THE TORSION OF A SEMI-INFINITE ELASTIC SOLID BY AN ELLIPTICAL STAMP

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1. Introduction. The problem of determining, within the limits of the classical theory of elasticity, the displacements and stresses in the interior of a semi-infinite solid ($z \geqq 0$) when a part of the boundary surface ($z = 0$) is forced to rotate through a given angle ω about an axis which is normal to the undeformed plane surface of the solid, has been discussed by several authors [**7, 8, 9, 1, 11**, and others]. All of this work is concerned with rotating a circular area of the boundary surface and the field equation to be solved is, essentially, J. H. Mitchell's equation for the torsion of bars of varying circular cross-sections.

Mindlin [**5**] obtains the elastic compliance of two bodies which have been pressed together and subjected to a small torsional couple in the plane of the contact surface. In [**5**], the solution is derived by integrating the half-space potential functions of Boussinesq and Cerruti (due to a point force) over the elliptical contact area in the plane $z = 0$. The weighting functions (in this case, the tangential shear stresses) are assumed to be generalizations of the ones pertaining to the limiting case of circular contact area. Such a method, unfortunately, leads to a series of intractable integrals for obtaining the components of displacement and stress in the interior of the half-space. A simpler method of analysis consists of writing down the solution of the governing field equations in terms of harmonic functions which identically frees the boundary surface $z = 0$ from the normal stress. These harmonic functions are, then, determined from the mixed boundary conditions on the plane $z = 0$. Such a method has been used by Lure [**4**] to examine the state of stress in an elastic half-space when there are prescribed shearing stresses on the boundary surface (first basic problem). Kassir and Sih [**2**] use a closely related analysis to determine the stress distribution in an infinite elastic solid containing an elliptical crack under uniform shear stress. The solution given in [**2**] is also suitable for the investigation of the state of stress in an elastic half-space, when there are prescribed tangential displacements on the boundary surface (second basic problem). An outline of this approach is given in [**3**].

In the present paper, we use a slightly modified form of the solution given in [**4**] and a set of orthogonal ellipsoidal coordinates to determine the local effects of the torsional distortion of a semi-infinite elastic solid when an elliptical stamp, rigidly attached to the free boundary surface, is forced to rotate about its normal axis. Expressions are derived, in terms of complete and incomplete elliptic integrals of the first and second kind, for the components of of displacement and stress across the boundary surface of the solid. Finally, we indicate briefly how the solution of the first basic problem may be derived by the proposed method.

2. Method of analysis. We consider a semi-infinite elastic solid $z \geqq 0$ with a rigid elliptical stamp attached to the boundary surface $z = 0$. The base of the stamp is defined by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} \leqq 1, \qquad z = 0, \tag{1}$$

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where a and b are the major and minor semi-axes of the ellipse, respectively. The centre of the ellipse is located at the origin of the cartesian coordinate system (x, y, z). The stamp is forced to rotate through an angle ω about the negative part of the z-axis, which coincides with the normal axis of the stamp.

In the absence of body forces, the field equations of the linear theory of elasticity for a homogeneous and isotropic body assume the vector form

$$\nabla^2\mathbf{u}+\frac{1}{1-2\nu}\nabla\nabla.\mathbf{u}=0. \tag{2}$$

Here ν is Poisson's ratio and $\mathbf{u}$ the displacement vector with components (u_x, u_y, u_z). The gradient and Laplacian operators in three dimensions are denoted by ∇ and ∇^2, respectively, while the dot indicates scalar multiplication.

The boundary conditions for the problem at hand can be written in the form

$$\left.\begin{aligned} u_x &= \omega y \\ u_y &= -\omega x \end{aligned}\right\} \quad \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\leqq 1, \quad z=0\right), \tag{3a, 3b}$$

$$\sigma_{xz}=\sigma_{yz}=0 \qquad \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}>1, \quad z=0\right), \tag{4}$$

$$\sigma_{zz}=0 \qquad (z=0). \tag{5}$$

$(\sigma_{xz}, \sigma_{yz}, \sigma_{zz})$ denote the stresses across a surface whose normal is in the z-direction. Further, as we are studying the effects of localized disturbances, the displacements and stresses must all tend to zero at large distances from the origin.

