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Linear and nonlinear stability criteria for compressible MHD flows in a gravitational field

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Abstract. The equilibrium and stability properties of ideal magnetohydrodynamics (MHD) of compressible flow in a gravitational field with a translational symmetry are investigated. Variational principles for the steady-state equations are formulated. The MHD equilibrium equations are obtained as critical points of a conserved Lyapunov functional. This functional consists of the sum of the total energy, the mass, the circulation along field lines (cross helicity), the momentum, and the magnetic helicity. In the unperturbed case, the equilibrium states satisfy a nonlinear second-order partial differential equation (PDE) associated with hydrodynamic Bernoulli law. The PDE can be an elliptic or a parabolic equation depending on increasing the poloidal flow speed. Linear and nonlinear Lyapunov stability conditions under translational symmetric perturbations are established for the equilibrium states.

1. Introduction

The study of magnetohydrodynamics (MHD) – electrically conducting fluids in the presence of a magnetic field – is an important part of modern fluid mechanics. The field of MHD was initiated in 1950 by Alfvén (1950), who established the basics of MHD, and developed by Alfvén and Falthammar (1963). The principles of MHD were described in detail in many textbooks of general content (e.g. Cowling 1957; Shercliff 1965; Hughes and Young 1966). MHD flows play a considerable role in many applied and fundamental studies. First, MHD flows of fluids in channels are the basis for different industrial technologies. Examples include electromagnetic flow measurements of conducting fluids (Shercliff 1962; Hofmann 2003), electromagnetic pumps used in metallurgy and systems of liquid-metal cooling for nuclear reactors (Branover and Unger 1993), and MHD power generators – the devices for direct conversion of thermal energy to electricity (Rosa 1968; Mitchner and Kruger 1973).

The investigation of MHD equilibria is one of the most important problems in MHD and arises in a number of fields including thermonuclear fusion reactors, astrophysics and solar physics. At present, there are many difficulties surrounding the description of fully three-dimensional configurations and so it is necessary to consider configurations with additional symmetry. Symmetric configurations of plasmas with steady mass flow occur both in laboratory experiments (generally axial symmetric) and in a great variety of astrophysical situations. In many astrophysical situations such as stellar and extra-galactic winds or collimated outflows, axial symmetry is important, while in solar physics translational symmetry is common in models of arcades and coronal loops.

MHD flows are common phenomena in modern experimental devices for confinement of fusion plasma, such as tokamaks. The plasma flows in tokamaks are developed in the regimes with additional heating, for example, as a result of unbalanced neutral beam injections (Suckewer et al. 1979; Asakura et al. 1993) or ion cyclotron resonance frequency heating (Eriksson et al. 1997; Rice et al. 1998). Theoretical studies of flowing plasmas were considered for the first time in Chandrasekhar (1956), Frieman and Rotenberg (1960) and Grad (1960). After the formalism of the magnetic flux function had been introduced by Shafranov (1957) for static equilibria, Woltjer (1959a,b) was the first to apply this formalism to the dynamic problem, deriving some integrals of the system. His most complicated case was an axisymmetric configuration with constant entropy. More general equations of state were treated by Hamieri (1983). A systematic review of the equations of MHD equilibria in the three different coordinates (Cartesian, cylindrical and spherical coordinates) can be found in the series of papers by Tsinganos (1981, 1982a, b, c), whereas the relativistic case is treated in Lovelace et al. (1986) and Bogovalov (1994). Unified treatments in a general curvilinear coordinate system were given by Edenstrasser (1980a, b) for the static case and by Agim and Tataronis (1985) for the case of equilibria with flow. Exact equilibria for MHD equations were obtained in Del Zanna and Chiuderi (1996), Vlahakis and Tsinganos (1998, 1999) and Cheviakov and Bogoyavlenskij (2004).

In a series of papers (Throumoulopoulos and Pantis 1989, 1996; Throumoulopoulos and Tasso 1997, 1999, 2001; Tasso and Throumoulopoulos 1998; Simintzis et al. 2001; Throumoulopoulos et al. 2003, 2009), the MHD equilibrium of ideal plasmas with incompressible flows and translational as well as axial symmetry were investigated. The main conclusions of these papers are as follows. (a) If the equilibrium flows of cylindrical plasmas with arbitrary cross-sectional shapes are purely poloidal, they should be incompressible. (b) Exact equilibria were constructed for constant poloidal flow Mach numbers for both cylindrical and axisymmetric configurations. (c) For the physically appealing class of cylindrical equilibria with isothermal magnetic surfaces, their cross-sections must be circular at large, while no restriction is imposed on the magnetic surfaces of axisymmetric equilibria, apart from the vicinity of the magnetic axis where they should be circular.

The effect of flows on stability properties of plasma configuration is not completely understood. Numerous experimental observations show that the plasma flows in tokamaks can improve the overall plasma confinement by stabilizing kink and resistive wall modes and suppressing turbulence (Taylor et al. 1995; Garofalo et al. 1999; Takechi et al. 2007). A stability analysis of MHD configurations with flows constitutes a very challenging mathematical problem. The linear stability analysis of such systems was first carried out in Frieman and Rotenberg (1960). The effects of poloidal and toroidal flows on tokamak plasma equilibria were examined by McClements and Hole (2010) in the MHD limit. Explicit sufficient conditions for the linear and nonlinear stability of equilibrium solutions of a variety of fluid and plasma problems in one, two and three dimensions were established by Holm et al. (1985). In their analysis, they used the development of the Lyapunov technique for Hamiltonian systems due to Arnold (1966). Stability analyses were performed on a general class of vortex flows with density inhomogeneity and magnetic fields (Fung 1984). Ideal MHD theory and its applications to magnetic fusion systems were reviewed and the equilibrium behavior of the currently toroidal magnetic fusion concepts and stability of such equilibria were investigated (Freidberg 1982). Lyapunov stability conditions for ideal MHD plasmas with mass flow in axisymmetric toroidal geometry (Almaguer et al. 1988) and in cylindrical geometry with arbitrary cross-section (Khater and Moawad 2003) were determined in the Eulerian representation. Exact equilibria of nonlinear two-dimensional cases (Khater and Moawad 2009a,b) and stability analysis of a barotropic compressible flow in the plane (Moawad 2012) were constructed.

In the present paper, we study the equilibrium and stability of an ideal plasma with compressible mass flow in the presence of a gravitational field. The paper is organized as follows.

In Sec. 2, we review the governing and steady-state equations of ideal compressible MHD flows. In Sec. 3, we investigate the constants of motion for the MHD system introduced in Sec. 2. We formulate variational principles and associate the steady-state MHD equations with critical points of a nonlinear conserved Lyapunov functional. In Sec. 4, we establish sufficient conditions of linearized stability for plasma equilibria given in Sec. 2. In Sec. 5, we establish a nonlinear stability criterion for plasma equilibria given in Sec. 2. In Sec. 6, we investigate physical interpretations for our obtained results. Finally, we summarize the results in Sec. 7.

