Glasgow Math. J. **63** (2021) 343–362. © The Author(s), 2020. Published by Cambridge University Press on behalf of Glasgow Mathematical Journal Trust.
doi:10.1017/S0017089520000221.

DISTAL ACTIONS OF AUTOMORPHISMS OF NILPOTENT GROUPS G ON SUB_G AND APPLICATIONS TO LATTICES IN LIE GROUPS

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(Received 29 November 2019; revised 23 April 2020; accepted 11 May 2020;
first published online 23 June 2020)

Abstract. For a locally compact group G, we study the distality of the action of automorphisms T of G on Sub_G, the compact space of closed subgroups of G endowed with the Chabauty topology. For a certain class of discrete groups G, we show that T acts distally on Sub_G if and only if T^n is the identity map for some $n \in \mathbb{N}$. As an application, we get that for a T-invariant lattice Γ in a simply connected nilpotent Lie group G, T acts distally on Sub_G if and only if it acts distally on Sub_Γ. This also holds for any closed T-invariant co-compact subgroup Γ in G. For a lattice Γ in a simply connected solvable Lie group, we study conditions under which its automorphisms act distally on Sub_Γ. We construct an example highlighting the difference between the behaviour of automorphisms on a lattice in a solvable Lie group and that in a nilpotent Lie group. We also characterise automorphisms of a lattice Γ in a connected semisimple Lie group which act distally on Sub_Γ. For torsion-free compactly generated nilpotent (metrisable) groups G, we obtain the following characterisation: T acts distally on Sub_G if and only if T is contained in a compact subgroup of $\mathrm{Aut}(G)$. Using these results, we characterise the class of such groups G which act distally on Sub_G. We also show that any compactly generated distal group G is Lie projective.

2010 *Mathematics Subject Classification*. Primary 37B05; Secondary 22E25, 22E40, 22D45

1. Introduction. Distal actions were introduced by David Hilbert to study the dynamics of non-ergodic actions on compact spaces (cf. Moore [23]). Let X be a (Hausdorff) topological space. A semigroup $\mathfrak{S}$ of homeomorphisms of X is said to act *distally* on X if for every pair of distinct elements $x, y \in X$, the closure of $\{(T(x), T(y)) \mid T \in \mathfrak{S}\}$ does not intersect the diagonal $\{(d, d) \mid d \in X\}$. Let Homeo(X) denote the set of homeomorphisms of X. The map $T \in$ Homeo(X) is said to be *distal* if the group $\{T^n\}_{n\in\mathbb{Z}}$ acts distally on X. If X is compact, then T is distal if and only if the semigroup $\{T^n\}_{n\in\mathbb{N}}$ acts distally (cf. [8]). Let G be a locally compact (Hausdorff) group with the identity e and let $T \in \mathrm{Aut}(G)$. Then T is distal if and only if $e \notin \overline{\{T^n(x) \mid n \in \mathbb{Z}\}}$ whenever $x \neq e$.

Distal actions on compact spaces have been studied extensively by Ellis [13], who obtained a characterisation, and Furstenberg [14], who has a deep structure theorem for distal maps on compact metric spaces. Distal actions by automorphisms on Lie groups and locally compact groups have been studied by many mathematicians (see Abels [1, 2], Jaworski-Raja [19], Raja-Shah [26, 27], Reid [28], Shah [30] and the references cited therein).

A locally compact (Hausdorff) group G is said to be *distal* if the conjugacy action of G on G is distal. Equivalently, $e \notin \overline{\{gxg^{-1} \mid g \in G\}}$, for every $x \neq e$. All discrete groups, compact groups and nilpotent groups are distal. It is well known that a connected locally compact group G is distal if and only if it has polynomial growth; and such a G is a compact extension of a connected solvable normal subgroup (see [**29**] and [**20**]). In [**28**], Reid has shown that any compactly generated totally disconnected distal group is Lie projective. We extend this to all compactly generated locally compact distal groups (see Theorem 2.1).

For a locally compact group G, let Sub_G denote the space of all closed subgroups of G equipped with the Chabauty topology (cf. [**11**]). Then Sub_G is compact. It is metrisable if G is so (cf. [**7**]). Note that Sub_G has been identified for certain groups G; for example, $\mathrm{Sub}_{\mathbb{R}}$ is isomorphic to $[0, \infty]$, $\mathrm{Sub}_{\mathbb{Z}}$ is isomorphic to $\{0\} \cup \{1/n \mid n \in \mathbb{N}\}$ and $\mathrm{Sub}_{\mathbb{R}^2}$ is isomorphic to $\mathbb{S}^4$. For the study of various aspects of Sub_G for different groups G, we refer the reader to Abert et al. [**4**], Baik and Clavier [**5**, **6**], Bridson et al. [**10**], Pourezza and Hubbard [**24**] and the references cited therein.

There is a natural action of $\mathrm{Aut}(G)$, the group of automorphisms of G, on Sub_G as follows:

$$\mathrm{Aut}(G) \times \mathrm{Sub}_G \to \mathrm{Sub}_G,\ \ H \mapsto T(H),\ \ T \in \mathrm{Aut}(G), H \in \mathrm{Sub}_G.$$

Each $T \in \mathrm{Aut}(G)$ defines a homeomorphism of Sub_G and the corresponding map from $\mathrm{Aut}(G) \to \mathrm{Homeo}(\mathrm{Sub}_G)$ is a group homomorphism.

For automorphisms T of connected Lie groups G, Shah and Yadav in [**32**] have studied and characterised the distality of the T-action on Sub_G under certain conditions on T or on G; for example, T is unipotent or the largest connected central subgroup of G is torsion-free. Our main aim is to study the distality of this action for some disconnected metrisable groups G, namely, a certain class of discrete groups, compact groups and compactly generated nilpotent groups.

A discrete (closed) subgroup Γ in a locally compact group G is said to be a *lattice* in G if G/H carries a finite G-invariant measure. We refer the reader to [**25**] for generalities on lattices. Note that a lattice Γ in a simply connected nilpotent group G is a finitely generated discrete nilpotent co-compact subgroup and any automorphism T of Γ extends to a unique automorphism of G (cf. [**25**], Theorem 2.11 and Corollary 1 following it). Note also that Sub_Γ is much smaller than Sub_G; for example, $\mathrm{Sub}_{\mathbb{Z}^n}$ is countable, while $\mathrm{Sub}_{\mathbb{R}^n}$ is not. We are motivated by the following question: Whether it is enough to study the T-action on Sub_Γ to determine the distality of the T-action on Sub_G. We show that it is in fact enough to assume the distality of the T-action on Sub^c_Γ, the set of cyclic subgroups of Γ, to show that $T^n = \mathrm{Id}$, the identity map on G and, hence, it acts distally on Sub_G (more generally, see Corollary 3.9). We also get a suitable generalisation of this for a closed co-compact Lie subgroup Γ in G (see Corollary 4.5). For a lattice Γ in a simply connected solvable group G, if $T \in \mathrm{Aut}(G)$ and $T(\Gamma) = \Gamma$, the distality of the action of T on Sub_Γ implies that $T^n|_\Gamma = \mathrm{Id}$, but T need not act distally on Sub_G (more generally, see Theorem 3.10 and Example 3.11). We also get a characterisation for automorphisms of a lattice Γ in a connected semisimple Lie group which act distally on Sub_Γ (see Theorem 3.16). For locally compact compactly generated nilpotent (metrisable) groups G such that G^0 is torsion-free, we get that $T \in \mathrm{Aut}(G)$ acts distally on Sub_G if and only if T is contained in a compact subgroup of $\mathrm{Aut}(G)$ (see Theorem 4.4). This also holds for any compact totally disconnected metrisable group (see Proposition 4.3). We also characterise metrisable locally compact compactly generated nilpotent groups G whose inner automorphisms act distally on Sub_G (see Theorem 4.6); this is an analogue of Corollary 4.5 of [**32**].

https://doi.org/10.1017/S0017089520000221 Published online by Cambridge University Press

Some of the results about the actions of automorphisms on Sub_G are proved under weaker assumptions such as either T acts distally on Sub^a_G, the set of closed abelian subgroups of G, or on a smaller class Sub^c_G, the set of closed cyclic subgroups of G. In [10], Bridson, de la Harpe and Kleptsyn describe the structure of $\mathrm{Sub}^a_{\mathbb{H}}$ and various other subspaces of $\mathrm{Sub}^a_{\mathbb{H}}$, for the three-dimensional Heisenberg group $\mathbb{H}$, and they also study and describe the action of $\mathrm{Aut}(G)$ on some of these spaces in detail. Baik and Clavier have identified Sub^a_G for $G = \mathrm{PSL}(2, \mathbb{C})$ in [6] and, they also give a description of the space which is the closure of Sub^c_G in Sub_G, where G is either $\mathrm{PSL}(2, \mathbb{R})$ or $\mathrm{PSL}(2, \mathbb{C})$ (cf. [5, 6]). We give conditions on discrete groups G under which Sub^c_G is closed in Sub_G and study the distality of the action of automorphisms of G on Sub^c_G. We also prove certain results for the automorphisms in the class (NC) introduced in [32], which contains those that act distally on Sub^a_G or on the closure of Sub^c_G.

Throughout, let G be a locally compact (Hausdorff) group with the identity e. For a subgroup H of G, let H^0 denote the connected component of the identity e in H, $[H, H]$ denote the commutator subgroup of H, $Z(H)$ denote the centre of H and let $Z_G(H)$ denote the centraliser of H in G. For $B \subset G$, let $\overline{B}$ denote the closure of B in G. If B is a group, so is $\overline{B}$. For any $T \in \mathrm{Aut}(G)$, T^0 is the identity map of G.

2. Compactly generated distal groups. Recall that a locally compact group G is distal if the conjugation action of G on G is distal, i.e. for every $x \in G$ such that $x \neq e$, the closure of $\{gxg^{-1} \mid g \in G\}$ does not contain the identity e. Compact groups, nilpotent groups and discrete groups are all distal. A locally compact group is said to be Lie projective if it has compact normal subgroups K_α, such that $\bigcap_\alpha K_\alpha = \{e\}$ and G/K_α is a Lie group for each α. Note that any connected, more generally any almost connected locally compact group is Lie projective (G is almost connected if G/G^0 is compact).

It is shown by Willis in [34] that any compactly generated totally disconnected locally compact nilpotent group is Lie projective. This was extended by Reid to all compactly generated totally disconnected locally compact distal groups (cf. [28], Corollary 1.9). We generalise this to all locally compact groups as follows.

