J. Inst. Math. Jussieu (2021) **20**(1), 225–276 225
doi:10.1017/S1474748019000161

ON Sp-DISTINGUISHED REPRESENTATIONS OF THE QUASI-SPLIT UNITARY GROUPS

A. MITRA[1] AND OMER OFFEN[2]

[1] *Indian Institute of Science Education and Research, Tirupati, India* (00.arnab.mitra@gmail.com)

[2] *Brandeis University, Waltham, MA, USA* (offen@brandeis.edu)

(Received 18 June 2018; revised 4 March 2019; accepted 10 March 2019; first published online 17 April 2019)

Abstract We study $\mathrm{Sp}_{2n}(F)$-distinction for representations of the quasi-split unitary group $U_{2n}(E/F)$ in $2n$ variables with respect to a quadratic extension E/F of p-adic fields. A conjecture of Dijols and Prasad predicts that no tempered representation is distinguished. We verify this for a large family of representations in terms of the Mœglin–Tadić classification of the discrete series. We further study distinction for some families of non-tempered representations. In particular, we exhibit L-packets with no distinguished members that transfer under base change to $\mathrm{Sp}_{2n}(E)$-distinguished representations of $\mathrm{GL}_{2n}(E)$.

Keywords: representations of p-adic reductive groups; distinguished representations; quadratic base change

2010 *Mathematics subject classification*: Primary 22E50
Secondary 11F70

Contents

1. Introduction

Let G be a p-adic reductive group and H a closed subgroup. A smooth, complex-valued representation (π, V) of G (henceforth simply a representation π of G) is called H-distinguished if there exists a non-zero linear functional ℓ on V such that

$$\ell(\pi(h)v) = \ell(v), \quad h \in H, \ v \in V.$$

Distinguished representations play a central role in harmonic analysis of homogeneous spaces (see, for instance, [5]) and in the study of period integrals of automorphic forms, special values of L-functions and characterization of the image of a functorial transfer. Inspired by the work of Sakellaridis and Venkatesh [36], much attention is given to the case where G/H is spherical and in particular, if G/H is a p-adic symmetric space.

In this work we focus on the symmetric space G/H where $G = U_{2n} = U_{2n}(E/F)$ is the quasi-split unitary group in $2n$ variables with respect to a quadratic extension E/F of p-adic fields and $H = \mathrm{Sp}_{2n}(F)$. The following is conjectured in [6, Conjecture 3].

Conjecture 1 (Dijols–Prasad). An L-packet of irreducible representations of U_{2n} that is associated to an Arthur packet on U_{2n} contains an $\mathrm{Sp}_{2n}(F)$-distinguished member if and only if its base change is an irreducible representation of $\mathrm{GL}_{2n}(E)$ that is $\mathrm{Sp}_{2n}(E)$-distinguished.

We remark that base change is a functorial transfer (in particular, a finite to one map) from irreducible representations of U_{2n} to irreducible representations of $\mathrm{GL}_{2n}(E)$ that takes tempered representations to tempered representations. (See § 10.2 for the definition. In [32] the transfer is referred to as standard base change.) The fibers of the base change map form the L-packets of irreducible representations of U_{2n}.

In particular, it is expected that no tempered representation of U_{2n} is $\mathrm{Sp}_{2n}(F)$-distinguished. (It is well known that no tempered representations of $\mathrm{GL}_{2n}(E)$ are $\mathrm{Sp}_{2n}(E)$-distinguished.) For cuspidal representations this is proved independently in [6] and in [25]. In fact, more generally, if π is an irreducible representation of U_{2n} with non-trivial partial cuspidal support then π is not $\mathrm{Sp}_{2n}(F)$-distinguished. (To say that π has trivial partial cuspidal support means that for a realization of π as a subrepresentation of a representation induced from cuspidal representations on a standard parabolic subgroup, this parabolic subgroup is contained in the Siegel parabolic subgroup.)

Since a similar vanishing result holds for period integrals of cuspidal automorphic representations, a natural approach to local non-distinction, at least for the discrete series, is a globalization argument of invariant linear forms as in [36, § 16]. It would

https://doi.org/10.1017/S1474748019000161 Published online by Cambridge University Press

be interesting to see if invariant linear forms on discrete series representations can be globalized in our context (i.e., realized as local components of the period integral of some cuspidal automorphic representation). However, we do not know how to generalize the argument of loc. cit. to our setting. Instead, we apply the Mœglin–Tadić classification of discrete series [28] and show that many discrete series (and tempered non-discrete) representations of U_{2n} are not $\mathrm{Sp}_{2n}(F)$-distinguished. Exhausting the entire tempered spectrum, however, will require different methods.

We further study distinction for an interesting family of non-tempered representations, namely the base change fiber of the class of Speh representations of $\mathrm{GL}_{2n}(E)$. Recall that Speh representations are the building blocks of the unitary dual of general linear groups [39, Theorem D]. In particular, we exhibit L-packets of irreducible representations of U_{2n} with no $\mathrm{Sp}_{2n}(F)$-distinguished member that transfer under base change to irreducible representations of $\mathrm{GL}_{2n}(E)$ that are $\mathrm{Sp}_{2n}(E)$-distinguished. This does not directly contradict Conjecture 1 since those L-packets are not expected to be associated to Arthur packets.

We now formulate the main results of this work. Mœglin and Tadić classified in [28] the discrete series representations of classical groups in terms of certain combinatorial data. In particular, to an irreducible discrete series representation π of U_{2n} they attach a triple $(\pi_{\mathrm{cusp}}, \mathrm{Jord}(\pi), \epsilon_\pi)$. If the partial cuspidal support π_{cusp} of π is non-trivial, as remarked above, π is not $\mathrm{Sp}_{2n}(F)$-distinguished. We therefore focus our attention on representations π with trivial partial cuspidal support. For the definition of the other components of the triple in this case, see § 6.1. We only remark here that $\mathrm{Jord}(\pi)$ stands for a certain finite set of pairs (ρ, a) attached to π where ρ is a conjugate (with respect to E/F) self-dual cuspidal representation of $\mathrm{GL}_m(E)$ for some $m \in \mathbb{N}$ and $a \in \mathbb{N}$ and with our assumption that π has trivial partial cuspidal support we may think of ϵ_π as a function from $\mathrm{Jord}(\pi)$ to $\{\pm 1\}$.

Theorem 1. *Let π be an irreducible discrete series representation of U_{2n} with trivial partial cuspidal support. Assume that there exists ρ such that the set*

$$\mathrm{Jord}_\rho(\pi) = \{a \in \mathbb{N} : (\rho, a) \in \mathrm{Jord}(\pi)\}$$

is not empty and at least one of the following three properties holds:

(1) *t is odd;*

(2) *$x_{2i-1} > x_{2i} + 2$ for some $i \leqslant t/2$;*

(3) *$\epsilon_\pi(\rho, x_{t+2}) = \epsilon_\pi(\rho, x_{t+1})$.*

Here, we write $\mathrm{Jord}_\rho(\pi) = (x_1, \ldots, x_k)$ where $x_1 > \cdots > x_k$, and let $t = 1$ if $k = 1$, and $t < k$ be such that $\epsilon_\pi(\rho, x_i) \neq \epsilon_\pi(\rho, x_{i+1})$, $i = 1, \ldots, t-1$ and $\epsilon_\pi(\rho, x_t) = \epsilon_\pi(\rho, x_{t+1})$, otherwise. Then π is not $\mathrm{Sp}_{2n}(F)$-distinguished.

As explained in § 10.2.6, the L-packet of an irreducible discrete series representation π of U_{2n} is determined by $\mathrm{Jord}(\pi)$. We emphasize that in the above theorem condition (2) gives non-distinction for entire L-packets.

We further remark that in light of [14, Proposition 3.1] we obtain, in particular, that no irreducible generic discrete series representation of U_{2n} is $\mathrm{Sp}_{2n}(F)$-distinguished and the same holds for the strongly positive discrete series. (In fact, a strongly positive discrete series with generic partial cuspidal support is generic.) However, it was proved in [18] more generally that no generic irreducible representation of U_{2n} is $\mathrm{Sp}_{2n}(F)$-distinguished applying standard techniques of invariant distributions.

To formulate the next result we recall the definition of a Speh representation. Let d, $m \in \mathbb{N}$, $n = dm$ and δ an irreducible essentially square-integrable representation of $\mathrm{GL}_d(E)$. The Speh representation $U(\delta, m)$ is the Langlands quotient of the representation of $\mathrm{GL}_n(E)$ parabolically induced from

$$|\det|^{\frac{m-1}{2}}\delta \otimes |\det|^{\frac{m-3}{2}}\delta \otimes \cdots \otimes |\det|^{\frac{1-m}{2}}\delta.$$

We recall that when n is even, the Speh representation $U(\delta, m)$ is $\mathrm{Sp}_n(F)$-distinguished if and only if m is even (see, for instance, [26, Theorem 10.3]).

For a representation π of $\mathrm{GL}_n(E)$ we view π as a representation of the Siegel Levi subgroup of U_{2n} and denote by $\pi \rtimes \mathbf{1}_0$ the representation of U_{2n} parabolically induced from π.

We remark that if an irreducible representation of $\mathrm{GL}_{2n}(E)$ is in the image of base change then it is, in particular, conjugate self-dual.

Theorem 2. *Let $\pi = U(\delta, m)$ be a Speh representation of $\mathrm{GL}_{2n}(E)$ that is conjugate self-dual.*

(1) *If m is odd (i.e., if π is not $\mathrm{Sp}_{2n}(E)$-distinguished) then π is in the image of base change if and only if δ is so (if and only if the Langlands parameter of δ is conjugate symplectic in the sense of [10, § 3]). In this case, there is a unique irreducible representation of U_{2n} that transfers to π under base change and it is not $\mathrm{Sp}_{2n}(F)$-distinguished.*
(2) *If m is even (i.e., if π is $\mathrm{Sp}_{2n}(E)$-distinguished) then there is a unique representation π_0 of U_{2n} that transfers to π under base change. Suppose further that $\pi \rtimes \mathbf{1}_0$ is irreducible. Then, π_0 is $\mathrm{Sp}_{2n}(F)$-distinguished if and only if m is divisible by 4.*

The condition that $\pi \rtimes \mathbf{1}_0$ is irreducible is explicated in Corollary 6. A Speh representation is a special case of a *ladder* representation (see § 8.3). In fact, Proposition 9 is a generalization of Theorem 2(2) to ladder representations.

Since any irreducible representation of U_{2n} occurs as a unique irreducible quotient of its standard module, a step toward classification of the distinguished smooth dual, is the classification of distinguished standard modules. We treat the special case where the standard module is induced from a Levi subgroup of the Siegel Levi subgroup of U_{2n} and the GL-part is irreducible.