A solution of equations (2) which identically satisfies condition (5) may be obtained by expressing the displacement components u_x, u_y and u_z in terms of two harmonic functions $f_1(x, y, z)$ and $f_2(x, y, z)$ [4]:†

$$u_x=-2\frac{\partial f_1}{\partial z}+2\nu\int_z^\infty\frac{\partial F}{\partial x}dz+z\frac{\partial F}{\partial x}, \tag{6a}$$

$$u_y=-2\frac{\partial f_2}{\partial z}+2\nu\int_z^\infty\frac{\partial F}{\partial y}dz+z\frac{\partial F}{\partial y}, \tag{6b}$$

$$u_z=-(1-2\nu)F+z\frac{\partial F}{\partial z}, \tag{6c}$$

where

$$F(x,y,z)=\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}. \tag{7}$$

† It is interesting to note that when we add the particular solution

$$u_x=2\nu\left(\frac{\partial f_1}{\partial z}-\int_z^\infty\frac{\partial F}{\partial x}dz\right), \quad u_y=2\nu\left(\frac{\partial f_2}{\partial z}-\int_z^\infty\frac{\partial F}{\partial y}dz\right), \quad u_z=0 \quad (*)$$

to equations (6) above, we obtain the solution derived in [2]. It can be easily verified that equations (*) give rise to zero dilation, i.e. $\nabla.\mathbf{u}=0$, and hence the expression for the normal stress σ_{zz} is identical in both solutions.

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The corresponding stress components are given by

$$\frac{\sigma_{xz}}{2\mu} = -\frac{\partial^2 f_1}{\partial z^2} + z\frac{\partial^2 F}{\partial x\,\partial z}, \tag{8a}$$

$$\frac{\sigma_{yz}}{2\mu} = -\frac{\partial^2 f_2}{\partial z^2} + z\frac{\partial^2 F}{\partial y\,\partial z}, \tag{8b}$$

$$\frac{\sigma_{zz}}{2\mu} = z\frac{\partial^2 F}{\partial z^2}, \tag{8c}$$

where μ is the shear modulus of the material. For the mere purpose of determining the potential functions $f_j\,(x, y, z)$ $(j = 1, 2)$, the remaining stress components σ_{xx}, σ_{yy} and σ_{xy} are not needed.

To determine the unknown harmonic functions f_1 and f_2, the symmetrical form of ellipsoidal coordinates ξ, η, ζ will be employed. The rectangular coordinates x, y, z of any point may be expressed in terms of the triply orthogonal system ξ, η, ζ by the relations **[10]**

$$a^2(a^2-b^2)x^2 = (a^2+\xi)(a^2+\eta)(a^2+\zeta), \tag{9a}$$

$$b^2(b^2-a^2)y^2 = (b^2+\xi)(b^2+\eta)(b^2+\zeta), \tag{9b}$$

$$a^2b^2z^2 = \xi\eta\zeta, \tag{9c}$$

where

$$\infty > \xi \geqq 0 \geqq \eta \geqq -b^2 \geqq \zeta \geqq -a^2.$$

In the plane $z = 0$, $\xi = 0$ corresponds to points inside the two-sided area enclosed by the ellipse $x^2/a^2 + y^2/b^2 = 1$, and $\eta = 0$ to the outside. The boundary of the ellipse is identified by $\xi = 0$ and $\eta = 0$.

