

CONVOLUTION SUMS INVOLVING THE DIVISOR FUNCTION

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Abstract The series

$$\begin{aligned}L_{r,4}(q) &= \sum_{n=0}^{\infty} \sigma(4n+r)q^{4n+r}, \quad r = 0, 1, 2, 3, \\M_{r,4}(q) &= \sum_{n=0}^{\infty} \sigma_3(4n+r)q^{4n+r}, \quad r = 0, 1, 2, 3, \\N_{r,4}(q) &= \sum_{n=0}^{\infty} \sigma_5(4n+r)q^{4n+r}, \quad r = 0, 1, 2, 3,\end{aligned}$$

are evaluated and used to prove convolution formulae such as

$$\sum_{m \leq n} \sigma(4m-3)\sigma(4n-(4m-3)) = 4\sigma_3(n) - 4\sigma_3\left(\frac{1}{2}n\right).$$

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1. Introduction

For $f, n \in \mathbb{N}$, we let $\sigma_f(n)$ denote the sum of the f th powers of the positive divisors of n :

$$\sigma_f(n) = \sum_{d|n} d^f.$$

If $n \notin \mathbb{N}$, we set $\sigma_f(n) = 0$. We set $\sigma_1(n) = \sigma(n)$.

In 1916 Ramanujan (see [7] and [9, pp. 136–162]) introduced the three functions $L(q)$, $M(q)$ and $N(q)$ defined for $q \in \mathbb{C}$ with $|q| < 1$ by

$$L(q) = 1 - 24 \sum_{n=1}^{\infty} \sigma(n)q^n, \tag{1.1}$$

$$M(q) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n, \quad (1.2)$$

$$N(q) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^n. \quad (1.3)$$

It is known that the series $L(q)$, $M(q)$ and $N(q)$ are algebraically independent [10, p. 69]. Ramanujan (see [7] and [9, pp. 136–162]) proved (among many others) the two formulae

$$L^2(q) = M(q) + 12q \frac{dL}{dq} \quad (1.4)$$

and

$$L(q)M(q) = N(q) + 3q \frac{dM}{dq}, \quad (1.5)$$

from which follow, by equating the coefficients of q^n on both sides, the arithmetic identities

$$\sum_{m < n} \sigma(m)\sigma(n-m) = \frac{5}{12}\sigma_3(n) + \frac{1}{12}\sigma(n) - \frac{1}{2}n\sigma(n) \quad (1.6)$$

and

$$\sum_{m < n} \sigma(m)\sigma_3(n-m) = \frac{7}{80}\sigma_5(n) - \frac{1}{8}n\sigma_3(n) + \frac{1}{24}\sigma_3(n) - \frac{1}{240}\sigma(n). \quad (1.7)$$

In all Ramanujan obtained nine identities of the type (1.6) and (1.7). For the history of such formulae see [4].

In this paper we consider the related series

$$L_{r,4}(q) = \sum_{n=0}^{\infty} \sigma(4n+r)q^{4n+r}, \quad r = 0, 1, 2, 3, \quad (1.8)$$

$$M_{r,4}(q) = \sum_{n=0}^{\infty} \sigma_3(4n+r)q^{4n+r}, \quad r = 0, 1, 2, 3, \quad (1.9)$$

$$N_{r,4}(q) = \sum_{n=0}^{\infty} \sigma_5(4n+r)q^{4n+r}, \quad r = 0, 1, 2, 3. \quad (1.10)$$

In §2 we obtain formulae for $L_{r,4}(q)$, $M_{r,4}(q)$ and $N_{r,4}(q)$ ($r = 0, 1, 2, 3$) similar to those given by Ramanujan (see [7] and [9, pp. 136–162]) for $L(q)$, $M(q)$ and $N(q)$ (see Theorem 2.1). In §3 we use these formulae to determine which products $L_{r,4}(q)L_{s,4}(q)$ and $L_{r,4}(q)M_{s,4}(q)$ ($0 \leq r \leq s \leq 3$) can be expressed in terms of the functions L , M and N and their derivatives (see Theorem 3.1). As a consequence of these identities we obtain, in §4, a number of arithmetic identities analogous to (1.6) and (1.7) (see Theorem 4.1). In §5 we prove the formulae

$$L(q)L(q^2) = \frac{1}{5}(M(q) + 4M(q^2)) + 3\left(q \frac{dL(q)}{dq} + 2q \frac{dL(q^2)}{dq}\right) \quad (1.11)$$

and

$$L(q)L(q^4) = \frac{1}{20}(M(q) + 3M(q^2) + 16M(q^4)) + \frac{1}{2}\left(3q\frac{dL(q)}{dq} + 12q\frac{dL(q^4)}{dq}\right) \tag{1.12}$$