Theorem 3. *Let δ_i be an irreducible essentially square-integrable representation of $\mathrm{GL}_{n_i}(E)$, $i = 1, \ldots, t$ such that $\exp(\delta_1) \geqslant \cdots \geqslant \exp(\delta_t) > 0$ (see § 8.1.10) and the representation π of $\mathrm{GL}_n(E)$ (with $n = n_1 + \cdots + n_t$) parabolically induced from $\delta_1 \otimes \cdots \otimes \delta_t$ is irreducible. Then the standard module $\pi \rtimes \mathbf{1}_0$ of U_{2n} is $\mathrm{Sp}_{2n}(F)$-distinguished if and only if the irreducible generic representation $\nu^{-1/2}\pi$ of $\mathrm{GL}_n(E)$ is $\mathrm{GL}_n(F)$-distinguished. When this is the case, if in addition $\exp(\delta_i) \in \frac{1}{2}\mathbb{Z}$, $i = 1, \ldots, t$, then $\nu^{-1/2}\pi$ is tempered.*

Next we outline the structure of the paper and the methods we use to prove the above theorems. An explicit version of the geometric lemma [4, Theorem 5.2] (adapted in [34] to restriction to H of parabolically induced representations of G when G/H is a p-adic symmetric space) plays a key role in the proof of all the above mentioned results.

After setting up the notation in § 2 and recalling some preliminary results in § 3 we explicate in § 4 certain necessary conditions for a parabolically induced representation of U_{2n} to be $\mathrm{Sp}_{2n}(F)$-distinguished.

In § 6 we prove Theorem 1. More precisely, we prove in Theorem 7 that if π is a representation of U_{2n} that satisfies the hypothesis of Theorem 1 then its contragredient $\pi^\vee$ is not $\mathrm{Sp}_{2n}(F)$-distinguished. However, based on the realization of the contragredient via the principal involution of U_{2n} [29, § 4, II.1], we observe in Corollary 1 that an irreducible representation π of U_{2n} is $\mathrm{Sp}_{2n}(F)$-distinguished if and only if $\pi^\vee$ is so. Theorem 1 immediately follows.

Our proof of Theorem 7 is based on the results of [28] that we recall in § 6.1. For an irreducible discrete series representation π of U_{2n} with trivial partial cuspidal support, Mœglin and Tadić provide a recipe to realize π as a subrepresentation of a certain representation parabolically induced from an essentially square-integrable representation of a Levi subgroup of the Siegel Levi of U_{2n}. The realization is read off the triple $(\pi_{\mathrm{cusp}}, \mathrm{Jord}(\pi), \epsilon_\pi)$. In fact, there are different ways to realize π in this way each one depends on a certain path with origin ϵ_π in a graph that we discuss in § 5.

For distinction problems, it is more convenient to realize π as a quotient. Hence, we dualize. It is also important to consider the possible different realizations (of $\pi^\vee$ as a quotient). We prove that $\pi^\vee$ is not $\mathrm{Sp}_{2n}(F)$-distinguished by proving that it is the quotient of some induced representation that is not $\mathrm{Sp}_{2n}(F)$-distinguished. For this sake, we study in § 5 a graph that underlies the combinatorics behind ϵ_π and the different realizations of π in induced representations. This section is purely combinatorial and independent of the rest of this work. We hope that it will be useful for other problems where explicit realizations of discrete series representations of classical groups play a role.

In § 7 we treat tempered representations that are not in the discrete series. In Proposition 5 we prove that many tempered irreducible representations of U_{2n} are not $\mathrm{Sp}_{2n}(F)$-distinguished. Our treatment is based on the description of the tempered spectrum in [28, § 13]. We remark, that in any case treated in Proposition 5 non-distinction is proved for an entire L-packet of tempered representations. This observation is based on the description of tempered L-packets that we recall in §§ 10.2.6 and 10.2.7.

In order to address the other results of the paper we recall some further preliminaries. In § 8 we recall the representation theory of $\mathrm{GL}_n(E)$, Zelevinsky's segment notation, Langlands parameters and the reciprocity map. In § 9 we deduce further applications of the geometric lemma that are applied in the sequel. In particular, we provide in Lemma 14 a sufficient condition for a representation of U_{2n} to be $\mathrm{Sp}_{2n}(F)$-distinguished. In § 10 we recall Mok's reciprocity map and the properties of the base change transfer.

We prove Theorem 2 in § 11 (see Theorem 10). It is an immediate consequence of Propositions 9 and 10 that are formulated more generally in the language of ladder representations. Finally, in § 12 we prove Theorem 3 (see Theorem 11 and Remark 7).

Remark 1. Throughout the paper we assume that F has characteristic zero. This assumption is made since it is used in some of the results we rely on, namely the Mœglin–Tadić classification (and in particular, their basic assumption) [28], the structure of the discrete L-packet [27] and Mok's reciprocity map [32].

2. Notation

We set the general notation in this section. More particular notation is defined in the section where it first occurs.

2.1. The relevant groups

2.1.1. Let F be a non-Archimedean local field of characteristic zero with normalized absolute value $|\cdot|_F$ and E/F a quadratic extension. Thus,

$$|a|_E = |a|_F^2, \quad a \in F.$$

We denote by $a \mapsto \bar{a}$ the action of non-trivial element of $\mathrm{Gal}(E/F)$.

2.1.2. We use bold letters such as $\mathbf{X}$ to denote varieties defined over F and the corresponding usual font to denote the topological space of its rational points. That is, $X = \mathbf{X}(F)$.

2.1.3. For a variety $\mathbf{Y}$ defined over E let $\mathbf{Res_{E/F}(Y)}$ denote its restriction of scalars from E to F.

2.1.4. Let $\mathbf{U_{2n}} = \mathbf{U_{2n}(E/F)}$ be the quasi-split unitary group in $2n$ variables associated to the anti-Hermitian form defined by

$$J_n = \begin{pmatrix} & w_n \\ -w_n & \end{pmatrix}$$

where $w_n = (\delta_{i,n+1-j}) \in \mathrm{GL}_n$. Explicitly,

$$\mathbf{U}_{2n} = \{g \in \mathbf{Res_{E/F}}((\mathbf{GL_{2n}})_\mathbf{E}) \mid {}^t\bar{g}\, J_n g = J_n\}.$$

2.1.5. Let $\mathbf{Sp_{2n}}$ denote the symplectic group of rank n defined by the skew-symmetric matrix J_n, i.e.,

$$\mathbf{Sp_{2n}} = \{g \in \mathbf{GL_{2n}} \mid {}^t g J_{2n} g = J_{2n}\}.$$

Note that

$$\mathbf{Sp_{2n}} = \{g \in \mathbf{U_{2n}} \mid \bar{g} = g\}.$$

2.1.6. Standard parabolic subgroups of $\mathbf{U_{2n}}$ are in bijection with tuples $\alpha = (n_1, \ldots, n_r; m)$ such that $n_1 + \cdots + n_r + m = n$, r, $m \in \mathbb{Z}_{\geqslant 0}$ and $n_1, \ldots, n_r \in \mathbb{N}$. The standard parabolic subgroup $\mathbf{Q}_\alpha = \mathbf{L}_\alpha \ltimes \mathbf{V}_\alpha$ with standard Levi subgroup $\mathbf{L}_\alpha$ and unipotent radical $\mathbf{V}_\alpha$ is the subgroup consisting of block upper triangular matrices in $\mathbf{U_{2n}}$ so that

$$\mathbf{L}_\alpha = \{\mathrm{diag}(g_1, \ldots, g_r, h, g_r^*, \ldots, g_1^*) \mid g_i \in \mathbf{Res_{E/F}}((\mathbf{GL_{n_i}})_\mathbf{E}),\ i = 1, \ldots, r,\ h \in \mathbf{U_{2m}}\}.$$

https://doi.org/10.1017/S1474748019000161 Published online by Cambridge University Press

Here $g \mapsto g^*$ is the involution on $\mathbf{Res_{E/F}}((\mathbf{GL_k})_\mathbf{E})$ defined for any $k \in \mathbb{N}$ by $g^* = w_k{}^t\bar{g}^{-1}w_k$. In particular, $\mathbf{U_{2n}} = \mathbf{Q_{(\ ;n)}} = \mathbf{L_{(\ ;n)}}$ corresponds to $r = 0$ and $m = n$ while $\mathbf{Q_{(n;0)}}$ is the Siegel parabolic subgroup.

Let $\iota_\alpha : \mathrm{Res}_{\mathbf{E/F}}((\mathrm{GL}_{\mathbf{n_1}})_\mathbf{E}) \times \cdots \times \mathrm{Res}_{\mathbf{E/F}}((\mathrm{GL}_{\mathbf{n_r}})_\mathbf{E}) \times \mathbf{U_{2m}} \to L_\alpha$ be the isomorphism defined by

$$\iota_\alpha(g_1, \ldots, g_r, h) = \mathrm{diag}(g_1, \ldots, g_r, h, g_r^*, \ldots, g_1^*).$$

If $m = 0$ (i.e., if $\mathbf{Q}_\alpha$ is contained in the Siegel parabolic subgroup) we simply write $\iota_\alpha(g_1, \ldots, g_r)$ and when clear from the context we often omit the subscript α.

2.1.7. If $\mathbf{P} = \mathbf{M} \ltimes \mathbf{U}$ is a standard parabolic subgroup of $\mathbf{U_{2n}}$ with its standard Levi decomposition, by taking rational points we refer to P (respectively M) as a standard parabolic (reps. Levi) subgroup of U_{2n}.

2.1.8. Let

$$T = \{\mathrm{diag}(a_1, \ldots, a_n, a_n^{-1}, \ldots, a_1^{-1}) \mid a_1, \ldots, a_n \in F^*\}$$

be the diagonal maximal split torus in U_{2n}.

For a standard Levi subgroup M of U_{2n} let $R(M, T)$ be the corresponding root system and let $R^+(M, T)$ (respectively Δ^M) be the set of positive (respectively simple) roots with respect to the standard Borel subgroup $M \cap Q_{(1^{(n)};0)}$ of M consisting of upper triangular matrices in M. Here $1^{(n)}$ is the n-tuple of ones.

2.1.9. More generally, let $P = M \ltimes U \subseteq Q = L \ltimes V$ be standard parabolic subgroups of U_{2n} with their standard Levi decomposition and T_M the connected component of the center of M. We denote by $R(L, T_M)$ the roots of L on T_M and by $R^+(L, T_M)$ (respectively Δ_M^L) the subset of positive (respectively simple) roots with respect to $L \cap P$. The set Δ_M^L consists of non-zero restrictions to T_M of elements of Δ^L. For $\alpha \in R(L, T_M)$ we write $\alpha > 0$ if $\alpha \in R^+(L, T_M)$ and $\alpha < 0$ otherwise. When $L = U_{2n}$ we also set $\Delta_M = \Delta_M^L$.

2.1.10. For a standard Levi subgroup M of U_{2n} let

$$W^M = N_M(T)/C_M(T)$$

be the Weyl group of M. In particular, the Weyl group $W^{U_{2n}}$ of U_{2n} is isomorphic to the signed permutation group in n letters.