In terms of (ξ, η, ζ), the Laplace equation assumes the form

$$\nabla^2 W = \sum_{\xi,\eta,\zeta}\left[(\eta-\zeta)\sqrt{\{Q(\xi)\}}\frac{\partial}{\partial\xi}\left(\sqrt{\{Q(\xi)\}}\frac{\partial W}{\partial\xi}\right)\right] = 0, \tag{10}$$

where

$$Q(s) = s(a^2+s)(b^2+s). \tag{11}$$

A suitable solution of (10) may be taken as **[3]**

$$W(x, y, z) = \int_\xi^\infty \phi[\rho(s)]\frac{ds}{\sqrt{\{Q(s)\}}}, \tag{12}$$

where

$$\rho(s) = 1 - \frac{x^2}{a^2+s} - \frac{y^2}{b^2+s} - \frac{z^2}{s}, \tag{13a}$$

which can be put in the equivalent form

$$\rho(s) = \frac{(s-\xi)(s-\eta)(s-\zeta)}{Q(s)}. \tag{13b}$$

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$\phi(\rho)$ is a twice differentiable function in the interval (0, 1) with finite derivatives at the boundaries of the interval. Upon differentiating (12) with respect to z and noting that **[10]**

$$\frac{\partial \xi}{\partial z} = \frac{2zQ(\xi)}{\xi(\xi-\eta)(\xi-\zeta)},$$

it is found that

$$\frac{\partial W}{\partial z} = -\frac{2}{ab}(\xi\eta\zeta)^{\frac{1}{2}} \int_{\xi}^{\infty} \phi'(\rho) \frac{ds}{s\sqrt{\{Q(s)\}}} - 2\phi(0) \frac{[\eta\zeta(a^2+\xi)(b^2+\xi)]^{\frac{1}{2}}}{ab(\xi-\eta)(\xi-\zeta)}, \tag{14}$$

where the prime in the integrand indicates differentiation with respect to the argument.

For $\eta = 0$, equation (14) gives $\partial W/\partial z = 0$ regardless of the form of the unknown function $\phi(\rho)$. Hence, by taking

$$\frac{\partial f_j}{\partial z} = \int_{\xi}^{\infty} \phi_j[\rho(s)] \frac{ds}{\sqrt{\{Q(s)\}}} \qquad (j = 1, 2), \tag{15}$$

the boundary conditions (4) are automatically satisfied. The unknown functions ϕ_1 and ϕ_2 are found from the remaining boundary conditions (3).

Equations (7) and (15) yield

$$F(x, y, z) = \int_{\xi}^{\infty} [z - z(s)] \left[\frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_2}{\partial y}\right] \frac{ds}{\sqrt{\{Q(s)\}}} + \int_{\xi}^{\infty} \left[\frac{x\phi_1(\rho)}{a^2+s} + \frac{y\phi_2(\rho)}{b^2+s}\right] \frac{ds}{[\rho_0(s)(a^2+s)(b^2+s)]^{\frac{1}{2}}}, \tag{16}$$

in which

$$z(s) = [s\rho_0(s)]^{\frac{1}{2}}, \quad \rho_0(s) = [\rho(s)]_{z=0},$$

and the remaining boundary conditions (3) may be shown to be satisfied by choosing

$$\frac{\partial f_1}{\partial z} = C_1 \frac{\partial I}{\partial y}, \tag{17a}$$

$$\frac{\partial f_2}{\partial z} = C_2 \frac{\partial I}{\partial x}. \tag{17b}$$

Here the C_j $(j = 1, 2)$ are arbitrary constants and

$$I(x, y, z) = -\tfrac{1}{2} \int_{\xi}^{\infty} \rho(s) \frac{ds}{\sqrt{\{Q(s)\}}}. \tag{18}$$

Equation (18), except for a multiplying constant, represents the gravitational potential of an elliptic disk at an external point [6].