(see Theorem 5.1), from which we deduce the arithmetic identities

$$\sum_{m < n/2} \sigma(m)\sigma(n - 2m) = \frac{1}{24}\{2\sigma_3(n) + 8\sigma_3(\frac{1}{2}n) + \sigma(n) + \sigma(\frac{1}{2}n) - 3n\sigma(n) - 6n\sigma(\frac{1}{2}n)\} \tag{1.13}$$

and

$$\sum_{m < n/4} \sigma(m)\sigma(n - 4m) = \frac{1}{48}\{\sigma_3(n) + 3\sigma_3(\frac{1}{2}n) + 16\sigma_3(\frac{1}{4}n) + 2\sigma(n) + 2\sigma(\frac{1}{4}n) - 3n\sigma(n) - 12n\sigma(\frac{1}{4}n)\} \tag{1.14}$$

(see Theorem 5.2), which are due to Melfi [5, 6] for odd positive integers n and to Huard and co-workers [4] for all positive integers n .

2. Formulae for $L_{r,4}(q)$, $M_{r,4}(q)$ and $N_{r,4}(q)$

Let q be a real number satisfying

$$0 < q < 1. \tag{2.1}$$

Then

$$0 < -\log q < \infty. \tag{2.2}$$

The derivative y' of the function

$$y = y(x) = \frac{\pi_2 F_1(\frac{1}{2}, \frac{1}{2}; 1; 1 - x)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)} \tag{2.3}$$

is given by

$$y' = -\frac{x^{-1}(1 - x)^{-1}}{\{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)\}^2} \tag{2.4}$$

(see, for example, Berndt [1, p. 87]). For $0 < x < 1$ we have

$${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x) = \sum_{n=0}^{\infty} \frac{1}{2^{4n}} \binom{2n}{n} x^n > 0, \tag{2.5}$$

so that by (2.4) and (2.5) we see that

$$y' < 0 \quad \text{for } 0 < x < 1. \tag{2.6}$$

By (2.6) y is a strictly decreasing function of x for $0 < x < 1$. As $y(0) = \infty$ and $y(1) = 0$, the function y decreases from ∞ to 0 as x increases from 0 to 1. Hence there exists a unique value of x between 0 and 1 such that

$$y = \frac{\pi_2 F_1(\frac{1}{2}, \frac{1}{2}; 1; 1 - x)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)} = -\log q. \tag{2.7}$$

Therefore,

$$q = \exp(-y) = \exp\left(-\frac{\pi_2 F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-x)}{2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)}\right). \quad (2.8)$$

We also set

$$w = {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x). \quad (2.9)$$

Ramanujan gave in his notebooks [8] the following formulae for $L(q)$, $M(q)$ and $N(q)$:

$$L(q) = (1 - 5x)w^2 + 12x(1 - x)w \frac{dw}{dx}, \quad (2.10)$$

$$M(q) = (1 + 14x + x^2)w^4, \quad (2.11)$$

$$N(q) = (1 + x)(1 - 34x + x^2)w^6. \quad (2.12)$$

Formulae (2.10)–(2.12) are proved in [2, pp. 127, 129].

It is shown in Berndt [2, p. 125] that if

$$\Omega(x, q, w) = 0 \quad (2.13)$$

is a relationship between x , q and w —where q satisfies (2.1), x is given in terms of q by (2.7) (or (2.8)) and w is given by (2.9)—then the relationship

$$\Omega\left(\left(\frac{1 - \sqrt{1-x}}{1 + \sqrt{1-x}}\right)^2, q^2, \frac{1}{2}w(1 + \sqrt{1-x})\right) = 0 \quad (2.14)$$

also holds. The second formula is said to be obtained from the first formula by duplication since q is changed to q^2 .