Note that $C_M(T) = L_{(1^{(n)};0)}$ is the standard Levi subgroup of the standard Borel subgroup of U_{2n} corresponding to the decomposition $(1^{(n)}; 0) = (1, \ldots, 1; 0)$ of n. Therefore, W^M is a subgroup of W^L whenever $M \subseteq L$ are standard Levi subgroups of U_{2n}.

There is a unique element of maximal length in the set of $w \in W^L$ satisfying

- w is of minimal length in wW^M;
- wMw^{-1} is a standard Levi subgroup of L.

We denote this element by w_M^L. It also satisfies $w_M^L\alpha < 0$ for $\alpha \in R^+(L, T_M)$.

2.1.11. Let $W = W^{U_{2n}}$. For standard Levi subgroups M and L of U_{2n} any double coset in $W^M\backslash W/W^L$ admits a unique element of minimal length. Denote by ${}_MW_L$ the set of $w \in W$ such that w is of minimal length in $W^M wW^L$. The set ${}_MW_L$ consists of all left

https://doi.org/10.1017/S1474748019000161 Published online by Cambridge University Press

M-reduced and right L-reduced elements in W. That is, $w\alpha \in R^+(U_{2n}, T)$ for all $\alpha \in \Delta^L$ and $w^{-1}\alpha \in R^+(U_{2n}, T)$ for all $\alpha \in \Delta^M$. The inclusion ${}_M W_L \subseteq W$ gives a natural bijection

$$ {}_M W_L \simeq W^M \backslash W / W^L. $$

2.1.12. Let $P = M \ltimes U$ and $Q = L \ltimes V$ be standard parabolic subgroups of U_{2n} with their standard Levi decompositions. The Bruhat decomposition is the bijection

$$ w \mapsto PwQ : {}_M W_L \xrightarrow{\sim} P \backslash U_{2n} / Q. $$

For every $w \in {}_M W_L$ the group

$$ P(w) = M \cap wQw^{-1} $$

is a standard parabolic subgroup of M (that is, $P(w) \supseteq M \cap Q_{(1^{(n)};0)}$). Its standard Levi decomposition is given by $P(w) = M(w) \ltimes U(w)$, where

$$ M(w) = M \cap wLw^{-1} \quad \text{and} \quad U(w) = M \cap wVw^{-1}. $$

2.2. Representations

Let $\mathbf{G}$ be a linear algebraic group defined over F. By a representation of G we always mean a smooth complex-valued representation.

Let $\Pi(G)$ be the subcategory of representations of G of finite length and $\mathrm{Irr}(G)$ the class of irreducible representations in $\Pi(G)$.

If in addition $\mathbf{G}$ is a connected reductive group let $\mathrm{Cusp}(G)$ be the set of all cuspidal representations in $\mathrm{Irr}(G)$.

Let $\pi^\vee$ denote the contragredient of a representation $\pi \in \Pi(G)$. Then $(\pi^\vee)^\vee \cong \pi$ and $\pi \in \mathrm{Irr}(G)$ if and only if $\pi^\vee \in \mathrm{Irr}(G)$.

2.2.1. Let δ_G be the modulus function of G with the convention that $\delta_G(g)dg$ is a right-invariant Haar measure if dg is a left-invariant Haar measure on G.

2.2.2. Let $\mathbf{H}$ be a subgroup of $\mathbf{G}$ and σ a smooth, complex-valued representation of H. We denote by $\mathrm{Ind}_H^G(\sigma)$ the normalized induced representation. It is the representation of G by right translations on the space of functions f from G to the space of σ satisfying

$$ f(hg) = (\delta_H \delta_G^{-1})^{1/2}(h)\sigma(h) f(g), \quad h \in H,\ g \in G $$

and f is right-invariant by some open subgroup of G.

The representation of G on the subspace of functions with compact support modulo H is denoted by $\mathrm{ind}_H^G(\sigma)$.

2.2.3. Let $\mathbf{P} = \mathbf{M} \ltimes \mathbf{U}$ be a parabolic subgroup of $\mathbf{G}$ with Levi part $\mathbf{M}$ and unipotent radical $\mathbf{U}$. The functor $\mathbf{i}_{G,M} : \Pi(M) \to \Pi(G)$ of normalized parabolic induction is defined as follows. For $\rho \in \Pi(M)$ we consider ρ as a representation of P by composing with the projection $P/U \to M$ and set

$$ \mathbf{i}_{G,M}(\rho) = \mathrm{Ind}_P^G(\rho). $$

The functor $\mathbf{i}_{G,M}$ is exact and we have

$$ \mathbf{i}_{G,M}(\rho)^\vee \cong \mathbf{i}_{G,M}(\rho^\vee). \tag{1} $$

2.2.4. In the notation of § 2.2.3, the functor $\mathbf{i}_{G,M}$ admits a left adjoint, namely, the normalized Jacquet functor $\mathbf{r}_{M,G} : \Pi(G) \to \Pi(M)$. For $\sigma \in \Pi(G)$, $\mathbf{r}_{M,G}(\sigma)$ is the representation of M on the space of U-coinvariants of σ induced by the action $\delta_P^{-1/2}\sigma$. It is also an exact functor and for $\sigma \in \Pi(G)$ and $\rho \in \Pi(M)$ we have the natural linear isomorphism (Frobenius reciprocity):

$$\mathrm{Hom}_G(\sigma, \mathbf{i}_{G,M}(\rho)) \cong \mathrm{Hom}_M(\mathbf{r}_{M,G}(\sigma), \rho). \tag{2}$$

When G is either $\mathrm{GL}_n(E)$ or U_{2n} we will further use the following standard notation for parabolic induction.

2.2.5. Let $G = \mathrm{GL}_n(E)$, $n = n_1 + \cdots + n_k$ and

$$M = \{\mathrm{diag}(g_1, \ldots, g_k) \mid g_i \in \mathrm{GL}_{n_i}(E),\ i = 1, \ldots, k\}.$$

For representations $\rho_i \in \Pi(\mathrm{GL}_{n_i}(E))$, $i = 1, \ldots, k$ let $\rho = \rho_1 \otimes \cdots \otimes \rho_k \in \Pi(M)$ and

$$\rho_1 \times \cdots \times \rho_k = \mathbf{i}_{G,M}(\rho).$$

2.2.6. Let $G = U_{2n}$, $\alpha = (n_1, \ldots, n_k; m)$ a composition of n as in § 2.1.6 and $M = L_\alpha$. For representations $\rho_i \in \Pi(\mathrm{GL}_{n_i}(E))$, $i = 1, \ldots, k$ and $\sigma \in \Pi(U_{2m})$ let $\rho = \rho_1 \otimes \cdots \otimes \rho_k \otimes \sigma \in \Pi(M)$ and

$$\rho_1 \times \cdots \times \rho_k \rtimes \sigma = \mathbf{i}_{G,M}(\rho).$$

2.2.7. We state basic properties of parabolic induction for the group U_{2n} which we will use several times in the sequel, often without further reference. For their proof, see, for instance, [40, Propositions 4.1 and 4.2].[1] Let $[\pi]$ denote the semi-simplification of a representation $\pi \in \Pi(U_{2n})$.

Proposition 1. *Let* $\pi_i \in \Pi(\mathrm{GL}_{n_i}(E))$, $i = 1, 2$ *and* $\sigma \in \Pi(U_{2n})$ *for some* $n_1, n_2, n \in \mathbb{N}$. *We have*

(1) $(\pi_1 \times \pi_2) \rtimes \sigma \cong \pi_1 \rtimes (\pi_2 \rtimes \sigma)$,
(2) $(\pi_1 \rtimes \sigma)^\vee \cong \pi_1^\vee \rtimes \sigma^\vee$,
(3) $[\overline{\pi_1}^\vee \rtimes \sigma] = [\pi_1 \rtimes \sigma]$.

2.2.8. This paper is concerned with distinction of representations in the following sense.

Definition 1. Let π be a representation of G and $\mathbf{H}$ a subgroup of $\mathbf{G}$.

- We say that π is H-*distinguished* if there exists a non-zero H-invariant linear form ℓ on the space of π, i.e., $\ell(\pi(h)v) = \ell(v)$ for all $h \in H$ and v in the space of π. We denote by $\mathrm{Hom}_H(\pi, 1)$ the space of H-invariant linear forms on π.
- More generally, for a character χ of H we say that π is (H, χ)-distinguished if the space $\mathrm{Hom}_H(\pi, \chi)$ of H-equivariant linear forms on π is non-zero.

[1] Strictly speaking, this reference is only for symplectic groups but the proof works for every classical group. See also [28].

https://doi.org/10.1017/S1474748019000161 Published online by Cambridge University Press

By Frobenius reciprocity [3, Theorem 2.28] we have a natural linear isomorphism

$$\mathrm{Hom}_H(\pi, \chi\delta_H^{1/2}\delta_G^{-1/2}) \cong \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G(\chi)). \tag{3}$$

2.2.9. We set some conventions for particular cases of distinction relevant to us. We primarily consider $\mathrm{Sp}_{2n}(F)$-distinguished representations of U_{2n} and $\mathrm{Sp}_{2n}(E)$-distinguished representations of $\mathrm{GL}_{2n}(E)$. In both cases, for simplicity, we say that the representation is Sp-distinguished.

Again for the sake of notational simplification, we say that a representation of $\mathrm{GL}_n(E)$ is $\mathrm{GL}(F)$-distinguished if it is $\mathrm{GL}_n(F)$-distinguished.

3. Preliminaries

3.1. Involutions on Weyl groups

Let W be a Weyl group of a root system with a basis Δ of simple roots. Let $s_\alpha \in W$ be the simple reflection associated to $\alpha \in \Delta$. Denote by $W[2] = \{w \in W : w^2 = e\}$ the set of involutions in W.

Based on Springer's treatment of involutions in [38] we define a directed Δ-labeled graph $\mathfrak{G}_W$ with vertices $W[2]$ and labeled edges $w \overset{\alpha}{\to} w'$ whenever $w' = s_\alpha w s_\alpha$ and $w\alpha \neq \pm\alpha$. Note that if $w \overset{\alpha}{\to} w'$ then also $w' \overset{\alpha}{\to} w$. If $w_0, \dots, w_k \in W[2]$ and $\alpha_1, \dots, \alpha_k \in \Delta$ are such that $w_{i-1} \overset{\alpha_i}{\to} w_i$, $i = 1, \dots, k$ we also write $w_0 \overset{\sigma}{\curvearrowright} w_k$ with $\sigma = s_{\alpha_k} \cdots s_{\alpha_1}$ for a path on the graph (the notation suppresses the dependence on the chosen word for σ). Note that if $w \overset{\sigma}{\curvearrowright} w'$ then in particular $\sigma w \sigma^{-1} = w'$.

Definition 2. An involution $w \in W[2]$ is called minimal if it is the longest element of the Weyl group generated by $\{s_\alpha : \alpha \in \Pi\}$ for some subset $\Pi \subseteq \Delta$ and $w\alpha = -\alpha$ for $\alpha \in \Pi$.