Equation (16) simplifies to

$$F(x, y, z) = (C_1 + C_2)xy \int_{\xi}^{\infty} \frac{ds}{[\rho_0(s)]^{\frac{1}{2}}(a^2+s)^{\frac{3}{2}}(b^2+s)^{\frac{3}{2}}}. \tag{19}$$

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In the plane $z = 0$, it is found that

$$\int_z^\infty \frac{\partial F}{\partial x} dz = (C_1 + C_2)\left[yJ - \frac{2x^2 y \xi^{\frac{1}{2}}}{(a^2+\xi)^{\frac{1}{2}}(b^2+\xi)^{\frac{3}{2}}(\xi-\eta)(\xi-\zeta)} \right], \tag{20a}$$

$$\int_z^\infty \frac{\partial F}{\partial y} dz = (C_1 + C_2' \left[xJ - \frac{2xy^2 \xi^{\frac{1}{2}}}{(a^2+\xi)^{\frac{1}{2}}(b^2+\xi)^{\frac{3}{2}}(\xi-\eta)(\xi-\zeta)} \right], \tag{20b}$$

where

$$J = \int_\xi^\infty \frac{s^{\frac{1}{2}} ds}{[(a^2+s)(b^2+s)]^{\frac{3}{2}}}. \tag{21}$$

This elliptic integral may be evaluated by putting

$$\xi = a^2 \frac{\mathrm{cn}^2 u}{\mathrm{sn}^2 u}, \qquad s = a^2 \frac{\mathrm{cn}^2 t}{\mathrm{sn}^2 t}; \tag{22}$$

we find that

$$J = \frac{2}{a^3 k^4}\left[(1+k'^2)u - 2E(u) + k^2 \frac{\mathrm{sn}\, u \,\mathrm{cn}\, u}{\mathrm{dn}\, u} \right],$$

where

$$E(u) = \int_0^u \mathrm{dn}^2 t \, dt,$$

and the quantities $\mathrm{sn}\, u$, $\mathrm{cn}\, u$, $\mathrm{dn}\, u$ are the Jacobian elliptic functions; $\mathrm{sn}\, u$ has real and imaginary periods $4K$, $2iK'$ corresponding to the modulus k and the complementary modulus k' respectively, where $ak = (a^2 - b^2)^{\frac{1}{2}}$, $ak' = b$ $(b < a)$.

Now, inserting equations (17), (20) in (6), we obtain from (3) a system of two algebraic equations for the two unknown constants C_1 and C_2, which yields

$$C_1 = -\tfrac{1}{4}\left[\frac{k^2 B - 2\nu(B - k'^2 A)}{k^2 k'^2 AB - \nu E(k)(B - k'^2 A)} \right] \omega a b^2, \tag{23a}$$

$$C_2 = \tfrac{1}{4}\left[\frac{k^2 A - 2\nu(B - k'^2 A)}{k^2 k'^2 AB - \nu E(k)(B - k'^2 A)} \right] \omega a b^2. \tag{23b}$$

In equations (23), the following abbreviations have been introduced:

$$k^2 k'^2 A = E(k) - k'^2 K(k),$$

$$k^2 B = K(k) - E(k).$$

Here $K(k)$ and $E(k)$ are respectively, Legendre's complete elliptic integrals of the first and second kind with the argument k.

When C_1 and C_2 are known, the displacements and stresses at any point of the solid can be calculated in a straightforward manner. The higher derivatives of the function (18) are given in [2]. Across the plane $z = 0$ outside the stamp ($\eta = 0$) the rectangular components of displacement are

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$$u_x = \frac{4y}{a^3k^4}\left\{-\left(\frac{k}{k'}\right)^2 C_1L(u)+\nu(C_1+C_2)\left[u-E(u)-L(u)-\frac{x^2M(\xi)}{a^2+\xi}\right]\right\}, \tag{24a}$$

$$u_y = \frac{4x}{a^3k^4}\left\{-k^2C_2[u-E(u)]+\nu(C_1+C_2)\left[u-E(u)-L(u)-\frac{y^2M(\xi)}{b^2+\xi}\right]\right\}, \tag{24b}$$