THEOREM 2.1. *Any compactly generated locally compact distal group is Lie projective.*

Proof. Let G be a compactly generated locally compact distal group. By Corollary 3.4 of [26], G/G^0 is distal. Since G/G^0 is also compactly generated, by Corollary 1.9 of [28], there exists a neighbourhood basis of the identity consisting of compact open normal subgroups in G/G^0, and hence there exist open normal subgroups H_α in G, such that H_α/G^0 is compact and $\bigcap_\alpha H_\alpha = G^0$. Let α be fixed. Note that the maximal compact normal subgroup K of H_α is characteristic in H_α and hence normal in G. Let $H = KG^0$. Then H is normal in G, K is the maximal compact normal subgroup of H and H/K is a Lie group. As H_α is Lie projective, we have that H is an open normal subgroup of G. Therefore, G/H is discrete, and it is finitely generated, since G is compactly generated. Let $x_1, \ldots, x_n \in G$ be such that their images in G/H generate G/H. Let L be the subgroup generated by $x_1, \ldots, x_n$ in G. Then L is countable. Since the conjugation action of L on K is distal, K has compact normal subgroups K_β such that K_β is L-invariant and K/K_β is a Lie group for each β, and $\bigcap_\beta K_\beta = \{e\}$ (cf. [18], Theorem 2.6 and Corollary 2.7). Let β be fixed. Since G^0 normalises K, by Theorem 1′ of [17], the action of G^0 (by inner automorphisms) on K is the same as the conjugation action of K^0 on K. Therefore, every normal subgroup of K is normalised by G^0. In particular, K_β is normal in $H = KG^0$. Since

https://doi.org/10.1017/S0017089520000221 Published online by Cambridge University Press

L also normalises K_β and $LH = G$, we get that K_β is normal in G. As K/K_β and H/K are Lie groups, so is H/K_β. Moreover, G/H is discrete. Therefore, G/K_β is a Lie group. Since this is true for every β and since $\bigcap_\beta K_\beta = \{e\}$, G is Lie projective. □

Note that in Theorem 2.1, both the conditions that the group is compactly generated and distal are necessary. Willis in [34] has given an example of a locally compact nilpotent (distal) group which is not Lie projective. For a compactly generated locally compact group which is not distal, one can take $G = \mathbb{Z} \ltimes (\mathbb{T}^2)^{\mathbb{Z}}$, where $\mathbb{T}^2$ is the (compact) two-dimensional torus, and the action of $1 \in \mathbb{Z}$ on $(\mathbb{T}^2)^{\mathbb{Z}}$ is given by the shift action. Here, G is compactly generated and locally compact, but it is not distal as the shift action on $(\mathbb{T}^2)^{\mathbb{Z}}$ is ergodic. It is easy to see that G is not Lie projective.

A locally compact group G is said to be Λ-Lie projective for a subgroup $\Lambda \subset \mathrm{Aut}(G)$, if it admits compact open normal Λ-invariant subgroups $\{K_\alpha\}$ such that G/K_α is a Lie group for each α and $\bigcap_\alpha K_\alpha = \{e\}$. Note that Λ-Lie projective groups were introduced in [27] and they are obviously Lie projective. G is said to be T-Lie projective for some $T \in \mathrm{Aut}(G)$ if it is $\{T^n\}_{n\in\mathbb{Z}}$-Lie projective. A group G is Λ-Lie projective for a finitely generated group Λ of $\mathrm{Aut}(G)$ if and only if $\Lambda \ltimes G$ is Lie projective, where Λ is endowed with the discrete topology. Similarly, G is T-Lie projective for some $T \in \mathrm{Aut}(G)$ if and only if $\mathbb{Z} \ltimes_T G$ is Lie projective, where the action of $n \in \mathbb{Z}$ on G is given by the action of T^n on G, and $\mathbb{Z}$ is endowed with the discrete topology.

We say that a locally compact group Λ acts (continuously) on G by automorphisms, if there exists a group homomorphism $\psi : \Lambda \to \mathrm{Aut}(G)$ such that the corresponding map $\Lambda \times G \to G$ given by $(\lambda, g) \mapsto \psi(\lambda)(g)$, $\lambda \in \Lambda$, $g \in G$, is continuous.

For a compact group G, if $T \in \mathrm{Aut}(G)$ is distal, it follows from Lemma 2.5 of [27], that G is T-Lie projective. As any compactly generated nilpotent group is a generalised $\overline{FC}$ group (cf. [22]), the following useful corollary follows easily from Corollary 3.7 of [27] and Theorem 2.1.

COROLLARY 2.2. *Let G be a compactly generated locally compact distal group. If Λ is a compactly generated nilpotent group which acts distally on G by automorphisms, then G is Λ-Lie projective. In particular, if $T \in \mathrm{Aut}(G)$ acts distally on G, then G is T-Lie projective; i.e. $\mathbb{Z} \ltimes_T G$ is Lie projective.*

The next corollary will be useful, it is well known and it can be easily deduced from Theorem 2 of [21] and Lemma 3.1 of [12]. Since nilpotent groups are distal, one can also use Theorem 2.1 instead of Theorem 2 of [21] to prove it.

COROLLARY 2.3. *Any locally compact compactly generated nilpotent group admits a unique maximal compact subgroup.*

The following group theoretic result, which may be known, will be useful in proving Theorem 4.6. It implies for a special case of compactly generated G below that the unique maximal compact subgroup centralises G^0. In particular, it implies that G^0 is central in any compact nilpotent group G.

PROPOSITION 2.4. *Let G be a locally compact nilpotent group. Then any compact subgroup of G centralises G^0.*

Proof. Let K be a compact subgroup of G. Since G^0 is normal and K is compact, we get that KG^0 is a closed subgroup. Also, it is compactly generated since G^0 is so. By Corollary 2.3, KG^0 has a unique maximal compact subgroup. To prove the assertion, we may replace G by KG^0 and also assume that K is the unique maximal compact subgroup of G, and show that it centralises G^0.

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As K is a unique maximal compact subgroup, it is characteristic in G, and hence it is normal in G. For $g \in G$, let $\mathrm{inn}(g)$ denote the inner automorphism of G by the element g, i.e. $\mathrm{inn}(g)(x) = gxg^{-1}$, $x \in G$. As G^0 is connected, $\mathrm{inn}(g)|_K \in [\mathrm{Aut}(K)]^0$ for all $g \in G^0$. By Theorem 1′ of [17], $[\mathrm{Aut}(K)]^0 = [\mathrm{Inn}(K)]^0 = \{\mathrm{inn}(k) \mid k \in K^0\}$. That is, given $g \in G^0$, there exists $k \in K^0$ such that $\mathrm{inn}(g)|_K = \mathrm{inn}(k)|_K$. To show that G^0 centralises K, it is enough to show that K^0 is central in K. Now we may assume that G is a compact nilpotent group and show that G^0 is central in G. Since G is compact, it is Lie projective, hence it is enough to prove this for a compact nilpotent Lie group G. As G^0 is compact and nilpotent, it is abelian (cf. [17], Lemma 2.2). Moreover, G/G^0 is finite.

We prove by induction on the length $l(G)$ of the central series of the compact nilpotent Lie group G that G^0 is central in G. If $l(G) = 1$, then G is abelian. Suppose for some $k \in \mathbb{N}$, the above statement holds for all such G with $l(G) \leq k$. Now let G be such that $l(G) = k + 1$. Let $Z(G)$ be the centre of G. Then $l(G/Z(G)) = k$. By the induction hypothesis, $(G/Z(G))^0$ is central in $G/Z(G)$. Let $x \in G$, $g \in G^0$ and let $x_g = xgx^{-1}g^{-1}$. Since $(G^0Z(G))/Z(G) = (G/Z(G))^0$, from the preceding assertion we have that $x_g \in Z(G)$. Since G/G^0 is finite, we get that $x^n \in G^0$ for some $n \in \mathbb{N}$. Since $x_g \in Z(G)$, we get that $x_g^n = xg^nx^{-1}g^{-n} = x^ngx^{-n}g^{-1} = e$ as G^0 is abelian. Therefore, x centralises g^n. Since this holds for all $g \in G^0$, which is connected and abelian, we get that x centralises G^0. Since this is true for all $x \in G$, we have that G^0 is central in G. Now the proof is complete by induction. □

3. Distal actions of automorphisms on Sub_G for discrete groups G and applications to lattices. Let G be a locally compact (metrisable) group. A sub-basis of the Chabauty topology on Sub_G is given by the sets $\mathcal{O}_1(K) = \{A \in \mathrm{Sub}_G \mid A \cap K = \emptyset\}$, $\mathcal{O}_2(U) = \{A \in \mathrm{Sub}_G \mid A \cap U \neq \emptyset\}$, where K is a compact and U is an open subset of G. As observed earlier, Sub_G is compact and metrisable. For details on the Chabauty topology, see [10] and [24].

We first state a criterion for convergence of sequences in Sub_G (cf. [7]).

LEMMA 3.1. *Let G be a locally compact first countable (metrisable) group. A sequence $\{H_n\}$ converges to H in Sub_G if and only if the following hold:*

(*i*) *For any $h \in H$, there exists a sequence $\{h_n\}$ with $h_n \in H_n$, $n \in \mathbb{N}$, such that $h_n \to h$.*
(*ii*) *For any unbounded sequence $\{n_k\} \subset \mathbb{N}$, if $\{h_{n_k}\}_{k\in\mathbb{N}}$ is such that $h_{n_k} \in H_{n_k}$, $k \in \mathbb{N}$, and $h_{n_k} \to h$, then $h \in H$.*

We define Sub_G^c as the space of all closed cyclic subgroups of G. In general, Sub_G^c need not be closed in Sub_G; for example, $\mathrm{Sub}_{\mathbb{R}}^c$ is dense in $\mathrm{Sub}_{\mathbb{R}}$ and $\mathrm{Sub}_{\mathbb{R}} = \mathrm{Sub}_{\mathbb{R}}^c \cup \{\mathbb{R}\}$. We will show that for a certain class of groups G, which include discrete finitely generated nilpotent groups, Sub_G^c is closed. We first state a useful lemma about the limits of sequences in a discrete group. We give a short proof for the sake of completeness.

LEMMA 3.2. *Let G be a discrete group. If $\{H_n\} \subset \mathrm{Sub}_G$ is such that $H_n \to H$ in Sub_G, then $H = \liminf H_n$.*

Proof. By Lemma 3.1 (*i*), we get that for every $h \in H$, there exist $h_n \in H_n$, $n \in \mathbb{N}$, such that $h_n \to h$. Since G is discrete, we get that $h_n = h$ for all large n. Therefore, $h \in \liminf H_n$, and hence $H \subset \liminf H_n$. Conversely, suppose $h \in \liminf H_n$. Then there exists $k \in \mathbb{N}$ such that $h \in H_n$, for all $n \geq k$. Now by Lemma 3.1 (*ii*), we get that $h \in H$. Hence

$$H = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} H_n = \liminf H_n.$$

□

https://doi.org/10.1017/S0017089520000221 Published online by Cambridge University Press

As any discrete subgroup of a connected solvable Lie group is finitely generated (cf. [**25**], Corollary 3.9), the following will be useful for results on lattices of a connected solvable Lie group (see Theorem 3.10).