Applying the process of duplication to (2.10)–(2.12), we obtain

$$L(q^2) = (1 - 2x)w^2 + 6x(1 - x)w \frac{dw}{dx}, \quad (2.15)$$

$$M(q^2) = (1 - x + x^2)w^4, \quad (2.16)$$

$$N(q^2) = (1 + x)(1 - \frac{1}{2}x)(1 - 2x)w^6 \quad (2.17)$$

(see [2, pp. 126, 127]). Applying duplication to (2.15)–(2.17), we obtain

$$L(q^4) = (1 - \frac{5}{4}x)w^2 + 3x(1 - x)w \frac{dw}{dx}, \quad (2.18)$$

$$M(q^4) = (1 - x + \frac{1}{16}x^2)w^4, \quad (2.19)$$

$$N(q^4) = (1 - \frac{1}{2}x)(1 - x - \frac{1}{32}x^2)w^6 \quad (2.20)$$

(see [2, pp. 126, 127]). Again by duplication from (2.18)–(2.20), we have with $g = (1 - x)^{1/4}$

$$L(q^8) = \frac{1}{16}(-1 + 6g^2 + 11g^4)w^2 + \frac{3}{2}g^4(1 - g^4)w \frac{dw}{dx}, \quad (2.21)$$

$$M(q^8) = \frac{1}{256}(1 + 60g^2 + 134g^4 + 60g^6 + g^8)w^4, \quad (2.22)$$

$$N(q^8) = -\frac{1}{4096}(1 + 6g^2 + g^4)(1 - 132g^2 - 250g^4 - 132g^6 + g^8)w^6. \quad (2.23)$$

Berndt [2, p. 126] has also described the process of obtaining a new formula from (2.13) by changing the sign of q . If (2.13) holds, then the formula

$$\Omega\left(\frac{x}{x-1}, -q, w\sqrt{1-x}\right) = 0 \tag{2.24}$$

also holds. This result is attributed to Jacobi by Berndt [2, p. 126].

Applying Jacobi’s change-of-sign procedure to (2.10)–(2.12), we obtain

$$L(-q) = (1 - 2x)w^2 + 12x(1 - x)w\frac{dw}{dx}, \tag{2.25}$$

$$M(-q) = (1 - 16x + 16x^2)w^4, \tag{2.26}$$

$$N(-q) = (1 - 2x)(1 + 32x - 32x^2)w^6. \tag{2.27}$$

In Cheng [3, pp. 195–208] the process of obtaining a new formula from (2.13) by rotation, that is, $q \rightarrow iq$, is given and proved. It is shown that if (2.13) holds, then so does the formula

$$\Omega\left(\frac{-8i(1-x)^{1/4}(1-\sqrt{1-x})}{(1-i(1-x)^{1/4})^4}, iq, i\frac{1}{2}w(1-i(1-x)^{1/4})^2\right) = 0. \tag{2.28}$$

Applying this process to (2.10)–(2.12), we obtain with $g = (1-x)^{1/4}$

$$L(iq) = -\frac{1}{4}(1 + 12ig + 18g^2 - 12ig^3 - 23g^4)w^2 + 12g^4(1 - g^4)w\frac{dw}{dx}, \tag{2.29}$$

$$M(iq) = \frac{1}{16}(1 - 120ig - 540g^2 + 840ig^3 + 1094g^4 - 840ig^5 - 540g^6 + 120ig^7 + g^8)w^4, \tag{2.30}$$

$$N(iq) = \frac{1}{64}(1 - 12ig - 6g^2 + 12ig^3 + g^4) \times (-1 - 264ig - 996g^2 + 1848ig^3 + 1978g^4 - 1848ig^5 - 996g^6 + 264ig^7 - g^8)w^6. \tag{2.31}$$