2. The governing and steady-state equations

The ideal MHD plasma flows are governed by the following set of equations, written in standard notations and convenient units: the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{1}$$

the entropy equation

$$\frac{\partial s}{\partial t} + \mathbf{v} \cdot \nabla s = 0, \tag{2}$$

the momentum equation

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = -\nabla p + \mathbf{J}\wedge\mathbf{B} - \rho\nabla\Omega, \tag{3}$$

Faraday's law

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \wedge \mathbf{E}, \tag{4}$$

the divergence-free Gauss law

$$\nabla \cdot \mathbf{B} = 0, \tag{5}$$

and the ideal Ohm's law

$$\mathbf{E} + \mathbf{v}\wedge\mathbf{B} = 0, \tag{6}$$

with the thermodynamics relations

$$p = \rho^2 e_\rho(\rho, s), \quad T = e_s(\rho, s), \tag{7}$$

where $\rho, t, \mathbf{v}, p, \Omega, s, e(\rho, s), \mathbf{B}, \mathbf{E}, \mathbf{J}$ and T stands as usual for the mass density, time, fluid velocity, gas pressure, gravitational potential, specific entropy, specific internal energy, magnetic field induction, electric field, electric current density and temperature, respectively. The system under consideration is a compressible MHD plasma in a cylindrical domain D with boundary ∂D and arbitrary cross-sectional shape. The plasma pressure and temperature are determined as functions of ρ and s from a prescribed equation of state for the specific internal energy via the first law of thermodynamics:

$$de = \frac{\partial e}{\partial \rho}d\rho + \frac{\partial e}{\partial s}ds = \frac{p}{\rho^2}d\rho + T\,ds. \tag{8}$$

The boundary conditions are taken to be

$$\mathbf{n}\cdot\mathbf{v} = 0, \quad \mathbf{n}\cdot\mathbf{B} = 0 \quad \text{on } \partial D, \tag{9}$$

where $\mathbf{n}$ is the outward unit vector normal to ∂D.

The divergence-free field $\mathbf{B}$ can be expressed as

$$\mathbf{B} = \nabla\wedge(\psi\mathbf{e}_z) + B_z\mathbf{e}_z, \tag{10}$$

where $\psi = \psi(x, y, t)$ is the scalar magnetic potential, B_z is the z-component of the magnetic field and $\mathbf{e}_z$ is the unit vector in the z-direction. Since $\mathbf{B} \cdot \nabla\psi = 0$, the magnetic surface is characterized by $\psi(x, y, t) = \text{const}$.
The electric current density is given by

$$\mathbf{J} = \frac{1}{\mu_0}\nabla \wedge \mathbf{B} = \frac{1}{\mu_0}(\nabla B_z \wedge \mathbf{e}_z - \nabla^2\psi\mathbf{e}_z), \tag{11}$$

where μ_0 is the permeability of free space. In what follows, for simplicity, we take the permeability of free space to be the unity. This is possible in the cgs units.
In the steady state ($\partial/\partial t = 0$), equations (1)–(4) become

$$\nabla \cdot (\rho\mathbf{v}) = 0, \tag{12}$$

$$\mathbf{v} \cdot \nabla s = 0, \tag{13}$$

$$\rho(\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + \mathbf{J} \wedge \mathbf{B} - \rho\nabla\Omega, \tag{14}$$

$$\nabla \wedge \mathbf{E} = 0. \tag{15}$$

In this case, the velocity field $\mathbf{v}$ can be expressed as

$$\rho\mathbf{v} = \nabla \wedge (\phi\mathbf{e}_z) + \rho v_z\mathbf{e}_z, \tag{16}$$

where $\phi = \phi(x, y)$ is the stream function of the flow and v_z is the z-component of the velocity field. Since $\mathbf{v} \cdot \nabla\phi = 0$, the flow surface is characterized by $\phi(x, y) = \text{const}$.
The scalar product of Ohm's law (6) by $\mathbf{B}$ yields

$$\mathbf{B} \cdot \mathbf{E} = 0. \tag{17}$$

From (15), the electric field can be expressed as $\mathbf{E} = -\nabla\Theta$. Hence, Ohm's law is projected along $\mathbf{e}_z$ yielding

$$\mathbf{e}_z \cdot (\nabla\phi \wedge \mathbf{e}_z) \wedge (\nabla\psi \wedge \mathbf{e}_z) = 0. \tag{18}$$

Equations (17) and (18) imply that $\phi = \phi(\psi)$ and $\Theta = \Theta(\psi)$. Hence, (13) implies that $s = s(\psi)$. Two additional surface quantities are found from the component of (6) perpendicular to a magnetic surface:

$$v_z - \frac{\phi'}{\rho}B_z = \Theta', \tag{19}$$

and from the component of the momentum conservation (14) along $\mathbf{e}_z$:

$$\phi' v_z - B_z = \chi(\psi). \tag{20}$$

The prime denotes differentiation with respect to ψ. Solving the set of (19) and (20) for B_z and v_z, one obtains

$$B_z = \frac{\rho(\phi'\Theta' - \chi)}{\rho - \phi'^2}, \tag{21}$$

$$v_z = \frac{\rho\Theta' - \phi'\chi}{\rho - \phi'^2}. \tag{22}$$

Using (10), (16) and (19), one can get the following relation between the velocity and magnetic field:

$$\mathbf{v} = \frac{\phi'}{\rho}\mathbf{B} + \Theta'\mathbf{e}_z. \tag{23}$$

From (10) and (11), (14) takes the form

$$\rho(\mathbf{v} \cdot \nabla)\mathbf{v} = \rho T\nabla s - \rho\nabla h - \nabla\psi \wedge \nabla B_z - \nabla^2\psi\nabla\psi - B_z\nabla B_z - \rho\nabla\Omega, \tag{24}$$

where we have used the following relation of the specific enthalpy h:

$$h = e + \frac{p}{\rho}. \tag{25}$$

Using (16) and the vector identity

$$(\mathbf{v} \cdot \nabla)\mathbf{v} = \nabla\left(\frac{v^2}{2}\right) - \mathbf{v} \wedge (\nabla \wedge \mathbf{v}), \tag{26}$$

equation (24) becomes

$$\rho\nabla\left(\frac{v^2}{2}\right) - (\nabla\phi \wedge \mathbf{e}_z) \wedge \nabla \wedge \left(\frac{\nabla\phi}{\rho} \wedge \mathbf{e}_z\right) + \nabla v_z \wedge \nabla\phi - \rho v_z\nabla v_z = \rho T\nabla s - \rho\nabla h - \nabla\psi \wedge \nabla B_z - \nabla^2\psi\nabla\psi - B_z\nabla B_z - \rho\nabla\Omega. \tag{27}$$

Using the vector identity

$$\nabla \wedge (\mathbf{a} \wedge \mathbf{b}) = \mathbf{a}\nabla \cdot \mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} - \mathbf{b}\nabla \cdot \mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}, \tag{28}$$

equation (27) becomes

$$(\phi'\nabla v_z - \nabla B_z) \wedge \nabla\psi + \rho\nabla\left(\frac{v^2}{2} + h + \Omega\right) - \rho v_z\nabla v_z + B_z\nabla B_z = \left[\phi'\nabla \cdot \left(\frac{\nabla\phi}{\rho}\right) - \nabla^2\psi + \rho T s'\right]\nabla\psi. \tag{29}$$

Using (20) into (29), we have

$$\rho\nabla\left(\frac{v^2}{2} + h + \Omega\right) - \rho v_z\nabla v_z + B_z\nabla B_z = \left[\phi'\nabla \cdot \left(\frac{\nabla\phi}{\rho}\right) - \nabla^2\psi + \rho T s'\right]\nabla\psi. \tag{30}$$

Using (19) and (20) in the last term of the left-hand side of (30), we get

$$\rho\nabla\left(\frac{v^2}{2} - v_z\Theta' + h + \Omega\right) = \left[\phi'\nabla \cdot \left(\frac{\nabla\phi}{\rho}\right) - \nabla^2\psi - \rho v_z\Theta'' + B_z\chi' - v_z B_z\phi'' + \rho T s'\right]\nabla\psi. \tag{31}$$

The components of (31) along $\mathbf{B}$ and perpendicular to a magnetic surface yield

$$\frac{v^2}{2} - v_z\Theta' + h + \Omega = G(\psi), \tag{32}$$

$$\phi'\nabla \cdot \left(\frac{\nabla\phi}{\rho}\right) - \nabla^2\psi - \rho v_z\Theta'' + B_z\chi' - v_z B_z\phi'' + \rho T s' = \rho G'(\psi), \tag{33}$$

where $G(\psi)$ is an arbitrary function of ψ. Equation (32) represents the hydrodynamic Bernoulli law and (33) represents a generalized Grad–Shafranov equation in which it is a generalization to the known Grad–Shafranov equation in the static equilibrium (Lüst and Schlüter 1957; Grad and Rubin 1958).