LEMMA 3.3. *Let G be a discrete finitely generated group such that all its subgroups are also finitely generated. Then* Sub^c_G *is closed.*

Proof. Let $\{H_n\}$ be a sequence in Sub^c_G such that $H_n \to H$. By Lemma 3.2, $H = \bigcup_{k=1}^{\infty} G_k$, where $G_k = \bigcap_{n=k}^{\infty} H_n$. From the hypothesis, H is finitely generated. Let $\{x_1, \ldots, x_m\}$ be the set of generators for H. Since H is an increasing union of cyclic groups G_k, there exists $n_0 \in \mathbb{N}$ such that $x_1, \ldots, x_m \in G_{n_0}$. Therefore $H = G_{n_0}$, and hence H is cyclic. □

A locally compact group G is said to be *strongly root compact* if for every compact subset C of G, there exists a compact subset C_0 of G with the property that for every $n \in \mathbb{N}$, the finite sequences $\{x_1, \ldots, x_n\}$ of G with $x_n = e$, satisfying $Cx_iCx_j \cap Cx_{i+j} \neq \emptyset$ for all $i + j \leq n$, are contained in C_0 (see [**15**], Definition 3.1.10). All compact groups and compactly generated nilpotent groups are strongly root compact (cf. [**15**], Theorem 3.1.17).

For any $g \in G$, let $R_g = \{x \in G \mid x^n = g \text{ for some } n \in \mathbb{N}\}$, the set of roots of g in G. If G is strongly root compact, then by Theorem 3.1.13 of [**15**], R_g is relatively compact for every $g \in G$.

LEMMA 3.4. *Let G be a discrete group. If for every* $g \in G$*, the set* R_g *of roots of g is finite, then* Sub^c_G *is closed. In particular, if G is strongly root compact, then* Sub^c_G *is closed.*

Proof. For a discrete group G, suppose R_g is finite for every $g \in G$. Let $\{H_n\}$ be a sequence in Sub^c_G such that $H_n \to H$. By Lemma 3.2, we get $H = \liminf H_n$. Let $G_k = \bigcap_{n=k}^{\infty} H_n$, $k \in \mathbb{N}$. Then for all $k \in \mathbb{N}$, $G_k \subset G_{k+1} \subset H_{k+1}$, $G_k \subset H$ and each G_k is cyclic. If $H = \{e\}$, then there is nothing to prove. Suppose G_k is finite for all $k \in \mathbb{N}$. Since $H = \bigcup_{k=1}^{\infty} G_k$, it consists of finite-order elements. Hence $H \subset R_e$ which is finite, so H is finite. Therefore, $H = G_k$ for some k, and hence H is cyclic. Now suppose there exists $m \in \mathbb{N}$ such that G_m is infinite. Replacing $\{H_n\}$ by $\{H_{n+m}\}$, we may assume that G_1 is infinite, and hence that G_k is an infinite cyclic group, $k \in \mathbb{N}$.

Let x_k be a generator of G_k, $k \in \mathbb{N}$. As $G_k \subset G_{k+1}$, $k \in \mathbb{N}$, replacing x_{k+1} by its inverse if necessary, we get that there exists $l_k \in \mathbb{N}$ such that $x_k = x_{k+1}^{l_k}$. Hence $x_1 = x_k^{n_k}$ for all $k \geq 2$, where $n_k = l_1 \cdots l_{k-1}$. From the hypothesis, R_{x_1} is finite, and hence $\{x_k\}_{k \in \mathbb{N}}$ is finite. As each x_k generates an infinite cyclic group G_k and $G_k \subset G_{k+1}$, $k \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that $G_k = G_{k+1}$ for all $k \geq n_0$, and hence $H = G_{n_0} = \bigcap_{n=n_0}^{\infty} H_n$. Therefore, H is cyclic. This proves that Sub^c_G is closed.

Now suppose G is a strongly root compact discrete group. By Theorem 3.1.13 of [**15**], R_g is relatively compact, and hence, finite for every $g \in G$. Now it follows from the first statement that Sub^c_G is closed. □

The class (NC) of automorphisms is defined in [**32**]. An automorphism T of a locally compact metrisable group G belongs to class (NC) if given any closed cyclic subgroup A of G, $T^{n_k}(A) \not\to \{e\}$ for any unbounded sequence $\{n_k\} \subset \mathbb{Z}$. For $x \in G$, let G_x denote the cyclic group generated by x in G. Then either G_x is closed (and hence discrete) or $\overline{G_x}$ is compact.

Note that if T acts distally on Sub^a_G, then $T \in$ (NC). It is easy to see that $T^n \in$ (NC) for some $n \in \mathbb{Z} \setminus \{0\}$ if and only if $T^n \in$ (NC) for all $n \in \mathbb{Z}$. We now state and prove an elementary result about the class (NC) for the discrete quotient groups.

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LEMMA 3.5. *Let G be a locally compact first countable (metrisable) group, $T \in \mathrm{Aut}(G)$ and let H be an open normal T-invariant subgroup of G. Let $\overline{T} : G/H \to G/H$ be the automorphism of G/H corresponding to T. For any $x \in G$, let $\bar{x}$ denote the element $xH \in G/H$. Suppose $T \in$ (NC). If $x \in G \setminus H$ is such that $G_{\bar{x}}$ is infinite, then $\overline{T}^{n_k}(G_{\bar{x}}) \not\to \{\bar{e}\}$, for any unbounded sequence $\{n_k\} \subset \mathbb{Z}$. In particular, if G/H is torsion-free, then $\overline{T} \in$ (NC).*

Proof. As H is open in G, it is also closed and G/H is a discrete group. Let $x \in G$ be such that $G_{\bar{x}}$ is infinite. If possible, suppose there exists an unbounded sequence $\{n_k\} \subset \mathbb{Z}$ such that $T^{n_k}(G_{\bar{x}}) \to \{\bar{e}\}$. Let G_x be the group generated by $x \in G$. Since $G_{\bar{x}}$ is closed, discrete and infinite, we have that G_x is also closed, discrete and infinite. As Sub_G is compact, we may choose a subsequence of $\{n_k\}$, and denote it by $\{n_k\}$ again, such that $T^{n_k}(G_x) \to L$ for some $L \in \mathrm{Sub}_G$. Since $T \in$ (NC), $L \neq \{e\}$. Let $g \in L \setminus \{e\}$, then $T^{n_k}(x^{m_k}) \to g$, for some $\{m_k\} \subset \mathbb{Z}$. Now $T^{n_k}(\bar{x}^{m_k}) \to \bar{g}$. As $T^{n_k}(G_{\bar{x}}) \to \{\bar{e}\}$, we have that $g \in H$, and hence $T^{n_k}(\bar{x}^{m_k}) = \bar{e}$ for all large k, since G/H is discrete. This implies that $\bar{x}$ has finite order, which leads to a contradiction. Therefore, $\overline{T}^{n_k}(G_{\bar{x}}) \not\to \{\bar{e}\}$. If G/H is torsion-free, every non-trivial element of G/H generates a discrete infinite group, and hence the last assertion follows easily. □

For a locally compact group G and $T \in \mathrm{Aut}(G)$, let $M(T) = \{x \in G \mid \{T^n(x)\}_{n\in\mathbb{Z}}$ is relatively compact$\}$. It is a T-invariant subgroup of G. The following basic lemma about automorphisms of strongly root compact groups in the class (NC) will be very useful.

LEMMA 3.6. *Let G be a locally compact first countable (metrisable) strongly root compact group. Let $T \in \mathrm{Aut}(G)$ be such that $T \in$ (NC). Then $\{x \in G \mid G_x$ is closed$\} \subset M(T)$.*

Proof. Let $x \in G$ be such that G_x is closed and let $O_x = \{T^n(x)\}_{n\in\mathbb{Z}}$. Since G is locally compact and metrisable, and $\overline{O_x}$ is separable, it is second countable, and hence $\overline{O_x} \subset \bigcup_{n\in\mathbb{N}} V_n$, for some open relatively compact sets V_n, and we may also assume that $V_n \subset V_{n+1}$ for all $n \in \mathbb{N}$. For some $n \in \mathbb{N}$, if $O_x \subset V_n$, then $\overline{O_x} \subset \overline{V_n}$ and hence $\overline{O_x}$ is compact.

If possible, suppose $O_x \not\subset V_n$, $n \in \mathbb{N}$. There exists $k_n \in \mathbb{Z}$ such that $|k_n| \geq n$ and $T^{k_n}(x) \notin V_n$, $n \in \mathbb{N}$. As $G_x \in \mathrm{Sub}_G$ and the latter is compact, passing to a subsequence if necessary, we get that $T^{k_n}(G_x) \to H$ in Sub_G for some closed subgroup H in G. Since $T \in$ (NC), it implies that $H \neq \{e\}$. Let $a \in H$ be such that $a \neq e$. By Lemma 3.1 (i), there exists a sequence $\{m_n\} \subset \mathbb{Z}$ such that $T^{k_n}(x^{m_n}) \to a$, and hence $\{T^{k_n}(x^{m_n})\}$ is relatively compact. Replacing a by a^{-1} if necessary, and passing to a subsequence, we may assume that $\{m_n\} \subset \mathbb{N}$. As G is strongly root compact, by Theorem 3.1.13 of [**15**], we get that $\{T^{k_n}(x)\}$ is relatively compact and hence, it has a limit point (say), b. Then $b \in \overline{O_x}$. As V_n is increasing, we get that for every $m \in \mathbb{N}$, $\{T^{k_n}(x)\}_{n\geq m} \subset G \setminus V_m$ which is closed. It follows that $b \notin V_m$ for every $m \in \mathbb{N}$, and hence $b \notin \bigcup_{m\in\mathbb{N}} V_m$; this leads to a contradiction since $b \in \overline{O_x}$. Therefore, $O_x \subset V_n$ for some $n \in \mathbb{N}$ and hence $\overline{O_x}$ is compact and $x \in M(T)$. This proves the assertion. □

In a discrete group, every element generates a discrete (closed) cyclic group. The next corollary follows easily from the proof of Lemma 3.6 as a sequence in a discrete group converges if and only if it is eventually constant and, discrete compact sets are finite.

COROLLARY 3.7. *Let G be a discrete group such that the set R_g of roots of g is finite for every $g \in G$. Let $T \in \mathrm{Aut}(G)$ be such that $T \in$ (NC). Then $G = M(T)$. That is, for every $x \in G$, the T-orbit of x, $\{T^n(x)\}_{n\in\mathbb{Z}}$ is finite.*

For any discrete group, all automorphisms are distal. For strongly root compact discrete groups G, or more generally for discrete groups G in which the set of roots of every element is finite, the following proposition shows that only finite-order automorphisms of G act distally on Sub_G. Note that for such groups G, Sub^c_G is closed by Lemma 3.4. The proposition holds in particular for discrete finitely generated nilpotent groups as they are strongly root compact (cf. [15]).

