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Two-dimensional incompressible viscous flow around a thin obstacle tending to a curve

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Building on a recent work, we consider a two-dimensional viscous fluid in the exterior of a thin obstacle shrinking to a curve, proving convergence to a solution of the Navier–Stokes equations in the exterior of a curve. The uniqueness of the limit solution is also shown.

1. Introduction

We study the influence of a thin material obstacle on the behaviour of two-dimensional incompressible viscous flow. The study of flow past a slender body is a classical problem in fluid mechanics and it presents a rich literature on experiments and simulations, particularly around a flat plane (see, for example, [1, 3, 4, 14, 16, 17]). The goal of this work is to establish existence and uniqueness outside a curve. The mathematical study of the problem of small obstacles in incompressible flows was initiated by Iftimie *et al*. [5–7, 12] and continued in [9].

Let Ω_ε be a small connected and simply connected bounded open set in $\mathbb{R}^2$. In all the above-mentioned papers, the initial data consist in the initial vorticity ω_0 and the circulation γ of the initial velocity around the boundary of the obstacle. Both ω_0 (supposed to be smooth and compactly supported) and γ are assumed to be independent of ε. Given the geometry of the obstacle Ω_ε, the two previous quantities uniquely determine the initial velocity field u_0^ε (divergence-free, tangent to the boundary and vanishing at infinity). With these initial data, the problem we consider here is determining the limit of the solutions of the Navier–Stokes equations in the exterior of Ω_ε when the obstacle Ω_ε shrinks to a curve as $\varepsilon \to 0$. The vanishing obstacle problem for incompressible, ideal, two-dimensional flow when the obstacle homothetically shrinks to a point was studied in [5]. It was proved therein that the limit velocity satisfies a modified Euler equation containing an additional term, which is a fixed Dirac mass of strength γ at the point to which the obstacle shrinks.

In [9], the author treated the same problem in the case when the obstacle shrinks to a curve Γ instead of a point. In this case, the additional term is of the form $g_\omega \delta_\Gamma$, where δ_Γ is the Dirac mass of the curve. The density g_ω is explicitly computed in [9] and depends on the vorticity and the circulation γ. It can be seen as the jump across Γ of the velocity field that is divergence free, tangent to Γ, vanishing at infinity

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and with curl ω in $\mathbb{R}^2 \setminus \Gamma$. The case of several obstacles, one of them shrinking to a point, was treated in [12]. The two-dimensional viscous case where the obstacle shrinks homothetically to a point was studied in [6], where it is proved that in the case of small circulation the limit equations are always the Navier–Stokes equations, and where the additional Dirac mass appears only in the initial data. This is due to the fact that the circulation of the initial velocity on the boundary of the obstacle vanishes for $t > 0$ when we consider the no-slip boundary condition.

Here we assume that the obstacle shrinks to a curve and we pass to the limit in the Navier–Stokes equations in the exterior of this obstacle. We prove that the limit equations are the Navier–Stokes equations in the exterior of the curve and they have a unique solution in a suitable sense. As we shall see in § 2.2, the initial datum for the limit velocity is not square-integrable since it behaves as $x^{\perp}/2\pi|x|^2$ at infinity. For such an initial datum we define a solution of the Navier–Stokes equations as a vector field verifying the equation in the sense of distributions and such that the difference between the solution and a fixed smooth vector field behaving like $x^{\perp}/2\pi|x|^2$ at infinity has the regularity expected from a Leray solution (see definition 4.4 for the precise definition).

More precisely, let Ω_ε be a simply connected smooth bounded domain such that Ω_ε shrinks to a curve Γ as $\varepsilon \to 0$ in the sense of § 2.2. The aim of this paper is to prove the following theorem.

THEOREM 1.1. *Let ω_0 and γ be independent of ε as defined above. Let u^ε be the solution of the Navier–Stokes equations on $\Pi_\varepsilon \equiv \mathbb{R}^2 \setminus \bar{\Omega}_\varepsilon$ with initial velocity u_0^ε (see (2.5)) and denote by Eu^ε the extension of u^ε to $\mathbb{R}^2$ with values 0 on Ω_ε. Then $\{Eu^\varepsilon\}$ converges in $L^2_{\mathrm{loc}}([0,\infty) \times (\mathbb{R}^2 \setminus \Gamma))$ to a solution of the Navier–Stokes equations in $\mathbb{R}^2 \setminus \Gamma$ (in the sense of definition 4.4).*

The initial vorticity of this limit solution is $\omega_0 + g_\omega \delta_\Gamma$ and the initial velocity is given by the relation

$$u_0 = K[\omega_0] + \alpha H,$$

with K and H defined in (2.9) and (2.10) depending only on the Γ shape, and with $\alpha = \gamma + \int \omega_0$. Then, this initial velocity is explicitly given in terms of ω_0 and γ and can be viewed as the divergence-free vector field which is tangent to Γ, vanishing at infinity, with curl in $\mathbb{R}^2 \setminus \Gamma$ equal to ω and with circulation around the curve Γ equal to γ. This velocity blows up at the end points of the curve Γ as the inverse of the square root of the distance and has a jump across Γ. In fact, one can also characterize g_ω as the jump of the tangential velocity across Γ.

Moreover, for such initial data, we also show that a solution of the Navier–Stokes equations in $\mathbb{R}^2 \setminus \Gamma$ (in the sense of definition 4.4, which means that the difference between the solution and a fixed smooth vector field behaving like $x^{\perp}/2\pi|x|^2$ at infinity has the regularity expected from a Leray solution) is unique (see proposition 5.1 for the precise statement).

The existence of solutions in the Navier–Stokes equations has been studied in general domains in [2] for dimensions 2 or 3 for square-integrable data, and in [13] for the dimension-3 and $H^{1/2}$ initial data. Kozono and Yamazaki [8] treated the case of $L^{2,\infty}$ data, but for exterior domains which are smooth. A by-product of theorem 1.1 is the existence and uniqueness of solutions of the Navier–Stokes equations on $\mathbb{R}^2 \setminus \Gamma$

in a case which is not covered in previous work. Indeed, the result of [2] does not apply, because the initial datum of our limit velocity is not square-integrable at infinity. Our extension from square-integrable velocities to velocities that decay like $1/|x|$ is physically meaningful: it allows non-vanishing initial circulation around the obstacle, something which can happen in impulsively started motion. On the other hand, our initial datum u_0 satisfies the smallness condition of Kozono and Yamazaki [8] (see (2.6)), but the domain $\mathbb{R}^2 \setminus \Gamma$ is not smooth, as required in [8].

The remainder of this work is organized as follows. In § 2 we introduce a family of conformal mappings between the exterior of Ω_ε and the exterior of the unit disc, allowing the use of explicit formulae for basic harmonic fields and the Biot–Savart law. Moreover, we formulate the flow problem in the exterior of a vanishing obstacle and we study the asymptotic behaviour of the initial data. In § 3 we find *a priori* estimates which will be used in § 4 to prove compactness in space-time and perform the passage to the limit stated in theorem 1.1. In § 5 the uniqueness of the Navier–Stokes equations on the exterior of a curve is established.

For the sake of clarity, the main notation is listed in an appendix.

2. Flow in an exterior domain

2.1. Conformal mapping

Let $D = B(0, 1)$ and $S = \partial D$. In what follows, we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$.

We begin this section by recalling some basic definitions on the curve.

Definition 2.1. We call a *Jordan arc* a curve C given by a parametric representation $C : \varphi(s)$, $0 \leqslant s \leqslant 1$, with φ an injective (one-to-one) function, continuous on $[0, 1]$. An *open Jordan arc* has a parametrization $C : \varphi(s)$, $0 < s < 1$, with φ continuous and injective on $(0, 1)$.

The Jordan arc is of class C^n $(n \in \mathbb{N}^*)$ if its parametrization φ is n-times continuously differentiable, satisfying $\varphi'(s) \neq 0$ for all s.

Let $\Gamma : \Gamma(s)$, $0 \leqslant s \leqslant 1$, be a Jordan arc. Then the subset $\mathbb{R}^2 \setminus \Gamma$ is connected and we will denote it by Π. The purpose of the following proposition is to give some properties of a biholomorphism $T : \Pi \to \operatorname{int} D^{\mathrm{c}}$. After applying a homothetic transformation, a rotation and a translation, we can suppose that the end points of the curve are $-1 = \Gamma(0)$ and $1 = \Gamma(1)$.