3. Constants of motion and variational formulations

A Hamiltonian system referred to as a conservative system describes a motion involving constraints and forces which have a potential. It is a mathematical formalism developed by Hamilton to describe the evolution equations of a physical system. The advantage of this description is that it gives important insight about the dynamics, even if the initial-value problem cannot be solved analytically. The equations of motion can be expressed in Hamiltonian form as

$$\frac{dF}{dt}(u) = \frac{\partial F}{\partial t}(u) + \{F, H\}(u), \tag{34}$$

where H is the Hamiltonian, F is any functional of a set of dynamical variables u, and $\{,\}$ is Poisson bracket.
The constants of motion for the Hamiltonian system (34) are conserved functionals C, so that

$$\frac{dC}{dt} = \frac{\partial C}{\partial t} + \{C, H\} = 0. \tag{35}$$

The conserved quantity C, Casimir functional, corresponds to the symmetry under Lagrangian relabeling of fluid particles. The Hamiltonian viewpoint of fluid mechanical systems with finite and infinite number of degrees of freedom was described by Morrison (1998), where the action principle for the ideal compressible fluid was described in terms of Lagrangian or material variables. Also, the Casimir's energy method was discussed and a variant of it that depends upon the notion of dynamical accessibility was described. An analytical approach based on Hamiltonian field theory was introduced in Morrison and Hazeltine (1984). It was shown that the nonlinear ideal reduced MHD system in both high-beta and low-beta versions can be expressed in Hamiltonian form (Morrison and Hazeltine 1984). The original idea that MHD in Eulerian variables is a Hamiltonian system in coordinates that are not canonical, and thus can be written in terms of a noncanonical Poisson bracket, was introduced in Morrison and Greene (1980,1982). Also, an extended discussion of such brackets was given in Morrison (1982). A procedure for obtaining Casimir invariants from noncanonical Poisson brackets was described by Andreussi et al. (2010) and earlier a practical use of such a procedure was given in Andreussi and Pegoraro (2008).

Requiring that a solution u_e be a constrained minimum of the Hamiltonian, $\delta(H + C)[u_e] =: \delta\mathfrak{R}[u_e] = 0$, gives an equilibrium solution. The solutions u_e is then said to be formally stable if $\delta^2\mathfrak{R}[u_e]$ is definite. This is related to δW energy principles, which extremize the potential energy.

Variational principle based on minimizing the potential energy (the sum of the magnetic and internal energies) subject to constancy of the topological invariants of the ideal magnetostatic equations was formulated by Kruskal and Kulsrud (1958) for characterizing the equilibrium and stability properties of static ideal plasmas in toroidal geometry. The linear stability of these static equilibria was investigated by Bernstein et al. (1958) by using the energy principle. The linear stability of stationary MHD equilibria with mass flow was investigated by Frieman and Rotenberg (1960), where the energy principle method was extended to that purpose. Constrained energy variational principles in the Eulerian representation were applied to characterize axisymmetric MHD equilibria as critical points of conserved quantities for astrophysical plasmas in Chandrasekhar and Predergast (1956), Chandrasekhar (1957), Chandrasekhar and Woltjer (1958) and Woltjer (1959c).
The ideal MHD equations introduced in Sec. 2 conserve the Hamiltonian

$$H = \int_D \left[\frac{1}{2}\left(\rho v^2 + B^2\right) + \rho e(\rho, s) + \rho\Omega\right] d\tau, \tag{36}$$

where $d\tau = dxdydz$ and $B = |\mathbf{B}|$.
Consider now the functional

$$\mathfrak{R} = \int_D \left[\frac{1}{2}\left(\rho v^2 + B^2\right) + \rho e(\rho, s) + \rho\Omega + \rho F_1 + \rho v_z F_2 + B_z F_3 + (\mathbf{v}\cdot\mathbf{B})F_4\right] d\tau. \tag{37}$$

The functional $\mathfrak{R}$ is a conserved quantity and it represents the invariants for ideal MHD in the cylindrical case. The fifth, sixth, seventh and eighth parts of $\mathfrak{R}$ represent the mass, momentum, magnetic helicity and cross-helicity, respectively. The first variation of $\mathfrak{R}$ is given by

$$\delta\mathfrak{R} = \int_D \Bigg[(\rho\mathbf{v} + F_4\mathbf{B})\cdot\delta\mathbf{v} + (\mathbf{B} + F_4\mathbf{v})\cdot\delta\mathbf{B} + \rho F_2\delta v_z + F_3\delta B_z + \left(\frac{v^2}{2} + h + \Omega + v_z F_2 + F_1\right)\delta\rho + \left[\rho F_1' + \rho v_z F_2' + B_z F_3' + (\mathbf{v}\cdot\mathbf{B})F_4' + \rho T s'\right]\delta\psi\Bigg] d\tau, \tag{38}$$

where we assume that there is no perturbation for the gravitational potential. By standard manipulations using (10) and (16) and the Gauss divergence theorem, (38) is

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converted into

$$\delta\mathfrak{R} = \int_D \Big\{(\rho v_z + F_4 B_z + \rho F_2)\delta v_z + (B_z + F_4 v_z + F_3)\,\delta B_z + \left(\frac{v^2}{2} + h + \Omega + v_z F_2 + F_1 - \frac{1}{\rho^2}|\nabla\phi|^2 - \frac{F_4}{\rho^2}\nabla\psi\cdot\nabla\phi\right)\delta\rho + \Big[\rho F_1' + \rho v_z F_2' + B_z F_3' + (\mathbf{v}\cdot\mathbf{B})F_4' + \rho T s' - \nabla^2\psi - \nabla\cdot\left(\frac{F_4\nabla\phi}{\rho}\right)\Big]\delta\psi + \frac{1}{\rho}(\nabla\phi + F_4\nabla\psi)\cdot\nabla\delta\phi\Big\}d\tau + \int_{\partial D}\delta\psi\mathbf{n}\cdot\left(\frac{F_4}{\rho}\nabla\phi + \nabla\psi\right)dS. \tag{39}$$

The surface integral in (39) vanishes with the boundary condition

$$\delta\psi = 0 \quad \text{on } \partial D. \tag{40}$$

The boundary condition (40) is consistent with the second condition of (9), which implies that $\psi = \text{const.}$ on ∂D.