Example 1. The minimal involutions in the symmetric group S_n (with respect to the standard basis $\Delta = \{e_i - e_{i+1} : i = 1, \dots, n-1\}$) are the products of disjoint simple reflections. That is, $w = s_{i_1} \cdots s_{i_k}$ where $s_i = (i, i+1)$ and $\{i_v, i_{v+1}\}$ is disjoint from $\{i_u, i_u + 1\}$ for any $1 \leqslant u \neq v \leqslant k$.

For the following result of Springer see [38, Proposition 3.3].

Lemma 1. *For any $w \in W[2]$ there exists a minimal involution $w' \in W[2]$ and $\sigma \in W$ such that $w \overset{\sigma}{\curvearrowright} w'$.*

Let $\mathfrak{W}_n$ be the signed permutation group in n variables. We realize it as

$$\mathfrak{W} = \mathfrak{W}_n = S_n \ltimes \Xi_n$$

where Ξ_n is the group of subsets of $\{1, \dots, n\}$ with symmetric difference as multiplication. Clearly, $\Xi_n \cong (\mathbb{Z}/2\mathbb{Z})^n$ and we may take $\sigma_i = \{i\}$, $i = 1, \dots, n$ as generations of Ξ_n.

The action of S_n on Ξ_n is given by

$$\tau\mathfrak{c}\tau^{-1} = \tau(\mathfrak{c}), \quad \tau \in S_n,\ \mathfrak{c} \in \Xi_n.$$

It is easy to see that the set of involutions in $\mathfrak{W}$ is

$$\mathfrak{W}[2] = \{\tau\mathfrak{c} : \tau \in S_n[2],\ \tau(\mathfrak{c}) = \mathfrak{c}\}.$$

We think of $\mathfrak{W}$ as a Weyl group of a root system of type C_n with the standard basis of simple roots $\Delta_n = \{\alpha_1, \dots, \alpha_n\}$ where $\alpha_i = e_i - e_{i+1}$, $i = 1, \dots, n-1$ and $\alpha_n = 2e_n$. Here $\mathfrak{W}$ acts by

$$\tau(e_i) = e_{\tau(i)}, \quad \tau \in S_n \quad \text{and} \quad \mathfrak{c}(e_i) = \begin{cases} e_i & i \notin \mathfrak{c} \\ -e_i & i \in \mathfrak{c} \end{cases}, \quad \mathfrak{c} \in \Xi_n,\ i = 1, \dots, n.$$

Here we examine certain properties of the graph $\mathfrak{G}_{\mathfrak{W}}$. The set of minimal involutions in $\mathfrak{W}$ is

$$\{\rho\mathfrak{c}_k : 0 \leqslant k \leqslant n,\ \rho \text{ is a minimal involution of } S_k,\ \mathfrak{c}_k = \mathfrak{c}_{k,n} = \{k+1, \dots, n\}\}. \tag{4}$$

Here S_k is embedded in S_n as the subgroup of permutations fixing $k+1, \dots, n$ point-wise. Also, by convention, $\mathfrak{c}_{n,n} = \emptyset$. Thus, the minimal involutions in S_n are the minimal involutions of $\mathfrak{W}$ associated to $k = n$.

Let $w = \tau\mathfrak{c} \in \mathfrak{W}$ with $\tau \in S_n$ and $\mathfrak{c} \in \Xi_n$. Define the following subsets of $\{1, \dots, n\}$:

- $\mathfrak{c}_+(w) = \{i \in \mathfrak{c} \mid \tau(i) = i\}$;
- $\mathfrak{c}_-(w) = \{i \notin \mathfrak{c} \mid \tau(i) = i\}$;
- $\mathfrak{c}_{\neq}(w) = \{i \mid \tau(i) \neq i\}$;
- $\mathfrak{c}_<(w) = \{i \mid i < \tau(i)\}$.

Note that $\mathfrak{c}_{\neq}(w)$ and $\mathfrak{c}_<(w)$ depend only on τ.

Example 2. If $w = \rho\mathfrak{c}_{k,n} \in \mathfrak{W}$ is a minimal involution as in (4) then according to Example 1 we have $\mathfrak{c}_+(w) = \mathfrak{c}_{k,n}$ while $\{1, \dots, k\}$ is the disjoint union

$$\{1, \dots, k\} = \mathfrak{c}_-(w) \sqcup \bigsqcup_{i \in \mathfrak{c}_<(w)} \{i, i+1\}$$

and

$$\rho = \prod_{i \in \mathfrak{c}_<(w)} s_i.$$

We make the following simple observations.

Lemma 2. *Let $w, \sigma \in \mathfrak{W}$ and $w' = \sigma w \sigma^{-1}$. Then*

(1) $\sigma\mathfrak{c}_+(w)\sigma^{-1} = \mathfrak{c}_+(w')$ *(in particular, $|\mathfrak{c}_+(w')| = |\mathfrak{c}_+(w)|$),*
(2) $\sigma\mathfrak{c}_-(w)\sigma^{-1} = \mathfrak{c}_-(w')$ *and*
(3) $\sigma\mathfrak{c}_{\neq}(w)\sigma^{-1} = \mathfrak{c}_{\neq}(w')$.

On the left hand side of each equation multiplication is in $\mathfrak{W}$.

https://doi.org/10.1017/S1474748019000161 Published online by Cambridge University Press

Proof. Let $w = \tau\mathfrak{c}$ and $w' = \tau'\mathfrak{c}'$ with $\tau, \tau' \in S_n$ and $\mathfrak{c}, \mathfrak{c}' \in \Xi_n$. It is enough to prove the lemma for σ in a set of generators for $\mathfrak{W}$; hence in the two cases: $\sigma \in S_n$ and $\sigma \in \Xi_n$.

Assume first that $\sigma \in S_n$. Then $\tau' = \sigma\tau\sigma^{-1}$ and therefore $\mathfrak{c}_{\neq}(w') = \sigma(\mathfrak{c}_{\neq}(w)) = \sigma\mathfrak{c}_{\neq}(w)\sigma^{-1}$ (hence the equality (3)) and $\mathfrak{c}_+(w') \cup \mathfrak{c}_-(w') = \sigma(\mathfrak{d}) = \sigma\mathfrak{d}\sigma^{-1}$ where $\mathfrak{d} = \mathfrak{c}_+(w) \cup \mathfrak{c}_-(w) = \{i \mid \tau(i) = i\}$. Since furthermore, $\mathfrak{c}' = \sigma(\mathfrak{c})$ and $\mathfrak{c}_+(w) = \mathfrak{c} \cap \mathfrak{d}$ the equalities (1) and (2) also follow.

Assume now that $\sigma \in \Xi_n$. Then $\tau' = \tau$ and it follows immediately that $\mathfrak{c}_{\neq}(w') = \mathfrak{c}_{\neq}(w) = \sigma\mathfrak{c}_{\neq}(w)\sigma^{-1}$ (hence the equality (3)). Furthermore, $\mathfrak{c}' = \tau^{-1}(\sigma)\sigma\mathfrak{c}$. Since the product in Ξ_n is the symmetric difference it is clear that the set $\tau^{-1}(\sigma)\sigma$ contains no fixed points of τ. Therefore, $\mathfrak{c}_+(w') = \mathfrak{c}_+(w) = \sigma\mathfrak{c}_+(w)\sigma^{-1}$ which gives the equality (1). The equality (2) follows from (3) and (1). □

Lemma 3. *Let $\sigma \in \mathfrak{W}$ and $w, w' \in \mathfrak{W}[2]$ be such that $w \overset{\sigma}{\curvearrowright} w'$. Then*

$$\sigma\mathfrak{c}_<(w)\sigma^{-1} = \mathfrak{c}_<(w').$$

Proof. It follows from Lemma 2(3) that the two sets have the same cardinality and it is therefore enough to show that $\sigma\mathfrak{c}_<(w)\sigma^{-1} \subseteq \mathfrak{c}_<(w')$. By induction on the length of the path defined by σ it is enough to prove the lemma in the case that σ is a simple reflection. Assume $\sigma = s_i = s_{\alpha_i}$ for $i < n$ and assume by contradiction that $s_i\mathfrak{c}_<(w)s_i \not\subseteq \mathfrak{c}_<(w')$. Writing $w = \tau\mathfrak{c}$ with $\tau \in S_n$ and $\mathfrak{c} \in \Xi_n$ we have $w' = \tau'\mathfrak{c}'$ where $\tau' = s_i\tau s_i$ and $\mathfrak{c}' = s_i(\mathfrak{c})$. Our assumption by contradiction (together with Lemma 2(3)) means that there exists $j < \tau(j)$ such that $s_i(j) > \tau'(s_i(j))$. Since $\tau'(s_i(j)) = s_i(\tau(j))$, this means that $j < \tau(j)$ but $s_i(j) > s_i(\tau(j))$. Hence $j = i$ and $\tau(j) = i + 1$. Thus, $i \in \mathfrak{c}$ if and only if $i + 1 \in \mathfrak{c}$ and therefore

$$w(e_i - e_{i+1}) = \tau\mathfrak{c}(e_i - e_{i+1}) = \begin{cases} e_i - e_{i+1} & i \in \mathfrak{c} \\ e_{i+1} - e_i & i \notin \mathfrak{c} \end{cases}$$

which contradicts $w \xrightarrow{\alpha_i} w'$.

Assume now that $\sigma = \sigma_n = s_{\alpha_n}$ and recall that $\mathfrak{c}_<(w) = \sigma_n\mathfrak{c}_<(w)\sigma_n$. Writing $w = \tau\mathfrak{c}$ with $\tau \in S_n$ and $\mathfrak{c} \in \Xi_n$ we have $w' = \tau\mathfrak{c}'$ where $\mathfrak{c}' = \tau^{-1}(\sigma_n)\sigma_n\mathfrak{c}$ and in particular $\mathfrak{c}_<(w) = \mathfrak{c}_<(w')$. The lemma follows. □

3.2. The symmetric space

To a connected reductive group $\mathbf{G}$ and an involution θ on $\mathbf{G}$ both defined over F we associate the symmetric space

$$\mathbf{X} = \mathbf{X}(\mathbf{G}, \theta) = \{g \in \mathbf{G} : g\theta(g) = e\}$$

with the $\mathbf{G}$-action by θ-twisted conjugation

$$(g, x) \mapsto g \cdot x = gx\theta(g)^{-1}, \quad g \in \mathbf{G},\ x \in \mathbf{X}.$$

For every $x \in \mathbf{X}$, let $\mathbf{G_x} = \mathrm{Stab}_{\mathbf{G}}(x)$ be its stabilizer, an algebraic group defined over F. More generally, for any subgroup $\mathbf{Q}$ of $\mathbf{G}$ let $\mathbf{Q_x} = \{g \in \mathbf{Q} \mid g \cdot x = x\}$.

https://doi.org/10.1017/S1474748019000161 Published online by Cambridge University Press

Example 3. We provide two well-known examples where X consists of a unique G-orbit.

