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On equicontinuous factors of flows on locally path-connected compact spaces

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Abstract. We consider a locally path-connected compact metric space K with finite first Betti number $\mathrm{b}_1(K)$ and a flow (K, G) on K such that G is abelian and all G-invariant functions $f \in \mathrm{C}(K)$ are constant. We prove that every equicontinuous factor of the flow (K, G) is isomorphic to a flow on a compact abelian Lie group of dimension less than $\mathrm{b}_1(K)/\mathrm{b}_0(K)$. For this purpose, we use and provide a new proof for Theorem 2.12 of Hauser and Jäger [Monotonicity of maximal equicontinuous factors and an application to toral flows. *Proc. Amer. Math. Soc.* **147** (2019), 4539–4554], which states that for a flow on a locally connected compact space the quotient map onto the maximal equicontinuous factor is *monotone*, i.e., has connected fibers. Our alternative proof is a simple consequence of a new characterization of the monotonicity of a quotient map $p\colon K \to L$ between locally connected compact spaces K and L that we obtain by characterizing the local connectedness of K in terms of the Banach lattice $\mathrm{C}(K)$.

Key words: locally connected, montone, equicontinuous
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1. *Introduction*

The study of topological dynamical systems via their maximal equicontinuous factors plays an important role for, e.g., tiling dynamical systems (see [**KLS15**, Ch. 5]), Toeplitz flows (see [**Dow05**]), or the Furstenberg structure theorem for minimal distal flows. One reason is that, for group actions, the maximal equicontinuous factor coincides with the Kronecker factor. The latter is highly structured, is minimal if and only if it is isomorphic to a minimal rotation on a homogeneous space of some compact group, and also captures spectral-theoretic information. In light of this, it is important to understand how the specific structure and properties of a system can be used to determine its maximal equicontinuous factor. For example, it is known that if (M, G) is a distal minimal flow on a compact manifold M, then its maximal equicontinuous factor is a flow on a homogeneous space of some compact Lie group; see [**Ree77**, Theorem 1.2] or [**IM84**, Theorem 1.2]. If, additionally, the acting group G is abelian, the maximal equicontinuous factor is in fact isomorphic to a flow on a compact abelian Lie group. For non-distal systems, however, few results in this spirit seem to exist. Notably, Hauser and Jäger recently proved the following.

THEOREM. [**HJ19**, Theorem 3.1] *Suppose that f is a homeomorphism of the two-torus. If the maximal equicontinuous factor of $(\mathbb{T}^2, f)$ is minimal, then it must be one of the following three:*

(i) *an irrational translation on the two-torus;*
(ii) *an irrational rotation on the circle;*
(iii) *the identity on a singleton.*

Thus, the geometric properties of the two-torus imply that the maximal equicontinuous factor of a flow on it must have a relatively simple structure if it is minimal: it is a rotation on a compact abelian Lie group of dimension less than two. As it turns out, this is representative of the following general phenomenon, which is the main result of this article. Recall that every compact abelian Lie group is isomorphic to the product $F \times \mathbb{T}^m$ of a finite abelian group F and a torus $\mathbb{T}^m$.

THEOREM. *Let (K, G) be a flow such that K is locally path-connected with finite first Betti number $\mathrm{b}_1(K)$, G is abelian, and K is metrizable. If all G-invariant functions $f \in \mathrm{C}(K)$ are constant, then every equicontinuous factor of (K, G) is isomorphic to a minimal flow on some compact abelian Lie group of dimension less than $\mathrm{b}_1(K)/\mathrm{b}_0(K)$.*

This provides a bound on the complexity of the maximal equicontinuous factor in terms of topological invariants of the underlying space K and applies in particular to minimal systems on compact manifolds. As a corollary, we obtain in Corollary 4.4 and Corollary 4.5 that the above-cited [**HJ19**, Theorem 3.1] holds analogously for tori of arbitrary dimension and, in an appropriate version, more generally for quotients H/Γ of connected, simply connected Lie groups H by discrete, cocompact subgroups $\Gamma \subset H$. Examples for such spaces are in particular given by nilmanifolds; see [**HK18**, Ch. 10].

Another result of Hauser and Jäger [**HJ19**] and a key element in their proof of [**HJ19**, Theorem 3.1] is that for a large class of flows the factor map onto the maximal equicontinuous factor is *monotone*, meaning that preimages of points are connected. Similar results were, it seems, first obtained in [**MW76**], where the authors proved that for each extension $p\colon (K, G) \to (L, G)$ of minimal flows that decomposes into a tower of equicontinuous extensions, the quotient map $K \to K/\mathrm{S}(p)$ is monotone, where $\mathrm{S}(p)$ denotes the relativized equicontinuous structure relation. Hence, for a distal minimal flow its Furstenberg tower consists entirely of monotone quotient maps, see [**Gre14**, Proposition 2.2], and in particular the map onto the maximal equicontinuous factor is monotone. Without such structural assumptions, the monotonicity of the maximal equicontinuous factor does not hold in general (take, e.g., the extension of the shift $\tau\colon \mathbb{Z} \to \mathbb{Z}$, $x \mapsto x + 1$ to the one-point compactification of $\mathbb{Z}$). If, however, the underlying space is locally connected, it is shown in [**HJ19**, Theorem 2.12] that the quotient map onto

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the maximal equicontinuous factor is indeed monotone, which is notable since monotone quotient maps relate the geometry of a space to that of a quotient: every monotone quotient map between suitable spaces induces a surjective homomorphism on the level of fundamental groups; see [**CGM12**, Theorem 1.1] (stated below as Theorem 4.1). Since this idea will be crucial for the proof of Theorem 4.2, we also provide an alternative proof for the monotonicity of the maximal equicontinuous factor under the assumption of local connectedness. This is based on two results that are of interest by themselves: we characterize the local connectedness of a compact space K in terms of the Banach lattice $\mathrm{C}(K)$ and then use this to give a new characterization for the monotonicity of a quotient map between locally connected compact spaces. (For background information on Banach lattices, see [**EFHN15**, §7.1].) The above-mentioned monotonicity result then is a simple application of these characterizations. We prove these results in § 3 after collecting some preliminaries in § 2. The main result is proved in § 4.

1.1. *Notation and terminology.* By a *topological dynamical system* (K, S, Φ) we mean a continuous action $\Phi\colon S \times K \to K$ of a topological semigroup S on a compact space K. (We always assume compact spaces to be Hausdorff.) We usually drop Φ from the notation and write sx instead of $\Phi(s, x)$ for $s \in S$ and $x \in K$. We simply call (K, S) a *flow* if S is a group. If we refer to a pair (K, φ) of a compact space K and a continuous map $\varphi\colon K \to K$ as a topological dynamical system, we regard the $\mathbb{N}$- or $\mathbb{Z}$-action on K given by the powers of φ, depending on whether φ is explicitly specified to be invertible or not. By an *extension* $p\colon (K, S) \to (L, S)$ of topological dynamical systems we mean a continuous, surjective, S-equivariant function $p\colon K \to L$. Given such an extension, we also call p a *factor map* and (L, S) a *factor* of (K, S). A system (K, S) is called *equicontinuous* if the family $\{\Phi(s, \cdot) \mid s \in S\} \subset \mathrm{C}(K, K)$ is equicontinuous.

If $\varphi\colon K \to L$ is a continuous function between compact spaces, we denote by T_φ its *Koopman operator*

$$T_\varphi\colon \mathrm{C}(L) \to \mathrm{C}(K),$$
$$f \mapsto f \circ \varphi.$$

We assume the reader to be familiar with Koopman operators and the theory of commutative C^*-algebras and refer to [**EFHN15**, Ch. 4] for background information. If H is an abelian group, then $\operatorname{rank}(H)$ denotes its torsion-free rank, i.e., the dimension of the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes H$. For a topological space X and $i \in \mathbb{N}_0$, we denote by $\mathrm{b}_i(X)$ the ith Betti number of X. Note that for a compact, locally path-connected space X, $\mathrm{b}_0(X)$ is the (finite) number of connected components of X.

