (16) holds at $i = i_{before}(k)$; we do not need the first part of Theorem 8 in this section otherwise. We do not use martingale arguments here, but rather we show that the distribution of the sequence of time steps at which a vertex of degree $k$ is chosen from the $d$-process is similar to a uniform distribution over all possible such sequences. Theorem 8, (10), and (15) tell us that w.h.p. we will have $\sim \frac{2(d-1)!}{k!}\ln(n)^{0.2}$ vertices of degree at most $k$ (or degree equal to $k$; they are the same here) left when there are $\lfloor \ln(n)^{d-0.8-k}\rfloor$ steps left; hence, the average distance between steps at which we remove vertices of degree $k$ is $\frac{k!}{2(d-1)!}\ln(n)^{d-1-k}$. When there are this many steps left times $r$, we expect $r$ such vertices to remain, and for the probability that there are no vertices of degree $k$ to be $e^{-r}$. Most of this section will build towards proving the following Theorem:

**Theorem 12.** *Let $L(n)$ be an integer-valued function so that $L(n) = \Theta(\ln(n)^{0.2})$ and let $J(n) = \lfloor \frac{dn}{2} \rfloor - i_{before}(k) \sim \ln(n)^{d-0.8-k}$. Let $H$ be any graph with $i_{before}(k)$ edges which satisfies (16) at $i = i_{before}(k)$, has no vertices of degree at most $k-1$, and has $L(n)$ vertices of degree $k$. Also, let $r \in \mathbb{R}^+$ be arbitrary. Then*

$$\mathbb{P}\left[ S^{(k)}_{\lfloor \frac{dn}{2} - \frac{rJ(n)}{L(n)} \rfloor} = 0 \,\middle|\, G_{i_{before}(k)} = H \right] \to e^{-r}.$$

First, we note that, given that (16) holds for $i = i_{before}(k)$ and by (1), that w.h.p. $dn - 2i_{before}(k) - S^{(d-1)}_{i_{before}(k)} = O\left(\frac{dn - 2i_{before}}{\ln(n)}\right)$ $\left(\text{consider } S^{(d-2)}_{i_{before}(k)}\right)$; hence, for all $i \in \left[i_{before}(k), i_{after}(k)\right]$:

$$S^{(d-1)}_i = dn - 2i + O\left(\frac{dn - 2i_{before}}{\ln(n)}\right) = (dn - 2i)\left(1 + O\left(\frac{1}{\ln(n)^{0.79}}\right)\right). \tag{19}$$

Let $t_{start} = i_{before}(k)$ and $t_{end} = \lfloor dn/2 - rJ(n)/L(n) \rfloor$. Consider the $d$-process between $t_{start}$ and $t_{end}$, given that $G_{t_{start}} = H$. At each step two vertices are chosen; now assume that the pair at each step is ordered uniformly at random, so that a sequence of $2(t_{end} - t_{start})$ vertices is generated. We also generate a *binary* sequence simultaneously, each digit corresponding to a vertex: after a pair of vertices is picked for the vertex sequence, for each of the two vertices (in the order that they are randomly shuffled) append a "1" to the binary sequence if the corresponding vertex had degree $k$ just before it was picked, and append a "0" otherwise. Let $\mathcal{P}: \{0, 1\}^{2(t_{end} - t_{start})} \to [0, 1]$ be the corresponding probability function that arises from this process (note that, if $\gamma$ is a string with more than $L(n)$ "1"'s, then $\mathcal{P}(\gamma) = 0$). Note that $\mathcal{P}$ depends on the graph $H$. We compare this to a second probability function $\mathcal{Q}: \{0, 1\}^{2(t_{end} - t_{start})} \to [0, 1]$, which is defined by picking a binary string with $L(n)$ 1's and $2J(n) - L(n)$ 0's uniformly at random, then taking the first $2(t_{end} - t_{start})$ digits.