Proposition 2.2. *If Γ is a C^2 Jordan arc, such that the intersection with the segment $[-1, 1]$ is a finite union of segments and points, then there exists a biholomorphism $T : \Pi \to \operatorname{int} D^{\mathrm{c}}$ which verifies the following properties:*

(i) *T^{-1} and DT^{-1} extend continuously up to the boundary, and T^{-1} maps S to Γ;*

(ii) *DT^{-1} is bounded;*

(iii) *T and DT extend continuously to Γ with different values on each side of Γ, except at the end points of the curve, where DT behaves like the inverse of the square root of the distance;*

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(iv) *DT is bounded in the exterior of any disc $B(0,R)$, with $\Gamma \subset B(0,R)$;*

(v) *DT is $L^p(\Pi \cap B(0,R))$ for all $p<4$ and $R>0$.*

The proof of this proposition can be found in [9]. Reading it, one understands why we need the condition that the curve is supposed to be more than C^1 (in fact, $C^{1,\alpha}$ can be sufficient). Indeed, we need to find some continuity properties of the first derivative of T, which is possible by the Kellogg–Warschawski theorem only if Γ is sufficiently regular. We also recall the following remark from [9].

Remark 2.3. If we have a biholomorphism H between the exterior of a bounded set A and D^{c}, such that $H(\infty)=\infty$, then there exist a non-zero real number β and a bounded holomorphic function $h : \Pi \to \mathbb{C}$ such that

$$H(z) = \beta z + h(z),$$

with

$$h'(z) = O\left(\frac{1}{|z|^2}\right) \quad \text{as } |z| \to \infty.$$

This property can be applied for the T above, observing that T sends the exterior of a bounded set B to $\operatorname{int} B(0,2)^{\mathrm{c}}$; hence $\frac{1}{2}T = \beta z + h(z)$.

2.2. The evanescent obstacle

In this subsection we formulate a precise statement of the thin obstacle problem. Many of the key issues regarding the small obstacle limit and incompressible flow have been discussed in detail in [9], so we briefly recall some properties.

As in [9], we fix $\omega_0 \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^2 \setminus \Gamma)$. Next, we introduce a family of problems, parametrized by the size of the obstacle. We consider a family of smooth domains Ω_ε, connected, simply connected and containing Γ, with sufficiently small ε, such that the support of ω_0 does not intersect Ω_ε. Let T_ε be a biholomorphism between $\Pi_\varepsilon \equiv \mathbb{R}^2 \setminus \bar{\Omega}_\varepsilon$ and D^{c}, satisfying the following assumption.

Assumption 2.4. The biholomorphism family $\{T_\varepsilon\}$ verifies that

(i) $\|(T_\varepsilon - T)/|T|\|_{L^\infty(\Pi_\varepsilon)} \to 0$ as $\varepsilon \to 0$,

(ii) $\det(DT_\varepsilon^{-1})$ is bounded on D^{c} independently of ε,

(iii) for any $R>0$, $\|DT_\varepsilon - DT\|_{L^3(B(0,R)\cap\Pi_\varepsilon)} \to 0$ as $\varepsilon \to 0$,

(iv) for $R>0$ large enough, there exists $C_R>0$ such that $|DT_\varepsilon(x)| \leqslant C_R$ on $B(0,R)^{\mathrm{c}}$,

(v) for $R>0$ large enough, there exists $C_R>0$ such that $|D^2T_\varepsilon(x)| \leqslant C_R/|x|$ on $B(0,R)^{\mathrm{c}}$.

Remark 2.5. We can observe that property (iii) implies that, for any R, DT_ε is bounded in $L^p(B(0,R)\cap\Pi_\varepsilon)$ independently of ε, for $p \leqslant 3$. Moreover, condition (i) means that $T_\varepsilon \to T$ uniformly on $B(0,R)\cap\Pi_\varepsilon$ for any $R>0$.

Assumption 2.4 corresponds to [9, assumption 3.1], adding part (v) and strengthening property (i) therein. Before continuing, we give an example of an obstacle family.

EXAMPLE 2.6. We consider $\Omega_\varepsilon \equiv T^{-1}(B(0,1+\varepsilon)\setminus D)$. In this case, $T_\varepsilon = T/(1+\varepsilon)$, which verifies the previous assumption. In fact, taking proposition 2.2 into account, $\|DT_\varepsilon - DT\|_{L^p(B(0,R)\cap\Pi_\varepsilon)} \to 0$ for all $p < 4$, and, using remark 2.5, $|D^2T_\varepsilon(x)| \leqslant C_R/|x|^3$ on $B(0,R)^{\mathrm{c}}$, but we will not need such strong estimates. If Γ is a segment, then Ω_ε is the interior of an ellipse around the segment.

We define $\Gamma_\varepsilon \equiv \partial\Omega_\varepsilon$. Moreover, we define by $G^\varepsilon = G^\varepsilon(x,y)$ Green's function of the Laplacian in Π_ε, by $K^\varepsilon(x,y) = \nabla_x^\perp G^\varepsilon(x,y)$ the kernel of the Biot–Savart law on Π_ε and we denote the associated integral operator by

$$f \mapsto K^\varepsilon[f] = \int_{\Pi_\varepsilon} K^\varepsilon(x,y)f(y)\,\mathrm{d}y.$$

Let $H^\varepsilon(x)$ be the unique harmonic vector field on Π_ε which verifies the condition

$$\oint_{\Gamma_\varepsilon} H^\varepsilon \cdot \mathrm{d}\boldsymbol{s} = 1,$$

where the contour integral is taken in the counterclockwise sense. Both K^ε and H^ε depend on T^ε, and we recall the following explicit formulae found [9, § 3.2]:

$$K^\varepsilon = \frac{1}{2\pi} DT_\varepsilon^t(x)\left(\frac{(T_\varepsilon(x)-T_\varepsilon(y))^\perp}{|T_\varepsilon(x)-T_\varepsilon(y)|^2} - \frac{(T_\varepsilon(x)-T_\varepsilon(y)^*)^\perp}{|T_\varepsilon(x)-T_\varepsilon(y)^*|^2}\right) \tag{2.1}$$

and

$$H^\varepsilon = \frac{1}{2\pi} DT_\varepsilon^t(x)\left(\frac{(T_\varepsilon(x))^\perp}{|T_\varepsilon(x)|^2}\right), \tag{2.2}$$

where $T_\varepsilon(y)^* = T_\varepsilon(y)/|T_\varepsilon(y)|^2$.

We recall from [5] that, given $\omega_0 \in C_{\mathrm{c}}^\infty(\mathbb{R}^2 \setminus \Gamma)$ and $\gamma \in \mathbb{R}$, for $\varepsilon > 0$, there exists a unique u_0^ε such that

$$\operatorname{div} u_0^\varepsilon = 0, \qquad \operatorname{curl} u_0^\varepsilon = \omega_0, \qquad \oint_{\Gamma_\varepsilon} u_0^\varepsilon \cdot \mathrm{d}\boldsymbol{s} = \gamma,$$

u_0^ε is tangent to Γ_ε and vanishes at infinity. Moreover, there exists a unique α such that

$$u_0^\varepsilon = K^\varepsilon[\omega_0^\varepsilon] + \alpha H^\varepsilon. \tag{2.3}$$

By Stokes's theorem, we have that

$$\alpha = \gamma + m \quad \text{with } m \equiv \int_{\mathbb{R}^2} \omega_0\,\mathrm{d}x$$

(see the proof [5, lemma 3.1]).

Now, we require information on far-field behaviour. We know from [9, § 2.2] that

$$|u_0^\varepsilon(x)| \leqslant \frac{|DT_\varepsilon(x)|}{2\pi}\int_{\operatorname{supp}\omega_0} \frac{|T_\varepsilon(y)-T_\varepsilon(y)^*|}{|T_\varepsilon(x)-T_\varepsilon(y)||T_\varepsilon(x)-T_\varepsilon(y)^*|}|\omega_0(y)|\,\mathrm{d}y + \frac{|DT_\varepsilon(x)|}{2\pi|T_\varepsilon(x)|}.$$

By assumption 2.4(i) and (iv), and the form of $T(x)$ at infinity (see remark 2.3), there exist $R > 0$ and $C > 0$ independent of ε such that

$$|K^\varepsilon[\omega_0](x)| \leqslant C/|x|^2 \quad \text{and} \quad |H^\varepsilon(x)| \leqslant C/|x| \quad \text{for all } |x| \geqslant R, \tag{2.4}$$

since $\omega_0 \in C_c^\infty(\Pi_\varepsilon)$.