PROPOSITION 3.8. *Let G be a discrete finitely generated group and let $T \in \mathrm{Aut}(G)$. Suppose the set R_g of roots of g is finite for every $g \in G$. Then the following are equivalent:*

1. $T \in (\mathrm{NC})$.
2. T *acts distally on* Sub^c_G.
3. T *acts distally on* Sub_G.
4. $T^n = \mathrm{Id}$, *where* Id *is the identity map.*

In particular, if G is strongly root compact, then (1–4) *above are equivalent.*

Proof. $4 \implies 3 \implies 2 \implies 1$ is obvious. It is enough to show that $1 \implies 4$. Suppose $T \in (\mathrm{NC})$. Then by Corollary 3.7, $G = M(T)$. Now as G is discrete, for every $x \in G$, the T-orbit of x, $\{T^n(x)\}_{n\in\mathbb{Z}}$ is finite, and hence $T^m(x) = x$ for some $m \in \mathbb{N}$. As G is finitely generated, there exist $x_1, \ldots, x_l \in G$ which generate G. Let $n_1, \ldots, n_l \in \mathbb{N}$ be such that $T^{n_i}(x_i) = x_i$, $1 \le i \le l$. Let $n = \mathrm{lcm}(n_1, \ldots, n_l)$. Then $T^n(x) = x$ for all $x \in G$, i.e. $T^n = \mathrm{Id}$.

If G is strongly root compact, then by Theorem 3.1.13 of [15], R_g is finite for every $g \in G$, and (1–4) are equivalent from above. □

Proposition 3.8, in particular, implies that if $T \in \mathrm{GL}(n, \mathbb{Z})$, $(n \ge 2)$, does not have finite order, then T does not act distally on $\mathrm{Sub}_{\mathbb{Z}^n}$; for example, T is any non-trivial strictly upper triangular matrix in $\mathrm{GL}(n, \mathbb{Z})$ with all its diagonal entries equal to 1. In fact, since $\mathrm{GL}(n, \mathbb{Z})$ is virtually torsion-free by Selberg's Lemma, there exists a subgroup (say) L of finite index in $\mathrm{GL}(n, \mathbb{Z})$ which is torsion-free, and hence every non-trivial $T \in L$ does not acts distally on $\mathrm{Sub}_{\mathbb{Z}^n}$. As an application of the proposition, we get the following corollary which relates the behaviour of an automorphism of a lattice Γ in a simply connected nilpotent group G in terms of the distality of its action on Sub_Γ and on Sub_G. Note that any automorphism of such a Γ extends to a unique automorphism of G (cf. [25]). Note also that such a Γ is finitely generated and strongly root compact, and hence by Lemma 3.4, Sub^c_Γ is closed.

COROLLARY 3.9. *Let G be a connected simply connected nilpotent Lie group and let Γ be a lattice in G. Let $T \in \mathrm{Aut}(G)$ be such that $T(\Gamma) = \Gamma$. Then the following are equivalent:*

(1) $T|_\Gamma \in (\mathrm{NC})$.
(2) T *acts distally on* Sub^a_Γ.
(3) T *acts distally on* Sub_Γ.
(4) $T \in (\mathrm{NC})$.
(5) T *acts distally on* Sub^a_G.
(6) T *acts distally on* Sub_G.
(7) T *acts distally on* Sub^c_Γ.
(8) $T^n = \mathrm{Id}$, *where* Id *is the identity map.*

Proof. $(8) \implies (6) \implies (5) \implies (4) \implies (1)$ and $(6) \implies (3) \implies (2) \implies (7) \implies (1)$ are obvious. It is enough to show that $(1) \implies (8)$. Since Γ is a lattice in a simply connected nilpotent group G, it is finitely generated, nilpotent and discrete.

https://doi.org/10.1017/S0017089520000221 Published online by Cambridge University Press

Therefore, Γ is strongly root compact and by Proposition 3.8, $T^n|_\Gamma = (T|_\Gamma)^n = \text{Id}$. By Theorem 2.11 of [25] and Corollary 1 following it, $T^n = \text{Id}$. □

Example 3.11 shows that Corollary 3.9 does not hold for lattices in a general connected simply connected solvable Lie group and it also illustrates that the following theorem is the best possible result for lattices Γ in a connected simply connected solvable Lie group. Note that such a lattice Γ is torsion-free and every subgroup of it is finitely generated, and hence by Lemma 3.3, Sub^c_Γ is closed. The theorem can be viewed as a generalisation of Corollary 3.9 as any automorphism of a lattice in a simply connected nilpotent group G extends uniquely to that of G. Example 3.11 also shows that not all the statements in the following theorem are equivalent.

THEOREM 3.10. *Let G be a connected simply connected solvable Lie group. Let N be the nilradical of G, the largest connected nilpotent normal subgroup of G. Suppose G admits a lattice Γ and an automorphism $T \in \text{Aut}(G)$ such that $T(\Gamma) = \Gamma$. Then* (1–2) *are equivalent as well as* (3–6) *are equivalent.*

(1) $T|_\Gamma \in (\text{NC})$.
(2) $T^n|_{\Gamma'} = (T|_{\Gamma'})^n = \text{Id}$ *for some $n \in \mathbb{N}$, where Γ' is a normal subgroup of finite index in Γ containing $\Gamma \cap N$, and* Id *is the identity map on Γ'.*
(3) *T acts distally on* Sub^c_Γ.
(4) *T acts distally on* Sub^a_Γ.
(5) *T acts distally on* Sub_Γ.
(6) $T^n|_\Gamma = (T|_\Gamma)^n = \text{Id}$, *for some $n \in \mathbb{N}$.*

Proof. Suppose (2) holds. Let $S = T^n$ and let m be the index of Γ' in Γ. Let $x \in \Gamma$. Then $x^m \in \Gamma'$ and $S(x^m) = x^m$ and, $x^m \neq e$ as G is torsion-free. It follows that any limit point H of $\{S^i(G_x)\}$ contains a subgroup generated by x^m. Therefore, $S|_\Gamma \in (\text{NC})$, and hence $T|_\Gamma \in (\text{NC})$ and (1) holds.

Now suppose (1) holds, i.e. $T|_\Gamma \in (\text{NC})$. As G is connected, solvable and simply connected, $[G, G]$ is a closed connected nilpotent normal subgroup. Also, the nilradical N is simply connected, $T(N) = N$, $[G, G] \subset N$ and G/N is abelian. Moreover, $\Gamma \cap N$ (resp. $(\Gamma N)/N$) is a lattice in N (resp. in G/N) (cf. [25], Corollary 3.5). Since T keeps $\Gamma \cap N$ invariant and T acts distally on $\text{Sub}^c_{\Gamma\cap N}$, by Corollary 3.9, $T^{n_1} = \text{Id}$ on $\Gamma \cap N$ for some $n_1 \in \mathbb{N}$. Note that $(\Gamma N)/N$ is a lattice in G/N which is simply connected and abelian. Note also that $\Gamma/(\Gamma \cap N)$ is isomorphic to $(\Gamma N)/N$, therefore it is finitely generated, abelian and torsion-free. By Lemma 3.5, we get that $\overline{T} \in (\text{NC})$, where $\overline{T} \in \text{Aut}(\Gamma/(\Gamma \cap N))$ is the automorphism corresponding to T. By Proposition 3.8, $\overline{T}^{n_2} = \text{Id}$ on $\Gamma/(\Gamma \cap N)$ for some $n_2 \in \mathbb{N}$. Let $n = \text{lcm}(n_1, n_2)$ and let $S = T^n$. Then $S|_{\Gamma\cap N} = \text{Id}$ and S acts trivially on $\Gamma/(\Gamma \cap N)$. Now suppose $x \in \Gamma$ is such that $S(x^j) \neq x^j$ for all $j \in \mathbb{N}$. Then $S(x) = xy$ for some non-trivial $y \in \Gamma \cap N$. As $\Gamma \cap N$ is torsion-free, by Lemma 3.12 of [32], we get that $S \notin (\text{NC})$. Hence $T \notin (\text{NC})$, which contradicts the hypothesis. Therefore, for every $x \in \Gamma$, there exists j which depends on x such that $S(x^j) = x^j$. As Γ is finitely generated, there exist $x_1, \ldots, x_k$ which are generators for Γ. Let j_i be such that $S(x_i^{j_i}) = x_i^{j_i}$, $1 \leq i \leq k$. Let $m = \text{lcm}(j_1, \ldots, j_k)$ and let Γ' be the subgroup of Γ generated by $\{x_1^{j_1}, \ldots, x_k^{j_k}\} \cup (\Gamma \cap N)$. Since Γ/Γ' is a finitely generated abelian group consisting of elements of finite order, it is finite. Therefore, Γ' is a subgroup of finite index in Γ. Also, Γ' is normal in Γ, since $[\Gamma, \Gamma] \subset \Gamma \cap N \subset \Gamma'$. As $S|_{\Gamma\cap N} = \text{Id}$, we get that $S|_{\Gamma'} = \text{Id}$. As $S = T^n$, it follows that (2) holds.

It is easy to see that (6) $\implies$ (5) $\implies$ (4) $\implies$ (3). It is enough to show that (3) $\implies$ (6). Suppose (3) holds, i.e. T acts distally on Sub^c_Γ. Since (3) $\implies$ (1), we have that (2) holds, i.e. there exists a normal subgroup Γ' of finite index in Γ containing $\Gamma \cap N$ and $n \in \mathbb{N}$ such that $T^n|_{\Gamma'} = \text{Id}$, where N is the nilradical of G. As in the proof above, we can choose Γ' and n such that T^n acts trivially on $\Gamma/(\Gamma \cap N)$.

Let $S = T^n$. Then we show that $S|_\Gamma = \text{Id}$. If possible, suppose $x \in \Gamma$ is such that $S(x) \neq x$. As $x^m \in \Gamma'$, for some $m \in \mathbb{N}$, we get that $S(x^m) = x^m$. Let k be the smallest positive integer such that $S(x^k) = x^k$. Then $k \geq 2$. Now $S(x^l) = x^l y_l$ for some $y_l \in \Gamma \cap N$, $y_l \neq e$ and $S^i(x^l) = x^l y_l^i$ for all for $1 \leq l \leq k-1$, $i \in \mathbb{N}$. Since G is torsion-free, $\{S^{i_j}(x^l)\}$ has no limit point if $i_j \to \infty$ and $1 \leq l \leq k-1$. Now it is easy to show that $S^i(G_x) \to G_{x^k}$ in Sub^c_Γ, as $i \to \infty$, where G_x (resp. G_{x^k}) is the cyclic group generated by x (resp. x^k) in Γ. As $S(G_{x^k}) = G_{x^k}$ and $k \geq 2$, it implies that S does not act distally on Sub^c_Γ. Since $S = T^n$, we get that T does not act distally on Sub^c_Γ. This contradicts (3). Therefore, $S|_\Gamma = \text{Id}$, and hence (6) holds. □

The following is an example of a connected simply connected solvable Lie group G which admits a non-trivial automorphism T and a lattice Γ_1 such that $T|_{\Gamma_1} = \text{Id}$ and $T \notin$ (NC). This is unlike the case of simply connected nilpotent groups (see Corollary 3.9). The example also shows that there exists a lattice Γ_2 in G such that $T|_{\Gamma_2} \in$ (NC) but it does not act distally on $\text{Sub}^c_{\Gamma_2}$.