The first variation in (39) vanishes at the stationary solution, provided

$$\rho v_z + F_4 B_z + \rho F_2 = 0, \tag{41a}$$

$$B_z + F_4 v_z + F_3 = 0, \tag{41b}$$

$$\frac{v^2}{2} + h + \Omega + v_z F_2 + F_1 - \frac{1}{\rho^2}|\nabla\phi|^2 - \frac{F_4}{\rho^2}\nabla\psi\cdot\nabla\phi = 0, \tag{41c}$$

$$\rho F_1' + \rho v_z F_2' + B_z F_3' + (\mathbf{v}\cdot\mathbf{B})F_4' + \rho T s' - \nabla^2\psi - \nabla\cdot\left(\frac{F_4\nabla\phi}{\rho}\right) = 0, \tag{41d}$$

$$\nabla\phi + F_4\nabla\psi = 0. \tag{41e}$$

Taking the cross product of (41e) by $\mathbf{e}_z$ and using (41a), we find that the velocity and magnetic field are related by

$$\mathbf{v} = -\frac{F_4}{\rho}\mathbf{B} - F_2\mathbf{e}_z. \tag{42}$$

The cross product of (41e) by $\nabla\psi$ implies that the stream function is a surface quantity, $\phi = \phi(\psi)$, and $F_4 = -\phi'$. Hence, (42) coincides with (23) by taking $F_2 = -\Theta'$. The substitution of (41e) in (41c) yields the Bernoulli law:

$$\frac{v^2}{2} + h + \Omega + v_z F_2 = -F_1, \tag{43}$$

which coincides with (32) by taking $F_1 = -G$.

Now, we show that the critical point conditions for $\mathfrak{R}$, (41a)–(41e), imply the equilibrium relations of the steady-state equations of an ideal MHD system in Sec. 2 (12)–(15). Relation (42) between the velocity and magnetic field at equilibrium gives immediately

$$\nabla\cdot(\rho\mathbf{v}) = -\nabla\cdot(F_4\mathbf{B} + \rho F_2\mathbf{e}_z) = 0, \tag{44}$$

$$\mathbf{v}\cdot\nabla s = 0, \tag{45}$$

$$\nabla\wedge(\mathbf{v}\wedge\mathbf{B}) = -\nabla\wedge(F_2\mathbf{e}_z\wedge\mathbf{B}) = -\nabla\wedge(F_2\nabla\psi) = 0, \tag{46}$$

which satisfy the continuity equation, the entropy equation and Faraday's law ((12), (13) and (15) in Sec. 2). Since $\mathbf{v}\cdot\mathbf{B} = \nabla\phi\cdot\nabla\psi/\rho + v_z B_z$, then (41d) takes the form

$$\rho F_1' + \rho v_z F_2' + B_z F_3' + v_z B_z F_4' + \rho T s' - \nabla^2\psi - F_4\nabla\cdot\left(\frac{\nabla\phi}{\rho}\right) = 0, \tag{47}$$

which coincides with (33) by using $F_1 = -G, F_2 = -\Theta', F_3 = \chi$ and $F_4 = -\phi'$. This shows that (41d) represents the component of the momentum (14) perpendicular to a magnetic surface. Also, scalar multiplication of (42) by ∇v_z yields

$$\rho\mathbf{v}\cdot\nabla v_z = -F_4\mathbf{B}\cdot\nabla v_z = -\mathbf{e}_z\cdot[\nabla(F_4 v_z)\wedge\nabla\psi], \tag{48}$$

which by using (41b) gives

$$\rho\mathbf{v}\cdot\nabla v_z = \mathbf{e}_z\cdot[\nabla(F_3 + B_z)\wedge\nabla\psi] = \mathbf{e}_z\cdot(\nabla B_z\wedge\nabla\psi) = \mathbf{e}_z\cdot(\mathbf{J}\wedge\mathbf{B}). \tag{49}$$

Equation (49) represents the z-component of (14). Moreover, (41a), (41b) and (43) coincide with (19), (20) and (32), respectively. Therefore, the critical point conditions (41a)–(41e) of the functional $\mathfrak{R}$ in (37) imply the equilibrium relations for the ideal MHD equations.

4. Linear stability criterion

In this section, we formulate variational principles and establish sufficient conditions for the linearized stability of MHD equilibria given in Sec. 2.

The Hamiltonian function for any Hamiltonian system linearized about equilibrium is one-half the second variation, i.e. $\frac{1}{2}\delta^2\mathfrak{R}$. Consequently, the quadratic form $\delta^2\mathfrak{R}$ is preserved in time by the dynamics of the linearized flow. Moreover, if $\delta^2\mathfrak{R}$ is definite in sign, then it provides a conserved norm for measuring deviation from equilibrium of an initial disturbance under the linearized dynamics. Thus, the conditions on the equilibrium flow for $\delta^2\mathfrak{R}$ to be definite are sufficient conditions for linearized Lyapunov stability (see Holm et al. 1985 and Spies 1980 and references therein for additional discussions of Lyapunov methods in plasma physics).

The second variation of the functional $\Re$ gives

$$\delta^2\Re = \int_D \{\rho|\delta \mathbf{v}|^2 + 2\mathbf{v}\cdot\delta\mathbf{v}\delta\rho + 2F_4\delta\mathbf{B}\cdot\delta\mathbf{v} + 2F_4'\delta\psi(\mathbf{B}\cdot\delta\mathbf{v} + \mathbf{v}\cdot\delta\mathbf{B}) + |\delta\mathbf{B}|^2 + 2F_2\delta\rho\delta v_z + 2\rho F_2'\delta\psi\delta v_z + 2F_3'\delta\psi\delta B_z + 2(F_1' + v_zF_2' + s'\frac{\partial h}{\partial s})\delta\psi\delta\rho + \frac{\partial h}{\partial \rho}(\delta\rho)^2 + (\rho F_1'' + \rho v_z F_2'' + B_zF_3'' + \mathbf{v}\cdot\mathbf{B}F_4'' + \rho s'^2\frac{\partial^2 e}{\partial s^2} + \rho T s'')(\delta\psi)^2\}d\tau. \tag{50}$$

Let $(\delta v_1, \delta v_2)$ and $(\delta B_1, \delta B_2)$ be the (x, y) components of $\delta\mathbf{v}$ and $\delta\mathbf{B}$, respectively, the second variation (50) can be rearranged into matrix quadratic from as follows:

$$\delta^2\Re = \int_D \begin{pmatrix}\delta v_1 & \delta v_2 & \delta v_z & \delta B_1 & \delta B_2 & \delta B_z & \delta\rho & \delta\psi\end{pmatrix} \begin{pmatrix} \rho & 0 & 0 & F_4 & 0 & 0 & v_1 & F_4'B_1 \\ 0 & \rho & 0 & 0 & F_4 & 0 & v_2 & F_4'B_2 \\ 0 & 0 & \rho & 0 & 0 & F_4 & g_1 & g_2 \\ F_4 & 0 & 0 & 1 & 0 & 0 & 0 & F_4'v_1 \\ 0 & F_4 & 0 & 0 & 1 & 0 & 0 & F_4'v_2 \\ 0 & 0 & F_4 & 0 & 0 & 1 & 0 & g_3 \\ v_1 & v_2 & g_1 & 0 & 0 & 0 & \frac{\partial h}{\partial \rho} & g_4 \\ F_4'B_1 & F_4'B_2 & g_2 & F_4'v_1 & F_4'v_2 & g_3 & g_4 & g_5 \end{pmatrix} \begin{pmatrix}\delta v_1 \\ \delta v_2 \\ \delta v_z \\ \delta B_1 \\ \delta B_2 \\ \delta B_z \\ \delta\rho \\ \delta\psi\end{pmatrix} d\tau, \tag{51}$$

where

$$g_1 = v_z + F_2, \quad g_2 = B_zF_4' + \rho F_2',$$
$$g_3 = v_zF_4' + F_3', \quad g_4 = F_1' + v_zF_2' + s'\frac{\partial h}{\partial s},$$
$$g_5 = \rho F_1'' + \rho v_zF_2'' + B_zF_3'' + \mathbf{v}\cdot\mathbf{B}F_4'' + \rho s'^2\frac{\partial^2 e}{\partial s^2} + \rho T s''. \tag{52}$$