$$u_z = \frac{2(1-2\nu)(C_1+C_2)xy}{a^2-b^2}\left[-\frac{\tanh^{-1}\{(b^2+\zeta)/(b^2+\xi)\}^{\frac{1}{2}}}{(b^2+\xi)^{\frac{1}{2}}(b^2+\zeta)^{\frac{1}{2}}}+\frac{\tanh^{-1}\{(a^2+\zeta)/(a^2+\xi)\}^{\frac{1}{2}}}{(a^2+\xi)^{\frac{1}{2}}(a^2+\zeta)^{\frac{1}{2}}}\right], \tag{24c}$$

where

$$L(u) = E(u)-k'^2u-k^2\frac{\operatorname{sn}u\operatorname{cn}u}{\operatorname{dn}u},$$

$$M(\xi) = \frac{a^3k^4\xi^{\frac{1}{2}}}{(a^2+\xi)^{\frac{1}{2}}(b^2+\xi)^{\frac{1}{2}}(\xi-\eta)(\xi-\zeta)},$$

while, for $\xi = 0$ (under the stamp), equations (3) are recovered and the normal displacement is given by

$$u_z = \frac{2(1-2\nu)(C_1+C_2)xy}{a^2-b^2}\left[-\frac{\tanh^{-1}\alpha-\tanh^{-1}(\alpha(\eta/\zeta)^{\frac{1}{2}})}{(b^2+\eta)^{\frac{1}{2}}(b^2+\zeta)^{\frac{1}{2}}}+\frac{\tanh^{-1}\beta-\tanh^{-1}(\beta(\eta/\zeta)^{\frac{1}{2}})}{(a^2+\eta)^{\frac{1}{2}}(a^2+\zeta)^{\frac{1}{2}}}\right], \tag{24d}$$

in which

$$\alpha^2 = \frac{b^2+\zeta}{b^2+\eta}, \quad \beta^2 = \frac{a^2+\zeta}{a^2+\eta}.$$

The resulting values for the shearing stresses are

$$\sigma_{xz} = \left\{\begin{matrix} \dfrac{4\mu C_1}{ab^3}y\left[1-\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}\right]^{-\frac{1}{2}} & \text{for} \quad \xi = 0 \\ 0 & \text{for} \quad \eta = 0 \end{matrix}\right\}, \tag{25a}$$

$$\sigma_{yz} = \left\{\begin{matrix} \dfrac{4\mu C_1}{a^3b}x\left[1-\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}\right]^{-\frac{1}{2}} & \text{for} \quad \xi = 0 \\ 0 & \text{for} \quad \eta = 0 \end{matrix}\right\}. \tag{25b}$$

The torque T which must be applied to maintain the rotation is given by

$$T = \iint_{\Sigma} [x\sigma_{yz}(x,y,0)-y\sigma_{xz}(x,y,0)]\,dx\,dy,$$

where Σ is the region bounded by the ellipse (1). Using equations (25) we find that

$$T = \tfrac{8}{3}\mu\pi(C_2-C_1). \tag{26}$$

The stress field in the vicinity of the base of the stamp is of great practical interest, mainly because of questions of mechanical failure. It is convenient to introduce a rectangular cartesian coordinate system n, t, z such that the origin of this system traverses the periphery of

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the ellipse. The zn-, nt-, and tz- planes are known, respectively, as the normal, rectifying and osculating planes. In this system of coordinates the local shear stresses are found† to be

$$\sigma_{nz} = \frac{N_1}{\sqrt{(2R)}} + O(R^0), \tag{27a}$$

$$\sigma_{tz} = \frac{N_2}{\sqrt{(2R)}} + O(R^0), \tag{27b}$$

where

$$N_1 = \frac{4\mu(C_1+C_2)\sin\phi\cos\phi}{(ab)^{\frac{1}{2}}(a^2\sin^2\phi + b^2\cos^2\phi)^{\frac{3}{4}}},$$

$$N_2 = \frac{4\mu(C_2 b^2\cos^2\phi - C_1 a^2\sin^2\phi)}{(ab)^{\frac{3}{2}}(a^2\sin^2\phi + b^2\cos^2\phi)^{\frac{3}{4}}},$$

and ϕ is the angle appearing in the parametric equations of the ellipse; i.e.

$$x = a\cos\phi, \quad y = b\sin\phi.$$

In equation (27), R is assumed to be small in comparison with a (or b). R is the radial distance in the nz-plane, measured from the edge of the ellipse.