2. *Factors and invariant subalgebras*

Consider the categories **CTop** of compact topological spaces and $\mathbf{C}^*_{\mathbf{com},\mathbf{1}}$ of commutative unital C^*-algebras as well as the contravariant functor

$$\mathrm{C}\colon \mathbf{CTop} \to \mathbf{C}^*_{\mathbf{com},\mathbf{1}}$$

given by $K \mapsto \mathrm{C}(K)$ and $\varphi \mapsto T_\varphi$ for compact spaces K and L and continuous functions $\varphi\colon K \to L$. It is a consequence of the classical Gelfand representation theorem (see

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[**EFHN15**, Theorem 4.23]) that C is an antiequivalence of categories. This allows us to use operator-theoretic concepts to understand topological dynamical systems (cf. Theorem 3.7) but also, conversely, to use geometric tools to obtain results about operators (cf. Corollary 4.7). A particular consequence of this antiequivalence that we will use throughout the article is the relationship between factors of a topological dynamical system (K, S) and S-invariant unital C^*-subalgebras of $\mathrm{C}(K)$: suppose that $p\colon (K, S) \to (L, S)$ is a factor map of topological dynamical systems. Then the Koopman operator

$$\begin{aligned} T_p\colon \mathrm{C}(L) &\to \mathrm{C}(K), \\ f &\mapsto f \circ p \end{aligned}$$

is an S-equivariant C^*-embedding and its image $\mathcal{A}_L := T_p(\mathrm{C}(L)) \subset \mathrm{C}(K)$ is an S-invariant unital C^*-subalgebra of $\mathrm{C}(K)$, i.e., $T_s(\mathcal{A}_L) \subset \mathcal{A}_L$ for each $s \in S$. Moreover, if $q\colon (K, S) \to (M, S)$ is another factor map such that $T_q(\mathrm{C}(M)) = T_p(\mathrm{C}(L))$, then

$$\Psi := T_p^{-1} \circ T_q\colon \mathrm{C}(M) \to \mathrm{C}(L)$$

defines an S-equivariant C^*-isomorphism and so there is a unique S-equivariant homeomorphism $\eta\colon L \to M$ such that $\Psi = T_\eta$, making the following diagram commutative.

$$\begin{array}{ccccc} & & (K, S) & & \\ & \swarrow^{p} & & \searrow^{q} & \\ (L, S) & & \xrightarrow{\quad \eta \quad} & & (M, S) \end{array}$$

This shows that a factor of (K, S) is, up to isomorphy, uniquely determined by its corresponding C^*-subalgebra of $\mathrm{C}(K)$. Conversely, every S-invariant unital C^*-subalgebra of $\mathrm{C}(K)$ canonically corresponds to a factor of (K, S) via the Gelfand representation theorem and one thereby obtains, again up to isomorphy, a one-to-one correspondence between factors of (K, S) and S-invariant unital C^*-subalgebras of $\mathrm{C}(K)$. For the convenience of the reader, Remark 2.3 below explains a more elementary approach to this correspondence.

Given that the C^*-subalgebras of many factors such as the maximal trivial factor, the maximal equicontinuous factor, the maximal tame factor, the Kronecker factor, or the Abramov factor admit relatively simple descriptions, it is convenient to study factors and factor maps via their corresponding subalgebras. We do so in Corollary 3.6 to give a simple criterion characterizing the monotonicity of a factor via its corresponding C^*-subalgebra. We will then see that this criterion is readily verified for the C^*-subalgebra of the maximal equicontinuous factor.

Example 2.1. Let (K, S) be a topological dynamical system.

(a) The factor consisting of a single point corresponds to the subalgebra $\mathbb{C}\mathbb{1}_K \subset \mathrm{C}(K)$ of constant functions.

(b) Let $p\colon (K, S) \to (L, S)$ be a factor map onto a *trivial* factor, i.e., one on which S acts trivially. Then $T_p(\mathrm{C}(L))$ is a subalgebra of

$$\mathcal{A}_{\mathrm{triv}} := \{f \in \mathrm{C}(K) \mid \forall s \in S\colon T_s f = f\}\,.$$

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This 'fixed' algebra corresponds to the *maximal trivial factor* (K_{triv}, S) of (K, S) through which every factor map onto another trivial factor factorizes.

(c) Similarly, the subalgebra

$$\mathcal{A}_{\text{eq}} := \{f \in \mathrm{C}(K) \mid \{T_s f \mid s \in S\} \text{ is equicontinuous}\}$$

corresponds to the *maximal equicontinuous factor* (K_{eq}, S) of (K, S) since a topological dynamical system (M, S) is equicontinuous if and only if for each $g \in \mathrm{C}(M)$ the orbit $\{T_s g \mid s \in S\}$ is equicontinuous. By the Arzelà–Ascoli theorem, this is equivalent to the orbits $\{T_s g \mid s \in S\}$ being relatively compact for each $g \in \mathrm{C}(M)$. If S acts via homeomorphisms, this means that the maximal equicontinuous factor (K_{eq}, S) coincides with the *Kronecker factor*, which corresponds, for abelian S, to the C^*-subalgebra

$$\mathcal{A} := \overline{\text{lin}}\{f \in \mathrm{C}(K) \mid \forall s \in S \ \ \exists \lambda_s \in \mathbb{T}\colon T_s f = \lambda_s f\}$$

spanned by the eigenfunctions of the action of S; see [**EFHN15**, Corollary 16.32].

Remark 2.2. Let (M, G) be an equicontinuous flow. Then each orbit closure is a minimal subset of M (see [**Aus88**, Lemma 2.3]) and so M decomposes into minimal subsets. Moreover, the orbit closure relation

$$\sim_G := \{(x, y) \in M \times M \mid \overline{Gx} = \overline{Gy}\}$$

is a closed equivalence relation (see [**Aus88**, Exercise 2.6]) and a moment's thought reveals that, hence, $M/{\sim_G}$ together with the trivial G-action is the maximal trivial factor of (M, G). In particular, (M, G) is minimal if and only if the maximal trivial factor of (M, G) is a point. We note for the proof of Theorem 4.2 that as a consequence, given an arbitrary system (K, G), its maximal equicontinuous factor is minimal if and only if every G-invariant function $f \in \mathrm{C}(K)$ is constant.

Remark 2.3. Given a compact space K, one can describe the relationship between compact quotients L of K and unital C^*-subalgebras $\mathcal{A}$ of $\mathrm{C}(K)$ without using the Gelfand representation theory: if $p\colon K \to L$ is a continuous surjective map onto a compact space L, define the unital C^*-subalgebra

$$\mathcal{A}_L := \{f \in \mathrm{C}(K) \mid \forall l \in L,\ \forall x, y \in p^{-1}(l)\colon f(x) = f(y)\}$$

of functions constant on fibers of p and note that $\mathcal{A}_L = T_p(\mathrm{C}(L))$. Conversely, if $\mathcal{A} \subset \mathrm{C}(K)$ is a unital C^*-subalgebra, define the closed equivalence relation

$$\sim_{\mathcal{A}} := \{(x, y) \in K \times K \mid \forall f \in \mathcal{A}\colon f(x) = f(y)\}$$

and set $L_{\mathcal{A}} := K/{\sim_{\mathcal{A}}}$. Then the assignments $L \mapsto \mathcal{A}_L$ and $\mathcal{A} \mapsto L_{\mathcal{A}}$ are, up to isomorphy of the compact spaces, mutually inverse. Analogously, one obtains the above-explained correspondence between factors and invariant subalgebras if one considers topological dynamical systems (K, S) on K.

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3. *Local connectedness and monotonicity of factors*

As noted in the introduction, one cannot expect the factor map onto the maximal equicontinuous factor of a flow to be monotone in general, i.e., its preimages of points are not generally connected. Therefore, we focus on quotient maps $p\colon K \to L$ between locally connected compact spaces and characterize their monotonicity in terms of the subalgebra $\mathcal{A}_L \subset \mathrm{C}(K)$ of functions constant on fibers of p. We then apply this to the maximal equicontinuous factor. Recall the following elementary results on locally connected spaces.