The purely algebraic quadratic form is positive definite if and only if each of its subdeterminants along the principal diagonal (principal minors) is positive definite. The eight principal minors of the symmetric 8×8 matrix in (51) are

$$\mu_1 = \rho, \quad \mu_2 = \rho^2, \quad \mu_3 = \rho^3, \quad \mu_4 = \rho^2\left(\rho - F_4^2\right),$$
$$\mu_5 = \rho\left(\rho - F_4^2\right)^2, \mu_6 = \left(\rho - F_4^2\right)^3,$$
$$\mu_7 = \left(\rho - F_4^2\right)^2\left[\left(\rho - F_4^2\right)\frac{\partial h}{\partial \rho} - v_1^2 - v_2^2 - g_1^2\right],$$
$$\mu_8 = \alpha\mu_7 + G^*\Big[(v_1Y - v_2X)^2 + (v_1Z - g_1X)^2 + (v_2Z - g_1Y)^2 + 2G^*g_4(v_1X + v_2Y + g_1Z) - G^{*2}g_4^2 - G^*\frac{\partial h}{\partial \rho}(X^2 + Y^2 + Z^2)\Big],$$

where

$$\alpha = g_5 - g_3^2 - F_4'^2(v_1^2 + v_2^2), \quad G^* = \rho - F_4^2,$$
$$X = F_4'(B_1 - v_1F_4), Y = F_4'(B_2 - v_2F_4),$$
$$Z = g_2 - F_4g_3. \tag{53}$$

The term $(v_1Y - v_2X)^2$ in μ_8 vanishes by using relation (42), then we get

$$\mu_8 = \alpha\mu_7 + G^*\Big[(v_1Z - g_1X)^2 + (v_2Z - g_1Y)^2 + 2G^*g_4(v_1X + v_2Y + g_1Z) - G^{*2}g_4^2 - G^*\frac{\partial h}{\partial \rho}(X^2 + Y^2 + Z^2)\Big]. \tag{54}$$

The subdeterminants from μ_1 to μ_6 are positive definite if $\rho - F_4^2 > 0$; hence, the sufficient conditions for linear stability are

$$\rho > F_4^2, \tag{55a}$$

$$\left(\rho - F_4^2\right)\frac{\partial h}{\partial \rho} > v_1^2 + v_2^2 + g_1^2, \tag{55b}$$

$$\mu_8 > 0. \tag{55c}$$

5. Nonlinear stability criterion

In this section, we establish nonlinear stability conditions for the MHD equilibria given in Sec. 2. We use the stability algorithm introduced in Holm et al. (1985) in the sense of the Lyapunov definition of nonlinear stability which states that in terms of a norm $\|\,\|$, an equilibrium point u_e of a dynamical system is said to be nonlinearly stable if for every $\epsilon > 0$ there is a $\delta > 0$ such that if $\|u(0) - u_e\| < \delta$, then $\|u(t) - u_e\| < \epsilon$ for $t > 0$ (t is the time).

Briefly, we list the nonlinear stability algorithm of Holm et al. (1985).

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5.1. Stability algorithm

(a) Choose a Banach space $\mathbb{U}$ of fields u and write the equations of motion on $\mathbb{U}$ as

$$\frac{\partial u}{\partial t} = X(u), \qquad (56)$$

for a nonlinear operator X mapping a domain in $\mathbb{U}$ to $\mathbb{U}$.

(b) Find a conserved functional H for (56), usually representing the total energy; that is find a map $H : \mathbb{U} \longrightarrow \mathbb{R}$ (the real numbers) such that $dH(u)/dt = 0$ for a continuously differentiable solution u of (56).

(c) Find a family of constants of motion for (56). That is, find a collection of functionals C on $\mathbf{U}$ such that $dC(u)/dt = 0$ for any continuously differentiable solution u of (56).

(d) Relate an equilibrium solution u_e of (56) to the constant of motion C by requiring that $\mathfrak{R} := H + C$ have a critical point at u.

(e) Find quadratic forms (convexity estimates) Q_1 and Q_2 on $\mathbb{U}$ such that

$$Q_1(\Delta u) \leqslant H(u_e + \Delta u) - H(u_e) - DH(u_e) \cdot \Delta u, \qquad (57a)$$

$$Q_2(\Delta u) \leqslant C(u_e + \Delta u) - C(u_e) - DC(u_e) \cdot \Delta u, \qquad (57b)$$

for all $\Delta u = u - u_e$ in $\mathbb{U}$. Then, require that

$$Q_1(\Delta u) + Q_2(\Delta u) > 0, \quad \text{for all}\, \Delta u \neq 0 \,\text{in}\, \mathbb{U}. \qquad (58)$$

(f) If steps from (a) to (e) have been carried out, then for any solution u of (56) the following *a priori* estimate on Δu holds:

$$Q_1(\Delta u) + Q_2(\Delta u) \leqslant \mathfrak{R}(u(0)) - \mathfrak{R}(u_e). \qquad (59)$$

(g) Set $||\Delta u(t)||^2 = Q_1(\Delta u)+Q_2(\Delta u)$, so $||\Delta u(t)||$ defines a norm on $\mathbb{U}$ and the functional $\mathfrak{R}$ is continuous in this norm at u_e, then u_e is nonlinearly stable.

Now we apply the above procedure to the MHD equilibria considered in Sec. 2. All the steps (a)–(d) have been carried out in Secs. 2 and 3, where the conserved functional H in step (b) was taken to be the Hamiltonian in (36), and the collection of a family of constants of motion in step (c) is

$$C = \int_D [\rho F_1 + \rho v_z F_2 + B_z F_3 + (\mathbf{v} \cdot \mathbf{B}) F_4] \, d\tau. \qquad (60)$$

What remains is to apply steps (e)–(g). For stationary solutions $(\rho_e, \psi_e \mathbf{v}_e, \mathbf{B}_e)$, we have

$$\begin{aligned}\hat{H} &:= H(\rho_e + \Delta\rho, \psi_e + \Delta\psi, \mathbf{v}_e + \Delta\mathbf{v}, \mathbf{B_e} + \Delta\mathbf{B}) \\ &\quad - H(\rho_e, \psi_e, \mathbf{v}_e, \mathbf{B}_e) - DH(\rho_e, \psi_e, \mathbf{v}_e, \mathbf{B}_e) \\ &\quad \cdot (\Delta\rho, \Delta\psi, \Delta\mathbf{v}, \Delta\mathbf{B}) = \int_D \left\{ \frac{1}{2} \left[(\rho_e + \Delta\rho)(\Delta\mathbf{v})^2 \right.\right. \\ &\quad \left. + 2\mathbf{v}_e \cdot \Delta\mathbf{v}\Delta\rho + (\Delta B)^2 \right] + \rho_e \left[e(\rho_e + \Delta\rho, s_e + \Delta s) \right. \\ &\quad \left. - e(\rho_e, s_e) - \left(\frac{\partial e}{\partial \rho}(\rho_e, s_e)\Delta\rho + T s' \Delta\psi \right) \right] \\ &\quad \left. + \left[e(\rho_e + \Delta\rho, s_e + \Delta s) - e(\rho_e, s_e) \right] \Delta\rho \right\} d\tau, \qquad (61)\end{aligned}$$