Let $u^\varepsilon = u^\varepsilon(x,t) = (u_1^\varepsilon(x_1,x_2,t), u_2^\varepsilon(x_1,x_2,t))$ be the velocity of an incompressible, viscous flow in Π_ε. We assume that u^ε verifies the no-slip condition at any positive time and $u^\varepsilon \to 0$ when $|x| \to \infty$. The evolution of such a flow is governed by the Navier–Stokes equations:

$$\left.\begin{aligned} \partial_t u^\varepsilon - \nu\Delta u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon &= -\nabla p^\varepsilon && \text{in } \Pi_\varepsilon \times (0,\infty), \\ \operatorname{div} u^\varepsilon &= 0 && \text{in } \Pi_\varepsilon \times [0,\infty), \\ u^\varepsilon &= 0 && \text{in } \Gamma_\varepsilon \times (0,\infty), \\ \lim_{|x|\to\infty} |u^\varepsilon| &= 0 && \text{for } t \in [0,\infty), \\ u^\varepsilon(x,0) &= u_0^\varepsilon(x) && \text{in } \Pi_\varepsilon. \end{aligned}\right\} \tag{2.5}$$

As u_0^ε is smooth, and therefore locally bounded, the behaviour at infinity given in (2.4) allows us to observe that $u_0^\varepsilon \in L^{2,\infty}(\Pi_\varepsilon) \cap L^p(\Pi_\varepsilon)$ with $p > 2$. Global-in-time well-posedness for problem (2.5) was established by Kozono and Yamazaki [8]. The existence part of their result requires that the initial velocity u_0^ε satisfies a smallness condition of the form

$$\limsup_{R\to\infty} R|\{x \in \Pi_\varepsilon \mid |u_0^\varepsilon(x)| > R\}|^{1/2} \ll 1. \tag{2.6}$$

Since u_0^ε is bounded, the supremum limit is always zero, for any $\varepsilon > 0$. Uniqueness holds without any additional conditions.

We conclude this subsection with the definition of a cut-off function family. Let $\Phi \in C^\infty(\mathbb{R})$ be a non-decreasing function such that $0 \leqslant \Phi \leqslant 1$, $\Phi(s) = 1$ if $s \geqslant 2$ and $\Phi(s) = 0$ if $s \leqslant 1$. Then, for $\lambda \geqslant 2$, we introduce

$$\Phi^{\varepsilon,\lambda} = \Phi^{\varepsilon,\lambda}(x) \equiv \Phi\left(\frac{|T_\varepsilon(x)| - 1}{\lambda}\right). \tag{2.7}$$

Due to the uniform convergence of T_ε to T on bounded sets (see assumption 2.4(i)), we note that the cut-off function $\Phi^{\varepsilon,\lambda}$ vanishes in a ball of radius $C_1\lambda$ and it is identically equal to 1 outside a larger ball of radius $C_2\lambda$, with C_1 and C_2 independent of ε. Furthermore, the radii of the annulus where $\Phi^{\varepsilon,\lambda}$ is not constant can be made independent of ε.

2.3. Asymptotic initial data

The aim of this section is to study the convergence, as $\varepsilon \to 0$, of the initial velocity fields u_0^ε. First, we introduce some notation. For each function f defined on Π_ε, we denote by Ef the extension of f to $\mathbb{R}^2$, by setting $Ef \equiv 0$ in Π_ε. If f is sufficiently regular and vanishes on $\partial\Omega_\varepsilon$, one has that $\nabla Ef = E\nabla f$ in $\mathbb{R}^2$. If v is a sufficiently regular vector field defined on Π_ε and tangent to $\partial\Omega_\varepsilon$, then $\operatorname{div} Ev = E \operatorname{div} v$ in $\mathbb{R}^2$. In particular, we have $\operatorname{div} Eu_0^\varepsilon = 0$ in $\mathbb{R}^2$.

The following lemmas are consequences of the case of an ideal fluid treated in [9].

LEMMA 2.7. *For $2 < p \leqslant 3$, there exists $C_p > 0$, which depends only on the shape of Γ and ω_0, such that $\|Eu_0^\varepsilon\|_{L^p(\mathbb{R}^2)} \leqslant C_p$.*

Proof. By [9, theorem 4.4], we can state that $\|Eu_0^\varepsilon\|_{L^p(S)} \leqslant C\|EDT_\varepsilon\|_{L^p(S)}$ for any $S \subset \mathbb{R}^2$. Then we can use remark 2.5 to observe that, for any $R > 0$, we can find a constant C_p such that $\|Eu_0^\varepsilon\|_{L^p(B(0,R))} \leqslant C_p$ for $p \leqslant 3$. Recalling (2.3) and (2.4), the desired conclusion follows, since the function $x \mapsto 1/|x|$ is L^p at infinity for $p > 2$. □

LEMMA 2.8. *We have that $Eu_0^\varepsilon \to K[\omega_0] + \alpha H$ strongly in $L^2_{\text{loc}}(\mathbb{R}^2)$ as $\varepsilon \to 0$, where K and H are defined as K^ε and H^ε, respectively (see (2.1) and (2.2)) by replacing T_ε by T.*

Proof. This result is a consequence of [9, § 5.1], where it is shown that, in the case of an ideal flow, $\Phi^\varepsilon u^\varepsilon \to u \equiv K[\omega] + \alpha H$ strongly in $L^2_{\text{loc}}([0,T] \times \mathbb{R}^2)$ with $\Phi^\varepsilon \equiv \Phi^{\varepsilon,\varepsilon}$. This was done in two steps: first proving that $\Phi^\varepsilon u^\varepsilon \to u$ strongly in $L^2_{\text{loc}}(\mathbb{R}^2)$ for each $t \geqslant 0$, and then, by the dominated convergence theorem, obtaining the convergence in $L^2_{\text{loc}}([0,T] \times \mathbb{R}^2)$. Here the first step is sufficient to complete the proof. □

Henceforth, we set

$$u_0 = K[\omega_0] + \alpha H, \tag{2.8}$$

with

$$K = \frac{1}{2\pi} DT^t(x)\left(\frac{(T(x) - T(y))^\perp}{|T(x) - T(y)|^2} - \frac{(T(x) - T(y)^*)^\perp}{|T(x) - T(y)^*|^2}\right) \tag{2.9}$$

and

$$H = \frac{1}{2\pi} DT^t(x)\left(\frac{(T(x))^\perp}{|T(x)|^2}\right). \tag{2.10}$$

By [9, proposition 5.7], we know that

(i) u_0 is continuous on $\mathbb{R}^2 \setminus \Gamma$,

(ii) u_0 is continuous up to $\Gamma \setminus \{-1; 1\}$, with different values on each side of Γ,

(iii) u_0 blows up at the end points of the curve like $C/\sqrt{|x-1||x+1|}$, which belongs to L^p_{loc} for $p < 4$,

(iv) u_0 is tangent to the curve.

Moreover, the [9, § 5.2] states also that u_0 is a divergence-free vector field, vanishing at infinity, with $\operatorname{curl} u_0 = \omega_0 + g_{\omega_0}(s)\delta_\Gamma$ in $\mathbb{R}^2$, where δ_Γ is the Dirac function of the curve Γ, and the g_{ω_0} depends on ω_0 and the circulation γ. One can also characterize g_{ω_0} as the jump of the tangential velocity across Γ. Then we know that u_0 is bounded, except at the end points, where it is equivalent to the inverse of the square root of the distance, and so u_0 verifies the smallness condition (2.6).

3. Velocity estimates

We start by introducing some functional spaces which embed the divergence-free and no-slip conditions.

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Definition 3.1. Let Ω be an open set in $\mathbb{R}^2$. We denote by $V(\Omega)$ the space of divergence-free vector fields, the components of which belong to $C_c^\infty(\Omega)$. The closure of $V(\Omega)$ in $H^1(\Omega)$ is denoted by $\mathcal{V}(\Omega)$, and its dual space by $\mathcal{V}'(\Omega)$. Finally, we denote by $\mathcal{H}(\Omega)$ the closure of $V(\Omega)$ in $L^2(\Omega)$. To simplify the notation, we also set $\mathcal{V}_\Gamma \equiv \mathcal{V}(\mathbb{R}^2 \setminus \Gamma)$ and $\mathcal{H}_\Gamma \equiv \mathcal{H}(\mathbb{R}^2 \setminus \Gamma)$.