EXAMPLE 3.11. Let $G = \mathbb{R} \ltimes \mathbb{R}^2$ where the group operation is given by $(s, x)(t, y) = (s + t, e^{2i\pi t}x + y)$, $s, t \in \mathbb{R}$ and $x, y \in \mathbb{R}^2$. Then G is a connected simply connected solvable Lie group. Let T be an inner automorphism by some $g \in \mathbb{Z}^2 \setminus \{0\}$, i.e. $T(t, y) = (t, y + e^{2i\pi t}g - g)$, for all (t, y) as above. Let $\Gamma_1 = \mathbb{Z} \times \mathbb{Z}^2$, where $\mathbb{Z}$ is a lattice in $\mathbb{R}$ and $\mathbb{Z}^2$ is a lattice in the normal subgroup $\mathbb{R}^2$. Then Γ_1 is a lattice in G and $T|_{\Gamma_1} = \text{Id}$. Also, $T|_{\mathbb{R}^2} = \text{Id}$ and the action on $G/\mathbb{R}^2$ corresponding to T is also trivial. Now choose an irrational number t in $\mathbb{R}$. Then $T(t) = (t, e^{2i\pi t}g - g)$, and hence $T(mt) \neq mt$ for all $m \in \mathbb{Z}$, i.e. T does not fix any non-trivial element in the discrete cyclic group G_t generated by t in $\mathbb{R}$. As $\mathbb{R}^2$ has no non-trivial compact subgroup, it is easy to show that $T^n(G_t) \to \{(0, 0)\}$ in Sub_G as $n \to \infty$. Therefore, $T \notin$ (NC) (this also follows from Lemma 3.12 of [32]).

Now choose $\Gamma_2 = \frac{1}{2}\mathbb{Z} \ltimes \mathbb{Z}^2$ and T is the inner automorphism by g as above, where $g \in \Gamma_1 \cap \mathbb{Z}^2 \setminus \{0\}$. Then Γ_2 is a T-invariant lattice in G, $\Gamma_1 \subset \Gamma_2$ and $T|_{\Gamma_1} = \text{Id}$. For any $x \in \Gamma_2$, $x^2 \in \Gamma_1$. Therefore, it is easy to see that $T|_{\Gamma_2} \in$ (NC). For $t = \frac{1}{2} \in \Gamma_2 \cap \mathbb{R}$, $T(t) = (t, -2g)$. As $g \neq 0$, it is easy to check that $T^n(G_t) \to \mathbb{Z} = \Gamma_1 \cap \mathbb{R}$ as $n \to \infty$, where G_t is the cyclic group generated by t in Γ_2. As $\Gamma_1 \cap \mathbb{R}$ is cyclic and $T(\Gamma_1 \cap \mathbb{R}) = \Gamma_1 \cap \mathbb{R} \neq G_t$, T does not act distally on $\text{Sub}^c_{\Gamma_2}$.

Now we study the action of automorphisms of a lattice Γ in a connected semisimple Lie group on Sub_Γ. We first give an example of an automorphism T of $\text{SL}(2, \mathbb{Z})$, which does not belong to (NC), and hence it does not act distally on $\text{Sub}^a_{\text{SL}(2,\mathbb{Z})}$.

EXAMPLE 3.12. Let $T = \text{inn}(g)$, the inner automorphism of $\text{SL}(2, \mathbb{Z})$, where

$$g = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \quad \text{For} \quad x = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

and, for $\{n_k\}$ and $\{l_k\}$ in $\mathbb{Z}$,

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$$T^{n_k}(x^{l_k}) = g^{n_k} x^{l_k} g^{-n_k} = \begin{bmatrix} 1+n_k l_k & -n_k^2 l_k \\ l_k & 1-n_k l_k \end{bmatrix}.$$

If $n_k \to \infty$ and $l_k \neq 0$, at least one of the entries of $T^{n_k}(x^{l_k})$ goes to ∞, and hence $\{T^{n_k}(x^{l_k})\}$ does not converge in $\mathrm{SL}(2, \mathbb{Z})$. This implies that for the cyclic group G_x generated by x in $\mathrm{SL}(2, \mathbb{Z})$, since $\{T^{n_k}(G_x)\}$ converges for some unbounded sequence $\{n_k\} \subset \mathbb{N}$, we have that $T^{n_k}(G_x) \to \{e\}$. Therefore, $T \notin$ (NC), and hence T does not act distally on $\mathrm{Sub}^a_{\mathrm{SL}(2,\mathbb{Z})}$.

The question arises for a lattice Γ in G, if the action of automorphisms of Γ on Sub_Γ for a connected semisimple group G behave in the same way or differently from the case when G is a simply connected nilpotent Lie group. Theorem 3.16 shows that an almost similar result as above holds in this case too.

We first state and prove some lemmas to use later. The following lemma may be known but we give a proof for the sake of completeness. An element $g \in \mathrm{GL}(n, \mathbb{R})$ is said to be *net* if the multiplicative group generated by the eigenvalues of g in $\mathbb{C} \setminus \{0\}$ is torsion-free. Note that g is net if and only if g_s is net, where g_s is the semisimple part of g in its multiplicative Jordan decomposition. A subgroup of $\mathrm{GL}(n, \mathbb{R})$ is said to be *net* if all its elements are net (see 17.1 in [9]).

LEMMA 3.13. *Let G be a connected semisimple Lie group and let Γ be a lattice in G. Then there exists a normal subgroup Γ' of finite index in Γ such that $R_g := \{x \in \Gamma' \mid x^n = g$ for some $n \in \mathbb{Z}\}$ is finite for all $g \in \Gamma'$. Moreover, the torsion elements in Γ' form a finite central subgroup in G.*

Proof. For the centre $Z(G)$ of G, $G/Z(G)$ is a linear subgroup of $\mathrm{GL}(n, \mathbb{R})$, for some $n \in \mathbb{N}$. Let $\pi : G \to G/Z(G)$ be the natural projection. Since Γ is a lattice in G, it is finitely generated. Hence $\pi(\Gamma)$ is finitely generated, and by Corollary 17.7 of [9], $\pi(\Gamma)$ has a subgroup of finite index (say) Γ_1 such that it is net. Let $\Gamma' = \pi^{-1}(\Gamma_1) \cap \Gamma$. Then Γ' is a subgroup of finite index in Γ.

For $x \in \Gamma$, let $\bar{x} = \pi(x)$ and let G_x (resp. $G_{\bar{x}}$) be the cyclic group generated by x in Γ (resp. $\bar{x}$ in $\pi(\Gamma)$). Then for any $x \in \Gamma'$, the Zariski closure $\tilde{G}_{\bar{x}}$ of $G_{\bar{x}}$ is connected (see the proof of Proposition 17.2 in [9]).

Let $g \in \Gamma'$. First suppose that $g \in Z(G)$. Then $\pi(R_g)$ is the group of torsion elements in Γ_1. Since Γ_1 is net, it is torsion-free, and hence $\pi(R_g) = \{\bar{e}\}$. Therefore, $R_g \subset Z(G)$. As $Z(G)$ is compactly generated and abelian, R_g is finite. This implies in particular that the set of torsion elements in Γ' is a subgroup of $Z(G)$, and hence it is finite since $Z(G)$ is compactly generated and abelian.

Now suppose $g \notin Z(G)$ and let $x \in R_g$. Then $\bar{g}, \bar{x} \in \Gamma_1$ are non-trivial and $G_{\bar{g}}$ is a subgroup of finite index in $G_{\bar{x}}$, and since each of them have connected Zariski closure, we get that $\tilde{G}_{\bar{g}} = \tilde{G}_{\bar{x}}$. That is, $\bar{x} \in \tilde{G}_{\bar{g}}$. Let $H_g = \pi^{-1}(\Gamma_1 \cap \tilde{G}_{\bar{g}})$. Since $\tilde{G}_{\bar{g}}$ is connected and abelian and since $Z(G)$ is finitely generated and central in G, we get that H_g is a finitely generated nilpotent group. From above, we have that $R_g \subset H_g$. Since H_g is finitely generated and nilpotent, it is strongly root compact and by Theorem 3.1.13 of [15], R_g is finite.

Replacing Γ' by a smaller subgroup of finite index if necessary, we may assume that Γ' is normal. □

Now we can deduce the following.

LEMMA 3.14. *Let Γ be a lattice in a connected semisimple Lie group G. Then Sub^c_Γ is closed.*

Proof. By Lemma 3.13, there exists a normal subgroup Γ' of finite index in Γ such that for every $g \in \Gamma'$, the set R_g of roots of g in Γ' is finite. By Lemma 3.4, we get that $\mathrm{Sub}^c_{\Gamma'}$ is closed. If $\Gamma = \Gamma'$, then $\mathrm{Sub}^c_{\Gamma} = \mathrm{Sub}^c_{\Gamma'}$ and it is closed. Now suppose Γ' is proper subgroup of Γ.

Let $\{H_n\}_{n\in\mathbb{N}}$ be a sequence in Sub^c_{Γ} such that $H_n \to H$ in Sub_{Γ}. We need to show that H is cyclic. Let $H'_n = H_n \cap \Gamma'$, $n \in \mathbb{N}$ and let $H' = H \cap \Gamma'$. Then $H'_n \in \mathrm{Sub}^c_{\Gamma'}$ and it follows by Lemma 3.2 that $H'_n \to H'$. From above, we have that H' is cyclic. Since $H\Gamma'/\Gamma'$ is finite and isomorphic to $H/(H \cap \Gamma')$, we have that $H' = H \cap \Gamma'$ is a normal subgroup of finite index in H, and hence H is finitely generated. Let $\{h_1, \ldots, h_m\}$ be a set of generators in H. Then there exists $k \in \mathbb{N}$ such that for $1 \leq i \leq m$, $h_i \in \bigcap_{n=k}^{\infty} H_n \subset H_k$. Therefore, $H \subset H_k$ and hence H is cyclic. □

The following lemma should be known; we give a short proof for the sake of completeness. Recall that for a group H, $Z(H)$ denotes the centre of H.