(1) Let $\mathbf{G} = \mathbf{GL_{2n}}$ and $\theta(g) = J_n {}^t g^{-1} J_n^{-1}$ (so that $\mathbf{G_e} = \mathbf{Sp_{2n}}$). Then

$$\mathbf{X} = \{g \in \mathbf{G} : {}^t(gJ_n) = -gJ_n\}$$

and it is well known that $X = G \cdot e$.

(2) Let $\mathbf{G} = \mathbf{Res_{E/F}}((\mathbf{GL_n})_\mathbf{E})$ and $\theta(g) = \bar{g}$ (so that $G = \mathrm{GL}_n(E)$ and $G_e = \mathrm{GL}_n(F)$). Then by Hilbert 90 we have $X = G \cdot e$.

In this paper, our focus is mainly on the symmetric space $\mathbf{X} = \mathbf{X}(\mathbf{U_{2n}}, \theta)$ where

$$\theta(g) = \bar{g}.$$

Recall that the stabilizer of the identity is $(\mathbf{U_{2n}})_\mathbf{e} = \mathbf{Sp_{2n}}$.

Lemma 4. *For* $\mathbf{X} = \mathbf{X}(\mathbf{U_{2n}}, \theta)$ *we have*

$$X = U_{2n} \cdot e.$$

Proof. Note that $\mathbf{G}(E) \simeq \mathrm{GL}_{2n}(E)$ and with this identification, over E (and in particular over an algebraic closure of F), $\mathbf{X}(E)$ is isomorphic to the symmetric space of Example 3(1) with F replaced by E. Thus it follows that $\mathbf{X} = \mathbf{G} \cdot e$. Since $H^1(F, \mathrm{Sp}_{\mathbf{2n}})$ is trivial we have $(\mathbf{G}/\mathbf{G_e})(F) = \mathbf{G}(F)/\mathbf{G_e}(F)$. The lemma follows. □

Remark 2. This subsection is valid for any field F.

3.3. Distinction and contragredients

Next, we show that an irreducible representation of U_{2n} is Sp_{2n}-distinguished if and only if its contragredient is. First we recall the following result of Mœglin, Vignéras and Waldspurger.

Let $V = E^{2n}$ (column vectors) and $\mathrm{GL}_F(V)$ the group of F-linear automorphism of V. We identify $\mathrm{GL}_{2n}(E)$ as a subgroup of $\mathrm{GL}_F(V)$ via matrix multiplication. In particular, U_{2n} is identified with the subgroup of $\mathrm{GL}_F(V)$ preserving the anti-Hermitian form

$$\langle v, v' \rangle = {}^t\bar{v} J_n v', \quad v, v' \in V.$$

Let

$$\delta \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \bar{v}_1 \\ -\bar{v}_2 \end{pmatrix}, \quad v_1, v_2 \in E^n.$$

Then $\delta \in \mathrm{GL}_F(V)$ satisfies $\delta^2 = e$ and

$$\langle v', v \rangle = \langle \delta(v), \delta(v') \rangle, \quad v, v' \in V.$$

In particular, $\delta(av) = \bar{a}\delta(v)$ for $a \in E$ and $v \in V$. Note that we have

$$\delta g \delta^{-1} = \epsilon \bar{g} \epsilon \in U_{2n}, \quad g \in U_{2n} \tag{5}$$

where $\epsilon = \mathrm{diag}(I_n, -I_n)$. In particular, for $\pi \in \Pi(U_{2n})$ the representation $\pi^\delta \in \Pi(U_{2n})$ is defined on the space of π by $\pi^\delta(g) = \pi(\delta g \delta^{-1})$. The following is the content of [29, § 4, II.1].

https://doi.org/10.1017/S1474748019000161 Published online by Cambridge University Press

Theorem 4 (Mœglin–Vignéras–Waldspurger). *For* $\pi \in \mathrm{Irr}(U_{2n})$ *we have*

$$\pi^\vee \simeq \pi^\delta.$$

Corollary 1. *For* $\pi \in \mathrm{Irr}(U_{2n})$ *we have that* π *is* Sp_{2n}*-distinguished if and only if* $\pi^\vee$ *is* Sp_{2n}*-distinguished.*

Proof. Clearly π is Sp_{2n}-distinguished if and only if π^δ is $\delta^{-1}\mathrm{Sp}_{2n}\delta$-distinguished. However, by (5) we have $\delta^{-1}h\delta = \epsilon h\epsilon$ for $h \in \mathrm{Sp}_{2n}$ and since $\epsilon \in \mathrm{GSp}_{2n}$ it normalizes Sp_{2n}. Therefore $\delta^{-1}\mathrm{Sp}_{2n}\delta = \mathrm{Sp}_{2n}$ and the corollary follows from Theorem 4. □

3.4. A necessary condition for distinction

For $\pi \in \mathrm{Irr}(U_{2n})$ there exist a unique $0 \leqslant m \leqslant n$ and a unique $\sigma \in \mathrm{Cusp}(U_{2m})$ up to isomorphism such that π is a quotient of $\tau \rtimes \sigma$ for some representation $\tau \in \Pi(\mathrm{GL}_{n-m}(E))$. The representation σ is called the partial cuspidal support of π and denoted by π_{cusp}.

We say that π has trivial partial cuspidal support if $m = 0$ (and then $\sigma = \mathbf{1}_0$ is the trivial representation of the trivial group so that $\tau \rtimes \mathbf{1}_0$ is induced from the Siegel parabolic subgroup of U_{2n}).

The following is one of the main results of [6]. It is also immediate from the main result of [25] and the double coset analysis in § 4.

Proposition 2. *Let* $\pi \in \mathrm{Irr}(U_{2n})$ *be* Sp_{2n}*-distinguished. Then* π *has trivial partial cuspidal support.*

4. The geometrical lemma for the symmetric space

Throughout this section let $G_n = U_{2n}$ and $H_n = \mathrm{Sp}_{2n}$ for $n \in \mathbb{N}$. Fix n and let $G = G_n$ and $H = H_n$. Let

$$X = \{g \in G \mid g\bar{g} = e\}$$

be considered as a G-space with action given by twisted conjugation $g \cdot x = gx\bar{g}^{-1}$, $g \in G$ and $x \in X$. Recall that by Lemma 4 we have that $X = G \cdot e$. The map $g \mapsto g\bar{g}^{-1} = g \cdot e$ defines a homeomorphism $G/H \simeq X$.

Let $P = M \ltimes U$ be a standard parabolic subgroup of G with its standard Levi decomposition. We write $P = Q_{(n_1,\ldots,n_k;m)}$ (so that $M = L_{(n_1,\ldots,n_k;m)}$ and $U = V_{(n_1,\ldots,n_k;m)}$).

We have a bijection $P\backslash G/H \simeq P\backslash X$. This set of orbits gives rise to a filtration of the restriction to H of a representation of G parabolically induced from P.

4.1. The filtration

Since P is stable under Galois conjugation we have $P \cdot x \subseteq PxP$, $x \in X$. Furthermore, note that $x\bar{x} = e$ and that every element of $P\backslash G/P$ is stable under Galois conjugation. Therefore, $PxP = Px^{-1}P$. Let $w \in {}_MW_M$ correspond to PxP under the Bruhat decomposition § 2.1.12. Then $W_MwW_M = W_Mw^{-1}W_M$. Since an element of minimal length in every W_M-doube coset of W is unique it follows that w is an involution. Let

$$\iota_P : P\backslash X \to W[2] \cap {}_MW_M$$

https://doi.org/10.1017/S1474748019000161 Published online by Cambridge University Press

be the map defined by the Bruhat decomposition. That is, $\iota_P(P \cdot x) = w$ where $PxP = PwP$. Then $L = M \cap wMw^{-1}$ is a standard Levi subgroup of M. Let Q be the standard parabolic subgroup of G with Levi subgroup L. Then Q (respectively L) is the parabolic (respectively Levi) subgroup of G associated to the P-orbit $P \cdot x$.

Let $\sigma \in \Pi(M)$. The restriction $\pi|_H$ to H of the induced representation $\pi = \mathbf{i}_{G,M}(\sigma)$ admits a natural filtration parameterized by $P\backslash G/H$.

The decomposition factor associated to a double coset $P\eta H$ is isomorphic to

$$I_\eta = \mathrm{ind}_{H_\eta^P}^H(\delta_{H_\eta^P}^{-1/2} \cdot (\delta_P^{1/2}\sigma)^\eta)$$

where $H_\eta^P = H \cap \eta^{-1}P\eta$. Let $x = \eta\bar{\eta}^{-1} \in X$.

Definition 3. We say that $P \cdot x$ (or w) is *relevant* to σ if I_η is H-distinguished.

We say that x is a *good* representative for its P-orbit if $x \in N_G(L)$ (in this case $xL = wL$ and $L = M \cap xMx^{-1}$). A good representative exists for every P-orbit (see [34, Lemma 3.2]).

Assume that x is a good representative of its P-orbit. By [34, Proposition 4.1] we have that I_η is H-distinguished if and only if $r_{L,M}(\sigma)$ is $(L_x, \delta_{Q_x}\delta_Q^{-1/2})$-distinguished.

By [34, Theorem 4.2] we have the following theorem.

Theorem 5. *With the above notation, if π is H-distinguished then there exists a P-orbit $\mathcal{O}$ in X that is relevant to σ. In particular, $r_{L,M}(\sigma)$ is $(L_x, \delta_{Q_x}\delta_Q^{-1/2})$-distinguished for a good representative $x \in \mathcal{O}$.*

In what follows, we make the condition that a P-orbit is relevant more explicit. With the above notation, for $x \in X$ a good representative of its P-orbit set

$$\delta_x = \delta_{Q_x}\delta_Q^{-1/2}|_{L_x}.$$

Let

$$[M] = \{L_{(n_{\sigma(1)},\ldots,n_{\sigma(k)};m)} : \sigma \in S_k\}$$

be the set of standard Levi subgroups of G that are conjugate to M. For $M' \in [M]$ let $W(M, M')$ be the set of $w \in W$ such that w is of minimal length in wW^M and $wMw^{-1} = M'$. Let

$$W(M) = \bigsqcup_{M' \in [M]} W(M, M')$$

and note that for $w \in W(M)$ and $w' \in W(wMw^{-1})$ we have that $w'w \in W(M)$.