Since the initial datum u_0^ε does not belong to L^2 ($u_0^\varepsilon = O(1/|x|)$ at infinity), we will remove the harmonic part at infinity. To this end, we define $W^\varepsilon(t,x) = u^\varepsilon(t,x) - v^\varepsilon(x)$, where $v^\varepsilon = \alpha H^\varepsilon \Phi^{\varepsilon,\lambda}$, with fixed λ, chosen to be sufficiently large so that the radii of the balls where $\Phi^{\varepsilon,\lambda}$ vanishes, for each $\varepsilon > 0$, are large enough to satisfy (iv) and (v) of assumption 2.4. This choice of λ is possible because the radii of these balls are $O(\lambda)$. Without any loss of generality, we may assume in addition that these balls contain $\bar{\Omega}_\varepsilon$. Due to assumption 2.4 and (2.4), we can give the following estimates on v^ε.

Lemma 3.2. *For λ fixed (sufficiently large, independent of ε),*

(a) *v^ε are bounded in $L^4(\mathbb{R}^2)$ independently of ε,*

(b) *∇v^ε are bounded in $L^2(\mathbb{R}^2)$ independently of ε,*

(c) *Δv^ε are bounded in $L^\infty(\mathbb{R}^2)$ independently of ε and supported in a compact set independent of ε.*

Proof. We recall the following explicit formula for v^ε:

$$v^\varepsilon(x) = \frac{\alpha}{2\pi}\Phi^{\varepsilon,\lambda}(x) DT_\varepsilon^t(x)\left(\frac{(T_\varepsilon(x))^\perp}{|T_\varepsilon(x)|^2}\right),$$

with $\Phi^{\varepsilon,\lambda}$ given in (2.7).

As $\Phi^{\varepsilon,\lambda}$ vanishes in a ball of radius $O(\lambda)$, conditions (i) and (iv) of assumption 2.4 guarantee that v^ε is uniformly bounded by $C\Phi^{\varepsilon,\lambda}(x)/|T(x)|$ for sufficiently large λ. Since the function T behaves like βx at infinity, the first estimate of the lemma is a consequence of the fact that $1/|x|$ is L^4 at infinity.

Using that $|T_\varepsilon| \geqslant 1$, we obtain that

$$|\nabla v^\varepsilon| \leqslant \frac{\alpha}{2\pi\lambda}\left|\Phi'\left(\frac{|T_\varepsilon(x)|-1}{\lambda}\right)\right||DT_\varepsilon|^2 + \frac{3\alpha}{2\pi}\Phi^{\varepsilon,\lambda}(x)\left(\frac{|D^2T_\varepsilon|}{|T_\varepsilon(x)|} + \frac{|DT_\varepsilon|^2}{|T_\varepsilon(x)|^2}\right).$$

Taking into account the fact that the radii of the annulus where $\Phi^{\varepsilon,\lambda}$ is not constant can be made independent of ε, assumption 2.4(iv) implies that the first term in the above inequality is uniformly bounded with respect to x and ε, and compactly supported in a compact set independent of ε. Parts (i), (iv) and (v) of assumption 2.4 allow us to state that, for sufficiently large λ, the second term is bounded by $C\Phi^{\varepsilon,\lambda}(x)/|x|^2$ (with a constant C independent of ε), which belongs to $L^2(\mathbb{R}^2)$. This proves the second assertion of the lemma.

Finally, we note that $\Delta H^\varepsilon = 0$ outside the balls where the $\Phi^{\lambda,\varepsilon}$ vanish, because $H^\varepsilon = \nabla^\perp \ln|T_\varepsilon(x)| = \nabla^\perp \operatorname{Re}(\ln T_\varepsilon(x))$, with $\ln T_\varepsilon$ a holomorphic function, so $\Delta \ln T_\varepsilon = 0$. Then, since $|T_\varepsilon(x)| \geqslant 1$, for some constant $C > 0$ we have

$$|\Delta v^\varepsilon| \leqslant C\left|\Phi'\left(\frac{|T_\varepsilon(x)|-1}{\lambda}\right)\right|(|DT_\varepsilon|^3 + |DT_\varepsilon||D^2T_\varepsilon|) + C\left|\Phi''\left(\frac{|T_\varepsilon(x)|-1}{\lambda}\right)\right||DT_\varepsilon|^3,$$

which is bounded in $L^\infty(\mathbb{R}^2)$ uniformly with respect to ε and compactly supported in a compact independent of ε. □

LEMMA 3.3. *We have that* $W_0^\varepsilon \equiv W^\varepsilon(\cdot,0) = K_\varepsilon[\omega_0] + \alpha(1-\Phi^{\varepsilon,\lambda})H_\varepsilon$ *is bounded in* L^p *independently of* ε *for* $1 < p \leqslant 3$.

Proof. This lemma can be established similarly to lemma 2.7 by using the fact that W_0^ε behaves like $1/|x|^2$ at infinity (see (2.4)), which belongs to L^p for $p > 1$. □

In particular, W_0^ε is bounded in L^2, which will be useful in obtaining *a priori* estimates for $W^\varepsilon \equiv u^\varepsilon - v^\varepsilon$.

LEMMA 3.4. *The vector fields* W^ε *are bounded independently of* ε *in*

$$L^\infty_{\mathrm{loc}}([0,\infty); L^2(\varPi_\varepsilon)) \cap L^2_{\mathrm{loc}}([0,\infty); H^1(\varPi_\varepsilon)).$$

Proof. We rewrite (2.5) for W^ε as

$$\begin{aligned}\partial_t W^\varepsilon - \nu\Delta W^\varepsilon - \nu\Delta v^\varepsilon + (W^\varepsilon + v^\varepsilon)\cdot\nabla W^\varepsilon \qquad&\\ + W^\varepsilon\cdot\nabla v^\varepsilon + v^\varepsilon\cdot\nabla v^\varepsilon = -\nabla p^\varepsilon \quad &\text{in } \varPi_\varepsilon\times(0,\infty),\\ \operatorname{div} W^\varepsilon = 0 \quad &\text{in } \varPi_\varepsilon\times[0,\infty),\\ W^\varepsilon(\cdot,t) = 0 \quad &\text{on } \varGamma_\varepsilon\times(0,\infty).\end{aligned}$$

Indeed,

$$\begin{aligned}\operatorname{div} W^\varepsilon &= -\operatorname{div} v^\varepsilon\\ &= \alpha H^\varepsilon\cdot\nabla\Phi^{\varepsilon,\lambda}\\ &= -\frac{\alpha}{2\pi\lambda}\biggl(\frac{T_\varepsilon}{|T_\varepsilon|^2}DT_\varepsilon\biggr)^{\perp}\cdot\Phi'\biggl(\frac{|T_\varepsilon|-1}{\lambda}\biggr)\biggl(\frac{T_\varepsilon}{|T_\varepsilon|}DT_\varepsilon\biggr) = 0.\end{aligned}$$

We multiply the equation above by W^ε and integrate to obtain

$$\begin{aligned}\mathcal{E} &\equiv \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|W^\varepsilon\|_{L^2}^2 + \nu\|\nabla W^\varepsilon\|_{L^2}^2\\ &= -\int_{\varPi_\varepsilon}[W^\varepsilon\cdot(W^\varepsilon\cdot\nabla v^\varepsilon) + W^\varepsilon\cdot(v^\varepsilon\cdot\nabla v^\varepsilon)]\,\mathrm{d}x + \nu\int_{\varPi_\varepsilon} W^\varepsilon\cdot\Delta v^\varepsilon\,\mathrm{d}x\\ &= \int_{\varPi_\varepsilon}[v^\varepsilon\cdot(W^\varepsilon\cdot\nabla W^\varepsilon) + v^\varepsilon\cdot(v^\varepsilon\cdot\nabla W^\varepsilon)]\,\mathrm{d}x + \nu\int_{\varPi_\varepsilon} W^\varepsilon\cdot\Delta v^\varepsilon\,\mathrm{d}x\\ &\leqslant \|W^\varepsilon\|_{L^4}\|\nabla W^\varepsilon\|_{L^2}\|v^\varepsilon\|_{L^4} + \|\nabla W^\varepsilon\|_{L^2}\|v^\varepsilon\|_{L^4}^2 + \nu\|W^\varepsilon\|_{L^2}\|\Delta v^\varepsilon\|_{L^2}.\end{aligned}$$