LEMMA 3.15. *Let G be a connected semisimple Lie group, Γ be a lattice in G and let Γ' be a subgroup of finite index in Γ. Then $Z(\Gamma) \cap \Gamma'$ is a subgroup of finite index in $Z(\Gamma')$.*

Proof. First suppose that G has no compact factors; that is, the maximal compact connected normal subgroup of G is trivial. By Corollary 5.18 of [**25**], $Z(\Gamma') \subset Z(G)$, the centre of G. Hence $Z(\Gamma') \subset Z(\Gamma)$. Now suppose G has a non-trivial compact factor. Let K be the largest compact connected normal subgroup of G. If $G = K$, then Γ and Γ' are finite and the assertion follows trivially. Suppose G is not compact. Then G/K is semisimple and it has no compact factors. Let $\psi : G \to G/K$ be the natural projection. Then $\psi(\Gamma)$ and $\psi(\Gamma')$ are lattices in G/K. From above, we have that $\psi(Z(\Gamma')) \subset Z(\psi(\Gamma')) \subset Z(G/K)$. Therefore, $xgx^{-1}g^{-1} \in K$ for all $g \in G$ and $x \in Z(\Gamma')$. Fix $x \in Z(\Gamma')$ and let $g \in \Gamma$.

We first assume that Γ' is normal in Γ. As $Z(\Gamma')$ is normal in Γ, $xgx^{-1}g^{-1} \in Z(\Gamma') \cap K$; this is a finite abelian group. Let m be the order of $Z(\Gamma') \cap K$. Since $Z(\Gamma')$ is abelian, we get that $(xgx^{-1}g^{-1})^m = x^m g x^{-m} g^{-1} = e$. Therefore, $x^m \in Z(\Gamma)$.

Note that the centre of any connected semisimple Lie group is compactly generated. Therefore, the centre of any lattice in G is compactly generated as its image in G/K is central in G/K, where K as above is compact. Since $Z(\Gamma')$ is compactly generated and abelian and $x^m \in Z(\Gamma)$ for every $x \in Z(\Gamma')$, we have that $Z(\Gamma')/(Z(\Gamma) \cap \Gamma')$ is finite.

Now suppose Γ' is not normal in Γ, there exists a normal subgroup Γ'' of finite index in Γ such that $\Gamma'' \subset \Gamma'$. From the above discussion, we have that $Z(\Gamma'')/(Z(\Gamma) \cap \Gamma'')$ and $Z(\Gamma'')/(Z(\Gamma') \cap \Gamma'')$ are finite. Since $Z(\Gamma')/(Z(\Gamma') \cap \Gamma'')$ is also finite, it is easy to deduce that $Z(\Gamma')/(Z(\Gamma) \cap \Gamma')$ is finite. □

Using the above lemmas, we can prove the following result which was suggested by an anonymous referee along with a sketch of a proof.

THEOREM 3.16. *Let G be a connected semisimple Lie group and let Γ be a lattice in G. Let $T \in \mathrm{Aut}(\Gamma)$. Then the following statements are equivalent:*

(1) $T \in (\mathrm{NC})$.
(2) *T acts distally on* Sub^c_{Γ}.
(3) *T acts distally on* Sub_{Γ}.
(4) $T^n = \mathrm{Id}$ *for some* $n \in \mathbb{N}$.
(5) $T^n|_{\Gamma'} = \mathrm{Id}$ *for some $n \in \mathbb{N}$, where Γ' is a subgroup of finite index in Γ.*

If $T = S|_{\Gamma}$ for some $S \in \mathrm{Aut}(G)$, then (1–5) are equivalent to each of the following statements:

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(6) *S acts distally on* Sub_G.

(7) *S is contained in a compact subgroup of* $\mathrm{Aut}(G)$.

Moreover, if G has no compact factors, then (1–7) *are equivalent to the following:*

(8) $S^n = \mathrm{Id}$ *for some* $n \in \mathbb{N}$.

Proof. (4) $\Longrightarrow$ (3) $\Longrightarrow$ (2) $\Longrightarrow$ (1) and (4) $\Longrightarrow$ (5) are obvious. Now suppose (5) holds. We show that (4) holds. Passing to a smaller subgroup of finite index if necessary, we may assume that Γ' is normal in Γ and that it is T-invariant.

Since Γ/Γ' is finite, replacing n by a larger number if necessary, we may assume that $T^n|_{\Gamma'} = \mathrm{Id}$ and T^n acts trivially on Γ/Γ'. Without loss of any generality, we replace T by T^n and assume that $T|_{\Gamma'}$ is trivial and T acts trivially on Γ/Γ'. We want to show that some power of T is the identity map.

Let $x \in \Gamma$. Then $T(x) = xy$ for some $y \in \Gamma'$. For any $g \in \Gamma'$, we have $xygy^{-1}x^{-1} = T(xgx^{-1}) = xgx^{-1}$, and hence $ygy^{-1} = g$. Therefore, $y \in Z(\Gamma')$, the centre of Γ'. By Lemma 3.15, $Z(\Gamma) \cap \Gamma'$ is a subgroup of finite index in $Z(\Gamma')$. Let m be the order of $Z(\Gamma')/(Z(\Gamma) \cap \Gamma')$ and let k be the order of Γ/Γ'. Then $y^m \in Z(\Gamma)$ and we get that $T^m(x) = xy^m \in xZ(\Gamma)$ and hence $T^m(x^k) = x^k y^{km}$. As $x^k \in \Gamma'$, we have that $T^m(x^k) = x^k$, and hence $y^{km} = e$. Therefore, $T^{km}(x) = x$. Thus $T^{km} = \mathrm{Id}$ and (4) holds.

Now we show that (1) $\Longrightarrow$ (5). Suppose $T \in (\mathrm{NC})$. Let Γ' be a normal subgroup of finite index in Γ as in Lemma 3.13. That is, the set R_g of roots of g in Γ' is finite for every $g \in \Gamma'$. Without loss of any generality, we may assume that Γ' is T-invariant and $T|_{\Gamma'} \in (\mathrm{NC})$. Note that Γ', being a subgroup of finite index in Γ, is a lattice in G. Hence Γ' is finitely generated, and we get from Proposition 3.8 that $T^n|_{\Gamma'} = \mathrm{Id}$ for some $n \in \mathbb{N}$ and (5) holds. That is, (1–5) are equivalent.

Let $S \in \mathrm{Aut}(G)$ and let $S|_\Gamma = T$. Then (7) $\Longrightarrow$ (6) (see Lemma 2.4 in [**32**] and the discussion before the lemma, or see Theorem 4.1 of [**32**], or Lemma 4.2). Note that (6) $\Longrightarrow$ (3) is obvious. Now we prove that (4) $\Longrightarrow$ (7). Suppose $T^n = \mathrm{Id}$ for some $n \in \mathbb{N}$. Since G is semisimple, some power of S is an inner automorphism of G. To prove (7), we may assume that S itself is an inner automorphism of G. Let $s \in G$ be such that $S = \mathrm{inn}(s)$. Now $s\Gamma s^{-1} = \Gamma$ and from (4), we get that s^l centralises Γ for some $l \in \mathbb{N}$. Replacing S by S^l, we may assume that $s \in Z_G(\Gamma)$, the centraliser of Γ in G. Let K be the largest compact connected normal subgroup of G which is the product of all compact factors of G.

If K is trivial, then by Theorem 5.18 of [**25**], $Z_G(\Gamma) = Z(G)$, and hence $S = \mathrm{Id}$. That is, if G has no compact factors, then (8) holds, and hence (7) also holds in this case. (Note that (4) $\Longrightarrow$ (8) also follows directly from the Borel Density Theorem if G has no compact factors.)

If G is compact, then $\mathrm{Aut}(G)$ is compact as $\mathrm{Inn}(G)$ is a subgroup of finite index in $\mathrm{Aut}(G)$, hence (7) holds. Now suppose G is not compact. Then $G = KG_1$ (almost direct product), where G_1 is a closed connected normal subgroup which is the product of all non-compact (simple) factors of G. Now $s = kh = hk$ for some $k \in K$ and $h \in G_1$. Let $\psi : G \to G/K$ be the natural projection. Then $\psi(\Gamma)$ is a lattice in G/K. As G/K has no compact factors and as $\psi(s)$ centralises $\psi(\Gamma)$, we get as above that $\psi(s) \in Z(G/K)$, and since $\psi(s) = \psi(h)$, $hgh^{-1}g^{-1} \in K$ for all $g \in G$. Since $h \in G_1$, which is normal in G, we get that $hgh^{-1}g^{-1} \in G_1 \cap K$ which is a finite (central) subgroup of G. As G is connected, the preceding assertion implies that $hgh^{-1}g^{-1} = e$ for all $g \in G$, and hence $h \in Z(G)$. Now $s = kh \in KZ(G)$ and $\mathrm{inn}(s) = \mathrm{inn}(k)$. As $k \in K$, we get that $\mathrm{inn}(s)$, and hence S is contained in a compact subgroup of $\mathrm{Aut}(G)$. Therefore, (7) holds. □

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Note that Example 3.11 shows that a connected simply connected solvable Lie group can admit an automorphism T and T-invariant lattices Γ_1 and Γ_2 such that Γ_1 is a subgroup of finite index in Γ_2 and $T|_{\Gamma_1} = \mathrm{Id}$ but $T^n|_{\Gamma_2} \neq \mathrm{Id}$ for any $n \in \mathbb{N}$. This is unlike the case of lattices in a connected semisimple Lie group as shown by (5) $\Longrightarrow$ (4) in Theorem 3.16.

It would be interesting to study the distality of the actions of automorphisms of Γ on Sub_Γ for a lattice Γ in a general connected Lie group.

4. Distal actions of automorphisms on Sub_G for certain compact groups and nilpotent groups. In this section, for certain locally compact metrisable groups G and $T \in \mathrm{Aut}(G)$, we characterise the distality of the T-action on Sub_G in terms of the compactness of the closure of the group generated by T in $\mathrm{Aut}(G)$. It is shown in [32] that if $\mathrm{Aut}(G)$ is endowed with the modified compact-open topology, then the map $\mathrm{Aut}(G) \times \mathrm{Sub}_G \to \mathrm{Sub}_G$ defined by $(T, H) \mapsto T(H)$, $T \in \mathrm{Aut}(G)$, $H \in \mathrm{Sub}_G$, is continuous, i.e. $\mathrm{Aut}(G)$ acts continuously on Sub_G by homeomorphisms (cf. [32], Lemma 2.4). For compact groups G, the modified compact-open topology is the same as the compact-open topology on $\mathrm{Aut}(G)$. For any connected Lie group G with the Lie algebra $\mathcal{G}$, for a $T \in \mathrm{Aut}(G)$, there is a unique Lie algebra automorphism $\mathrm{d}\,T$ in $\mathrm{GL}(\mathcal{G})$. Note that $\mathrm{Aut}(G)$ is isomorphic to a closed subgroup of $\mathrm{GL}(\mathcal{G})$ (isomorphism is given by the map $T \mapsto \mathrm{d}\,T$). Therefore, $\mathrm{Aut}(G)$ is a Lie group whose topology is the same as the compact-open topology as well as the modified compact-open topology, (cf. [3, 16]). In general, if $\mathrm{Aut}(G)$ is endowed with the compact-open topology, then $\mathrm{Aut}(G)$ is a topological semigroup and the natural map $\mathrm{Aut}(G) \times G \to G$ is continuous (see [33] for more details on topologies on $\mathrm{Aut}(G)$).