Although $W(M)$ is not a group it is in a natural bijection with the signed permutation group $\mathfrak{W}_k$. The set Δ_M of simple roots with respect to M is in natural bijection with the set Δ_k of simple roots associated with $\mathfrak{W}_k$ in § 3.1. This identifies the set of elementary symmetries in $W(M)$ (in the sense of [31, § I.1.7]) and the set of simple reflections in $\mathfrak{W}_k$ and defines a unique bijection $\jmath_M : W(M) \to \mathfrak{W}_k$ that satisfies

$$\jmath_M(w'w) = \jmath_{wMw^{-1}}(w')\jmath_M(w), \quad w \in W(M),\ w' \in W(wMw^{-1}).$$

https://doi.org/10.1017/S1474748019000161 Published online by Cambridge University Press

Explicitly, if $w \in W(M)$ and $\jmath_M(w) = \tau\mathfrak{c}$ then there exists an element $t_w \in N_G(T)$, unique up to multiplication by an element of the center of M, representing the Weyl element w and satisfying

$$t_w\iota(g_1, \ldots, g_k, h)t_w^{-1} = \iota(g_1', \ldots, g_k'; h), \quad g_i \in \mathrm{GL}_{n_i}(E),\ i = 1, \ldots, k,\ h \in G_m$$

where

$$g_i' = \begin{cases} g^*_{\tau^{-1}(i)} & i \in \mathfrak{c} \\ g_{\tau^{-1}(i)} & i \notin \mathfrak{c}. \end{cases}$$

Let

$$\mathcal{N}_M = \{g \in G : gMg^{-1} \in [M]\} = \bigsqcup_{w \in W(M)} t_w M.$$

Then $\kappa_M : \mathcal{N}_M/M \to W(M)$ given by $\kappa_M(t_w M) = w$ is a bijection satisfying

$$\kappa_{wMw^{-1}}(g')\kappa_M(g) = \kappa_M(g'g), \quad g \in \mathcal{N}_M,\ g' \in \mathcal{N}_{wMw^{-1}}.$$

Let $\mathfrak{p}_M : \mathcal{N}_M \to \mathfrak{W}_k$ be the composition $\mathfrak{p}_M = \jmath_M \circ \kappa_M$.

Similar maps can be defined with G replaced by H and in particular, each t_w can be chosen in H, that is, we may further assume that $\bar{t}_w = t_w$. Since also $\overline{M} = M$ it follows that

$$\bar{g}^{-1} g \in M, \quad g \in \mathcal{N}_M.$$

Note that restricted to $N_G(M)$, κ_M defines a group isomorphism $N_G(M)/M \cong W(M, M)$. Furthermore, for $w \in W(M)$ such that $\jmath_M(w) = \tau\mathfrak{c} \in \mathfrak{W}_k$ we have $w \in W(M, M)$ if and only if $n_{\tau(i)} = n_i$, $i = 1, \ldots, k$.

Remark 3. Let $x \in N_G(M) \cap X$ and $\rho = \mathfrak{p}_M(x) \in \mathfrak{W}_k[2]$. If $\sigma \in \mathfrak{W}_k$, $\rho' \in \mathfrak{W}_k[2]$ and $g \in \mathcal{N}_M$ are such that $\rho \overset{\sigma}{\curvearrowright} \rho'$ and $\mathfrak{p}_M(g) = \sigma$ then $(M, x) \overset{g}{\curvearrowright} (M', x')$ is a path in the graph defined in [34, § 6] where $M' = gMg^{-1} = gM\bar{g}^{-1}$ and $x' = gx\bar{g}^{-1}$.

4.2. Minimal involutions

In light of Lemma 1 the key for the study of M-orbits in $N_G(M) \cap X$ is to first understand the orbits that project under $\mathfrak{p}_M$ to minimal involutions.

Definition 4. An element $w \in W(M, M)$ is called M-minimal if $w = w_M^L$ for some standard Levi subgroup $L \supseteq M$ and $w\alpha = -\alpha$ for $\alpha \in \Delta_M^L$.

Note that $w \in W(M, M)$ is M-minimal if and only if $\jmath_M(w) \in \mathfrak{W}_k$ is a minimal involution.

Let $w \in W(M, M)$ be M-minimal. Let $\ell \in \{0, 1, \ldots, k\}$ and $\rho \in S_\ell$ a minimal involution so that $\jmath_M(w) = \rho\mathfrak{c}_{\ell,k}$ (see (4) and Example 2). Let

$$S = \mathfrak{c}_-(\jmath_M(w)) \quad \text{and} \quad R = \mathfrak{c}_<(\jmath_M(w)).$$

Lemma 5. *With the above notation, $wM \cap X$ is not empty if and only if n_i is even for all $\ell + 1 \leqslant i \leqslant k$ and in this case $wM \cap X$ is a unique M-orbit.*

https://doi.org/10.1017/S1474748019000161 Published online by Cambridge University Press

Proof. Assume that $x \in wM \cap X$. We can choose

$$t_w = \iota(t_1, \ldots, t_{r+s}; t) \in w$$

where

$$t_j = \begin{cases} I_{n_{i(j)}} & i(j) \in S \\ \begin{pmatrix} 0 & I_{n_{i(j)}} \\ I_{n_{i(j)}} & 0 \end{pmatrix} & i(j) \in R \end{cases}, \quad i(1) = 1, \quad i(j+1) = \begin{cases} i(j)+1 & i(j) \in S \\ i(j)+2 & i(j) \in R \end{cases} \tag{6}$$

for $j = 1, \ldots, r+s$ (note that $i(j) \in R \sqcup S$ for all j) and

$$t = \begin{pmatrix} & & & & & & I_{n_{\ell+1}} \\ & & & & & \iddots & \\ & & & & I_{n_k} & & \\ & & & I_{2m} & & & \\ & & -I_{n_k} & & & & \\ & \iddots & & & & & \\ -I_{n_{\ell+1}} & & & & & & \end{pmatrix}.$$

Then, by definition, $x \in t_w M \cap X$. Let $x = t_w \iota(g_1, \ldots, g_k; h)$ with $g_i \in \mathrm{GL}_{n_i}(E)$ and $h \in G_m$. The condition $x \in X$ is equivalent to

$$\begin{cases} g_i \bar{g}_i = I_{n_i} & i \in S \\ g_i \bar{g}_{i+1} = I_{n_i} & i \in R \\ g_i \bar{g}_i^* = -I_{n_i} & \ell+1 \leqslant i \leqslant k \\ h\bar{h} = I_{2m}. \end{cases} \tag{7}$$

In particular, for $\ell+1 \leqslant i \leqslant k$ the condition is that $g_i w_{n_i}$ is an alternating matrix and therefore n_i is even. For $d_i \in \mathrm{GL}_{n_i}(E)$, $i = 1, \ldots, k$ and $d' \in G_m$ let $d = \iota(d_1, \ldots, d_k; d') \in M$. Then

$$dx\bar{d}^{-1} = t_w \iota(g_1', \ldots, g_k'; h')$$

where

$$\begin{cases} g_i' = d_i g_i \bar{d}_i^{-1} & & i \in S \\ g_i' = d_{i+1} g_i \bar{d}_i^{-1}, & g_{i+1}' = d_i g_{i+1} \bar{d}_{i+1}^{-1} = \bar{g}_i'^{-1} & i \in R \\ g_i' = d_i^* g_i \bar{d}_i^{-1} & & \ell+1 \leqslant i \leqslant k \\ h' = d' h \bar{d}'^{-1}. & & \end{cases}$$

It follows from Example 3 and Lemma 4 that $t_w M \cap X$ is a unique M-orbit.

On the other hand, assuming that n_i is even whenever $\ell+1 \leqslant i \leqslant k$, with the above choice of t_w we have

$$x_w = t_w \iota(I_{n_1+\cdots+n_\ell}, \epsilon_{n_{\ell+1}}, \ldots, \epsilon_{n_k}; I_{2m}) \in wM \cap X \tag{8}$$

where $\epsilon_{2n} = \begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix}$. □

Next we compute the stabilizer and relevant modulus function for a minimal orbit.

https://doi.org/10.1017/S1474748019000161 Published online by Cambridge University Press

4.2.1. Let $x \in N_G(M) \cap X$ be M-minimal. Applying the notation of Lemma 5 and its proof we write $x = t_w \iota(g_1, \ldots, g_k; h)$ so that (7) holds. Following the proof of the lemma we see that M_x consists of elements of the form $d = \iota(d_1, \ldots, d_k; d')$ where

$$\begin{array}{lll} d_i \in \mathrm{GL}_{n_i}(E) & g_i = d_i g_i \bar{d}_i^{-1} & i \in S \\ d_i \in \mathrm{GL}_{n_i}(E) & d_{i+1} = g_i \bar{d}_i g_i^{-1} & i \in R \\ d_i \in \mathrm{GL}_{n_i}(E) & g_i = d_i^* g_i \bar{d}_i^{-1} & \ell+1 \leqslant i \leqslant k \\ d' \in G_m & h = d' h \bar{d'}^{-1} & \end{array}$$

and in particular

$$M_x \cong \left[\prod_{i \in S} \mathrm{GL}_{n_i}(F)\right] \times \left[\prod_{i \in R} \mathrm{GL}_{n_i}(E)\right] \times \left[\prod_{i=\ell+1}^{k} \mathrm{Sp}_{n_i}(E)\right] \times H_m.$$

The representative x can be chosen so that this isomorphism is the standard one. In fact, with the above notation, for $x = x_w$ (see (8)) it is easy to see that M_x consists of elements of the form $\iota(g_1, \ldots, g_k; h)$ where

- $g_i \in \mathrm{GL}_{n_i}(F)$, $i \in S$;
- $g_{i+1} = \bar{g}_i \in \mathrm{GL}_{n_i}(E)$, $i \in R$;
- $g_i \in \mathrm{Sp}_{n_i}(E)$, $\ell+1 \leqslant i \leqslant k$;
- $h \in H_m$.

4.2.2. We explicate the modulus functions δ_x associated to M-minimal elements when M is contained in the Siegel Levi subgroup. We freely use the notation of § 4.2.1.

Lemma 6. *Let $x \in N_G(M) \cap X$ be M-minimal and assume that M is contained in the Siegel Levi subgroup of G (i.e., that with the above notation $m = 0$). Then,*

$$\delta_x(\iota(d_1, \ldots, d_k)) = \prod_{i \in S} |\det d_i|_F \cdot \prod_{i \in R} |\det d_i|_E, \quad \iota(d_1, \ldots, d_k) \in M_x.$$

Proof. For $g \in M$ it is easy to see that the above equation holds for x if and only if it holds for $gx\bar{g}^{-1}$. By Lemma 5 we may therefore assume that $x = x_w$ as defined by (8). For this case we carry the computation out explicitly.

For an automorphism α of a group Q we denote by $\mathrm{mod}_Q(\alpha)$ the associated modulus function. For example, for $z \in \mathrm{GL}_a(F)$ and $y \in \mathrm{GL}_b(F)$ we have

$$\mathrm{mod}_{M_{a\times b}(F)}(z) = |\det z|_F^b, \quad \mathrm{mod}_{M_{a\times b}(F)}(y) = |\det y|_F^a \tag{9}$$

where the respective automorphism is given by left or right matrix multiplication. Let $\mathcal{S}_n(F) = \{z \in M_n(F) : {}^t z = z\}$ and $\mathcal{H}_n(E/F) = \{z \in M_n(E) : {}^t \bar{z} = z\}$. Then, with respect to the natural actions $g \cdot z = gz\,{}^t g$, $g \in \mathrm{GL}_n(F)$, $z \in \mathcal{S}_n(F)$ and $g \cdot z = gz\,{}^t \bar{g}$, $g \in \mathrm{GL}_n(E)$, $z \in \mathcal{H}_n(E/F)$ we have

$$\mathrm{mod}_{\mathcal{S}_n(F)}(g) = |\det g|_F^{n+1}, \quad g \in \mathrm{GL}_n(F), \quad \mathrm{mod}_{\mathcal{H}_n(E/F)}(g) = |\det g|_E^n, \quad g \in \mathrm{GL}_n(E). \tag{10}$$

In our modulus function computations below we freely use, without further mention, the fact that $\det g = 1$ for $g \in \mathrm{Sp}_{2m}(E)$, $m \in \mathbb{N}$.