Next, we use the interpolation inequality:

$$\|W^\varepsilon\|_{L^4} \leqslant C\|W^\varepsilon\|_{L^2}^{1/2}\|\nabla W^\varepsilon\|_{L^2}^{1/2},$$

with a constant $C > 0$ independent of ε. In the case of $\mathbb{R}^2$, this inequality can be found in [10, ch. 1]. To obtain the corresponding inequality in $\varPi_\varepsilon$, one simply extends W^ε to $\mathbb{R}^2$ by setting it identically zero inside $\varOmega_\varepsilon$. As W^ε vanishes on $\varGamma_\varepsilon$, the extension has an H^1-norm in the plane which is identical to the H^1 norm of

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W^ε in Π_ε. Moreover, Δv^ε is bounded in L^2 and v^ε is uniformly bounded in L^4 independently of ε due to lemma 3.2. Hence,

$$\begin{aligned}\mathcal{E} &\leqslant C\|W^\varepsilon\|_{L^2}^{1/2}\|\nabla W^\varepsilon\|_{L^2}^{3/2}\|v^\varepsilon\|_{L^4} + \|\nabla W^\varepsilon\|_{L^2}\|v^\varepsilon\|_{L^4}^2 + \nu\|W^\varepsilon\|_{L^2}\|\Delta v^\varepsilon\|_{L^2} \\ &\leqslant \tfrac{1}{2}\nu\|\nabla W^\varepsilon\|_{L^2}^2 + C_1\|W^\varepsilon\|_{L^2}^2 + C_2\end{aligned}$$

for some constants C_1 and C_2 independent of ε, so

$$\frac{\mathrm{d}}{\mathrm{d}t}\|W^\varepsilon\|_{L^2}^2 + \nu\|\nabla W^\varepsilon\|_{L^2}^2 \leqslant 2C_1\|W^\varepsilon\|_{L^2}^2 + 2C_2.$$

Gronwall's inequality now gives, for any $t > 0$,

$$\mathrm{e}^{-2C_1 t}\|W^\varepsilon\|_{L^2}^2 + \nu\int_0^t \mathrm{e}^{-2C_1 s}\|\nabla W^\varepsilon(s,\cdot)\|_{L^2}^2\,\mathrm{d}s \leqslant \frac{C_2}{C_1} + \|W^\varepsilon(0,\cdot)\|_{L^2}^2. \tag{3.1}$$

Using the fact that the $W^\varepsilon(0,\cdot)$ are bounded in L^2 independently of ε (see lemma 3.3), we can rewrite (3.1) as

$$\|W^\varepsilon\|_{L^2(\Pi_\varepsilon)}^2 + \nu\mathrm{e}^{2C_1 t}\int_0^t \mathrm{e}^{-2C_1 s}\|\nabla W^\varepsilon(s,\cdot)\|_{L^2(\Pi_\varepsilon)}^2\,\mathrm{d}s \leqslant \mathrm{e}^{2C_1 t}C, \tag{3.2}$$

with a constant C. This completes the proof. □

We now deduce the main result of this section.

THEOREM 3.5. *Let u^ε be the solution of (2.5). Then the following hold true.*

(1) *The family $\{Eu^\varepsilon - v^\varepsilon\}$ is bounded in*

$$L^\infty_{\mathrm{loc}}((0,\infty); L^2(\mathbb{R}^2)) \cap L^2_{\mathrm{loc}}([0,\infty); H^1(\mathbb{R}^2)).$$

(2) *The family $\{\nabla Eu^\varepsilon\}$ is bounded in $L^2_{\mathrm{loc}}([0,\infty); L^2(\mathbb{R}^2))$.*

(3) *The family $\{Eu^\varepsilon\}$ is bounded in*

$$L^\infty_{\mathrm{loc}}((0,\infty); L^2_{\mathrm{loc}}(\mathbb{R}^2)) \cap L^4_{\mathrm{loc}}([0,\infty); L^4(\mathbb{R}^2)).$$

Proof. The proof is based on lemmas 3.2 and 3.4. Indeed, part 1 follows from lemma 3.4, while part 2 is a consequence of the same lemma and of lemma 3.2(b). To prove part 3, we use again the interpolation inequality

$$\|W^\varepsilon\|_{L^4(L^4)} \leqslant C\|W^\varepsilon\|_{L^\infty(L^2)}^{1/2}\|\nabla W^\varepsilon\|_{L^2(L^2)}^{1/2},$$

which ensures that W^ε is uniformly bounded in $L^4_{\mathrm{loc}}([0,\infty); L^4(\mathbb{R}^2))$. It suffices now to use lemma 3.2(a), which gives the uniform boundedness in $L^4_{\mathrm{loc}}([0,\infty); L^4(\mathbb{R}^2))$ for u^ε (whereas Eu_0^ε is not uniformly bounded in $L^4_{\mathrm{loc}}(\mathbb{R}^2)$). □

For each $\varepsilon > 0$, we know that $\operatorname{div} EW^\varepsilon = \operatorname{div} Eu^\varepsilon = 0$ on $\mathbb{R}^2$. Moreover, since the supports of EW^ε and Eu^ε are contained in Π_ε, we can rewrite the previous theorem with the functional spaces of definition 3.1.

COROLLARY 3.6. *Let u^ε be the solution of (2.5). Then the following hold true.*

(1) *The family* $\{Eu^\varepsilon - v^\varepsilon\}$ *is bounded in* $L^\infty_{\mathrm{loc}}((0,\infty);\mathcal{H}_\Gamma)\cap L^2_{\mathrm{loc}}([0,\infty);\mathcal{V}_\Gamma)$.

(2) *The family* $\{\nabla Eu^\varepsilon\}$ *is bounded in* $L^2_{\mathrm{loc}}([0,\infty);\mathcal{H}_\Gamma)$.

We will later use the following proposition on regularization of functions in $L^2_{\mathrm{loc}}([0,\infty);\mathcal{V}_\Gamma)$.

PROPOSITION 3.7. *Let* $T\in[0,+\infty)$ *and* $f\in L^2([0,T];\mathcal{V}_\Gamma)$. *There exists a sequence* $\{f_n\}$ *of divergence-free functions belonging to* $C^\infty_{\mathrm{c}}((0,T)\times(\mathbb{R}^2\setminus\Gamma))$ *such that* $f_n\to f$ *in* $L^2([0,T],\mathcal{V}_\Gamma)$.

Proof. In order to find this family, we start by regularizing in time, as in [15]. To this end, we multiply f by the characteristic function $\chi_{[1/n,T-1/n]}$ and then regularize by a function $\rho_n(t)$ such that the size of the support of ρ_n is less than or equal to $1/2n$. Therefore, we obtain a family $\{\rho_n*(\chi_{[1/n,T-1/n]}f)\}$ which belongs to $C^\infty_{\mathrm{c}}((0,T),\mathcal{V}_\Gamma)$ and which tends to f in $L^2([0,T],\mathcal{V}_\Gamma)$. Now, we will approximate functions belonging to $C^\infty_{\mathrm{c}}(\mathcal{V}_\Gamma)$ by divergence-free functions $C^\infty_{\mathrm{c}}((0,T)\times(\mathbb{R}^2\setminus\Gamma))$, which will allow us to conclude due to a diagonal extraction of a subsequence.

As $\mathcal{V}_\Gamma$ is a separable Hilbert space for the scalar product $H^1(\mathbb{R}^2)$, $\mathcal{V}_\Gamma$ admits an orthonormal base $\{e_n\}$. Let $\varphi_{n,m}\in V(\mathbb{R}^2\setminus\Gamma)$ be a sequence tending to e_n in $H^1(\mathbb{R}^2)$ as $m\to\infty$. Clearly, the family $\{\varphi_{n,m}\}$ is countable, and the vector space generated by this family is dense in $\mathcal{V}_\Gamma$. Therefore, by the Gram–Schmidt process we can conclude that there exists an orthonormal base $\{\tilde e_n\}$ of $\mathcal{V}_\Gamma$ with $\tilde e_n\in V(\mathbb{R}^2\setminus\Gamma)$. So, if $f\in C^\infty_{\mathrm{c}}((0,T);\mathcal{V})$, we can write $f=\sum\alpha_n(t)\tilde e_n(x)$ with $\alpha_n\in C^\infty_{\mathrm{c}}((0,T))$, and we can choose

$$f_N=\sum_0^N\alpha_n(t)\tilde e_n(x).$$

These functions belong to $C^\infty_{\mathrm{c}}((0,T)\times(\mathbb{R}^2\setminus\Gamma))$. Moreover, $g_n(t)=\|f(\cdot,t)-f_n(\cdot,t)\|^2_{H^1}$ belongs to $L^1([0,T])$ (since $\|g_n\|_{L^1}\leqslant 4(\|f\|_{L^2([0,T],H^1)})^2$) and, for each $t\in[0,T]$, $\{g_n(t)\}$ is a non-increasing sequence which tends to zero. Then, by the Beppo Levi theorem, g_n tends to zero in $L^1([0,T])$, which means that f_n converges to f in $L^2([0,T],H^1(\mathbb{R}^2))$. □

4. Passing to the limit

In this section, we prove that $\{Eu^\varepsilon\}$ converges to a solution of the Navier–Stokes equations on $\mathbb{R}^2\setminus\Gamma$ in the sense of distributions. It suffices to find a strong convergence for the sequence $\{Eu^\varepsilon\}$ in $L^2_{\mathrm{loc}}([0,\infty)\times(\mathbb{R}^2\setminus\Gamma))$.