For a metric space X, a subset Ω of Homeo(X) is said to be equicontinuous at $x \in X$ if given $\epsilon > 0$, there exists $\delta > 0$ such that $\phi(B_\delta(x)) \subset B_\epsilon(\phi(x))$, $\phi \in \Omega$, where $B_r(x)$ is the ball of radius r centred at x in X for $r > 0$. Ω is said to be equicontinuous on X if Ω is equicontinuous at every $x \in X$. If G is a locally compact first countable (metrisable) group with the identity e, then G has a left invariant metric, and hence any $\Omega \subset \mathrm{Aut}(G)$ is equicontinuous at x if and only if it is equicontinuous at e. Therefore, Ω is equicontinuous on G if and only if given any neighbourhood U of e, there exists a neighbourhood V of e such that $\phi(U) \subset V$ for all $\phi \in \Omega$.

By Arzela–Ascoli Theorem (see, e.g., [33], Theorem 9.24), $\Omega \subset \mathrm{Aut}(G)$ is relatively compact in $\mathrm{Aut}(G)$ (with respect to the compact-open topology) if it is equicontinuous at e and $\{\phi(x) \mid \phi \in \Omega\}$, the Ω-orbit of x is relatively compact in G, for every $x \in G$. The converse also holds since G is locally compact and $\overline{\Omega}$ is compact, the action of $\overline{\Omega}$ on G is uniformly continuous. The following useful version of Arzela–Ascoli Theorem for locally compact metrisable groups easily follows from above.

LEMMA 4.1. [Arzela–Ascoli Theorem] *If G is a locally compact first countable (metrisable) group. Let* $\mathrm{Aut}(G)$ *be the group of automorphisms of G endowed with the compact-open topology. Let Ω be a subset of* $\mathrm{Aut}(G)$. *Then $\overline{\Omega}$ is compact in* $\mathrm{Aut}(G)$ *if and only if the following hold:*

(1) $\{\phi(x) \mid \phi \in \Omega\}$ is relatively compact in G, for every $x \in G$.
(2) Ω is equicontinuous at e.

The following lemma will be useful.

LEMMA 4.2. *Let G be a locally compact first countable (metrisable) group. Let $H \subset \mathrm{Aut}(G)$ be a subgroup. If H is relatively compact with respect to the compact-open topology, then $\overline{H}$ is a compact group and H acts distally on* Sub_G.

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Proof. Note that $\mathrm{Aut}(G)$ is a topological semigroup with respect to the compact-open topology (see, e.g., Lemma 9.5 of [33]). Therefore, $\overline{H}$ is a compact semigroup and hence $\overline{H}$ is a compact group in $\mathrm{Aut}(G)$ with respect to the compact-open topology (see, e.g., Theorem 30.6 of [33]). Note that on $\overline{H}$, the compact-open topology and the modified compact-open topology coincide. It follows from Lemma 2.4 of [32] that the natural map $\overline{H} \times \mathrm{Sub}_G \to \mathrm{Sub}_G$, defined by the action of automorphisms of G on Sub_G, is continuous. As $\overline{H}$ is compact, it follows that $\overline{H}$, and hence H acts distally on Sub_G. □

As observed in the proof of Lemma 4.2, it follows that a subgroup H of $\mathrm{Aut}(G)$ is compact with respect to the compact-open topology, if and only if it is compact with respect to the modified compact-open topology. Moreover, a subgroup H, which is compact (in the compact-open topology), is a compact topological group. Henceforth, $\mathrm{Aut}(G)$ is endowed with the compact-open topology, and for compact subgroups of $\mathrm{Aut}(G)$, we will not specify the topology.

For a totally disconnected locally compact group G, $T \in \mathrm{Aut}(G)$ is distal if and only if G has arbitrarily small compact open T-invariant subgroups (this follows from Proposition 2.1 of [19] together with the 'Note added in proof' in [19]). Moreover if G is metrisable, then the above implies that, T is distal if and only if $\{T^n\}_{n\in\mathbb{Z}}$ is equicontinuous (at e). We now get the following characterisation for compact totally disconnected groups.

PROPOSITION 4.3. *Let G be a compact totally disconnected first countable (metrisable) group and let $T \in \mathrm{Aut}(G)$. Then the following are equivalent:*

(1) *T acts distally on G.*
(2) *T acts distally on* Sub_G.
(3) *T is contained in a compact subgroup of* $\mathrm{Aut}(G)$.

Proof. Here, (3) $\Longrightarrow$ (2) follows from Lemma 4.2. As G is totally disconnected, (2) $\Longrightarrow$ (1) follows from Theorem 3.6 of [32]. It is enough to show that (1) $\Longrightarrow$ (3). Suppose T acts distally on G. Let $\Omega_T = \{T^n\}_{n\in\mathbb{Z}}$. By Proposition 2.1 of [19], Ω_T is equicontinuous (at e). Also, since G is compact, the Ω_T-orbit of x is relatively compact for every $x \in G$. By Lemma 4.1, Ω_T has compact closure in $\mathrm{Aut}(G)$. Hence $\overline{\Omega_T}$ is a compact group (see, e.g., Theorem 30.6 of [33]). □

Note that Proposition 4.3 also holds for a non-compact totally disconnected (additive) group $G = \mathbb{Q}_p^n$, ($n \in \mathbb{N}$), a p-adic vector space, and $T \in \mathrm{GL}(n, \mathbb{Q}_p)$ (where p is a prime). This follows from Lemma 2.1 of [31] and Lemma 4.2 above together with the fact that $\mathrm{GL}(n, \mathbb{Q}_p)$ is a (metrisable) topological group and its topology is the same as the (modified) compact-open topology.

The following generalises Theorem 4.1 of [32] in the case of connected nilpotent Lie groups to all compactly generated nilpotent groups. Note that in any connected nilpotent group G, the unique maximal compact subgroup K is connected, abelian and central in G (as G is Lie projective) and its torsion group is dense in K. Therefore, such a G is torsion-free if and only if it is a simply connected nilpotent Lie group (equivalently, it has no non-trivial compact subgroup). Any compactly generated nilpotent Lie group is torsion-free if and only if its maximal compact subgroup is trivial.

THEOREM 4.4. *Let G be a locally compact metrisable compactly generated nilpotent group such that G^0 is torsion-free and let $T \in \mathrm{Aut}(G)$. Then the following are equivalent.*

(1) *T acts distally on* Sub_G.
(2) *The closure of the group generated by T in* $\mathrm{Aut}(G)$ *is a compact group.*

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Moreover, if G as above is a Lie group (with not necessarily finitely many connected components), then the following are equivalent and they are also equivalent to statements (1–2) *above.*

(3) $T \in$ (NC).
(4) *T acts distally on* Sub^a_G.

Proof. Let $\Omega_T = \{T^n\}_{n\in\mathbb{Z}}$. If $\overline{\Omega_T}$ is compact, then it is a compact group (cf. [33], Theorem 30.6). Now using Lemma 4.2, we get that (2) $\Longrightarrow$ (1). We know that (1) $\Longrightarrow$ (4) $\Longrightarrow$ (3). Suppose $T \in$ (NC). Since G is strongly root compact, by Lemma 3.6, $\{x \in G \mid G_x \text{ is closed}\} \subset M(T)$. Since G is compactly generated and nilpotent, it has a unique maximal compact group K such that G/K is a compactly generated torsion-free Lie group and all its cyclic subgroups are discrete. Now if $x \notin K$, we get that G_x is closed, and hence $x \in M(T)$. As K is T-invariant, we have $K \subset M(T)$, and hence $G = M(T)$. This implies that Ω_T satisfies the condition (1) of Lemma 4.1.

As G^0 is a simply connected nilpotent Lie group, by Theorem 4.1 of [32], we get that $T|_{G^0}$ generates a relatively compact group in $\mathrm{Aut}(G^0)$. This implies that $\{(T|_{G^0})^n\}_{n\in\mathbb{Z}}$ is equicontinuous on G^0.

Suppose G is a Lie group. As G^0 is open, the preceding assertion implies that Ω_T is equicontinuous at e and Ω_T satisfies the condition (2) of Lemma 4.1. Therefore, (3) $\Longrightarrow$ (2), and hence (1–4) are equivalent for a Lie group G.

Suppose G is not a Lie group and suppose (1) holds. Then $T \in$ (NC) and Ω_T satisfies the condition (1) of Lemma 4.1 as shown above. As T acts distally on Sub_G, by Theorem 3.6 of [32], T is distal. As G is compactly generated and nilpotent, it is distal and by Corollary 2.2, G is T-Lie projective. Therefore, there exist compact open T-invariant normal subgroups K_n such that G/K_n is a Lie group, $K_n \subset K_{n+1}$, $n \in \mathbb{N}$, and $\bigcap_n K_n = \{e\}$. As G^0 has no non-trivial compact subgroup, $G_n := G^0 \times K_n$ are open T-invariant subgroups of G such that $T(G^0) = G^0$ and $T(K_n) = K_n$, $n \in \mathbb{N}$. We know from above that $\{(T|_{G^0})^n\}_{n\in\mathbb{Z}}$ is equicontinuous on G^0. Let $\{W_m\}_{m\in\mathbb{N}}$ be a neighbourhood basis of the identity e in G^0 such that $T^k(W_{m+1}) \subset W_m$ for all $k \in \mathbb{Z}$ and $m \in \mathbb{N}$. Then $\{K_n \times W_m \mid m, n \in \mathbb{N}\}$ is a neighbourhood basis of the identity e in G. As K_n are T-invariant, it follows that Ω_T is equicontinuous at e and it satisfies the condition (2) of Lemma 4.1. Hence $\overline{\Omega_T}$ is a compact group in $\mathrm{Aut}(G)$ and (2) holds. □

Note that if Γ is a locally compact compactly generated nilpotent group without any non-trivial compact subgroup, then Γ embeds in a connected simply connected nilpotent Lie group G as a closed co-compact subgroup and any automorphism of Γ extends to a unique automorphism of G (cf. [25]). Note also that any closed subgroup of a simply connected nilpotent group is compactly generated. We now have the following corollary which can be viewed as an extension of Corollary 3.9.