LEMMA 3.1. *Let X and Y be topological spaces.*

(a) *X is locally connected if and only if for every open set $O \subset X$ each connected component of O is open in X.*

(b) *X is locally connected if and only if for every basis $\mathcal{B}$ for the topology of X and every $O \in \mathcal{B}$ each connected component of O is open in X.*

(c) *If X is compact, X is locally connected if and only if it is uniformly locally connected, i.e., if each entourage $U \in \mathcal{U}_X$ contains an entourage $V \in \mathcal{U}_X$ such that $V[x]$ is connected for each $x \in X$.*

(d) *If X is locally connected and $f\colon X \to Y$ is a surjective quotient map, then Y is locally connected.*

(e) *If X is compact and locally connected, X has only finitely many connected components.*

Proof. For (a), (d), and (e), see Theorem 27.9, Corollary 27.11, and Theorem 27.12 of [**Wil04**] and for (c), see [**Jam99**, Proposition 9.39]. Part (b) follows from the definition of local connectedness and part (a). □

Since in a compact space K the sets of the form $[f \neq 0]$ for continuous, complex-valued functions $f \in \mathrm{C}(K)$ constitute a base of the topology, Lemma 3.1(a) and (b) provide a natural way to characterize the local connectedness of K purely in terms of the Banach lattice $\mathrm{C}(K)$. For this purpose, we call $f, g \in \mathrm{C}(K)$ *orthogonal* and write $f \perp g$ if they are orthogonal in the Banach lattice $\mathrm{C}(K)$. (That is, $f \perp g$ if and only if $|f| \wedge |g| = 0$, which is equivalent to $fg = 0$.) A decomposition $f = g + h$ is called *orthogonal* if g and h are orthogonal. A function $f \in \mathrm{C}(K)$ is called *reducible* if $f = 0$ or if there is an orthogonal decomposition $f = g + h$ with non-zero $g, h \in \mathrm{C}(K)$ and f is called *irreducible* otherwise. If $f = g + h$ is an orthogonal decomposition and g is irreducible, then g is called an *irreducible part* of f. For a function $f \in \mathrm{C}(K)$, define

$$\mathrm{irr}(f) := \{g \in \mathrm{C}(K) \mid g \text{ is an irreducible part of } f\}$$

and, for a subset $\mathcal{F} \subset \mathrm{C}(K)$, set

$$\mathrm{irr}(\mathcal{F}) := \bigcup_{f \in \mathcal{F}} \mathrm{irr}(f).$$

A decomposition $f = \sum_{g \in \mathcal{F}} f$ for some at most countable set $\mathcal{F} \subset \mathrm{C}(K)$ is called *irreducible* if all $g \in \mathcal{F}$ are irreducible and pairwise orthogonal and the sum converges uniformly to f.

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PROPOSITION 3.2. *Let K be a compact space.*

(a) *If $f \in \mathrm{C}(K)$ and $M \subset [f \neq 0]$, then $\mathbb{1}_M f \in \mathrm{C}(K)$ if and only if M is clopen in $[f \neq 0]$.*

(b) *If $f \in \mathrm{C}(K)$, then f is irreducible if and only if $[f \neq 0]$ is connected.*

(c) *For each $f \in \mathrm{C}(K)$,*

$$\mathrm{irr}(f) = \{\mathbb{1}_O f \mid O \text{ is an open connected component of } [f \neq 0]\}.$$

(d) *For each $f \in \mathrm{C}(K)$ and $\varepsilon > 0$, the set*

$$\{g \in \mathrm{irr}(f) \mid \|g\|_\infty > \varepsilon\}$$

is finite. In particular, $\mathrm{irr}(f)$ *is countable.*

(e) *K is locally connected if and only if each $f \in \mathrm{C}(K)$ admits a unique irreducible decomposition. In that case, the irreducible decomposition is given by*

$$f = \sum_{g \in \mathrm{irr}(f)} g.$$

Proof. For a fixed $f \in \mathrm{C}(K)$, the multiplication operator

$$\mathrm{C_b}([f \neq 0]) \to \mathrm{C}(K), \quad g \mapsto gf$$

is well defined and, if $M \subset [f \neq 0]$ is clopen, then $\mathbb{1}_M|_{[f\neq 0]} \in \mathrm{C_b}([f \neq 0])$. Therefore, $f\mathbb{1}_M \in \mathrm{C}(K)$. Conversely, if $M \subset [f \neq 0]$ is such that $\mathbb{1}_M f \in \mathrm{C}(K)$, the restriction $\mathbb{1}_M f|_{[f\neq 0]}$ is continuous and so dividing by $f|_{[f\neq 0]}$ yields the continuity of $\mathbb{1}_M|_{[f\neq 0]}$. This proves (a), which in turn yields (b).

If O is an open connected component of $[f \neq 0]$, then $\mathbb{1}_O f \in \mathrm{irr}(f)$ by (a) and (b). Conversely, take $g \in \mathrm{irr}(f)$. Then $g = \mathbb{1}_{[g\neq 0]} f$ and so, by (a), $[g \neq 0]$ is a clopen subset of $[f \neq 0]$ and hence a union of connected components of $[f \neq 0]$. However, since g is irreducible, $[g \neq 0]$ is connected and so it is an open connected component of $[f \neq 0]$, proving (c). Moreover, (c) yields that $\mathrm{irr}(f)$ is a bounded and equicontinuous set in $\mathrm{C}(K)$ and so, by the Arzelà–Ascoli theorem, $\mathrm{irr}(f)$ is relatively compact. Since, by (c), for every two $g, h \in \mathrm{irr}(f)$ with $g \neq h$, one has $\|g - h\| = \max\{\|g\|, \|h\|\}$, (d) follows from the relative compactness of $\mathrm{irr}(f)$.

Now suppose K to be locally connected and take $f \in \mathrm{C}(K)$. Since K is locally connected, each connected component of $[f \neq 0]$ is open and so, by (c) and (d), the sum $\sum_{g \in \mathrm{irr}(f)} g$ converges uniformly to f. Hence, f admits an irreducible decomposition, which is readily verified to be unique. Conversely, assume that each $f \in \mathrm{C}(K)$ admits an irreducible decomposition and let $x \in K$. To show that K is locally connected at x, let $U \subset K$ be an open neighborhood of x. Since K is completely regular, there exists an $f \in \mathrm{C}(K)$ with $x \in [f \neq 0] \subset U$. Moreover, since f admits a unique irreducible decomposition, there is a unique $g \in \mathrm{irr}(f)$ such that $g(x) = f(x) \neq 0$. In particular, $x \in [g \neq 0] \subset [f \neq 0] \subset U$ and, since g is irreducible, $[g \neq 0]$ is connected, showing that K is locally connected. □

After these preparatory notes on local connectedness, we now turn towards the notion of monotonicity and its characterizations. We restrict to compact spaces although many of the arguments are easily adapted to completely regular spaces.

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Definition 3.3. Let X and Y be topological spaces and $p\colon X \to Y$ a map. Then p is called *monotone* if for each $y \in Y$ the preimage $p^{-1}(y)$ is a connected subset of X.

If $p\colon K \to L$ is a continuous surjective map, then as noted in § 2 and in particular in Remark 2.3, L is, up to isomorphism, uniquely determined by the subalgebra $\mathcal{A}_L = T_p(\mathrm{C}(L)) \subset \mathrm{C}(K)$ of functions constant on the fibers of p. If K is locally connected, this allows us to use Proposition 3.2 to characterize the monotonicity of a quotient map $p\colon K \to L$ in terms of the subalgebra $\mathcal{A}_L$ and the Koopman operator T_p.