PROPOSITION 4.1. *Let* $T>0$ *and let* O *be a smooth open set relatively compact in* $\mathbb{R}^2\setminus\Gamma$. *Then the sequence* $\{Eu^\varepsilon\}$ *is precompact in* $L^\infty((0,T);H^{-3}(O))$.

Proof. We show that $\{Eu^\varepsilon\}$ is bounded in $L^\infty((0,T);L^2(O))$ and equicontinuous as a function of $(0,T)$ into $H^{-2}(O)$, which will allow us to apply the Arzelà–Ascoli theorem. Fix Ψ, a smooth divergence-free vector field, compactly supported in O. As the obstacle shrinks to the curve Γ, there exists $\varepsilon_O>0$ such that $\Omega_\varepsilon\cap\bar O=\emptyset$

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for all $0 < \varepsilon \leqslant \varepsilon_O$. For each interval $(t_1, t_2) \subset (0, T)$, using (2.5) we see that

$$\begin{aligned}\langle Eu^\varepsilon(t_2) - Eu^\varepsilon(t_1), \Psi\rangle &= \int_{\mathbb{R}^2} (Eu^\varepsilon(t_2) - Eu^\varepsilon(t_1))\Psi \, \mathrm{d}x \\ &= \int_{\mathbb{R}^2} \biggl(\int_{t_1}^{t_2} \partial_t Eu^\varepsilon \, \mathrm{d}t\biggr)\Psi \, \mathrm{d}x \\ &= -\int_{t_1}^{t_2}\int_{\mathbb{R}^2} Eu^\varepsilon \cdot \nabla u^\varepsilon \Psi \, \mathrm{d}x \, \mathrm{d}t - \nu \int_{t_1}^{t_2}\int_{\mathbb{R}^2} \nabla u^\varepsilon \nabla \Psi \, \mathrm{d}x \, \mathrm{d}t \\ &\equiv I_1 + I_2.\end{aligned}$$

We first estimate I_1. Using theorem 3.5, we deduce that

$$\begin{aligned}|I_1| &\leqslant \|Eu^\varepsilon\|_{L^\infty((0,T);L^2(O))} \|\nabla Eu^\varepsilon\|_{L^2([0,T];L^2(O))} \|\Psi\|_{L^\infty} \sqrt{|t_2 - t_1|} \\ &\leqslant C\|\Psi\|_{H^2} \sqrt{|t_2 - t_1|},\end{aligned}$$

due to the Sobolev embedding $H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$. Next, we treat I_2:

$$|I_2| \leqslant \nu \|\nabla u^\varepsilon\|_{L^2([0,T];L^2(O))} \|\nabla \Psi\|_{L^2} \sqrt{|t_2 - t_1|} \leqslant C\|\Psi\|_{H^2} \sqrt{|t_2 - t_1|}.$$

The above inequalities show that $\{Eu^\varepsilon\}$ is equicontinuous as a function of time into $H^{-2}(O)$.

Since $\{Eu^\varepsilon\}$ is bounded in $L^\infty((0,T); L^2(O))$ by theorem 3.5, it follows from the Arzelà–Ascoli theorem that there is a subsequence of Eu^ε which converges strongly in $L^\infty((0,T); H^{-3}(O))$. □

We now improve the space-time compactness result, which is a direct consequence of the previous proposition.

LEMMA 4.2. *There exists a sequence such that* $\{Eu^\varepsilon\}$ *converges strongly in*

$$L^2_{\mathrm{loc}}([0, \infty) \times (\mathbb{R}^2 \setminus \Gamma)).$$

Proof. We know from theorem 3.5 that $\{Eu^\varepsilon\}$ is bounded in $L^2([0,T]; H^1(O))$, and proposition 4.1 states that $\{Eu^\varepsilon\}$ is precompact in $L^\infty((0,T); H^{-3}(O))$. It follows by interpolation that there exists a subsequence such that $\{Eu^\varepsilon\}$ converges strongly in $L^2([0,T] \times O)$. By taking diagonal subsequences in the set of the compact subset of $\mathbb{R}^2 \setminus \Gamma$ and in the time, we may assume that there is a subsequence which converges strongly in $L^2_{\mathrm{loc}}([0,\infty) \times (\mathbb{R}^2 \setminus \Gamma))$. □

We will prove that the limits of the sequence $\{Eu^\varepsilon\}$ are solutions of the Navier–Stokes equations on the exterior of a curve in a suitable weak sense. The difficulty is that Eu^ε does not belong to $L^2(\mathbb{R}^2)$. So, as we did in corollary 3.6, we should keep the harmonic part v^ε. Since we previously obtained a limit for Eu^ε, now we look for a limit for v^ε. We recall that $v^\varepsilon = \alpha H_\varepsilon \Phi^{\varepsilon,\lambda}$, with H_ε and $\Phi^{\varepsilon,\lambda}$ given by (2.2) and (2.7). We also denote H and $\Phi^{0,\lambda}$ as H^ε and $\Phi^{\varepsilon,\lambda}$ by replacing T_ε by T.

LEMMA 4.3. *If we define* $v \equiv \alpha H \Phi^{0,\lambda}$, *then* $v_\varepsilon \to v$ *in* $L^2_{\mathrm{loc}}(\mathbb{R}^2)$.

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Proof. For any compact K of $\mathbb{R}^2$, using the explicit formula of v^ε and v, we have

$$
\begin{aligned}
&\|v^\varepsilon - v\|_{L^2(K)} \\
&\quad = \frac{\alpha}{2\pi}\left\|\Phi^{\varepsilon,\lambda}\left(DT_\varepsilon^t\frac{T_\varepsilon^\perp}{|T_\varepsilon|^2} - DT^t\frac{T^\perp}{|T|^2}\right) + (\Phi^{\varepsilon,\lambda} - \Phi^{0,\lambda})\left(DT^t\frac{T^\perp}{|T|^2}\right)\right\|_{L^2(K)} \\
&\quad \leqslant \frac{\alpha}{2\pi}\left\|\Phi^{\varepsilon,\lambda}\left(DT_\varepsilon^t\frac{T_\varepsilon^\perp}{|T_\varepsilon|^2} - DT^t\frac{T^\perp}{|T|^2}\right)\right\|_{L^2(K)} + \frac{\alpha}{2\pi}\|\Phi^{\varepsilon,\lambda} - \Phi^{0,\lambda}\|_{L^\infty}\|DT^t\|_{L^2(K)}.
\end{aligned}
$$

Recalling that $\Phi^{\varepsilon,\lambda} = 0$ on a ball of radius $C_1\lambda$, we can conclude from assumption 2.4(iii) and remark 2.5 that the first term tends to zero. For the second term, we note that the cut-off function Φ is Lipschitz, and by the explicit formula of $\Phi^{\varepsilon,\lambda}$ given in (2.7) we conclude that

$$
|\Phi^{\varepsilon,\lambda}(x) - \Phi^{0,\lambda}(x)| \leqslant (\sup|\Phi'|)\left|\frac{|T_\varepsilon(x)| - |T(x)|}{\lambda}\right|.
$$

Then, on the annulus (chosen independently of ε) where $\Phi^{\varepsilon,\lambda} - \Phi^{0,\lambda}$ is not zero, the previous term tends to zero due to remark 2.5. □

Therefore, we can formulate precisely the notion of weak solution to be used.