For any binary sequence $\gamma$ with $\ell$ digits, and $I \subset [\ell]$, let $\gamma_I$ be the subsequence with indices from $I$; for example, $\gamma_{[a]}$ would be the first $a$ digits of $\gamma$, and $\gamma_{\{a\}}$ would just be the $a$-th digit (for notation's sake, let "$\gamma_{[0]}$" be the empty string). Also let $\|\gamma\|$ denote the number of 1's in $\gamma$. We now present the following Lemma:

**Lemma 13.** *Let $\alpha$ be an arbitrary $2(t_{end} - t_{start})$ length binary sequence with at most $L(n)$ 1's, and let $\gamma$ be the random binary sequence according to either $\mathcal{P}$ or $\mathcal{Q}$. Let $i \in [2(t_{end} - t_{start})]$. Then (letting $a_{\{0\}} = 1$ for sake of notation):*

$$\frac{\mathbb{P}_{\mathcal{P}}[\gamma_{[i]} = \alpha_{[i]} \mid \gamma_{[i-1]} = \alpha_{[i-1]}]}{\mathbb{P}_{\mathcal{Q}}[\gamma_{[i]} = \alpha_{[i]} \mid \gamma_{[i-1]} = \alpha_{[i-1]}]} \begin{cases} = 1 + O\left(\frac{1}{J(n)\ln(n)^{0.39}}\right) & \text{if } \alpha_{\{i\}} = 0 \text{ and } \alpha_{\{i-1\}} = 0 \\ = 1 + O\left(\frac{\ln(n)^{0.4}}{J(n)}\right) & \text{if } \alpha_{\{i\}} = 0 \text{ and } \alpha_{\{i-1\}} = 1 \\ = 1 + O\left(\frac{1}{\ln(n)^{0.79}}\right) & \text{if } \alpha_{\{i\}} = 1 \text{ and } \alpha_{\{i-1\}} = 0 \\ \le 1 + O\left(\frac{1}{\ln(n)^{0.79}}\right) & \text{if } \alpha_{\{i\}} = 1 \text{ and } \alpha_{\{i-1\}} = 1. \end{cases}$$

**Proof.** First, we consider the cases where $\alpha_{\{i\}} = 1$. We have

$$\mathbb{P}_{\mathcal{Q}}[\gamma_{\{i\}} = 1 \mid \gamma_{[i-1]} = \alpha_{[i-1]}] = \frac{L(n) - \|\alpha_{[i-1]}\|}{2J(n) - (i-1)}. \tag{20}$$

For the probability space $\mathcal{P}$, we need to consider three subcases: we need to consider whether $i$ is even or odd, and if it is even, whether $\alpha_{\{i-1\}}$ is 0 or 1, since each step of the $d$-process outputs two digits of the binary string. Let's say that $\tau$ corresponds to the last step in the $d$-process before the $i$-th binary digit is generated (recall that pairs of digits are generated together). Then if $i$ is odd:

$$\begin{aligned}\mathbb{P}_{\mathcal{P}}[\gamma_{\{i\}}=1 \mid \gamma_{[i-1]}=\alpha_{[i-1]}] &= -\frac{1}{2}\mathbb{E}\left[S^{(k)}_{\tau+1}-S^{(k)}_{\tau}|G_\tau\right] \\ &= \frac{S^{(k)}_\tau}{S^{(d-1)}_\tau}\left(1+O\left(\frac{1}{dn-2\tau}\right)\right) \qquad \text{by (3)} \\ &= \frac{S^{(k)}_\tau}{2\lfloor dn/2-\tau\rfloor}\left(1+O\left(\frac{1}{\ln(n)^{0.79}}\right)\right) \qquad \text{by (19)} \\ &= \frac{L(n)-\|\alpha_{[i-1]}\|}{2J(n)-(i-1)}\left(1+O\left(\frac{1}{\ln(n)^{0.79}}\right)\right). \end{aligned} \tag{21}$$