PROPOSITION 3.4. *Let K and L be compact spaces, K locally connected, and $p\colon K \to L$ continuous and surjective. Then the following assertions are equivalent.*

(a) *p is monotone.*
(b) *For every connected set $C \subset L$ the preimage $p^{-1}(C)$ is connected.*
(c) *For every open, connected set $U \subset L$ the preimage $p^{-1}(U)$ is connected.*
(d) *For every irreducible $f \in \mathrm{C}(L)$ the set $p^{-1}([f \neq 0])$ is connected.*
(e) *T_p preserves irreducibility of functions.*
(f) *The subalgebra $\mathcal{A}_L = T_p(\mathrm{C}(L)) \subset \mathrm{C}(K)$ satisfies $\mathrm{irr}(\mathcal{A}_L) \subset \mathcal{A}_L$.*

Proof. For the implication (a) $\Longrightarrow$ (b), suppose $C \subset L$ to be connected and that $p^{-1}(C) = U \cup V$ for disjoint, open sets $U, V \subset p^{-1}(C)$. Then U and V are saturated, i.e., $p^{-1}(p(U)) = U$ and $p^{-1}(p(V)) = V$, since each fiber of p over C is connected. Hence, the open sets $p(U)$ and $p(V)$ form a cover of C by disjoint, open sets. Since C is connected, $U = \emptyset$ or $V = \emptyset$ and so $p^{-1}(C)$ is connected.

The implication (b) $\Longrightarrow$ (c) is trivial. For the implication (c) $\Longrightarrow$ (a), note that for $l \in L$

$$p^{-1}(l) = \bigcap_{U \in \mathcal{U}(l)} p^{-1}(U) = \bigcap_{\substack{U \in \mathcal{U}(l) \\ \text{closed, connected}}} p^{-1}(U)$$

as L is locally connected by Lemma 3.1. Since, in a compact space, the intersection of a decreasing family of closed, connected subsets is again connected, p is monotone.

The equivalence of (d) and (e) follows since $[T_p(f) \neq 0] = [f \circ p \neq 0] = p^{-1}([f \neq 0])$. Moreover, (e) and (f) are seen to be equivalent using the existence of irreducible decompositions for functions in $\mathrm{C}(L)$. Finally, the implication (c) $\Longrightarrow$ (d) is trivial and the converse implication follows analogously to (c) $\Longrightarrow$ (a) because, L being locally connected and completely regular, the sets of the form $[f \neq 0]$ for irreducible $f \in \mathrm{C}(L)$ form a basis of the topology of L, which allows us to copy the argument. □

Definition 3.5. Let K be a compact space. A unital C*-subalgebra $\mathcal{A} \subset \mathrm{C}(K)$ is called *monotone* if the canonical quotient map $K \to K/{\sim_{\mathcal{A}}}$ is monotone.

COROLLARY 3.6. *Let K be a locally connected compact space and $\mathcal{A} \subset \mathrm{C}(K)$ a unital C*-subalgebra. Then $\mathcal{A}$ is monotone if and only if it contains the irreducible parts of all its functions.*

As mentioned in §2, many abstractly defined factors in topological dynamics, including the maximal equicontinuous factor, naturally have corresponding C*-subalgebras that

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admit simple descriptions. Hence, Proposition 3.4 and Corollary 3.6 provide a useful way of verifying the monotonicity of factors. For the maximal equicontinuous factor, this yields the following.

THEOREM 3.7. *Let (K, S) be a topological dynamical system such that K is locally connected and the semigroup S acts on K via monotone maps. Then the factor map onto the maximal equicontinuous factor of (K, S) is monotone.*

Proof. By Example 2.1 and Corollary 3.6, it suffices to show that the subalgebra

$$\mathcal{A}_{\mathrm{eq}} = \{f \in \mathrm{C}(K) \mid \{T_s f \mid s \in S\} \text{ is equicontinuous}\}$$

satisfies $\mathrm{irr}(\mathcal{A}_{\mathrm{eq}}) \subset \mathcal{A}_{\mathrm{eq}}$. So, let $f \in \mathcal{A}_{\mathrm{eq}}$ and $g \in \mathrm{irr}(f)$. Since the connectedness of a set is preserved by taking preimages under monotone maps, $T_s g$ is irreducible for each $s \in S$ and so $T_s g \in \mathrm{irr}(T_s(f))$. Therefore,

$$\{T_s g \mid s \in S\} \subset \bigcup_{s \in S} \mathrm{irr}(T_s(f)) = \mathrm{irr}(\{T_s(f) \mid s \in S\}).$$

The latter set is equicontinuous by Lemma 3.8 below and so it follows that the orbit $\{T_s g \mid s \in S\}$ of g is equicontinuous as well. Hence, $g \in \mathcal{A}_{\mathrm{eq}}$. □

LEMMA 3.8. *Let K be a locally connected compact space and $\mathcal{F} \subset \mathrm{C}(K)$. Then $\mathcal{F}$ is equicontinuous if and only if $\mathrm{irr}(\mathcal{F})$ is.*

Proof. Suppose $\mathcal{F}$ to be equicontinuous and take $\varepsilon > 0$. Then there exists an entourage $V \in \mathcal{U}_K$ such that for each $f \in \mathcal{F}$ and $(x, y) \in V$ one has $|f(x) - f(y)| < \varepsilon/2$. By Lemma 3.1, we may assume that $V[x]$ is connected for each $x \in K$. Let $g \in \mathrm{irr}(\mathcal{F})$, i.e., $g \in \mathrm{irr}(f_0)$ for some $f_0 \in \mathcal{F}$. We claim that $|g(x) - g(y)| < \varepsilon$ for all $(x, y) \in V$, which would show that $\mathrm{irr}(\mathcal{F})$ is equicontinuous.

So, let $(x, y) \in V$. If $g(x) = g(y) = 0$, it holds trivially that $|g(x) - g(y)| < \varepsilon$, so assume without loss of generality that $g(x) \neq 0$. If $V[x]$ lies in $[f \neq 0]$, then it lies in the connected component of $[f \neq 0]$ containing x and so $y \in V[x] \subset [g \neq 0]$. Therefore,

$$|g(x) - g(y)| = |f(x) - f(y)| < \varepsilon/2.$$

If $V[x] \not\subseteq [f \neq 0]$, there is a $z \in V[x]$ with $f(z) = 0$ and so

$$|g(x) - g(y)| \leqslant |f(x)| + |f(y)| = |f(x) - f(z)| + |f(y) - f(z)| < \varepsilon.$$

Therefore, $\mathrm{irr}(\mathcal{F})$ is equicontinuous. The converse implication follows similarly. □

Of course, the most common examples of semigroups acting via monotone maps are given by group actions, so that we obtain a new proof for [**HJ19**, Theorem 2.12].

COROLLARY 3.9. *Let (K, G) be a flow on a locally connected compact space K. Then the factor map onto the maximal equicontinuous factor is monotone.*

It is known (see [**BOT05**, Theorem 3.16]) that if $(\mathbb{T}^2, \varphi)$ is a minimal topological dynamical system on the two-torus, then φ is necessarily monotone, and there do exist non-invertible examples for such systems with a non-trivial maximal equicontinuous

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factor; see [**KST01**, Theorem 3.3]. Therefore, there are examples in which Theorem 3.7 provides meaningful information that cannot be deduced from Corollary 3.9.

In preparation for the next section, we collect several properties of monotone subalgebras.