Next, by applying Jacobi’s change-of-sign procedure to (2.29)–(2.31), we deduce

$$L(-iq) = -\frac{1}{4}(1 - 12ig + 18g^2 + 12ig^3 - 23g^4)w^2 + 12g^4(1 - g^4)w\frac{dw}{dx}, \tag{2.32}$$

$$M(-iq) = \frac{1}{16}(1 + 120ig - 540g^2 - 840ig^3 + 1094g^4 + 840ig^5 - 540g^6 - 120ig^7 + g^8)w^4, \tag{2.33}$$

$$N(-iq) = \frac{1}{64}(1 + 12ig - 6g^2 - 12ig^3 + g^4) \times (-1 + 264ig - 996g^2 - 1848ig^3 + 1978g^4 + 1848ig^5 - 996g^6 - 264ig^7 - g^8)w^6. \tag{2.34}$$

A simple calculation shows that

$$L_{0,4}(q) = \frac{1}{96}(4 - L(q) - L(-q) - L(iq) - L(-iq)), \tag{2.35}$$

$$L_{1,4}(q) = \frac{1}{96}(-L(q) + L(-q) + iL(iq) - iL(-iq)), \tag{2.36}$$

$$L_{2,4}(q) = \frac{1}{96}(-L(q) - L(-q) + L(iq) + L(-iq)), \quad (2.37)$$

$$L_{3,4}(q) = \frac{1}{96}(-L(q) + L(-q) - iL(iq) + iL(-iq)), \quad (2.38)$$

$$M_{0,4}(q) = \frac{1}{960}(-4 + M(q) + M(-q) + M(iq) + M(-iq)), \quad (2.39)$$

$$M_{1,4}(q) = \frac{1}{960}(M(q) - M(-q) - iM(iq) + iM(-iq)), \quad (2.40)$$

$$M_{2,4}(q) = \frac{1}{960}(M(q) + M(-q) - M(iq) - M(-iq)), \quad (2.41)$$

$$M_{3,4}(q) = \frac{1}{960}(M(q) - M(-q) + iM(iq) - iM(-iq)), \quad (2.42)$$

$$N_{0,4}(q) = \frac{1}{2016}(4 - N(q) - N(-q) - N(iq) - N(-iq)), \quad (2.43)$$

$$N_{1,4}(q) = \frac{1}{2016}(-N(q) + N(-q) + iN(iq) - iN(-iq)), \quad (2.44)$$

$$N_{2,4}(q) = \frac{1}{2016}(-N(q) - N(-q) + N(iq) + N(-iq)), \quad (2.45)$$

$$N_{3,4}(q) = \frac{1}{2016}(-N(q) + N(-q) - iN(iq) + iN(-iq)). \quad (2.46)$$

Using (2.10)–(2.12), (2.25)–(2.27), (2.29)–(2.34) in (2.35)–(2.46), we obtain the following theorem.

Theorem 2.1.

$$L_{0,4}(q) = \frac{1}{192} \left(8 + (11 + 18g^2 - 37g^4)w^2 - 96g^4(1 - g^4)w \frac{dw}{dx} \right), \quad (2.47)$$

$$L_{1,4}(q) = \frac{1}{32}(1 - g)(1 + g)^3 w^2, \quad (2.48)$$

$$L_{2,4}(q) = \frac{3}{64}(1 - g)^2(1 + g)^2 w^2, \quad (2.49)$$

$$L_{3,4}(q) = \frac{1}{32}(1 - g)^3(1 + g)w^2, \quad (2.50)$$

$$M_{0,4}(q) = \frac{1}{7680}(-32 + (137 - 540g^2 + 838g^4 - 540g^6 + 137g^8)w^4), \quad (2.51)$$

$$M_{1,4}(q) = \frac{1}{64}(1 - g)(1 + g)^3(1 - 3g + 6g^2 - 3g^3 + g^4)w^4, \quad (2.52)$$

$$M_{2,4}(q) = \frac{9}{512}(1 - g)^2(1 + g)^2(1 + 6g^2 + g^4)w^4, \quad (2.53)$$

$$M_{3,4}(q) = \frac{1}{64}(1 - g)^3(1 + g)(1 + 3g + 6g^2 + 3g^3 + g^4)w^4, \quad (2.54)$$