If $i$ is even and $\alpha_{\{i-1\}}=1$, then $S^{(k)}_\tau=L(n)-\|\alpha_{[i-1]}\|+1$. At step $\tau$ there are $S^{(k)}_\tau\left(S^{(d-1)}_\tau+O(1)\right)$ ordered pairs of vertices whose first vertex has degree $k$, and *at most* $2\binom{S^{(k)}_\tau}{2}$ ordered pairs of vertices both with degree $k$; hence:

$$\begin{aligned}\mathbb{P}_{\mathcal{P}}[\gamma_{\{i\}}=1 \mid \gamma_{[i-1]}=\alpha_{[i-1]}] &\le \frac{S^{(k)}_\tau-1}{S^{(d-1)}_\tau+O(1)} \\ &= \frac{S^{(k)}_\tau-1}{2\lfloor dn/2-\tau\rfloor}\left(1+O\left(\frac{1}{\ln(n)^{0.79}}\right)\right) \qquad \text{by (19)} \\ &= \frac{L(n)-\|\alpha_{[i-1]}\|}{2J(n)-(i-1)}\left(1+O\left(\frac{1}{\ln(n)^{0.79}}\right)\right). \end{aligned} \tag{22}$$

Hence the final inequality of the Lemma holds by (21) and (22).

Next, consider the case where $i$ is even and $\alpha_{\{i-1\}}=0$; here, $S^{(k)}_\tau=L(n)-\|\alpha_{[i-1]}\|$ once again. At step $\tau$ there are $\left(S^{(d-1)}_\tau-S^{(k)}_\tau\right)\left(S^{(d-1)}_\tau+O(1)\right)$ ordered pairs of vertices whose first vertex has degree greater than $k$, and $S^{(k)}_\tau\left(S^{(d-1)}_\tau-S^{(k)}_\tau+O(1)\right)$ ordered pairs of vertices for which the first vertex has degree greater $k$ and the second vertex has degree $k$ (one can "pick the second vertex first" to see this). Hence:

$$\begin{aligned}\mathbb{P}_{\mathcal{P}}[\gamma_{\{i\}}=1 \mid \gamma_{[i-1]}=\alpha_{[i-1]}] &= \frac{S^{(k)}_\tau}{S^{(d-1)}_\tau}\left(1+O\left(\frac{1}{S^{(d-1)}_\tau}\right)\right) \qquad \text{since } S^{(d-1)}_\tau \gg S^{(k)}_\tau \text{ there} \\ &= \frac{S^{(k)}_\tau}{2\lfloor dn/2-\tau\rfloor}\left(1+O\left(\frac{1}{\ln(n)^{0.79}}\right)\right) \qquad \text{by (19)} \\ &= \frac{L(n)-\|\alpha_{[i-1]}\|}{2J(n)-(i-1)}\left(1+O\left(\frac{1}{\ln(n)^{0.79}}\right)\right), \end{aligned} \tag{23}$$

hence the third equality of the Lemma holds by (21) and (23).

Now consider $\alpha_{\{i\}}=0$. By modifying (20) to accommodate $\gamma_{\{i\}}=0$, we have

$$\mathbb{P}_{\mathcal{Q}}[\gamma_{\{i\}}=0 \mid \gamma_{[i-1]}=\alpha_{[i-1]}]=1-\frac{L(n)-\|\alpha_{[i-1]}\|}{2J(n)-(i-1)}. \tag{24}$$

Similarly, by modifying (21) and (23), if $a_{\{i-1\}}=0$ then

$$\mathbb{P}_{\mathcal{P}}[\gamma_{\{i\}}=0 \mid \gamma_{[i-1]}=\alpha_{[i-1]}]=1-\frac{L(n)-\|\alpha_{[i-1]}\|}{2J(n)-(i-1)}\left(1+O\left(\frac{1}{\ln(n)^{0.79}}\right)\right). \tag{25}$$

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By modifying (21) and (22), if $a_{\{i-1\}} = 1$, then

$$\mathbb{P}_{\mathcal{P}}[\gamma_{\{i\}} = 0 \mid \gamma_{[i-1]} = \alpha_{[i-1]}] \geq 1 - \frac{L(n) - \|\alpha_{[i-1]}\|}{2J(n) - (i-1)} \left(1 + O\left(\frac{1}{\ln (n)^{0.79}}\right)\right)$$
$$= 1 + O\left(\frac{L(n) - \|\alpha_{[i-1]}\|}{2J(n) - (i-1)}\right). \tag{26}$$