Definition 4.4. Let u_0 be such that $u_0 - v \in \mathcal{H}_\Gamma$. We say that u is a weak solution of the incompressible Navier–Stokes equations on $\mathbb{R}^+ \times (\mathbb{R}^2 \setminus \Gamma)$ with initial velocity u_0 if and only if $u - v$ belongs to the space

$$
C([0,\infty); \mathcal{H}_\Gamma) \cap L^2_{\mathrm{loc}}([0,\infty); \mathcal{V}_\Gamma)
$$

and for any divergence-free test vector field $\psi \in C_{\mathrm{c}}^\infty((0,\infty) \times (\mathbb{R}^2 \setminus \Gamma))$, the vector field u satisfies the following condition:

$$
\int_0^\infty \int_{\mathbb{R}^2\setminus\Gamma} (u\cdot\psi_t + [(u\cdot\nabla)\psi]\cdot u + \nu u\cdot\Delta\psi)\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{4.1}
$$

Furthermore, $\operatorname{div} u = 0$ in the sense of distributions, and $u(\cdot,t) \rightharpoonup u_0$ in the sense of distributions as $t \to 0^+$.

Remark 4.5. In fact, if we prove that the vector field u verifies (4.1) for all divergence-free test vector fields $\psi \in C_{\mathrm{c}}^\infty((0,\infty) \times (\mathbb{R}^2 \setminus \Gamma))$, with $u - v$ belonging to $L^2_{\mathrm{loc}}([0,\infty); \mathcal{V}_\Gamma) \cap L^\infty_{\mathrm{loc}}((0,\infty); \mathcal{H}_\Gamma)$, then

$$
\partial_t u \in L^2_{\mathrm{loc}}([0,\infty), \mathcal{V}'_\Gamma). \tag{4.2}
$$

Indeed, with lemma 3.2 and the interpolation inequality

$$
\|u - v\|_{L^4(L^4)} \leqslant C\|u - v\|^{1/2}_{L^\infty(L^2)}\|\nabla(u - v)\|^{1/2}_{L^2(L^2)},
$$

we note that u belongs to

$$
L^4_{\mathrm{loc}}([0,\infty); L^4(\mathbb{R}^2 \setminus \Gamma))
$$

and ∇u belongs to $L^2_{\mathrm{loc}}([0,\infty); L^2(\mathbb{R}^2 \setminus \Gamma))$. For each $T > 0$, using (4.1) and theorem 3.5 for each divergence-free function $\psi \in C^\infty_{\mathrm{c}}((0,T) \times (\mathbb{R}^2 \setminus \Gamma))$, we have

$$\begin{aligned}\langle \partial_t u, \psi \rangle &\leqslant (\|u\|^2_{L^4((0,T);L^4)} + \nu \|\nabla u\|_{L^2((0,T);L^2)}) \|\nabla \psi\|_{L^2((0,T);L^2)} \\ &\leqslant C \|\psi\|_{L^2((0,T);\mathcal{V}_\Gamma)}\end{aligned}$$

with a constant $C > 0$. As the set of divergence-free functions belonging to

$$C^\infty_{\mathrm{c}}((0,T) \times (\mathbb{R}^2 \setminus \Gamma))$$

is dense on $L^2([0,T], \mathcal{V}_\Gamma)$ (by proposition 3.7), the linear form

$$\psi \mapsto \iint \partial_t u \cdot \psi$$

is bounded on $L^2([0,T], \mathcal{V}_\Gamma)$, so (4.2) follows.

Theorem 4.6. *There exists one strong limit u of $\{Eu^\varepsilon\}$ in $L^2_{\mathrm{loc}}([0,\infty) \times (\mathbb{R}^2 \setminus \Gamma))$ which is a weak solution of the Navier–Stokes equations in $\mathbb{R}^2 \setminus \Gamma$ in the sense of definition 4.4, with initial velocity given by $u_0 = K[\omega_0] + \alpha H$.*

Proof. By lemmas 2.8 and 4.3, we know that $Eu^\varepsilon_0 - v^\varepsilon \to u_0 - v$ in $L^2_{\mathrm{loc}}(\mathbb{R}^2)$. According to theorem 3.5, $u_0 - v$ belongs to $L^2(\mathbb{R}^2)$. Moreover, $Eu^\varepsilon_0 - v^\varepsilon$ is supported in a smooth domain (Π_ε); thus, we can approximate it by functions belonging to V_Γ. Then, by a diagonal extraction, we obtain that $u_0 - v \in \mathcal{H}_\Gamma$.

Let $\psi \in C^\infty_{\mathrm{c}}((0,\infty) \times (\mathbb{R}^2 \setminus \Gamma))$ such that $\operatorname{div} \psi = 0$. If we consider ε small enough such that the support of ψ does not intersect Ω_ε, we can rewrite the integrals on Π_ε as full-planar integrals, using the extension operator and multiplying (2.5) by ψ. We obtain the following relation:

$$\int_0^\infty \int_{\mathbb{R}^2 \setminus \Gamma} (Eu^\varepsilon \cdot \psi_t + [(Eu^\varepsilon \cdot \nabla)\psi] \cdot Eu^\varepsilon + \nu Eu^\varepsilon \cdot \Delta \psi) \, \mathrm{d}x \, \mathrm{d}t = 0.$$

Due to the convergence of Eu^ε to a vector field u in $L^2_{\mathrm{loc}}([0,\infty) \times \mathbb{R}^2 \setminus \Gamma)$ (see lemma 4.2), we can pass to the limit $\varepsilon \to 0$ and obtain that u satisfies (4.1).

Moreover, v^ε tends to v (see lemma 4.3) so, passing to a subsequence if necessary, corollary 3.6 implies that $u - v$ belongs in $L^2_{\mathrm{loc}}([0,\infty); \mathcal{V}_\Gamma) \cap L^\infty_{\mathrm{loc}}((0,\infty); \mathcal{H}_\Gamma)$. The incompressible condition is a consequence of the strong convergence of divergence-free vector fields (lemma 4.2).

Now, we prove that $u - v$ belongs to $C([0,\infty); \mathcal{H}_\Gamma)$. We know from corollary 3.6 that $u - v$ belongs to $L^2([0,T]; \mathcal{V}_\Gamma)$ and from remark 4.5 that its derivative $\partial_t(u - v)$ belongs to $L^2([0,T]; \mathcal{V}'_\Gamma)$. As

$$\mathcal{V}_\Gamma \hookrightarrow \mathcal{H}_\Gamma \equiv \mathcal{H}'_\Gamma \hookrightarrow \mathcal{V}'_\Gamma,$$

lemma 1.2 of [15, ch. III] (see also the Lions–Magenes theorem of interpolation [11]) allows us to state that $u - v$ is almost everywhere equal to a function continuous from $(0,T)$ into $\mathcal{H}_\Gamma$ and we have the following equality, which holds in the scalar distribution sense on $(0,T)$:

$$\frac{\mathrm{d}}{\mathrm{d}t} |u - v|^2 = 2 \langle \partial_t (u - v), u - v \rangle. \tag{4.3}$$

Therefore, $u - v \in C([0,\infty); \mathcal{H}_\Gamma)$.

Furthermore, since Eu^ε converges to u uniformly in time with values in $H^{-3}_{\mathrm{loc}}(\mathbb{R}^2 \setminus \Gamma)$ (by proposition 4.1), one has that Eu_0^ε converges to $u_{t=0}$ in H^{-3}_{loc}. On the other hand, lemma 2.8 states that Eu_0^ε converges to $K[\omega_0] + \alpha H$ in $L^2_{\mathrm{loc}}(\mathbb{R}^2)$. By uniqueness of the limit in H^{-3}_{loc}, we conclude that $u_0 = K[\omega_0] + \alpha H$, which completes the proof. □

5. Uniqueness for the limit problem

We now state the uniqueness result that completes theorem 4.6.

PROPOSITION 5.1. *There exists at most one global solution in the sense of definition 4.4, verifying that the initial velocity is* $u_0 = K[\omega_0] + \alpha H$.

Proof. Let u_1 and u_2 be two global solutions of the Navier–Stokes equations around the curve Γ with the same initial velocity $u_0 = K[\omega_0] + \alpha H$. By remark 4.5 we have that the $\partial_t u_i$ belong to $L^2_{\mathrm{loc}}([0,\infty), \mathcal{V}'_\Gamma)$, for $i = 1, 2$.