$$N_{0,4}(q) = \frac{1}{64 \cdot 512}(128 + (1 + 6g^2 + g^4)(2081 - 8328g^2 + 12478g^4 - 8328g^6 + 2081g^8)w^6), \quad (2.55)$$

$$N_{1,4}(q) = \frac{1}{256}(1 - g)(1 + g)^3 \\ \times (8 - 15g + 30g^2 - 105g^3 + 172g^4 - 105g^5 + 30g^6 - 15g^7 + 8g^8)w^6, \quad (2.56)$$

$$N_{2,4}(q) = \frac{33}{1024}(1 - g)^2(1 + g)^2(1 + 2g + 2g^2 - 2g^3 + g^4)(1 - 2g + 2g^2 + 2g^3 + g^4)w^6, \quad (2.57)$$

$$N_{3,4}(q) = \frac{1}{256}(1 - g)^3(1 + g) \\ \times (8 + 15g + 30g^2 + 105g^3 + 172g^4 + 105g^5 + 30g^6 + 15g^7 + 8g^8)w^6, \quad (2.58)$$

where $g = (1 - x)^{1/4}$.

We conclude this section by noting a few results which will be used in §§ 3 and 5.

The function $w = {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)$ satisfies the differential equation

$$x(1-x)\frac{d^2w}{dx^2} + (1-2x)\frac{dw}{dx} - \frac{1}{4}w = 0$$

so that

$$\frac{d^2w}{dx^2} = \frac{w}{4x(1-x)} - \frac{(1-2x)}{x(1-x)} \frac{dw}{dx}. \tag{2.59}$$

By (2.4), (2.7) and (2.9), we have

$$\frac{1}{q} \frac{dq}{dx} = -\frac{dy}{dx} = \frac{1}{x(1-x)w^2}$$

so that

$$\frac{dq}{dx} = \frac{q}{x(1-x)w^2}. \tag{2.60}$$

From (2.10), (2.59) and (2.60), we obtain

$$\begin{aligned} \frac{dL(q)}{dq} &= \frac{dL(q)}{dx} \Big/ \frac{dq}{dx} \\ &= \frac{d}{dx} \left((1-5x)w^2 + 12x(1-x)w \frac{dw}{dx} \right) \Big/ \frac{q}{x(1-x)w^2} \\ &= \left(-5w^2 + (14-34x)w \frac{dw}{dx} + 12x(1-x) \left(\frac{dw}{dx} \right)^2 + 12x(1-x)w \frac{d^2w}{dx^2} \right) \Big/ \frac{q}{x(1-x)w^2} \\ &= \left(-2w^2 + (2-10x)w \frac{dw}{dx} + 12x(1-x) \left(\frac{dw}{dx} \right)^2 \right) \Big/ \frac{q}{x(1-x)w^2} \end{aligned}$$

so that

$$q \frac{dL(q)}{dq} = -2x(1-x)w^4 + 2x(1-x)(1-5x)w^3 \frac{dw}{dx} + 12x^2(1-x)^2w^2 \left(\frac{dw}{dx} \right)^2. \tag{2.61}$$

Similarly from (2.15), (2.18), (2.21), (2.59) and (2.60), we obtain

$$q \frac{dL(q^2)}{dq} = -\frac{1}{2}x(1-x)w^4 + 2x(1-x)(1-2x)w^3 \frac{dw}{dx} + 6x^2(1-x)^2w^2 \left(\frac{dw}{dx} \right)^2, \tag{2.62}$$

$$q \frac{dL(q^4)}{dq} = -\frac{1}{2}g^4(1-g^4)w^4 + g^4(1-g^4)\left(-\frac{1}{2} + \frac{5}{2}g^4\right)w^3 \frac{dw}{dx} + 3g^8(1-g^4)^2w^2 \left(\frac{dw}{dx} \right)^2 \tag{2.63}$$

and

$$\begin{aligned} q \frac{dL(q^8)}{dq} &= -\frac{1}{16}(1-g^4)(3g^2+5g^4)w^4 - \frac{1}{8}g^4(1-g^4)(1-6g^2-11g^4)w^3 \frac{dw}{dx} \\ &\quad + \frac{3}{2}g^8(1-g^4)^2w^2 \left(\frac{dw}{dx} \right)^2. \end{aligned} \tag{2.64}$$