where $s_e = s_e(\psi_e)$ and $\Delta s = s'(\psi_e)\Delta\psi$, and $\Delta(\cdot)$ refers to perturbations. Using the two dimensional Taylor's expansion with remainder in Lagrange's form, we have

$$\begin{aligned}\hat{H} &= \frac{1}{2} \int_D \left\{ (\rho_e + \Delta\rho)(\Delta\mathbf{v})^2 + 2\mathbf{v}_e \cdot \Delta\mathbf{v}\Delta\rho + (\Delta B)^2 \right. \\ &\quad + \rho_e \left[\frac{\partial^2 e}{\partial \rho^2}(\varphi_1, \chi_1)(\Delta\rho)^2 + 2s'(\chi_1) \frac{\partial^2 e}{\partial s \partial \rho}(\varphi_1, \chi_1)\Delta\rho\Delta\psi \right. \\ &\quad \left. + \left(T s'' + s'^2 \frac{\partial^2 e}{\partial s^2} \right) (\varphi_1, \chi_1)(\Delta\psi)^2 \right] \\ &\quad \left. + 2\frac{\partial e}{\partial \rho}(\varphi_2, \chi_2)(\Delta\rho)^2 + 2s'(\chi_2)\frac{\partial e}{\partial s}(\varphi_2, \chi_2)\Delta\rho\Delta\psi \right\} d\tau,\end{aligned} \qquad (62)$$

where $\rho_e < \varphi_1, \varphi_2 < \rho_e + \Delta\rho$, $\psi_e < \chi_1, \chi_2 < \psi_e + \Delta\psi$. From this analysis, we take the quadratic functional Q_1 in step (e) to be

$$\begin{aligned}Q_1 &= \frac{1}{2} \int_D \left[\rho_{\min}(\Delta\mathbf{v})^2 + 2\mathbf{v}_e \cdot \Delta\mathbf{v}\Delta\rho + (\Delta B)^2 \right. \\ &\quad \left. + a(\Delta\rho)^2 + 2b\Delta\rho\Delta\psi + c\rho_e(\Delta\psi)^2 \right] d\tau, \qquad (63)\end{aligned}$$

where $\rho_{\min}$, a, b and c are the minimum values of ρ, $\partial h/\partial\rho$, $\partial h/\partial\psi = s'\partial h/\partial s$ and $\partial^2 e/\partial\psi^2 = Ts'' + s'^2 \partial^2 e/\partial s^2$,

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respectively. Next, consider the functional

$$\begin{aligned}\hat{C} &:= C(\rho_e+\Delta\rho,\psi_e+\Delta\psi,\mathbf{v}_e+\Delta\mathbf{v},\mathbf{B_e}+\Delta\mathbf{B})\\ &\quad -C(\rho_e,\psi_e,\mathbf{v}_e,\mathbf{B}_e)-DC(\rho_e,\psi_e,\mathbf{v}_e,\mathbf{B}_e)\\ &\quad \cdot(\Delta\rho,\Delta\psi,\Delta\mathbf{v},\Delta\mathbf{B})=\int_D\Big\{(\rho_e+\Delta\rho)F_1(\psi_e+\Delta\psi)\\ &\quad +(\rho_e+\Delta\rho)(v_{z(e)}+\Delta v_z)F_2(\psi_e+\Delta\psi)+(B_{z(e)}\\ &\quad +\Delta B_z)F_3(\psi_e+\Delta\psi)+(\mathbf{v}_e+\Delta\mathbf{v})\cdot(\mathbf{B_e}+\Delta\mathbf{B})F_4(\psi_e\\ &\quad +\Delta\psi)-[F_1(\psi_e)+v_{z(e)}F_2(\psi_e)]\Delta\rho-\rho_eF_1(\psi_e)\\ &\quad -\rho_e(v_{z(e)}+\Delta v_{z(e)})F_2(\psi_e)-(B_{z(e)}+\Delta B_z)F_3(\psi_e)\\ &\quad -(\mathbf{v}_e\cdot\mathbf{B}_e+\mathbf{v}_e\cdot\Delta\mathbf{B}+\mathbf{B}_e\cdot\Delta\mathbf{v})F_4(\psi_e)\\ &\quad -\rho_e[F_1'(\psi_e)+v_{z(e)}F_2'(\psi_e)+B_{z(e)}F_3'(\psi_e)\\ &\quad +(\mathbf{v}_e\cdot\mathbf{B_e})F_4'(\psi_e)]\Delta\psi\Big\}\,d\tau. \qquad (64)\end{aligned}$$

Using the following notation:

$$\hat{F}_i=F_i(\psi_e+\Delta\psi)-F_i(\psi_e)-F_i'(\psi_e)\Delta\psi, \qquad (65a)$$

$$F_i^*=F_i(\psi_e+\Delta\psi)-F_i(\psi_e), \qquad (65b)$$

where $i=1,2,3,4$; thus (64) becomes

$$\begin{aligned}\hat{C}=\int_D&\big[\rho_e\hat{F}_1+\rho_ev_{z(e)}\hat{F}_2+B_{z(e)}\hat{F}_3+(\mathbf{v}_e\cdot\mathbf{B}_e)\hat{F}_4\\ &+F_1^*\Delta\rho+F_2^*(\rho_e\Delta v_z+v_{z(e)}\Delta\rho)+F_3^*\Delta B_z\\ &+F_4^*(\mathbf{v}_e\cdot\Delta\mathbf{B}+\mathbf{B}_e\cdot\Delta\mathbf{v})+F_2(\psi_e+\Delta\psi)\Delta\rho\Delta v_z\\ &+F_4(\psi_e+\Delta\psi)\Delta\mathbf{v}\cdot\Delta\mathbf{B}\big]\,d\tau. \qquad (66)\end{aligned}$$

Using the one-dimensional Taylor's expansion with remainder in Lagrange's form, we have

$$\hat{F}_i=\frac{1}{2}F_i''(\xi_i)(\Delta\psi)^2,\quad \xi_i=\psi_e+\theta_i\Delta\psi,\quad 0<\theta_i<1, \qquad (67a)$$

$$F_i^*=F_i'(\zeta_i)\Delta\psi,\quad \zeta_i=\psi_e+\lambda_i\Delta\psi,\quad 0<\lambda_i<1, \qquad (67b)$$

where $i=1,2,3,4$. Therefore, (66) reads

$$\begin{aligned}\hat{C}=\int_D&\Big\{\frac{1}{2}\big[\rho_eF_1''(\xi_1)+\rho_ev_{z(e)}F_2''(\xi_2)+B_{z(e)}F_3''(\xi_3)\\ &+(\mathbf{v}_e\cdot\mathbf{B}_e)F_4''(\xi_4)\big](\Delta\psi)^2+F_1'(\zeta_1)\Delta\rho\Delta\psi\\ &+F_2'(\zeta_2)(\rho_e\Delta v_z+v_{z(e)}\Delta\rho)\Delta\psi+F_3'(\zeta_3)\Delta B_z\Delta\psi\\ &+F_4'(\zeta_4)(\mathbf{v}_e\cdot\Delta\mathbf{B}+\mathbf{B}_e\cdot\Delta\mathbf{v})\Delta\psi+F_2(\psi_e+\Delta\psi)\Delta\rho\Delta v_z\\ &+F_4(\psi_e+\Delta\psi)\Delta\mathbf{v}\cdot\Delta\mathbf{B}\Big\}\,d\tau. \qquad (68)\end{aligned}$$