Since $L(n) = \Theta(\ln (n)^{0.2})$ and

$$2J(n) - (i-1) = 2\lfloor dn/2 - \tau \rfloor = \Omega(dn/2 - t_{end}) = \Omega(J(n)/\ln (n)^{0.2}),$$

then $\frac{L(n)-\|\alpha_{[i-1]}\|}{2J(n)-(i-1)} = O\left(\frac{\ln (n)^{0.4}}{J(n)}\right)$. Therefore the ratio of (25) and (24) is $1 + O\left(\frac{1}{J(n)\ln (n)^{0.39}}\right)$, verifying the first inequality of the Lemma, and the ratio of (26) and (24) is $1 + O\left(\frac{\ln (n)^{0.4}}{J(n)}\right)$, verifying the second inequality of the Lemma. □

**Proof of Theorem 12.** First, let $\alpha$ be an arbitrary string which satisfies the criteria in Lemma 13. By using the Lemma 13 recursively:

$$\frac{\mathbb{P}_{\mathcal{P}}[\gamma = \alpha]}{\mathbb{P}_{\mathcal{Q}}[\gamma = \alpha]} \leq \exp\left\{O\left(J(n)\frac{1}{J(n)\ln (n)^{0.39}} + L(n)\left(\frac{1}{\ln (n)^{0.79}} + \frac{\ln (n)^{0.4}}{J(n)}\right)\right)\right\}$$
$$= 1 + o(1), \tag{27}$$

and if $\alpha$ is an arbitrary string *with no two consecutive 1's* which satisfies the criteria in Lemma 13, then by similar logic,

$$\frac{\mathbb{P}_{\mathcal{P}}[\gamma = \alpha]}{\mathbb{P}_{\mathcal{Q}}[\gamma = \alpha]} = 1 + o(1). \tag{28}$$

Let $\mathcal{C}$ be the event that $\gamma$ has two consecutive 1's; we consider $\mathbb{P}[\mathcal{C} \mid G_{t_{start}} = H]$. We consider probability space $\mathcal{Q}$ first. Recall that $\alpha$ is a string that has $\sim 2J(n) = \Omega(\ln (n)^{1.2})$ characters and at most $L(n) = \Theta(\ln (n)^{0.2})$ 1's. Because $\mathcal{Q}$ is a truncation of a uniform distribution, the probability of having two consecutive 1's will be $O\left(\frac{(L(n))^2}{J(n)}\right) = O(\ln (n)^{-0.8})$. Hence, by (27) we must have

$$\mathbb{P}_{\mathcal{P}}[\mathcal{C} \mid G_{t_{start}} = H] = o(1) \text{ and } \mathbb{P}_{\mathcal{Q}}[\mathcal{C} \mid G_{t_{start}} = H] = o(1). \tag{29}$$

We now combine (28) and (29) to prove Theorem 12 (for ease of notation, assume we are given $G_{t_{start}} = H$):

$$\mathbb{P}\left[S^{(k)}_{\lfloor \frac{dn}{2} - \frac{rJ(n)}{L(n)} \rfloor} = 0\right] = \mathbb{P}_{\mathcal{P}}[\|\gamma\| = L(n)]$$
$$= \mathbb{P}_{\mathcal{P}}[\|\gamma\| = L(n) \mid \mathcal{C}]\mathbb{P}_{\mathcal{P}}[\mathcal{C}] + \mathbb{P}_{\mathcal{P}}[\|\gamma\| = L(n) \mid \overline{\mathcal{C}}]\mathbb{P}_{\mathcal{P}}[\overline{\mathcal{C}}]$$
$$= \mathbb{P}_{\mathcal{Q}}[||\gamma|| = L(n)] + o(1) \qquad \text{by (28) and (29)}$$
$$= \frac{\binom{2(t_{end}-t_{start})}{L(n)}}{\binom{2J(n)}{L(n)}} + o(1)$$
$$= \frac{\binom{2J(n)-2(\lfloor rJ(n)/L(n)\rfloor)}{L(n)}}{\binom{2J(n)}{L(n)}} + o(1)$$
$$= \left(1 - \frac{r(1+o(1))}{L(n)}\right)^{L(n)} + o(1)$$
$$= e^{-r} + o(1).$$