In a similar manner we find

$$q \frac{dM(q)}{dq} = 4x(1-x)(1+14x+x^2)w^5 \frac{dw}{dx} + 2x(1-x)(7+x)w^6, \quad (2.65)$$

$$q \frac{dM(q^2)}{dq} = -x(1-x)(1-2x)w^6 + 4x(1-x)(1-x+x^2)w^5 \frac{dw}{dx}, \quad (2.66)$$

$$q \frac{dM(q^4)}{dq} = -\frac{1}{8}x(1-x)(8-x)w^6 + \frac{1}{4}x(1-x)(16-16x+x^2)w^5 \frac{dw}{dx}, \quad (2.67)$$

$$q \frac{dM(q^8)}{dq} = \frac{1}{256}(1-g^4)(-30g^2 - 134g^4 - 90g^6 - 2g^8)w^6 + \frac{1}{64}g^4(1-g^4)(1+60g^2+134g^4+60g^6+g^8)w^5 \frac{dw}{dx}. \quad (2.68)$$

3. Products $L_{r,4}(q)L_{s,4}(q)$ and $L_{r,4}(q)M_{s,4}(q)$

Using the formulae given in § 2, a MAPLE program was run to determine which of the 10 products $L_{r,4}(q)L_{s,4}(q)$ ($0 \leq r \leq s \leq 3$) can be expressed as a linear combination of

$$L(q), L(q^2), L(q^4), L(q^8), M(q), M(q^2), M(q^4), M(q^8)$$

and the derivatives of

$$L(q), L(q^2), L(q^4) \text{ and } L(q^8),$$

and which of the 10 products $L_{r,4}(q)M_{s,4}(q)$ ($0 \leq r \leq s \leq 3$) can be expressed as a linear combination of

$$L(q), L(q^2), L(q^4), L(q^8), M(q), M(q^2), M(q^4), M(q^8), N(q), N(q^2), N(q^4), N(q^8)$$

and the derivatives of

$$L(q), L(q^2), L(q^4), L(q^8), M(q), M(q^2), M(q^4) \text{ and } M(q^8).$$

Five formulae were found.

Theorem 3.1.

$$L_{0,4}^2(q) = \frac{1}{576} - \frac{1}{288}(7L(q^4) - 6L(q^8)) + \frac{1}{2880}(161M(q^4) - 156M(q^8)) + \frac{1}{48} \left(7q \frac{dL(q^4)}{dq} - 6q \frac{dL(q^8)}{dq} \right), \quad (3.1)$$

$$L_{2,4}^2(q) = \frac{3}{80}(M(q^4) - M(q^8)), \quad (3.2)$$

$$L_{0,4}(q)L_{2,4}(q) = -\frac{1}{192}(L(q^2) - 3L(q^4) + 2L(q^8)) + \frac{1}{1920}(11M(q^2) - 99M(q^4) + 88M(q^8)) + \frac{1}{32} \left(q \frac{dL(q^2)}{dq} - 3q \frac{dL(q^4)}{dq} + 2q \frac{dL(q^8)}{dq} \right), \quad (3.3)$$

$$L_{1,4}(q)L_{3,4}(q) = \frac{1}{60}(M(q^4) - M(q^8)), \quad (3.4)$$

$$L_{2,4}(q)M_{2,4}(q) = -\frac{3}{56}(N(q^4) - N(q^8)). \quad (3.5)$$

We just give the proof of (3.1).

Proof of (3.1). By (2.18) and (2.21) we have

$$7L(q^4) - 6L(q^8) = -\frac{1}{8}(11 + 18g^2 - 37g^4)w^2 + 12g^4(1 - g^4)w \frac{dw}{dx},$$

and from (2.19) and (2.22)

$$161M(q^4) - 156M(q^8) = \frac{1}{64}(605 - 2340g^2 + 3790g^4 - 2340g^6 + 605g^8)w^4,$$

and from (2.63) and (2.64)

$$\begin{aligned} 7q \frac{dL(q^4)}{dq} - 6q \frac{dL(q^8)}{dq} &= \frac{1}{8}(9g^2 - 13g^4)(1 - g^4)w^4 - \frac{1}{8}g^4(1 - g^4)(22 + 36g^2 - 74g^4)w^3 \frac{dw}{dx} \\ &\quad + 12g^8(1 - g^4)^2w^2 \left(\frac{dw}{dx}\right)^2. \end{aligned}$$