Recall (see (6)) that $x = \iota(t_1, \dots, t_{r+s}; x')$ where

$$x' = -\begin{pmatrix} & & & & & \epsilon_{n_{\ell+1}} \\ & & & & \iddots & \\ & & & \epsilon_{n_k} & & \\ & & \epsilon_{n_k} & & & \\ & \iddots & & & & \\ \epsilon_{n_{\ell+1}} & & & & & \end{pmatrix}$$

and that M_x consists of elements of the form $\iota(g_1, \dots, g_k)$ where

$$\begin{cases} g_i \in \mathrm{GL}_{n_i}(F) & i \in S \\ g_{i+1} = \bar{g}_i \in \mathrm{GL}_{n_i}(E) & i \in R \\ g_i \in \mathrm{Sp}_{n_i}(E) & i = \ell+1, \dots, k. \end{cases} \tag{11}$$

Let $N_1 = n_1 + \cdots + n_\ell$, $N_2 = n_{\ell+1} + \cdots + n_k$ and $Q = L \ltimes V$ the maximal parabolic subgroup of G with $L = M_{(N_1;N_2)}$ and $V = U_{(N_1;N_2)}$. Let $Z = L \cap U$ so that $U = Z \ltimes V$ and consider the involution $\theta_x(g) = x\bar{g}x^{-1}$, $g \in G$. Note that $g \cdot x = x$ if and only if $\theta_x(g) = g$. Since $x \in L$, it normalizes V and therefore $\theta_x(V) = V$. It is easy to see that $U \cap \theta_x(U) = (Z \cap \theta_x(Z)) \ltimes V$ and since this decomposition is θ_x-stable we get that $U_x = Z_x \ltimes V_x$. Furthermore, each of the groups Z, V, Z_x and V_x is normalized by M_x. It follows that

$$\delta_P(h) = \mathrm{mod}_Z(h)\,\mathrm{mod}_V(h) \text{ and similarly } \delta_{P_x}(h) = \mathrm{mod}_{Z_x}(h)\,\mathrm{mod}_{V_x}(h), \quad h \in M_x. \tag{12}$$

We write a matrix $z \in Z$ in block form as $z = \iota(z_1; z_2)$ where $z_1 = (z_{i,j})$, with $1 \leqslant i, j \leqslant \ell$ and $z_{i,j} \in M_{n_i \times n_j}(E)$ satisfying

$$\begin{aligned} & z_{i,j} = 0 \qquad i > j \\ & z_{i,i} = I_{n_i} \end{aligned}$$

and $z_2 \in U_{(n_{\ell+1}, \dots, n_k)}$.

It is then easy to verify that $z \in Z_x$ if and only if $z_2 = I_{N_2}$ and whenever $1 \leqslant i < j \leqslant \ell$ we have

$$\begin{array}{ll} z_{i,i+1} = 0 & i \in R \\ z_{i,j} \in M_{n_i \times n_j}(F) & i, j \in S \\ z_{i,j+1} = \bar{z}_{i,j} & i \in S,\ j \in R \\ z_{i+1,j} = \bar{z}_{i,j} & i \in R,\ j \in S \\ z_{i+1,j+1} = \bar{z}_{i,j} \text{ and } z_{i+1,j} = \bar{z}_{i,j+1} & i, j \in R. \end{array}$$

Furthermore, for $h = \iota(g_1, \dots, g_k) \in M_x$ as in (11) and $z \in Z$ we have

$$hzh^{-1} = \iota(z_1'; z_2')$$

where $z_1' = (g_i z_{i,j} g_j^{-1})$ (and similarly $z_2' \in U_{(n_{\ell+1}, \dots, n_k)}$ is the conjugate of z_2 by $\iota(g_{\ell+1}, \dots, g_k)$).

Taking (9) into consideration, it is now easy to see that

$$\mathrm{mod}_Z^{1/2}|_{M_x} = \mathrm{mod}_{Z_x}|_{M_x}. \tag{13}$$

Next, let

$$V^{(1)} = \left\{ v^{(1)}(s) = \begin{pmatrix} I_{N_1} & s & \\ & I_{2N_2} & s' \\ & & I_{N_1} \end{pmatrix} \middle| s \in M_{N_1 \times 2N_2}(E),\ s' = J_{N_2}{}^t\bar{s}w_{N_1} \right\}$$

and

$$V^{(2)} = \left\{ v^{(2)}(y) = \begin{pmatrix} I_{N_1} & & y \\ & I_{2N_2} & \\ & & I_{N_1} \end{pmatrix} \middle| yw_{N_1} \in \mathcal{H}_{N_1}(E/F) \right\}.$$

Note that $V^{(2)}$ is a normal subgroup of V and $V^{(1)}$ is a set of representatives for $V/V^{(2)}$. Thus,

$$V = \{v(s, y) = v^{(1)}(s)v^{(2)}(y) \mid s \in M_{N_1 \times 2N_2}(E),\ yw_{N_1} \in \mathcal{H}_{N_1}(E/F)\}.$$

Note that if $v = v_1v_2$ with $v_i \in V^{(i)}$ then also $\theta_x(v_i) \in V^{(i)}$, $i = 1, 2$. By the uniqueness of decomposition it follows that $v \in V_x$ if and only if $v_1, v_2 \in V_x$. For $v = v(r, s) \in V$ write $s = (s_1, s_2)$ with $s_1, s_2 \in M_{N_1 \times N_2}(E)$ and $yw_{N_1} = (y_{i,j})$ with $y_{i,j} \in M_{n_i \times n_j}(E)$, $1 \leqslant i$, $j \leqslant \ell$. Then explicitly,

$$\begin{aligned} &y_{j,i} = {}^t\bar{y}_{i,j} \in M_{n_j \times n_i}(E) \quad i \neq j \\ &y_{i,i} \in \mathcal{H}_{n_i}(E/F). \end{aligned}$$

We then have $v \in V_x$ if and only if

$$s_2 = -\mathrm{diag}(t_1, \ldots, t_{r+s})\, \overline{s_1} \begin{pmatrix} & & \epsilon_{n_{\ell+1}} \\ & \iddots & \\ \epsilon_{n_k} & & \end{pmatrix}$$

and y satisfies

$$\begin{array}{ll} y_{i,i} \in \mathcal{S}_{n_i}(F) & i \in S \\ y_{i+1,i+1} = \bar{y}_{i,i} \text{ and } y_{i+1,i} = \bar{y}_{i,i+1} \in \mathcal{S}_{n_i}(E) & i \in R \\ \text{and for all } i < j & \\ y_{i,j+1} = \bar{y}_{i,j} & i \in S,\ j \in R \\ y_{i+1,j} = \bar{y}_{i,j} & i \in R,\ j \in S \\ y_{i+1,j+1} = \bar{y}_{i,j} \text{ and } y_{i+1,j} = \bar{y}_{i,j+1} & i,\ j \in R. \end{array}$$

Furthermore, for $h = \iota(g_1, \ldots, g_k) \in M_x$ as in (11) we have

$$hv^{(1)}(s)h^{-1} = v^{(1)}(s'')$$

where

$$s'' = \mathrm{diag}(g_1, \ldots, g_\ell)s\,\iota(g_{\ell+1}, \ldots, g_k)^{-1}$$

and

$$hv^{(2)}(y)h^{-1} = v^{(2)}(y''') \quad \text{where } (y''')_{i,j} = g_i y_{i,j}{}^t\bar{g}_j.$$

Thus,

$$hv(s,y)h^{-1} = h(s'', y''').$$

Taking (9) and (10) into consideration, it is now easy to see that

$$\mathrm{mod}_V^{-1/2}(h)\,\mathrm{mod}_{V_x}(h) = \mathrm{mod}_{V^{(2)}}^{-1/2}(h)\,\mathrm{mod}_{V_x^{(2)}}(h) = \left[\prod_{i\in S}|\det g_i|_F\right]\left[\prod_{i\in R}|\det g_i|_E\right]. \tag{14}$$

The lemma is now immediate from (13), (14) and (12). □

4.2.3. As an immediate consequence we formulate the following.

Corollary 2. *Assume that M is contained in the Siegel Levi subgroup of G. Let $x \in N_G(M)\cap X$ be M-minimal and $\mathrm{p}_M(x) = \rho\mathfrak{c}_{\ell,k}$, $\rho \in S_\ell$ (see (4)). With the above notation let π_i be a representation of $\mathrm{GL}_{n_i}(E)$, $i = 1,\dots,k$. The representation $\pi = \pi_1\otimes\cdots\otimes\pi_k$ of M is (M_x,δ_x)-distinguished if and only if*

$$\begin{cases} \nu^{-1/2}\pi_i \text{ is } \mathrm{GL}_{n_i}(F)\text{-distinguished} & i\in S\\ \pi_{i+1}\cong \nu\bar{\pi}_i^\vee & i\in R\\ \pi_i \text{ is } \mathrm{Sp}_{n_i}(E)\text{-distinguished} & \ell+1\leqslant i\leqslant k.\end{cases}$$

4.3. Admissible orbits

Let

$$\mathcal{W}_M = \mathrm{p}_M(N_G(M)\cap X).$$

Lemma 7. *We have*

$$\mathcal{W}_M = \{w\in\mathfrak{W}_k[2] \mid n_i \text{ is even for all } i\in\mathfrak{c}_+(w)\}.$$

Furthermore, p_M defines a bijection

$$M\backslash(N_G(M)\cap X)\simeq\mathcal{W}_M.$$

Proof. Let $w\in\mathfrak{W}_k[2]$ and let ℓ be such that $k-\ell = |\mathfrak{c}_+(w)|$. It follows from Lemma 1, (4) and Lemma 2(1) that there exist $\sigma\in\mathfrak{W}_k$ and $\rho\in S_\ell$ a minimal involution such that $w \overset{\sigma}{\curvearrowright} \rho\mathfrak{c}_{\ell,k}$.