COROLLARY 4.5. *Let G be a connected simply connected nilpotent Lie group. Let Γ be a closed co-compact subgroup of G. Let $T \in \mathrm{Aut}(G)$ be such that $T(\Gamma) = \Gamma$. Then* $(1-6)$ *of Corollary 3.9 are equivalent and they are also equivalent to the following: T is contained in a compact subgroup of* $\mathrm{Aut}(G)$.

Proof. Note that it is enough to show that if $T|_\Gamma \in$ (NC), then T is contained in a compact subgroup of $\mathrm{Aut}(G)$. Let $T|_\Gamma \in$ (NC). By Theorem 4.4, $T|_\Gamma$ is contained in a compact subgroup of $\mathrm{Aut}(\Gamma)$. Here, $\exp : \mathcal{G} \to G$ is a homeomorphism with log as its inverse. Let $\mathrm{d}\,T : \mathcal{G} \to \mathcal{G}$ be the Lie algebra automorphism corresponding to T. Since Γ is co-compact, it follows that $\log(\Gamma)$ generates $\mathcal{G}$ as a vector space and we also have that $\{\mathrm{d}\,T^n(g)\}_{n\in\mathbb{Z}}$

is relatively compact for all $g \in \log(\Gamma)$, and hence it is relatively compact for all $g \in \mathcal{G}$. This implies that d T is contained in a compact subgroup of $\mathrm{GL}(\mathcal{G})$. As $\mathrm{Aut}(G)$ is a closed subgroup of $\mathrm{GL}(\mathcal{G})$, T is contained in a compact subgroup of $\mathrm{Aut}(G)$. □

Note that the action of G on Sub_G is the same as the action of $\mathrm{Inn}(G)$ on Sub_G, where $\mathrm{Inn}(G)$ is the group of inner automorphisms of G. For subgroups H_1 and H_2 of G, let $[H_1, H_2]$ denote the subgroup generated by $\{h_1h_2h_1^{-1}h_2^{-1} \mid h_1 \in H_1, h_2 \in H_2\}$. Recall that for a subgroup H of G, $Z_G(H)$ denotes the centraliser of H in G. The following theorem is an analogue of Corollary 4.5 of [**32**] in the case of certain disconnected nilpotent groups.

THEOREM 4.6. *Let G be a locally compact metrisable compactly generated nilpotent group and let K be the unique maximal compact (normal) subgroup of G. Then the following are equivalent:*

(1) *Every inner automorphism of G acts distally on* Sub_G.
(2) *G acts distally on* Sub_G.
(3) $\overline{\mathrm{Inn}(G)}$ *is a compact subgroup of* $\mathrm{Aut}(G)$.
(4) *G/K is abelian and $G = Z_G(G^0)$.*

In case G is a torsion-free Lie group, then (1–4) *are equivalent to the following:*

(5) *G is abelian.*

Proof. Here (3) $\implies$ (2) follows from Lemma 4.2. It is obvious that (2) $\implies$ (1). We now show that (1) $\implies$ (4). Suppose (1) holds. Since K is the unique maximal compact subgroup of G, K is normal in G and G/K is a compactly generated nilpotent Lie group without any non-trivial compact subgroup. By Lemma 3.1 of [**32**], every inner automorphism of G/K acts distally on $\mathrm{Sub}_{G/K}$. To prove that G/K is abelian, we may assume that G is a Lie group without any non-trivial compact subgroup and show that it is abelian. If possible, suppose G is not abelian. Let $Z = Z(G)$, the centre of G. Then $Z \neq G$, and since G is nilpotent, there exists a closed subgroup $Z_1 = \{g \in G \mid xgx^{-1}g^{-1} \in Z \text{ for all } x \in G\}$ such that $Z_1 \supsetneq Z \supsetneq \{e\}$. Let $y \in Z_1$ be such that $y \notin Z$. Then there exists $x \in G$ such that $xyx^{-1} = yz$ for some non-trivial $z \in Z$. Now $\mathrm{inn}(x)(y) = yz$, and $\mathrm{inn}(x)$ acts trivially on Z which has no non-trivial compact subgroup. Let G_y be the subgroup generated by y in Z_1. Here, $xy^nx^{-1} = y^nz^n$ and since G is torsion-free, we have that no non-trivial element of G_y is stabilised by $\mathrm{inn}(x)$. By Lemma 3.12 of [**32**], $\mathrm{inn}(x) \notin (\mathrm{NC})$. In particular, $\mathrm{inn}(x)$ does not act distally on Sub_G. This contradicts the statement in (1), and hence G is abelian. This implies the first assertion in (4).

If G is a torsion-free Lie group, we have that K is trivial since the set of torsion elements is dense in K. Hence, the above shows that for such a G, (1) $\implies$ (5) and also (4) $\implies$ (5). Since (5) $\implies$ (4) and (5) $\implies$ (3), all the statements $(1-5)$ are equivalent for such a torsion-free Lie group G.

Now we show that $G = Z_G(G^0)$; that is, we show that G^0 is central in G. If G^0 is trivial, then $G = Z_G(G^0)$. Now suppose $G^0 \neq \{e\}$. We know from Proposition 2.4 that K centralises G^0.

Suppose G is a Lie group. We know that $C = K \cap G^0$ is the maximal compact subgroup of G^0, C is connected and central in G^0, and by Proposition 2.4, C is central in KG^0.

We first show that C is central in G. Since C is characteristic in G^0, C is normal in G. As G/K is abelian and G^0 is normal, we have that $[G, G^0] \subset K \cap G^0 = C$. Suppose C is not central in G and suppose $x \in G$ does not centralise C. Then $x \notin KG^0$. As G/K has no non-trivial compact subgroup, xK generates a discrete infinite subgroup in G/K, and hence

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the cyclic group G_x generated by x in G is discrete and infinite. From above, $[G_x, C] \neq \{e\}$. Let $C_0 = C$ and $C_k = \overline{[G_x, C_{k-1}]}$, $k \in \mathbb{N}$. Here, $C_1 \neq \{e\}$, $C_k \subset C_{k-1} \subset C$, $xC_kx^{-1} = C_k$ and C_k is a compact subgroup of C, $k \in \mathbb{N}$. Since C is connected, we get that C_1, and hence each C_k is connected. Moreover, inn(x) acts trivially on C_{k-1}/C_k, $k \in \mathbb{N}$. As G is nilpotent, $C_l \neq \{e\}$ and $C_{l+1} = \{e\}$ for some $l \in \mathbb{N}$.

Since C is a connected abelian Lie group, the above implies that the action of inn(x) on C is unipotent (i.e. the eigenvalues of Ad(x) on the Lie algebra of C are all equal to 1). As inn(x) acts distally on Sub$_C$, by Proposition 4.2 of [**32**], we get that inn(x) acts trivially on C, (one can also directly argue as in Step 1 of the proof of Proposition 4.2 of [**32**] to conclude that inn$(x)|_C = \text{Id}$). This leads to a contradiction, and hence we get that x centralises C. Since this holds for all $x \in G$, we get that $G = Z_G(C)$.

Now suppose $G^0 \neq C$. By Corollary 4.5 of [**32**], $G^0 = \mathbb{R}^n \times C$ for some $n \in \mathbb{N}$. Suppose $G \neq Z_G(G^0)$. There exists $x \in G$ which does not centralise G^0. Then $x \notin KG^0$ as KG^0 centralises G^0. Note that as G/K is abelian, we have that $[G, G^0] \subset C$ and $xgx^{-1} \in gC$ for all $g \in \mathbb{R}^n$. Therefore, inn(x) acts trivially on G^0/C, and hence the action of inn(x) on G^0 is unipotent. By Proposition 4.2 of [**32**], inn(x) acts trivially on G^0 (one can also directly conclude this by arguing as in the latter part of Step 2 of the proof of Proposition 4.2 of [**32**]). This leads to a contradiction, and hence $x \in Z_G(G^0)$. Since this holds for all $x \in G$, we have that $G = Z_G(G^0)$.

Now suppose G is not a Lie group. Since G is nilpotent, it is distal, and since it is compactly generated, locally compact and metrisable, by Theorem 2.1, G is a projective limit of Lie groups G/K_n, where $K_n \subset K$, $n \in \mathbb{N}$ and $\bigcap_n K_n = \{e\}$. By Lemma 3.1 of [**32**], every inner automorphism of G/K_n also acts distally on Sub$_{G/K_n}$, $n \in \mathbb{N}$. As $(G/K_n)^0 = G^0K_n/K_n$ is a Lie group, we get from above that $G/K_n = Z_{G/K_n}(G^0K_n/K_n)$. This implies that $[G, G^0] \subset K_n$ for all n, and hence $[G, G^0] \subset \bigcap_n K_n = \{e\}$. Therefore, $G = Z_G(G^0)$. This completes the proof of (1) $\implies$ (4).

Now suppose (4) holds. We show that (3) holds. Since G/K is abelian, for every $g \in G$, the Inn(G)-orbit of g is contained in gK. Therefore, (1) of Lemma 4.1 is satisfied for $\Omega = \text{Inn}(G)$. Now we show that Inn(G) is equicontinuous at e. By Theorem 2.1, G is Lie projective, and hence has compact normal subgroups K_n such that G/K_n is a Lie group, $n \in \mathbb{N}$, and $\bigcap_n K_n = \{e\}$. Therefore, G^0K_n is open in G. Let $\{U_n\}_{n\in\mathbb{N}}$ be a neighbourhood basis of the identity e in G^0. Since K centralises G^0, $K_nU_n = U_nK_n$, $n \in \mathbb{N}$, and $\{K_nU_n\}_{n\in\mathbb{N}}$ is a neighbourhood basis of the identity e in G. As $G = Z_G(G^0)$, we have that for all $x \in G$, $xK_nU_nx^{-1} = K_nU_n$, $n \in \mathbb{N}$. Therefore, Inn(G) is equicontinuous at e and by Lemma 4.1, $\overline{\text{Inn}(G)}$ is relatively compact in Aut(G), and hence it is a compact group and (3) holds. Therefore, (1–4) are equivalent. □

Note that in Theorem 4.6, (5) is not equivalent to (1–4) in general. There exist metrisable compact non-abelian totally disconnected nilpotent groups G; for example, take G to be a subgroup of strictly upper triangular matrices in SL$(3, \mathbb{Z}_p)$, where $\mathbb{Z}_p$ is the ring of p-dic integers in $\mathbb{Q}_p$ for a prime p. For such a G, the inner automorphisms act distally on Sub$_G$, and hence Theorem 4.6 (1–4) obviously hold for G due to Proposition 4.3.

ACKNOWLEDGEMENTS. R. Shah would like to acknowledge the MATRICS research grant from DST-SERB, Govt. of India. R. Palit would like to acknowledge the CSIR-JRF research fellowship from CSIR, Govt. of India. The authors would like to thank the anonymous referee for useful comments and for a sketch of the proofs for Lemma 3.13 and Theorem 3.16 for lattices in a connected semisimple Lie group.

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