PROPOSITION 3.10. *Let K be a locally connected compact space and $\mathcal{A} \subset \mathrm{C}(K)$ a unital C^*-subalgebra.*

(a) *If $(f_n)_{n \in \mathbb{N}}$ is a sequence in $\mathrm{C}(K)$ converging to $f \in \mathrm{C}(K)$ and $f_n = h_n + g_n$ is an orthogonal decomposition for each $n \in \mathbb{N}$, then there is a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ such that $(g_{n_k})_{k \in \mathbb{N}}$ and $(h_{n_k})_{k \in \mathbb{N}}$ converge uniformly. If f is irreducible, one of the sequences converges to 0 uniformly.*

(b) *If $(f_n)_{n \in \mathbb{N}}$ is a positive increasing sequence in $\mathrm{C}(K)$ converging uniformly to $f \in \mathrm{C}(K)$, then for each $g \in \mathrm{irr}(f)$ there is a positive increasing sequence $(g_n)_{n \in \mathbb{N}}$ such that $g_n \uparrow g$ and for each $n \in \mathbb{N}$ either $g_n = 0$ or $g_n \in \mathrm{irr}(f_n)$.*

(c) *If $f \in \mathcal{A}$, then $\mathrm{irr}(f) \subset \mathcal{A}$ if and only if $\mathrm{irr}(|f|) \subset \mathcal{A}$.*

(d) *If $D \subset \mathcal{A}$ is a dense $\mathbb{Q}$-vector sublattice containing $\mathbb{1}_K$, then $\mathrm{irr}(D) \subset D$ implies that $\mathrm{irr}(\mathcal{A}) \subset \mathcal{A}$.*

(e) *Let $\mathcal{S}$ be a system of unital C^*-subalgebras of $\mathrm{C}(K)$ closed under arbitrary intersections and containing $\mathrm{C}(K)$. Then there exists a smallest monotone subalgebra $\mathcal{A}_{\mathrm{mon}}^{\mathcal{S}} \in \mathcal{S}$ containing $\mathcal{A}$, called the* monotone hull *of $\mathcal{A}$ in $\mathcal{S}$.*

(f) *Furthermore, suppose that for every separable C^*-subalgebra $\mathcal{B} \subset \mathrm{C}(K)$ there is a separable C^*-subalgebra $\mathcal{B}^{\mathcal{S}} \in \mathcal{S}$ satisfying $\mathcal{B} \subset \mathcal{B}^{\mathcal{S}}$ and that if $(\mathcal{B}_n)_{n \in \mathbb{N}}$ is an increasing sequence in $\mathcal{S}$, then $\overline{\bigcup_{n \in \mathbb{N}} \mathcal{B}_n} \in \mathcal{S}$. Then the monotone hull $\mathcal{A}_{\mathrm{mon}}^{\mathcal{S}}$ is separable if and only if $\mathcal{A}$ is.*

Proof. For (a), it follows from the Arzelà–Ascoli theorem that the set

$$\{g_n \mid n \in \mathbb{N}\} \cup \{h_n \mid n \in \mathbb{N}\}$$

is relatively compact in $\mathrm{C}(K)$, which implies the existence of the sequence $(n_k)_{n \in \mathbb{N}}$. The limits g and h of $(g_{n_k})_{k \in \mathbb{N}}$ and $(h_{n_k})_{k \in \mathbb{N}}$ satisfy $f = g + h$ and $g \perp h$ and hence yield an orthogonal decomposition of f. Therefore, if f is irreducible, $g = 0$ or $h = 0$, which proves (a).

For (b), one can pass to $(\mathbb{1}_{[g \neq 0]} f_n)_{n \in \mathbb{N}}$ and $\mathbb{1}_{[g \neq 0]} f = g$ and hence assume that f itself is already irreducible. Pick $x \in [f \neq 0]$. Without loss of generality, we may assume that $f_1(x) > 0$. Then, for each $n \in \mathbb{N}$, there is a unique $g_n \in \mathrm{irr}(f_n)$ with $g_n(x) = f_n(x) > 0$. To see that $(g_n)_{n \in \mathbb{N}}$ is increasing, note that for each $n \in \mathbb{N}$

$$[g_n \neq 0] \subset [f_n \neq 0] \subset [f_{n+1} \neq 0] = [g_{n+1} \neq 0] \cup [f_{n+1} - g_{n+1} \neq 0].$$

This implies that

$$[g_n \neq 0] = ([g_n \neq 0] \cap [g_{n+1} \neq 0]) \cup ([g_n \neq 0] \cap [f_{n+1} - g_{n+1} \neq 0])$$

and, since this union is disjoint, g_n is irreducible, and $x \in [g_n \neq 0] \cap [g_{n+1} \neq 0]$, $[g_n \neq 0] \subset [g_{n+1} \neq 0]$. Since $(f_n)_{n \in \mathbb{N}}$ is increasing, this yields that $(g_n)_{n \in \mathbb{N}}$ is increasing. Now consider the orthogonal decomposition $f_n = g_n + (f_n - g_n)$. Since $g_n(x) \geqslant g_1(x) > 0$

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for each $n \in \mathbb{N}$, (a) yields that every subsequence of $(f_n - g_n)_{n \in \mathbb{N}}$ has a subsequence converging to 0, showing that $f_n - g_n \to 0$. Therefore, $g_n \uparrow f$.

For (c), note that if $f \in \mathcal{A}$, the absolute value yields a bijection $\mathrm{irr}(f) \to \mathrm{irr}(|f|)$ and that $\mathrm{irr}(f) \subset \mathcal{A}$ hence implies that $\mathrm{irr}(|f|) \subset \mathcal{A}$. So, suppose that, conversely, $\mathrm{irr}(|f|) \subset \mathcal{A}$ and let $g \in \mathrm{irr}(f)$. Then $|g| \in \mathrm{irr}(|f|) \subset \mathcal{A}$ and so $f|g|^{1/n} \in \mathcal{A}$ for each $n \in \mathbb{N}$. Without loss of generality, we may assume that $|g| \leqslant 1$, in which case $f|g|^{1/n}$ converges uniformly to g as $n \to \infty$ and so $\mathrm{irr}(f) \subset \mathcal{A}$.

Now let D be as in (d) and $f \in \mathcal{A}$. We need to show that $\mathrm{irr}(f) \subset \mathcal{A}$ and by (c) we may assume that f is positive. We want to use part (b) and therefore claim that D contains a positive, increasing sequence $(f_n)_{n \in \mathbb{N}}$ converging to f. To see this, let $(h_n)_{n \in \mathbb{N}}$ be a sequence in D converging to f. By passing to $(|h_n|)_{n \in \mathbb{N}}$, we may assume that the sequence is positive. Moreover, we can arrange that $\|f - h_n\| < 1/n$ for each $n \in \mathbb{N}$ and, by passing to $g_n' = (g_n - (1/n)\mathbb{1}_K)_+ = \sup(g_n - (1/n)\mathbb{1}_K, 0)$ we may therefore also assume that for each $n \in \mathbb{N}$

$$\left(f - \frac{2}{n}\mathbb{1}_K\right)_+ \leqslant g_n \leqslant \bigvee_{k=1}^{n} g_k \leqslant f.$$

Since $(f - (2/n)\mathbb{1}_K)_+$ converges uniformly to f as $n \to \infty$, we can set $f_n := \bigvee_{k=0}^{n} g_k$ for $n \in \mathbb{N}$ and obtain a sequence $(f_n)_{n \in \mathbb{N}}$ in D such that $f_n \uparrow f$. Now pick $g \in \mathrm{irr}(f)$. Then, by (b), there is a sequence $(g_n)_{n \in \mathbb{N}}$ with $g_n \uparrow g$ and $g_n \in \mathrm{irr}(f_n) \cup \{0\}$ for each $n \in \mathbb{N}$. Since by assumption $\mathrm{irr}(D) \subset D$, $(g_n)_{n \in \mathbb{N}}$ lies in $D \subset \mathcal{A}$ and so $g \in \mathcal{A}$.

Part (e) immediately follows from Corollary 3.6 by taking the intersection of all monotone C*-subalgebras in $\mathcal{S}$ that contain $\mathcal{A}$. For (f), it suffices to find a separable, monotone subalgebra in $\mathcal{S}$ that contains $\mathcal{A}$. To this end, define increasing $\mathbb{Q}$-vector sublattices $D_n \subset \mathrm{C}(K)$ and subalgebras $\mathcal{B}_n \subset \mathrm{C}(K)$ as follows: let $\mathcal{B}_0 \in \mathcal{S}$ be a separable subalgebra containing $\mathcal{A}$ and let $D_0 \subset \mathcal{B}_0$ be a countable dense $\mathbb{Q}$-vector sublattice containing $\mathbb{1}_K$. For $n \in \mathbb{N}_0$, then define $\mathcal{B}_{n+1}$ to be a separable C*-algebra in $\mathcal{S}$ containing the C*-algebra generated by $\mathrm{irr}(D_n)$ and $\mathcal{B}_n$ and let $D_{n+1} \subset \mathcal{B}_n$ be a countable dense $\mathbb{Q}$-vector sublattice containing D_n and $\mathrm{irr}(D_n)$. Then $\mathcal{B} := \overline{\bigcup_{n \in \mathbb{N}} \mathcal{B}_n}$ lies in $\mathcal{S}$ and $D := \bigcup_{n \in \mathbb{N}} D_n$ is a countable dense vector sublattice of $\mathcal{B}$ satisfying $\mathrm{irr}(D) \subset D$. Since $\mathcal{B}$ is therefore separable, contains $\mathcal{A}$, and is monotone by (d), $\mathcal{A}_{\mathrm{mon}}^{\mathcal{S}} \subset \mathcal{B}$ is separable. □