Finally,

$$\begin{aligned} &\frac{1}{576} - \frac{1}{288}(7L(q^4) - 6L(q^8)) + \frac{1}{2880}(161M(q^4) - 156M(q^8)) \\ &\quad + \frac{1}{48} \left(7q \frac{dL(q^4)}{dq} - 6q \frac{dL(q^8)}{dq} \right) \\ &= \frac{1}{576} - \frac{1}{288} \left(-\frac{1}{8}(11 + 18g^2 - 37g^4)w^2 + 12g^4(1 - g^4)w \frac{dw}{dx} \right) \\ &\quad + \frac{1}{2880} \left(\frac{1}{64} \right) (605 - 2340g^2 + 3790g^4 - 2340g^6 + 605g^8)w^4 \\ &\quad + \frac{1}{48} \left(\frac{1}{8} \right) (9g^2 - 13g^4)(1 - g^4)w^4 \\ &\quad - \frac{1}{48} \left(\frac{1}{8} \right) g^4(1 - g^4)(22 + 36g^2 - 74g^4)w^3 \frac{dw}{dx} + \frac{1}{48} 12g^8(1 - g^4)^2w^2 \left(\frac{dw}{dx}\right)^2 \\ &= \frac{1}{36864} \left(8 + (11 + 18g^2 - 37g^4)w^2 - 96g^4(1 - g^4)w \frac{dw}{dx} \right)^2 \\ &= L_{0,4}(q)^2 \end{aligned}$$

by (2.47).

This completes the proof of (3.1). The remaining formulae can be proved similarly. \square

4. Arithmetic identities

Equating the coefficients of q^n on both sides of the five formulae in Theorem 3.1, we obtain the following theorem.

Theorem 4.1.

$$\sum_{m < n} \sigma(4m)\sigma(4n - 4m) = \frac{1}{12} \{161\sigma_3(n) - 156\sigma_3(\frac{1}{2}n) + (1 - 24n)(7\sigma(n) - 6\sigma(\frac{1}{2}n))\}, \quad (4.1)$$

$$\sum_{m \leq n} \sigma(4m - 2)\sigma(4n - (4m - 2)) = 9\sigma_3(n) - 9\sigma_3(\frac{1}{2}n), \quad (4.2)$$

$$\sum_{m < n} \sigma(4m)\sigma(4n - 2 - 4m) = \frac{1}{8} \{11\sigma_3(2n - 1) + (13 - 24n)\sigma(2n - 1)\}, \quad (4.3)$$

$$\sum_{m \leq n} \sigma(4m - 3)\sigma(4n - (4m - 3)) = 4\sigma_3(n) - 4\sigma_3(\frac{1}{2}n), \quad (4.4)$$

$$\sum_{m \leq n} \sigma_3(4m - 2)\sigma(4n - (4m - 2)) = 27\sigma_5(n) - 27\sigma_5(\frac{1}{2}n). \quad (4.5)$$

Let a and b be integers satisfying $b \geq 1$ and $0 \leq a \leq b - 1$. Set

$$S(a, b) = \sum_{\substack{m=1 \\ m \equiv a \pmod{b}}}^{n-1} \sigma(m)\sigma(n - m).$$

Huard and co-workers [4, §5] have given results for $S(a, b)$ for $b = 1, 2, 3$ and 4 .

Formula (4.4) gives the value of $S(1, 4)$ for $n \equiv 0 \pmod{4}$ [4, Theorem 9]. Formula (4.3) gives the value of $S(0, 4)$ for $n \equiv 2 \pmod{4}$ [4, Theorem 9]. Formula (4.2) gives the value of $S(2, 4)$ for $n \equiv 0 \pmod{4}$. Formula (4.1) gives the value of $S(0, 4)$ for $n \equiv 0 \pmod{4}$. The latter two formulae extend the result given in [4, Theorem 9].