For finite-amplitude perturbations $\psi_{\min}\leqslant\xi_i,\zeta_i,\chi_i\leqslant\psi_{\max}$ and $\rho_{\min}\leqslant\varphi_j\leqslant\rho_{\max}$, where $i=1,2,3,4$, we take the quadratic functional Q_2 in step (e) to be

$$\begin{aligned}Q_2=\int_D&\Big[\frac{1}{2}k(\Delta\psi)^2+F_1'(\zeta_1)\Delta\rho\Delta\psi+F_2'(\zeta_2)(\rho_e\Delta v_z\\ &+v_{z(e)}\Delta\rho)\Delta\psi+F_3'(\zeta_3)\Delta B_z\Delta\psi+F_4'(\zeta_4)(\mathbf{v}_e\cdot\Delta\mathbf{B}\\ &+\mathbf{B}_e\cdot\Delta\mathbf{v})\Delta\psi+F_2(\chi_3)\Delta\rho\Delta v_z\\ &+F_4(\chi_4)\Delta\mathbf{v}\cdot\Delta\mathbf{B}\Big]\,d\tau, \qquad (69)\end{aligned}$$

where k is the minimum value of $\Lambda:=\rho_eF_1''(\xi_1)+\rho_ev_{z(e)}F_2''(\xi_2)+B_{z(e)}F_3''(\xi_3)+(\mathbf{v}_e\cdot\mathbf{B}_e)F_4''(\xi_4)$, and hence

$$\begin{aligned}(Q_1+Q_2)(\Delta\rho,\Delta\psi,\Delta\mathbf{v},\Delta\mathbf{B})=\frac{1}{2}\int_D&\Big[\rho_{\min}(\Delta\mathbf{v})^2+2\mathbf{v}_e\\ &\cdot\Delta\mathbf{v}\Delta\rho+(\Delta B)^2+a(\Delta\rho)^2+(k+c\rho_e)(\Delta\psi)^2\\ &+2[F_1'(\zeta_1)+v_{z(e)}F_2'(\zeta_2)+b]\Delta\rho\Delta\psi\\ &+2\left[\rho_eF_2'(\zeta_2)\Delta v_z+F_3'(\zeta_3)\Delta B_z\right.\\ &\left.+F_4'(\zeta_4)(\mathbf{v}_e\cdot\Delta\mathbf{B}+\mathbf{B}_e\cdot\Delta\mathbf{v})\right]\Delta\psi+2F_2(\chi_3)\Delta\rho\Delta v_z\\ &+2F_4(\chi_4)\Delta\mathbf{v}\cdot\Delta\mathbf{B}\Big]\,d\tau. \qquad (70)\end{aligned}$$

Equation (70) can be rearranged into matrix quadratic form as in (51), with the replacement of ρ, $\partial h/\partial\rho$, g_4 and g_5 by $\rho_{\min}$, a, $F_1'(\zeta_1)+v_{z(e)}F_2'(\zeta_2)+b$ and $k+c\rho_e$, respectively. Consequently, condition (58) holds if

$$\rho_{\min}>F_4^2, \qquad (71a)$$

$$a\left(\rho_{\min}-F_4^2\right)>v_1^2+v_2^2+g_1^2, \qquad (71b)$$

$$\mu_8>0. \qquad (71c)$$

Thus, the *a priori* estimate (59), in step (f), holds. Finally, to achieve step (g) we define the following norm:

$$\|(\Delta\rho,\Delta\psi,\Delta\mathbf{v},\Delta\mathbf{B})\|^2=(Q_1+Q_2)(\Delta\rho,\Delta\psi,\Delta\mathbf{v},\Delta\mathbf{B}). \qquad (72)$$

Then we have the following nonlinear stability criterion.

5.2. *Nonlinear stability criterion*

Let $(\rho_e,\psi_e,\mathbf{v}_e,\mathbf{B}_e)$ be an equilibrium solution of the system (1)–(6). Suppose the following.
(i) Equations (41a)–(41e) are satisfied for some twice continuously differentiable functions $F_1(\psi),F_2(\psi),F_3(\psi)$ and $F_4(\psi)$.
(ii) For $0<\rho_{\min}\leqslant\rho(x,y)\leqslant\rho_{\max}<\infty$, the functions $F_1(\psi)$, $F_2(\psi)$, $F_3(\psi)$ and $f(\psi)=F_1''+v_zF_2''+B_zF_3''+\omega_zF_4''$

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satisfy

$$-\infty < a \leqslant \frac{\partial h}{\partial \rho} \leqslant \left(\frac{\partial h}{\partial \rho}\right)_{\max} < \infty, \tag{73a}$$

$$-\infty < b \leqslant \frac{\partial h}{\partial \psi} \leqslant \left(\frac{\partial h}{\partial \psi}\right)_{\max} < \infty, \tag{73b}$$

$$-\infty < c \leqslant \frac{\partial^2 e}{\partial \psi^2} \leqslant \left(\frac{\partial^2 e}{\partial \psi^2}\right)_{\max} < \infty, \tag{73c}$$

$$-\infty < k \leqslant \Lambda \leqslant \Lambda_{\max} < \infty. \tag{73d}$$

(iii) For $-\infty < \psi_{\min} \leqslant \psi(x, y) \leqslant \psi_{\max} < \infty$,

$$\rho_{\min} > F_4^2(\psi), \tag{74a}$$

$$a\left(\rho_{\min} - F_4^2(\psi)\right) > v_1^2 + v_2^2 + g_1^2, \tag{74b}$$

$$\mu_8 > 0. \tag{74c}$$

then $(\rho_e, \psi_e, \mathbf{v}_e, \mathbf{B}_e)$ is nonlinearly stable relative to the norm (72).

6. Physical interpretations

In what follows, we explain physical interpretations of the stability criteria obtained in Secs. 4 and 5. On the basis of the definition $\mathbf{V}_{Ap} \equiv \mathbf{B_p}/\sqrt{\rho}$ for the poloidal Alfvén velocity $\mathbf{V}_{Ap}$ associated with the poloidal magnetic field $\mathbf{B}_p \equiv \nabla\psi \wedge \mathbf{e}_z$, we have $v_{Ap}^2 = B_p^2/(\rho)$ (where $v_{Ap} \equiv |\mathbf{V}_{Ap}|$ is the Alfvén speed). Hence, by using (42), we find that the square of the poloidal velocity $v_p^2 = F_4^2 v_{Ap}^2/\rho$, where v_p is the magnitude of the poloidal velocity $\mathbf{v}_p$, ($\mathbf{v}_p = \nabla\phi/\rho \wedge \mathbf{e}_z$). Thus, by the definition of the square of the Alfvén Mach number $M_A^2 \equiv v_p^2/v_{Ap}^2 = F_4^2/\rho$ of the poloidal flow, condition (55a) can be written as $M_A^2 < 1$, which requires the equilibrium flow to be sub-Alfvénic. This is also required by condition (74a) of the nonlinear stability criterion because $M_A^2 \leqslant F_4^2/\rho_{\min} < 1$. From condition (55b) we note that the specific enthalpy exceeds with increasing the fluid density. This is physically reasonable because the specific enthalpy is related to the sound speed by the relation $c_s^2 = \rho\partial h/\partial\rho$. Condition (55b) can be written as