Remark 3.11. Let $p\colon (K, G) \to (L, G)$ be a factor map of flows on locally connected spaces. Then the family $\mathcal{S}$ of G-invariant C*-subalgebras of $\mathrm{C}(K)$ satisfies the condition in Proposition 3.10(e) and so we can consider the monotone G-invariant hull of $\mathcal{A}_L = T_p(\mathrm{C}(L))$ in $\mathcal{S}$, which we denote by $T_p(\mathrm{C}(L))_{\mathrm{mon}}^{G}$. This subalgebra corresponds to a monotone factor map $q\colon (K, G) \to (L_{\mathrm{mon}}, G)$ of (K, G) and p factorizes over q:

$$(K, G) \xrightarrow{\;q\;} (L_{\mathrm{mon}}, G) \xrightarrow{\;\hat{p}\;} (L, G), \qquad p = \hat{p} \circ q.$$

Moreover, since (L_{mon}, G) corresponds to the monotone hull of $T_p(\mathrm{C}(L))$, it is the smallest monotone factor of (K, S) over which p factorizes. It is not difficult to see that (L_{mon}, G) is therefore isomorphic to the quotient of (K, G) by the G-invariant equivalence

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relation

$$\mathrm{Rc}(p) := \left\{ (x, y) \in K \times K \;\middle|\; \begin{array}{c} x \text{ and } y \text{ are in the same} \\ \text{connected component of } p^{-1}(p(x)) \end{array} \right\}$$

that was apparently first considered in [**MW76**, Definition 2.2] and is closed by [**MW76**, Proposition 2.3] or the more general [**HJ19**, Proposition 2.3]. We note for the next section that for separable G, Proposition 3.10(f) implies that L_{mon} is metrizable if and only if L is. Combined with Theorem 3.7, this means that if (L, G) is an equicontinuous metrizable factor of (K, G), then so is (L_{mon}, G).

4. *Equicontinuous factors*

Given a quotient map $p\colon X \to Y$ of topological spaces, it is generally very difficult to relate geometric properties of X to those of Y. The Hahn–Mazurkiewicz theorem illustrates how hopeless the situation is in general: it shows that every non-empty, connected, locally connected, compact metric space is the quotient of the unit interval. Considering that this includes, in particular, all compact, connected manifolds, it is clear that additional properties of p are needed in order to relate the geometric structure of X to that of Y. The following theorem shows that monotonicity is such a property.

THEOREM 4.1. [**CGM12**, Theorem 1.1] *Let $f\colon (X, x_0) \to (Y, y_0)$ be a quotient map of pointed topological spaces, where X is locally path-connected and Y is semilocally simply-connected. If each fiber $f^{-1}(y)$ is connected, then the induced homomorphism $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ of the fundamental groups is surjective.*

Combining this with the previous discussion, we obtain our main representation result for equicontinuous factors.

THEOREM 4.2. *Let (K, G) be a flow such that K is locally path-connected with finite first Betti number $\mathrm{b}_1(K)$, G is abelian, and K is metrizable or G is separable. If all G-invariant functions $f \in \mathrm{C}(K)$ are constant, then every equicontinuous factor of (K, G) is isomorphic to a minimal flow on some compact abelian Lie group of dimension less than $\mathrm{b}_1(K)/\mathrm{b}_0(K)$.*

More precisely, for every equicontinuous factor (L, G) of (K, G), there are a finite abelian group F of order $|F| \leqslant \mathrm{b}_0(K)$ and an $m \leqslant \mathrm{b}_1(K)/\mathrm{b}_0(K)$ such that (L, G) is isomorphic to a minimal action of G on $F \times \mathbb{T}^m$ via rotations.

Proof. We assume that (M, G) is a monotone equicontinuous factor of (K, G), which will imply the claim for every other equicontinuous factor. Denote by $\vartheta\colon (K, G) \to (M, G)$ the corresponding factor map and note that (M, G) is minimal since an equicontinuous system is minimal if and only if every continuous G-invariant continuous function on it is constant; see Remark 2.2. Since G is abelian and acts equicontinuously on M, the Ellis group $\mathrm{E}(M, G)$ is a compact abelian group and it is well known that a minimal equicontinuous flow (M, G) with an abelian group G is isomorphic to the minimal action $(\mathrm{E}(M, G), G)$ of G on the Ellis group $\mathrm{E}(M, G)$ via rotations (see [**Aus88**, Theorem 3.6]).

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Since $\mathrm{E}(M, G) \cong M$ is the quotient of a locally connected space, it follows by Lemma 3.1 that $\mathrm{E}(M, G)$ is locally connected too.

First, assume M and hence $\mathrm{E}(M, G)$ to be metrizable. It then follows from the classification of locally connected, second-countable, compact abelian groups that $\mathrm{E}(M, G) \cong F \times \mathbb{T}^I$ for a finite group F and an (at most countable) set I; see [**HM13**, Theorem 8.34] and [**HM13**, Theorem 8.46]. We again denote the induced map from K to $F \times \mathbb{T}^I$ by ϑ. Since ϑ is monotone by Theorem 3.7, K and $F \times \mathbb{T}^I$ have the same number of connected components and so F is of order $\mathrm{b}_0(K)$. Since G acts minimally on $F \times \mathbb{T}^I$ via the isomorphism $\mathrm{E}(M, G) \cong F \times \mathbb{T}^I$, it follows that G acts transitively on F and hence on the connected components of K. Therefore, if we fix the connected component $K_0 := \vartheta^{-1}(\{0\} \times \mathbb{T}^I)$, then $\mathrm{b}_1(K) = \mathrm{b}_0(K)\mathrm{b}_1(K_0)$.

Next, we show that I is finite by using Theorem 4.1 to show that $|I| \leqslant \mathrm{b}_1(K)/\mathrm{b}_0(K)$, though we need to be careful since $\mathbb{T}^I$ is semilocally simply connected if and only if I is finite. We therefore proceed by considering monotone finite-dimensional quotients: for $k \in \mathbb{N}$ with $k \leqslant |I|$, let $i_1, \ldots, i_k \in I$ be pairwise different and denote by $p_k \colon \mathbb{T}^I \to \mathbb{T}^k$ the canonical projection induced by the isomorphism $\mathbb{T}^k \cong \mathbb{T}^{\{i_1,\ldots,i_k\}}$. Moreover, let $\vartheta_0 \colon K_0 \to \mathbb{T}^I$ be the map canonically induced by ϑ. Then $p_n \circ \vartheta_0 \colon K_0 \to \mathbb{T}^k$ is monotone by Proposition 3.4(b), being the composition of monotone maps. Since $\mathbb{T}^k$ is semilocally simply connected, Theorem 4.1 shows that $p_k \circ \vartheta_0$ induces a surjective morphism $(p_k \circ \vartheta_0)_* \colon \pi_1(K_0) \to \pi_1(\mathbb{T}^n)$. Since $\pi_1(\mathbb{T}^n) \cong \mathbb{Z}^k$ is abelian, this morphism factorizes through the abelianization of $\pi_1(K_0)$, which is canonically isomorphic to $H_1(K_0)$ by the Hurewicz theorem. If we denote by $\eta \colon H_1(K_0) \to \pi_1(\mathbb{T}^k)$ the induced surjective group homomorphism, then

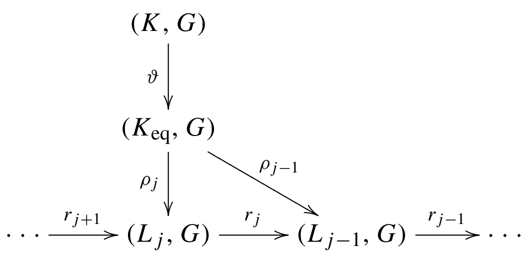
$$k = \operatorname{rank}(\pi_1(\mathbb{T}^k)) = \operatorname{rank}(\eta(H_1(K_0))) \leqslant \operatorname{rank}(H_1(K_0)) = \mathrm{b}_1(K_0) = \mathrm{b}_1(K)/\mathrm{b}_0(K).$$

Since $k \in \mathbb{N}$ was arbitrary with $k \leqslant |I|$, this shows that $|I| \leqslant \mathrm{b}_1(K)/\mathrm{b}_0(K)$.