5. Further relations

It is possible to derive many other relations similar to those in Theorems 3.1 and 4.1. We refer the reader to Cheng [3] for details. We just give two other identities.

Theorem 5.1.

$$L(q)L(q^2) = \frac{1}{5}(M(q) + 4M(q^2)) + 3\left(q\frac{dL(q)}{dq} + 2q\frac{dL(q^2)}{dq}\right), \quad (5.1)$$

$$L(q)L(q^4) = \frac{1}{20}(M(q) + 3M(q^2) + 16M(q^4)) + \frac{1}{2}\left(3q\frac{dL(q)}{dq} + 12q\frac{dL(q^4)}{dq}\right). \quad (5.2)$$

Proof. First we prove (5.1). By (2.11) and (2.16) we have

$$M(q) + 4M(q^2) = 5(1 + 2x + x^2)w^4$$

and from (2.61) and (2.62)

$$\begin{aligned} q\frac{dL(q)}{dq} + 2q\frac{dL(q^2)}{dq} &= -3x(1-x)w^4 + 6x(1-x)(1-3x)w^3\frac{dw}{dx} + 24x^2(1-x)^2w^2\left(\frac{dw}{dx}\right)^2. \end{aligned}$$

Hence

$$\begin{aligned} & \frac{1}{5}(M(q) + 4M(q^2)) + 3\left(q\frac{dL(q)}{dq} + 2q\frac{dL(q^2)}{dq}\right) \\ &= (1 - 2x)(1 - 5x)w^4 + 18x(1 - x)(1 - 3x)w^3\frac{dw}{dx} + 72x^2(1 - x)^2w^2\left(\frac{dw}{dx}\right)^2 \\ &= L(q)L(q^2) \end{aligned}$$

by (2.10) and (2.15).

Next we prove (5.2). By (2.11), (2.16) and (2.19) we have

$$M(q) + 3M(q^2) + 16M(q^4) = 5(4 - x + x^2)w^4$$

and from (2.61) and (2.63)

$$\begin{aligned} & 3q\frac{dL(q)}{dq} + 12q\frac{dL(q^4)}{dq} \\ &= -12x(1 - x)w^4 + 30x(1 - x)(1 - 2x)w^3\frac{dw}{dx} + 72x^2(1 - x)^2w^2\left(\frac{dw}{dx}\right)^2. \end{aligned}$$

Hence

$$\begin{aligned} & \frac{1}{20}(M(q) + 3M(q^2) + 16M(q^4)) + \frac{1}{2}\left(3q\frac{dL(q)}{dq} + 12q\frac{dL(q^4)}{dq}\right) \\ &= (1 - 5x)\left(1 - \frac{5}{4}x\right)w^4 + 15x(1 - x)(1 - 2x)w^3\frac{dw}{dx} + 36x^2(1 - x)^2w^2\left(\frac{dw}{dx}\right)^2 \\ &= L(q)L(q^4) \end{aligned}$$

by (2.10) and (2.18). This completes the proof of Theorem 5.1. □

Equating the coefficients of q^n on both sides of (5.1) and (5.2), we obtain the following theorem.

Theorem 5.2.

$$\begin{aligned} & \sum_{m < n/2} \sigma(m)\sigma(n - 2m) \\ &= \frac{1}{24}\{2\sigma_3(n) + 8\sigma_3(\frac{1}{2}n) + \sigma(n) + \sigma(\frac{1}{2}n) - 3n\sigma(n) - 6n\sigma(\frac{1}{2}n)\}, \end{aligned} \tag{5.3}$$

$$\begin{aligned} & \sum_{m < n/4} \sigma(m)\sigma(n - 4m) \\ &= \frac{1}{48}\{\sigma_3(n) + 3\sigma_3(\frac{1}{2}n) + 16\sigma_3(\frac{1}{4}n) + 2\sigma(n) + 2\sigma(\frac{1}{4}n) - 3n\sigma(n) - 12n\sigma(\frac{1}{4}n)\}. \end{aligned} \tag{5.4}$$

Formulae (5.3) and (5.4) are due to Melfi [5, 6] for n odd and to Huard and co-workers [4] for all n .

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