$$0 < M_A^2 + \frac{v_p^2 + g_1^2}{c_s^2} < 1, \tag{75}$$

where c_s^2 is the square of the sound speed. From inequality (75) we find that $v_p^2 < c_s^2$, which requires the equilibrium flow to be subsonic. Also, we have $v_p^2 < c_s^2 = \rho\partial h/\partial\rho = p/\rho$. This is physically reasonable from the fact that the pressure increases as the density of the fluid increases. The algebraic condition (55c) places a constraint on the fields $\mathbf{v}$ and $\mathbf{B}$. From inequality (75), we note that $M_A^2 < 1$, which coincides with condition (55a). Therefore, condition (55b) includes condition (55a) and the linear stability criterion, in Sec. 4, is reduced to

$$\left(\rho - F_4^2\right)\frac{\partial h}{\partial \rho} > v_1^2 + v_2^2 + g_1^2, \tag{76a}$$

$$\mu_8 > 0, \tag{76b}$$

which are equivalent to

$$M_A^2 + \frac{v_p^2 + g_1^2}{c_s^2} < 1, \tag{77a}$$

$$\mu_8 > 0. \tag{77b}$$

Conditions (76a) and (76b) or (77a) and (77b) are sufficient for the linear stability of ideal gravitating MHD flows. Condition (74b) of the nonlinear stability criterion can be rewritten in the form

$$0 < \frac{F_4^2}{\rho_{\max}} + \frac{v_p^2 + g_1^2}{c_s^2} < \frac{\rho_{\min}}{\rho_{\max}}, \tag{78}$$

which implies $F_4^2 < \rho_{\min}$. Therefore, condition (74a) of the nonlinear stability criterion can be omitted because it is contained into condition (74b). Conditions (74b) and (74c) are more stringent than conditions (76b) and (76c) of the linear stability criterion.

Returning to the generalized Grad–Shafranov equation (47), in which it can be written in the form

$$\left(1 - \frac{F_4^2}{\rho}\right)\nabla^2\psi - F_4\nabla\left(\frac{F_4}{\rho}\right)\cdot\nabla\psi$$
$$= (\rho F_1' + \rho v_z F_2' + B_z F_3' + v_z B_z F_4' + \rho T s'). \tag{79}$$

Equation (79) is a nonlinear second-order PDE of mixed type (Heinemann and Olbert 1978; Lovelace et al. 1986). It requires the specification of the five arbitrary functions F_1, F_2, F_3, F_4 and s. Somewhat less general forms of (79) have been derived and discussed previously in the axisymmetric case (Zehrfeld and Green 1972; Blandford and Payne 1982; Lovelace et al. 1986; Contopoulos and Lovelace 1994; McClements and Thyagaraja 2001). Equation (79) is an elliptic or hyperbolic equation depending on whether the discriminant $D \equiv A_{xy}^2 - A_{xx}A_{yy}$ is negative or positive, where A_{xx}, A_{xy} and A_{yy} represent the coefficients of $\partial^2\psi/\partial x^2, \partial^2\psi/(\partial x\partial y)$ and $\partial^2\psi/\partial y^2$. In general, with v_p increasing from zero, (79) is elliptic, hyperbolic, elliptic and ultimately hyperbolic at high flow speeds (Zehrfeld and Green 1972; Heinemann and Olbert 1978; Contopoulos and Lovelace 1994). The four flow regimes are separated by three critical points. Two of these critical points correspond to the flow speed equal to the speeds of slow and fast magnetosonic waves, $v_{\text{fms}}, v_{\text{sms}} \equiv [1/2(c_s^2 + v_A^2 \pm [(c_s^2 + v_A^2)^2 - 4c_s^2 v_{Ap}^2]^{1/2})]^{1/2}$, respectively, and one critical point is the speed of the cusp of the slow-mode wave front propagation, $v_c \equiv c_s v_{Ap}/(c_s^2 + v_A^2)^{1/2}$ (see e.g. Whitham 1974; Miyamoto 1976). Both fast and slow magnetosonic waves have been recently discovered in the solar corona in Chandrasekhar and Woltjer (1958), Woltjer (1959c), Zehrfeld and Green (1972), Whitham (1974), Miyamoto (1976), Heinemann and Olbert (1978), Spies (1980), Blandford and Payne

(1982), Contopoulos and Lovelace (1994), McClements and Thyagaraja (2001), Nakariakov et al. (2000), Nakariakov and Verwichte (2005) and Verwichte et al. (2006a,b), which created an observational foundation for the novel technique for the coronal plasma diagnostics, coronal seismology. The different four regimes of (79) are the following:

(*a*) elliptic for $v_p^2 < v_c^2$,

(*b*) hyperbolic for $v_c^2 < v_p^2 < v_{\rm sms}^2$,

(*c*) elliptic for $v_{\rm sms}^2 < v_p^2 < v_{\rm fms}^2$,

(*d*) hyperbolic for $v_{\rm fms}^2 < v_p^2$.

The condition (77a) requires the equilibrium flow to be in the first elliptic regime. This can be pointed out by putting condition (77a) in the form

$$\frac{v_p^2}{v_{Ap}^2} + \frac{v_p^2 + g_1^2}{c_s^2} < 1, \tag{80}$$

that is

$$v_p^2 < v_c^2 = \frac{c_s^2 v_{Ap}^2}{c_s^2 + v_A^2} < \frac{c_s^2 v_{Ap}^2}{c_s^2 + v_{Ap}^2}. \tag{81}$$

Thus, the poloidal speed of the flow is less than the speed of the cusp of the slow-mode wave front propagation.

7. Summary

In this paper, we have described the equilibrium properties and stability of an ideal MHD plasma with compressible mass flow in the presence of a gravitational field. In the unperturbed steady case, it has been shown that the equilibrium states satisfy a nonlinear PDE associated with a hydrodynamic Bernoulli law. If either the z-component of the velocity field or the z-component of the magnetic field is a function of the magnetic flux, then only incompressible equilibrium flows are possible. Variational principles for the above equilibrium states have been formulated where the equilibrium solutions are associated with critical points of a nonlinear conserved Lyapunov functional $\mathfrak{R}$. Sufficient conditions for the linear and nonlinear stability of these equilibria under translational symmetric perturbations have been established. They are interpreted, in the linear stability case, by using the general fact for Hamiltonian systems that the second variation of the nonlinear Lyapunov functional $\mathfrak{R}$ is conserved by the linearized dynamics around the relative equilibrium state for which $\mathfrak{R}$ has a critical point. In the nonlinear stability case, they are established in the sense of norm based on the Lyapunov definition of nonlinear stability.

The obtained conditions require the equilibrium flow to be sub-Alfvénic and subsonic in a rotating frame determined on each flux surface. According to Guazzotto et al. (2004), there are no sub-Alfvénic roots for the Bernoulli equation as it is solved for the flow density. That is, the poloidal flow must be reduced in order for sub-Alfvénic roots to exist. When the roots are found, they may be corresponding to subsonic, transonic and supersonic flows. The flow domain is divided into two regions. The inner region has subsonic flow and the outer region has super-Alfvénic flow. There is a discontinuity on a transonic surface where the poloidal flow varies from subsonic to supersonic speed (Betti and Freidberg 2000). It should be pointed out that subsonic property is a sufficient condition for the stability of flow, as we explained in Sec. 5. Condition (61) is an ellipticity condition for the equilibrium flow. As occurs in fluid dynamics with transonic flow, weak solutions may develop when the Grad–Bernoulli–Shafranov system for the ideal MHD flows leaves the first elliptic regime. The sufficient stability conditions obtained here require the equilibrium flow to be in the first elliptic regime for the Grad–Bernoulli–Shafranov system.

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