□

We can now complete the proof of the second statement of Theorem 8 at value $k$. Roughly speaking, we will use Theorem 12 with $L(n) \approx \frac{2(d-1)!\ln(n)^{0.2}}{k!}$, so $\frac{dn}{2} - i(r_k, k) \approx \frac{r_k J(n)}{L(n)}$. First, note that $S^{(k)}_{i_{after}(k)} = 0$ (w.h.p.) comes automatically when the rest of the statement is proved (by putting $r_\ell = 0$ for $\ell < k$ and having $r_k \to 0$). Let $\mathcal{G}_\ell$ be the event that $S^{(\ell)}_{\lfloor i(r_\ell, \ell)\rfloor} = 0$ and $\mathcal{G} = \bigcap_{\ell \le k} \mathcal{G}_\ell$, let $\mathcal{F}$ be the event that (16) holds for $i = i_{before}(k)$ and $S^{(j)}_{i_{before}(k)} = 0$ for $j < k$, and let $\mathcal{A} = \mathcal{F} \cap \bigcap_{\ell < k} \mathcal{G}_\ell$. Also, let

$$\mathcal{I} = [ns_k(i_{before}(k)/n) - 4E_k(i_{before}(k)), ns_k(i_{before}(k)/n) + 4E_k(i_{before}(k))].$$

Note that, by part 1 of Theorem 8, by induction on the second part Theorem 8, and since $i_{before}(k) > i_{after}(k-1)$, $\mathcal{F}$ happens with probability $1 - o(1)$. Therefore:

$$\begin{aligned}\mathbb{P}[\mathcal{G}] &= \mathbb{P}\left[\mathcal{G}_k \cap \mathcal{A}\right] + o(1)\\ &= \sum_{p\in\mathcal{I}} \mathbb{P}\left[\mathcal{G} \,\middle|\, \mathcal{A} \cap \left(S^{(k)}_{i_{before}(k)} = p\right)\right] \mathbb{P}\left[\mathcal{A} \cap \left(S^{(k)}_{i_{before}(k)} = p\right)\right] + o(1).\end{aligned}$$

We can now apply (10), (15), and Theorem 12 to get

$$\mathbb{P}\left[\mathcal{G} \,\middle|\, \mathcal{A} \cap \left(S^{(k)}_{i_{before}(k)} = p\right)\right] = e^{-r_\ell} + o(1)$$

for $p \in \mathcal{I}$. We note that all $o(1)$ functions in the sum can be made to be the same by carefully reviewing the proof of Theorem 12. Therefore:

$$\begin{aligned}\mathbb{P}[\mathcal{G}] &= \sum_{p\in\mathcal{I}} (e^{-r_\ell} + o(1))\mathbb{P}\left[\mathcal{A} \cap \left(S^{(k)}_{i_{before}(k)} = p\right)\right] + o(1)\\ &= e^{-r_\ell} \sum_{p\in\mathcal{I}} \mathbb{P}\left[\mathcal{A} \cap \left(S^{(k)}_{i_{before}(k)} = p\right)\right] + o(1)\\ &= e^{-r_\ell}\mathbb{P}\left[\bigcap_{\ell<k} \mathcal{G}_\ell\right] + o(1) \qquad \text{by Theorem 8}\\ &= \exp\left\{\sum_{\ell=0}^{k} e^{-r_\ell}\right\} + o(1) \qquad \text{by induction on Theorem 8,}\end{aligned}$$

proving Theorem 8.

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