In the limiting case of a stamp with circular cross-section ($a = b$), $K(k) = E(k) = \frac{1}{2}\pi$, equations (23) simplify to

$$C_2 = -C_1 = \frac{\omega a^3}{\pi}.$$

Hence, equation (19) gives $F = 0$, and the solution simplifies considerably to the following:

$$(u_x, u_y, u_z) = \frac{2\omega a^3}{\pi}\left(\frac{\partial I}{\partial y}, -\frac{\partial I}{\partial x}, 0\right),$$

$$(\sigma_{xz}, \sigma_{yz}, \sigma_{zz}) = 2\mu\frac{\omega a^3}{\pi}\frac{\partial}{\partial z}\left(\frac{\partial I}{\partial y}, -\frac{\partial I}{\partial x}, 0\right),$$

$$(\sigma_{xx}, \sigma_{yy}, \sigma_{xy}) = 4\mu\frac{\omega a^3}{\pi}\left(\frac{\partial^2 I}{\partial x\,\partial y}, -\frac{\partial^2 I}{\partial x\,\partial y}, \frac{\partial^2 I}{\partial y^2} - \frac{\partial^2 I}{\partial x^2}\right),$$

where $I(x, y, z)$ is given in (18) with $a = b$ and $x^2 + y^2 = r^2$. Similarly, the expressions (24) (25) and (26), when transferred to the usual system of polar coordinates (r, θ, z), agree with the corresponding values given by Sneddon [9].

3. Prescribed tangential stresses. In this section, we consider the problem of determining the components of stress and displacements in the interior of a semi-infinite elastic solid ($z \leq 0$) when there are prescribed tangential shearing stresses within an elliptical area of the boundary surface $z = 0$. We will consider the case of tangential stress parallel to the x-axis

† The detailed calculations are similar to those in [2].

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only. The case of shear stress parallel to the y-axis may be solved in a similar manner. The following conditions must be satisfied on the boundary $z = 0$.

$$\sigma_{xz} = t(x, y) \qquad (\xi = 0), \tag{28a}$$

$$\sigma_{xz} = 0 \qquad (\eta = 0), \tag{28b}$$

$$\sigma_{yz} = \sigma_{zz} = 0 \qquad (z = 0). \tag{29}$$

Here $t(x, y)$ is the specified shearing stress within the ellipse (1). For simplicity of future analysis let us assume that $t(x, y)$ can be expressed as a function $\tau(Z)$ of the variable Z, where

$$Z = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}. \tag{30}$$

In the basic solution (8), the boundary conditions (28b) and (29) may be satisfied by selecting

$$\frac{\partial f_1}{\partial z} = \int_\xi^\infty \phi[\rho(s)] \frac{ds}{\sqrt{\{Q(s)\}}}, \tag{31}$$

$$f_2 = 0, \tag{32}$$

the notation being the same as before. The boundary condition (28a) may be shown† to lead to an integral equation of Abel type for the unknown function $\phi(\rho)$, which yields

$$\phi(\rho) = \frac{ab}{4\pi\mu} \int_0^\rho \frac{\tau(Z)\,dZ}{\sqrt{(\rho - Z)}}, \tag{33}$$

and the problem is reduced to quadrature.