Now we show that (M, G) is necessarily metrizable, which we only need to check for the maximal equicontinuous factor (K_{eq}, G). By Example 2.1, K_{eq} is metrizable if and only if the subalgebra

$$\mathcal{A}_{\mathrm{eq}} = \{f \in \mathrm{C}(K) \mid \{T_\varphi^n f \mid n \in \mathbb{N}\} \text{ is equicontinuous}\}$$

is separable. If K is metrizable, this is the case for every C^*-subalgebra of $\mathrm{C}(K)$, so assume instead that G is separable. If $\mathcal{A}_{\mathrm{eq}}$ is not separable, Proposition 3.10(f) and Remark 3.11 yield a sequence $(\mathcal{A}_j)_{j \in \mathbb{N}}$ of strictly increasing, separable, monotone, G-invariant C^*-subalgebras of $\mathcal{A}_{\mathrm{eq}}$ which induces the following commutative diagram of factor maps.



Since each of the systems (L_j, G) is metrizable and the factor maps $\rho_j \circ \vartheta$ are monotone, the above discussion applies and we can therefore replace the diagram with the following.

$$\begin{array}{ccccc} & (K,\varphi) & & & \\ & \downarrow{\scriptstyle \vartheta} & & & \\ & (\mathrm{E}(K_{\mathrm{eq}},G),G) & & & \\ & \downarrow{\scriptstyle \rho_j} & \searrow^{\rho_{j-1}} & & \\ \cdots \xrightarrow{r_{j+1}} & (F_j\times\mathbb{T}^{m_j},G) & \xrightarrow{r_j} & (F_{j-1}\times\mathbb{T}^{m_{j-1}},G) & \xrightarrow{r_{j-1}} \cdots \end{array}$$

where $m_j \in \{1, \ldots, \mathrm{b}_1(K)/\mathrm{b}_0(K)\}$ and F_j is a finite abelian group for each $j \in \mathbb{N}$. Since each of the systems $(F_j \times \mathbb{T}^{m_j}, G)$ is minimal and G acts via rotations, each r_j is a surjective group homomorphism and so Lemma 4.3 below implies that $m_{j-1} \leqslant m_j$. Moreover, since $\rho_{j-1} \circ \vartheta$ is monotone by the choice of $\mathcal{A}_{j-1}$, it follows that ρ_{j-1} is monotone for each $j \in \mathbb{N}$. But, if $\rho_{j-1} = r_j \circ \rho_j$ is monotone, it also follows that r_j is monotone for each $j \in \mathbb{N}$.

Since $m_j \leqslant \mathrm{b}_1(K)/\mathrm{b}_0(K)$ for each $j \in \mathbb{N}$, there can be only finitely many j such that $m_j < m_{j+1}$. In particular, there is a $J \in \mathbb{N}$ such that $m_j = m_{j+1}$ for each $j > J$. However, if $m_j = m_{j+1}$, then $\mathcal{A}_j = \mathcal{A}_{j+1}$: a surjective group homomorphism on $F \times \mathbb{T}^{m_{j+1}}$ must have finite kernel by Lemma 4.3 and can therefore only be monotone if its kernel is trivial, i.e., if it is an isomorphism. This contradicts the strict inclusion $\mathcal{A}_j \subsetneq \mathcal{A}_{j+1}$ and shows that $\mathcal{A}_{\mathrm{eq}}$ must be separable. Hence, K_{eq} is metrizable.

Now suppose that (L, G) is an arbitrary equicontinuous factor of (K, G) and let (K_{eq}, G) be the maximal equicontinuous factor of (K, G). Then, as in the monotone case, it follows that $(L, G) \cong (\mathrm{E}(L, G), G)$ and since the factor map $(K_{\mathrm{eq}}, G) \to (L, G)$ induces a surjective group homomorphism $r\colon \mathrm{E}(K_{\mathrm{eq}}, G) \to \mathrm{E}(L, G)$, the claim follows from the monotone case via Lemma 4.3. □

LEMMA 4.3. *Let $m \in \mathbb{N}$, F be a finite abelian group, and $r\colon F \times \mathbb{T}^m \to H$ be a continuous, surjective group homomorphism onto a compact group H. Then $H \cong F' \times \mathbb{T}^n$ for some finite abelian group F' of order $|F'| \leqslant |F|$ and $n \leqslant m$. Moreover, $n = m$ if and only if the kernel of r is finite.*

Proof. If G is a Lie group and $K \subset G$ is a closed normal subgroup, then G/K carries a canonical differentiable structure turning G/K into a Lie group and $\pi : G \to G/K$ into a submersion; see [**Lee12**, Theorems 21.17 and 21.26]. Therefore, $H \cong F \times \mathbb{T}^m/\ker(r)$ is a Lie group of dimension less than m. Being the quotient of $F \times \mathbb{T}^m$, H is a compact abelian Lie group and it is well known (see [**Sep07**, Theorem 5.2]) that this implies that $H \cong F' \times \mathbb{T}^n$ for some finite abelian group F' and $n \in \mathbb{N}$, proving the first statement. If $n = m$, $r\colon F \times \mathbb{T}^m \to F' \times \mathbb{T}^n$ is a submersion between manifolds of equal dimension and it thus follows from the inverse function theorem that it is in fact a local diffeomorphism. Therefore, the kernel of r is discrete and, since $F \times \mathbb{T}^m$ is compact, the kernel of r must be finite. Conversely, if $\ker(r)$ is finite, r is a local diffeomorphism and so $n = m$. □

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Many spaces satisfy the conditions of Theorem 4.2, including compact manifolds and finite CW complexes, since by the Seifert–van Kampen theorem their first Betti number is finite. In particular, we obtain the following generalization of [**HJ19**, Theorem 3.1] to quotients of connected, simply connected Lie groups by discrete, cocompact subgroups. Important examples for such spaces are given by nilmanifolds; see [**HK18**, Ch. 10].

COROLLARY 4.4. *Let H be a connected, simply connected Lie group, $\Gamma \subset H$ a discrete, cocompact subgroup, G an abelian group, and $(H/\Gamma, G)$ a dynamical system on H/Γ such that every G-invariant function $f \in \mathrm{C}(H/\Gamma, G)$ is constant. Then every equicontinuous factor of $(H/\Gamma, G)$ is isomorphic to a minimal action of G on some torus $\mathbb{T}^m$, $m \leqslant \mathrm{rank}(\Gamma/[\Gamma,\Gamma])$, via rotations.*

Proof. The canonical map $H \to H/\Gamma$ is the universal cover of H/Γ and its kernel is thus isomorphic to the fundamental group of H/Γ, i.e., $\pi_1(H/\Gamma) \cong \Gamma$. Since, H being connected, $\mathrm{b}_0(H/\Gamma) = 1$ and

$$\mathrm{b}_1(H/\Gamma) = \mathrm{rank}(H_1(H/\Gamma)) = \mathrm{rank}(\pi_1(H/\Gamma)_{\mathrm{ab}}) = \mathrm{rank}(\Gamma_{\mathrm{ab}}) = \mathrm{rank}(\Gamma/[\Gamma,\Gamma]),$$

the claim follows by Theorem 4.2. □

Note that the action of G on H/Γ itself is not assumed to be via rotations. If H is compact and G acts via rotations, the statement can be proved much more directly.

In the special case $H = \mathbb{R}^n$ and $\Gamma = \mathbb{Z}^n$, one obtains the following consequence.