If we define $\tilde{u} = u_1 - u_2$, then, by proposition 3.7, for a fixed $T > 0$ there exists a divergence-free family $\{\psi_n\}$ in $C^\infty_{\mathrm{c}}((0,T) \times \mathbb{R}^2 \setminus \Gamma)$ such that $\psi_n \to \tilde{u}$ in $L^2([0,T]; \mathcal{V}_\Gamma)$.

Subtracting the equations satisfied by u_1 and u_2, and multiplying by the test function ψ_n, we see that

$$\int_0^T \int_{\mathbb{R}^2\setminus\Gamma} \partial_t \tilde{u} \cdot \psi_n \,\mathrm{d}x\,\mathrm{d}t - \nu \int_0^T \int_{\mathbb{R}^2\setminus\Gamma} \tilde{u} \cdot \Delta\psi_n \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{\mathbb{R}^2\setminus\Gamma} ([(\tilde{u}\cdot\nabla)\psi_n] \cdot u_1 + [(u_2\cdot\nabla)\psi_n]\cdot\tilde{u}) \,\mathrm{d}x\,\mathrm{d}t. \quad (5.1)$$

Using the interpolation inequality

$$\|u^\varepsilon\|_{L^4(L^4)} \leqslant C \|u^\varepsilon\|^{1/2}_{L^\infty(L^2)} \|\nabla u^\varepsilon\|^{1/2}_{L^2(L^2)},$$

the right-hand side term can be bounded by

$$\begin{aligned}\int_0^T & \|\tilde{u}\|_{L^4} (\|u_1\|_{L^4} + \|u_2\|_{L^4}) \|\nabla\psi_n\|_{L^2} \\ &\leqslant C \int_0^T \|\nabla\psi_n\|_{L^2} \|\nabla\tilde{u}\|^{1/2}_{L^2} \|\tilde{u}\|^{1/2}_{L^2} (\|u_1\|_{L^4} + \|u_2\|_{L^4}) \\ &\leqslant \frac{\nu}{2} \int_0^T \|\nabla\psi_n\|^2_{L^2} + \frac{\nu}{2} \int_0^T \|\nabla\tilde{u}\|^2_{L^2} + C_1 \int_0^T \|\tilde{u}\|^2_{L^2} (\|u_1\|^4_{L^4} + \|u_2\|^4_{L^4}),\end{aligned}$$

with constants C and C_1 independent of T. For the left hand-side term, by (4.3) and because $\tilde{u}(\cdot, 0) = 0$, we can write

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^2\setminus\Gamma} \partial_t\tilde{u}\cdot\psi_n \,\mathrm{d}x\,\mathrm{d}t &= \int_0^T \int_{\mathbb{R}^2\setminus\Gamma} \partial_t\tilde{u}\cdot\tilde{u} \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^2\setminus\Gamma} \partial_t\tilde{u}\cdot(\psi_n - \tilde{u}) \,\mathrm{d}x\,\mathrm{d}t \\ &= \tfrac{1}{2}\|\tilde{u}(\cdot,T)\|_{L^2(\mathbb{R}^2)} + \int_0^T \int_{\mathbb{R}^2\setminus\Gamma} \partial_t\tilde{u}\cdot(\psi_n - \tilde{u}) \,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

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The last double integral tends to 0 as $n \to \infty$ because $\partial_t \tilde{u}$ belongs to $L^2_{\mathrm{loc}}([0,\infty);\mathcal{V}'_\Gamma)$ and ψ_n converges to $\tilde{u}$ in $L^2([0,T];\mathcal{V}_\Gamma)$.

In the same way, we have

$$
\begin{aligned}
-\lim_{n\to\infty}\int_0^T\int_{\mathbb{R}^2\setminus\Gamma}\tilde{u}\cdot\Delta\psi_n\,\mathrm{d}x\,\mathrm{d}t
&=\lim_{n\to\infty}\int_0^T\int_{\mathbb{R}^2\setminus\Gamma}\nabla\tilde{u}\cdot\nabla\psi_n\,\mathrm{d}x\,\mathrm{d}t\\
&=\int_0^T\int_{\mathbb{R}^2\setminus\Gamma}\nabla\tilde{u}\cdot\nabla\tilde{u}\,\mathrm{d}x\,\mathrm{d}t+\lim_{n\to\infty}\int_0^T\int_{\mathbb{R}^2\setminus\Gamma}\nabla\tilde{u}\cdot(\nabla\psi_n-\nabla\tilde{u})\,\mathrm{d}x\,\mathrm{d}t\\
&=\|\nabla\tilde{u}\|^2_{L^2([0,T],L^2(\mathbb{R}^2))},
\end{aligned}
$$

because $\nabla\tilde{u}$ belongs to $L^2([0,T];\mathcal{H}_\Gamma)$ and $\nabla\psi_n$ converges to $\nabla\tilde{u}$ in $L^2([0,T];\mathcal{H}_\Gamma)$. This convergence also implies that

$$\lim_{n\to\infty}\|\nabla\psi_n\|^2_{L^2([0,T]\times\mathbb{R}^2)}=\|\nabla\tilde{u}\|^2_{L^2([0,T]\times\mathbb{R}^2)}.$$

Therefore, passing to the limit $n\to\infty$ in (5.1) yields

$$\|\tilde{u}(\cdot,T)\|^2_{L^2}\leqslant 2C_1\int_0^T\|\tilde{u}\|^2_{L^2}(\|u_1\|^4_{L^4}+\|u_2\|^4_{L^4}).$$

The latter equality holds for all $T>0$, with the constant C_1 independent of T. Noting that the functions $t\mapsto\|\tilde{u}(\cdot,t)\|^2_{L^2}$, $t\mapsto(\|u_1(\cdot,t)\|^4_{L^4}+\|u_2(\cdot,t)\|^4_{L^4})$ and $t\mapsto\|\tilde{u}(\cdot,t)\|^2_{L^2}(\|u_1(\cdot,t)\|^4_{L^4}+\|u_2(\cdot,t)\|^4_{L^4})$ are L^1_{loc}, we can apply Gronwall's lemma to obtain

$$\|\tilde{u}(\cdot,T)\|^2_{L^2}\leqslant 0,$$

which concludes the proof of uniqueness. □

Once the uniqueness of the limit velocity is established, and given that by theorem 4.6 we know that from every sequence of solutions u^ε we can extract a subsequence converging in $L^2_{\mathrm{loc}}([0,\infty)\times(\mathbb{R}^2\setminus\Gamma))$, we deduce with a standard argument that strong convergence in $L^2_{\mathrm{loc}}([0,\infty)\times(\mathbb{R}^2\setminus\Gamma))$ holds without the need to extract a subsequence. Theorem 1.1 is therefore completely proved.

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Appendix A. Notation

A.1. Domains

The following domains are used in this paper:

- $D\equiv B(0,1)$ is the unit disc;
- $S\equiv\partial D$;

- Γ is a Jordan arc (see proposition 2.2);
- $\Pi \equiv \mathbb{R}^2 \setminus \Gamma$;
- Ω_ε is a bounded, open, connected, simply connected subset of the plane, such as $\Omega_\varepsilon \to \Gamma$ as $\varepsilon \to 0$;
- $\Gamma_\varepsilon \equiv \partial\Omega_\varepsilon$ is a C^∞ Jordan curve;
- $\Pi_\varepsilon \equiv \mathbb{R}^2 \setminus \bar{\Omega}_\varepsilon$.

A.2. Functions

The following functions are used in this paper:

ω_0	the initial vorticity ($C_c^\infty(\Pi)$);
γ	the circulation of u_0^ε on Γ_ε (see § 1);
u^ε	the solution of the Navier–Stokes equations on Π_ε;
T	a biholomorphism between Π and int D^c (see proposition 2.2);
T_ε	a biholomorphism between Π_ε and int D^c (see assumption 2.4);
K^ε and H^ε	as given in (2.1) and (2.2); $K^\varepsilon[\omega_0](x) \equiv \int_{\Pi_\varepsilon} K^\varepsilon(x,y)\omega_0(y)\,\mathrm{d}y$;
$\Phi^{\varepsilon,\lambda}$	a cut-off function (see (2.7)).

$V(\Omega)$, $\mathcal{V}(\Omega)$, $\mathcal{V}'(\Omega)$, $\mathcal{H}(\Omega)$, $\mathcal{V}_\Gamma$ and $\mathcal{H}_\Gamma$ are some vector spaces defined in definition 3.1.

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