As an example of the use of this formula, let us consider the case

$$\tau(Z) = \tau_0 Z^n,$$

where τ_0 is constant. Equations (31) and (33) give

$$\frac{\partial f_1}{\partial z} = \frac{ab\tau_0 \Gamma(n+1)}{4\mu(\pi)^{\frac{1}{2}}\Gamma(n+\frac{3}{2})} \int_\xi^\infty [\rho(s)]^{n+\frac{1}{2}} \frac{ds}{\sqrt{\{Q(s)\}}} \qquad (n > -1), \tag{34}$$

in which $\Gamma(n)$ is the usual gamma function. In general, the integral appearing in (34) can be evaluated for $n = \frac{1}{2} + m$, where m is a non-negative integer. In the degenerate case of $a = b$, solutions may be found for any n. For $n = \frac{1}{2}$, using equation (22), we find that

$$\frac{\partial f_1}{\partial z} = \frac{b\tau_0}{4\mu k^2} \left\{ k^2 u - \left(\frac{x}{a}\right)^2 [u - E(u)] - \left(\frac{y}{b}\right)^2 L(u) - \left(\frac{kz}{b}\right)^2 \left[\frac{\operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u} - E(u)\right] \right\},$$

where $L(u)$ is given in (24). Similarly, from equations (6), (7), (22) and (34) we obtain the relations:

† The derivation is similar to a case considered in [3].

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$$u_x(x,y,0) = -\frac{b\tau_0}{2\mu k^4}\left\{k^2\left(k^2 u - \left(\frac{kx}{a}\right)^2[u-E(u)] - \left(\frac{ky}{b}\right)^2 L(u)\right)\right.$$
$$+\nu\left(k^2 k'^2 u - k^2 E(u) + \left(\frac{y}{a}\right)^2[u-E(u)-L(u)]\right.$$
$$\left.\left.-\left(\frac{x}{\alpha}\right)^2[2k'^2 u - (1+k'^2)E(u) + k^2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u]\right)\right\}, \tag{35a}$$

$$u_y(x,y,0) = \frac{\nu b x y \tau_0}{a^2 k^4}[u-E(u)-L(u)], \tag{35b}$$

$$u_z(x,y,0) = \frac{(1-2\nu)\tau_0 a b x(\eta-\zeta)}{8\mu[(\eta+a^2)^2-(\eta-\zeta)(\eta+b^2)]}\left\{\frac{1}{1-\beta^2} - \frac{\eta\zeta}{\zeta^2-\beta^2\eta^2}\right.$$
$$\left.+\frac{1}{2\beta}\log\left|\frac{(1+\beta)(\zeta-\beta\eta)}{(1-\beta)(\zeta+\beta\eta)}\right| - \frac{1}{\alpha}\log\left|\frac{(1+\alpha)(\zeta-\alpha\eta)}{(1-\alpha)(\zeta+\alpha\eta)}\right|\right\} \quad (\xi=0), \tag{35c}$$

$$u_z(x,y,0) = \frac{(1-2\nu)\tau_0 a b x(\xi-\zeta)}{8\mu[(\xi+a^2)^2-(\xi-\zeta)(\xi+b^2)]}$$
$$\times\left\{\frac{1}{1-\beta_1^2} + \frac{1}{2\beta_1}\log\left|\frac{1+\beta_1}{1-\beta_1}\right| - \frac{1}{\alpha_1}\log\left|\frac{1+\alpha_1}{1-\alpha_1}\right|\right\} \quad (\eta=0), \tag{35d}$$

where α and β are given in (24) and

$$\alpha_1^2 = \frac{b^2+\zeta}{b^2+\xi}, \qquad \beta_1^2 = \frac{a^2+\zeta}{a^2+\xi}.$$

For $\xi \to 0$, i.e., points (x, y) within the ellipse (1), $u \to K(k)$, $E(u) \to E(k)$, $\operatorname{sn} K = 0$, $\operatorname{dn} K = k$ and the results agree with those given by Luré [4]. Similar calculations may be carried out for the stresses.

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