COROLLARY 4.5. *Let $(\mathbb{T}^n, G)$ be a flow such that G is abelian and all G-invariant functions $f \in \mathrm{C}(\mathbb{T}^n)$ are constant. Then every equicontinuous factor of $(\mathbb{T}^n, G)$ is isomorphic to a minimal action of G on some torus $\mathbb{T}^m$, $m \leqslant n$, via rotations.*

Note, however, that in the case of the two-torus, Theorem 3.8 of Hauser and Jäger [**HJ19**, Theorem 3.8] is a lot more general: they showed that each monotone, minimal quotient of a strongly effective flow on $\mathbb{T}^2$ is isomorphic to a flow on $\mathbb{T}^2$, $\mathbb{T}$, or a point. They required neither that the acting group G be abelian nor that the factor under consideration be equicontinuous.

If $\mathrm{b}_1(K) = 0$, we also obtain the following for the maximal distal factor of a minimal homeomorphism.

COROLLARY 4.6. *Let $\varphi \colon K \to K$ be a minimal homeomorphism on a locally path-connected space with $\mathrm{b}_1(K) = 0$. Then the maximal distal factor of (K, φ) is trivial.*

Proof. The maximal equicontinuous factor and maximal distal factor of (K, φ) are trivial as a consequence of Theorem 4.2 and the Furstenberg structure theorem for distal minimal flows. □

In particular, there are no minimal distal transformations on such a space K. This includes simply connected spaces such as S^n for $n \geqslant 2$ (cf. [**Aus88**, Theorem 7.16]) but also spaces for which $\pi_1(K)$ or $H_1(K)$ are torsion groups, e.g., $\mathbb{R}\mathrm{P}^n$ for $n > 1$ as $H_1(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z}/2$ for $n > 1$. Note that a very similar result to Corollary 4.6 using Čech cohomology exists in [**KR69**, Theorem 3.5].

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Given a homeomorphism $\varphi\colon K \to K$ such that the Koopman operator T_φ on $\mathrm{C}(K)$ has one-dimensional fixed space, its point spectrum $\sigma_{\mathrm{p}}(T_\varphi)$ is a subgroup of the torus $\mathbb{T}$. As noted in Example 2.1(c), the maximal equicontinuous factor of (K, φ) is very closely related to the spectral theory of T_φ. For Theorem 4.2, this relationship has the following consequence.

COROLLARY 4.7. *Let $\varphi\colon K \to K$ be a homeomorphism on a locally path-connected, compact space with finite first Betti number $\mathrm{b}_1(K)$ and suppose that $\dim \operatorname{fix}(T_\varphi) = 1$. Then the point spectrum $\sigma_{\mathrm{p}}(T_\varphi)$ of the Koopman operator T_φ on $\mathrm{C}(K)$ is a subgroup of $\mathbb{T}$ generated by at most $\mathrm{b}_1(K)$ elements.*

Proof. Let (M, ψ) be the maximal equicontinuous factor of (K, φ). By the discussion in Example 2.1, the point spectrum of T_φ on $\mathrm{C}(K)$ is the same as that of T_ψ on $\mathrm{C}(M)$, so we only need to consider the system (M, ψ). By Theorem 4.2, (M, ψ) is isomorphic to a minimal rotation $(F \times \mathbb{T}^n, a)$ for an abelian group F of order $|F| \leqslant \mathrm{b}_0(K)$, a torus $\mathbb{T}^n$ of dimension $n \leqslant \mathrm{b}_1(K)/\mathrm{b}_0(K)$, and an $a \in F \times \mathbb{T}^n$. If we denote by $L_a\colon \mathrm{C}(F \times \mathbb{T}^n) \to \mathrm{C}(F \times \mathbb{T}^n)$ the Koopman operator corresponding to this rotation, then

$$\sigma_{\mathrm{p}}(L_a) = \widehat{F \times \mathbb{T}^n}(a) = \{\chi(a) \mid \chi \in \widehat{F \times \mathbb{T}^n}\},$$

where $\widehat{F \times \mathbb{T}^n}$ denotes the Pontryagin dual of $F \times \mathbb{T}^n$ (see [**EFHN15**, Proposition 14.24]). Since

$$\widehat{F \times \mathbb{T}^n} \cong \widehat{F} \times \widehat{\mathbb{T}^n} \cong F \times \mathbb{Z}^n,$$

the claim follows from the inequalities $|F| \leqslant \mathrm{b}_0(K)$ and $n \leqslant \mathrm{b}_1(K)/\mathrm{b}_0(K)$. □

Understanding the rank of point spectra is a question that goes back to Kolmogorov, see [**Lin75**, p. 300], and in fact the same bound in terms of Betti numbers has been previously established for more special systems in the context of classical mechanics, see [**AA68**, Theorem A 16.2].

Remark 4.8. Theorem 4.2 imposes constraints on the maximal equicontinuous factor if it is minimal. In the case of $K = \mathbb{T}^n$, these constraints are sharp: every torus $\mathbb{T}^m$ of dimension $m \leqslant n$ can be realized as the maximal equicontinuous factor of an invertible system $(\mathbb{T}^n, \varphi)$. To see this, let $(\mathbb{T}^m, \varphi_a)$ be a minimal rotation with $a \in \mathbb{T}^m$ and let $\psi\colon \mathbb{T}^{n-m} \to \mathbb{T}^{n-m}$ be the map which is, after the identification $\mathbb{T}^{n-m} \cong [0, 1)^{n-m}$, given by

$$[0, 1)^{n-m} \to [0, 1)^{n-m}, \quad (x_1, \ldots, x_{n-m}) \mapsto (x_1^2, \ldots, x_{n-m}^2).$$

Then $(\mathbb{T}^m, \varphi_a)$ is the maximal equicontinuous factor of $(\mathbb{T}^n, \varphi_a \times \psi)$.

A natural question now is whether the constraints on the maximal equicontinuous factor listed in Theorem 4.2 are sharp in general. The answer is negative: consider the wedge sum $K := \bigvee_{i=1}^n \mathbb{T}$. Then K is connected, locally connected, and $\mathrm{b}_1(K) = n$. However, if $k > 1$, there cannot be any monotone surjective map $\rho\colon K \to \mathbb{T}^k$ since $\mathbb{T}^k$ is the disjoint union of uncountably many connected non-singleton sets whereas K is not.

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In light of this example, one might look for other topological constraints on the maximal equicontinuous factor, and the covering dimension $\dim(K)$ of K presents itself as a possible candidate. Unfortunately, monotonicity by itself cannot yield such a bound: as observed by Hurewicz in [Hur30], every compact metric space embeds into a monotone image of S^3. In particular, S^3 has monotone quotients of infinite dimension. Therefore, one cannot, in general, conclude that if $p\colon K \to L$ is a monotone quotient map, then $\dim(L) \leqslant \dim(K)$. Positive results exist for factors of distal minimal flows for which this estimate does hold, as shown in [Ree77, Theorem 1.1]. Without additional assumptions, general results only exist in low dimensions: if $p\colon K \to L$ is a monotone quotient map of compact spaces and K is a two-manifold, then $\dim(L) \leqslant \dim(K)$; see [Zem77]. In higher dimensions, though, one cannot hope for dimension inequalities for factors without additional structural assumptions.

Remark 4.9. As mentioned in the introduction, the maximal equicontinuous factor of a distal minimal flow on a compact manifold is isomorphic to a compact abelian Lie group if the acting group is abelian, otherwise it is merely isomorphic to a flow on a homogeneous space of some compact Lie group. One could therefore ask whether Theorem 4.2 generalizes to non-abelian groups G in an analogous way. Unfortunately, the proof given above hinges on the fact that the dimension of a compact abelian Lie group H is precisely $\mathrm{b}_1(H)/\mathrm{b}_0(H)$ and is thus encoded in the first two homology groups, which is false for general compact Lie groups. However, an interesting question for further investigation is whether, instead of the maximal equicontinuous factor, there is a generalization of Theorem 4.2 to the maximal compact abelian group factor of a system (see [GGY18]).

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