For $g\in\mathcal{N}_M$ let $M' = gM\bar{g}^{-1} = gMg^{-1}\in[M]$. Then the map $x\mapsto gx\bar{g}^{-1}$ defines a bijection between M-orbits in $N_G(M)\cap X$ and M'-orbits in $N_G(M')\cap X$. By choosing g so that $\mathrm{p}_M(g)=\sigma$ we see that $w\in\mathcal{W}_M$ if and only if $\rho\mathfrak{c}_{\ell,k}\in\mathcal{W}_{M'}$. The lemma now follows from Lemma 5. □

4.3.1. Let $x\in N_G(M)\cap X$ and $w = \mathrm{p}_M(x)\in\mathcal{W}_M$. Let $\tau\in S_k$ and $\mathfrak{c}\in\Xi_k$ be such that $w=\tau\mathfrak{c}$. Based on the above analysis we can now describe M_x as follows. We have

$$M_x \cong \left[\prod_{i\in\mathfrak{c}_+(w)}\mathrm{GL}_{n_i}(F)\right]\times\left[\prod_{i\in\mathfrak{c}_<(w)}\mathrm{GL}_{n_i}(E)\right]\times\left[\prod_{i\in\mathfrak{c}_+(w)}\mathrm{Sp}_{n_i}(E)\right]\times H_m.$$

https://doi.org/10.1017/S1474748019000161 Published online by Cambridge University Press

More explicitly, a representative x as above can be chosen so that M_x consists of elements $\iota(g_1, \dots, g_k; h)$ such that

$$\begin{array}{ll} g_i \in \mathrm{GL}_{n_i}(F) & i \in \mathfrak{c}_-(w) \\ g_{\tau(i)} = \bar{g}_i \in \mathrm{GL}_{n_i}(E) & i \in \mathfrak{c}_<(w) \setminus \mathfrak{c} \\ g_i \in \mathrm{Sp}_{n_i}(E) & i \in \mathfrak{c}_+(w) \\ g_{\tau(i)} = \bar{g}_i^* \in \mathrm{GL}_{n_i}(E) & i \in \mathfrak{c}_<(w) \cap \mathfrak{c} \\ h \in H_m. & \end{array} \tag{15}$$

4.3.2. The following proposition follows from [34, Corollary 6.5] in light of Remark 3. It is a consequence of Springer's theory of twisted involutions as further developed in [22].

Proposition 3. *Let $x \in N_G(M) \cap X$ with $\mathrm{p}_M(x) = \rho \in \mathfrak{W}_k[2]$ and let $\sigma \in \mathfrak{W}_k$ and $\rho' \in \mathfrak{W}_k[2]$ be such that $\rho \overset{\sigma}{\curvearrowright} \rho'$. For $g \in \mathcal{N}_M$ such that $\mathrm{p}_M(g) = \sigma$ set $M' = gMg^{-1}$ and $x' = gx\bar{g}^{-1}$ (so that $\mathrm{p}_{M'}(x') = \rho'$). Then*

$$\delta_{x'}(gmg^{-1}) = \delta_x(m), \quad m \in M_x.$$

Corollary 3. *Assume that $m = 0$ (i.e., M is contained in the Siegel Levi subgroup of G) and let $w = \tau\mathfrak{c} \in \mathcal{W}_M$ with $\tau \in S_k$ and $\mathfrak{c} \in \Xi_k$. Then, for $x \in N_G(M) \cap X$ such that $\mathrm{p}_M(x) = w$ and $\iota(g_1, \dots, g_k) \in M_x$ we have*

$$\delta_x(\iota(g_1, \dots, g_k)) = \prod_{i \in \mathfrak{c}_-(w)} |\det g_i|_F \cdot \prod_{i \in \mathfrak{c}_<(w)} |\det g_i|_E = |\det g|_F \tag{16}$$

where $g = \mathrm{diag}(g_1, \dots, g_k)$.

Proof. The second equality of (16) is straightforward from (15). We prove the first equality. By Lemma 1 there exist $w' \in \mathfrak{W}_k[2]$ a minimal involution and $\sigma \in \mathfrak{W}$ such that $w \overset{\sigma}{\curvearrowright} w'$. Let $g \in \mathcal{N}_M$ be such that $\sigma = \mathrm{p}_M(g)$ and let $x' = gx\bar{g}^{-1}$ and $M' = gMg^{-1}$. Then x' is M'-minimal. It follows from Lemma 6 that (16) holds for x'. That it holds for x now follows from Proposition 3 and Lemmas 2 and 3. □

4.3.3. As a consequence of the explication of the stabilizer in (15) and Corollary 3 we have the following.

Corollary 4. *Assume that M is contained in the Siegel Levi subgroup of G and let $x \in N_G(M) \cap X$ and $\mathrm{p}_M(x) = w$. Let $\tau \in S_k$ and $\mathfrak{c} \in \Xi_k$ be such that $w = \tau\mathfrak{c}$. For representations π_i of $\mathrm{GL}_{n_i}(E)$, $i = 1, \dots, k$, the representation $\pi = \pi_1 \otimes \cdots \otimes \pi_k$ of M is (M_x, δ_x)-distinguished if and only if*

$$\begin{cases} \nu^{-1/2}\pi_i \text{ is } \mathrm{GL}_{n_i}(F)\text{-distinguished} & i \in \mathfrak{c}_-(w) \\ \pi_i \text{ is } \mathrm{Sp}_{n_i}(E)\text{-distinguished} & i \in \mathfrak{c}_+(w) \\ \pi_{\tau(i)} \cong \nu\bar{\pi}_i^\vee & i \in \mathfrak{c}_{\neq}(w) \setminus \mathfrak{c} \\ \pi_{\tau(i)} \cong \nu^{-1}(\pi_i^\iota)^\vee & i \in \mathfrak{c}_<(w) \cap \mathfrak{c} \end{cases}$$

where ι is the involution $g \mapsto g^\iota = w_n {}^t g^{-1} w_n$ on $\mathrm{GL}_n(E)$.

https://doi.org/10.1017/S1474748019000161 Published online by Cambridge University Press

Remark 4. By [11] we have $\pi^\vee \cong \pi^\iota$ if π is irreducible. Thus, if the π_i's are assumed to be irreducible then the condition above for $i \in \mathfrak{c}_<(w) \cap \mathfrak{c}$ becomes $\pi_{\tau(i)} \cong \nu^{-1}\pi_i$.

4.4. General orbits

Assume that M is contained in the Siegel Levi $(m = 0)$. We consider a general P-orbit $P \cdot x$ in X. Assume without loss of generality that x is a good representative. Let $w \in {}_M W_M \cap W[2]$ be such that $PxP = PwP$ so that $x \in wL \cap X$, where $L = M \cap wMw^{-1}$ is the standard Levi subgroup of M associated with the orbit $P \cdot x$. Let $Q = L \ltimes V$ be the associated standard parabolic subgroup.

Write $L = M_{(\gamma_1,\dots,\gamma_k;0)}$ where $\gamma_i = (a_{i,1}, \dots, a_{i,j_i})$ is a partition of n_i and let

$$\mathcal{I} = \{(i, j) : i = 1, \dots, k,\ j = 1, \dots, j_i\}.$$

Via lexicographic order on $\mathcal{I}$ (i.e., $(i, j) \prec (i', j')$ if either $i < i'$, or $i = i'$ and $j < j'$) we identify $\mathfrak{W}_{|\mathcal{I}|}$ with $S_{\mathcal{I}} \ltimes \Xi_{\mathcal{I}}$ where $S_{\mathcal{I}}$ is the permutation group on $\mathcal{I}$ and $\Xi_{\mathcal{I}}$ the group of subsets of $\mathcal{I}$ with respect to symmetric difference. Then $x \in N_G(L) \cap X$ and we can set $\mathfrak{p}_L(x) = \tau\mathfrak{c}$ with $\tau \in S_{\mathcal{I}}$ and $\mathfrak{c} \subseteq \mathcal{I}$.

Note that since w is L-admissible it acts as an involution on the set of roots

$$\Sigma_L = R(G, T_L).$$

This involution can be described as follows. We can write

$$\Sigma_L = \{\pm(e_\imath + e_\jmath),\ (e_\imath - e_\jmath) : \imath,\ \jmath \in \mathcal{I}\} \setminus \{0\}$$

with respect to a standard basis $\{e_\imath : \imath \in \mathcal{I}\}$. Then

$$we_\imath = \begin{cases} -e_{\tau(\imath)} & \imath \in \mathfrak{c} \\ e_{\tau(\imath)} & \imath \notin \mathfrak{c}. \end{cases}$$

Let $\Sigma_L^M = R(M, T_L)$ and note that $\Sigma_M^L = \{e_{(i,j)} - e_{(i,j')} : i = 1, \dots, k, 1 \leqslant j \neq j' \leqslant j_i\}$. The fact that $L = M \cap wMw^{-1}$ is equivalent to saying that $w\alpha \notin \Sigma_L^M$ for $\alpha \in \Sigma_L^M$. This can be explicated as follows.

- If $(i, j), (i, j') \in \mathfrak{c}$ with $j \neq j'$, $\tau(i, j) = (a, b)$ and $\tau(i, j') = (c, d)$ then $a \neq c$.
- If $(i, j), (i, j') \notin \mathfrak{c}$ with $j \neq j'$, $\tau(i, j) = (a, b)$ and $\tau(i, j') = (c, d)$ then $a \neq c$.

The fact that w is M-reduced means that $w\alpha > 0$ for every $\alpha \in \Delta_L^M$ (and therefore also for every $\alpha \in R(Q \cap M, T_L)$. This can be explicated as

- For $j \neq j'$ if $(i, j) \notin \mathfrak{c}$ and $(i, j') \in \mathfrak{c}$ then $j < j'$.
- if $(i, j), (i, j') \in \mathfrak{c}$ with $j < j'$ then $\tau(i, j') \prec \tau(i, j)$.
- if $(i, j), (i, j') \notin \mathfrak{c}$ with $j < j'$ then $\tau(i, j) \prec \tau(i, j')$.

Combined, we get that for every i there is $s_i \in \{0, 1, \dots, j_i\}$ such that

$$\mathfrak{c} = \{(i, j) : s_i < j \leqslant j_i\} \tag{17}$$

and if $\tau(\imath) = (c_\imath, d_\imath)$, $\imath \in \mathcal{I}$ then

$$\begin{cases} c_{(i,j)} < c_{(i,j')} & \text{for } j < j' \leqslant s_i \\ c_{(i,j)} > c_{(i,j')} & \text{for } s_i < j < j'. \end{cases} \tag{18}$$

https://doi.org/10.1017/S1474748019000161 Published online by Cambridge University Press

5. A graph of signs

In this section we describe a graph that underlies the combinatorics behind the classification of Mœglin and Tadić of the discrete series of classical groups [28]. The material of the section is purely combinatorial and can be read independently from the rest of this article. We hope that the combinatorial framework that we develop here would have applications more generally in questions where the Mœglin–Tadić classification plays a role.

Let $V = \bigsqcup_{k=0}^{\infty} \{\pm 1\}^k$ be the set of ordered tuples of signs. We consider the directed, labeled graph $\mathcal{E}$ with vertices V and labeled edges $\mathbf{e} \xrightarrow{i} \mathbf{e}'$ whenever $\mathbf{e} = (e_1, \ldots, e_k) \in \{\pm 1\}^k$, $i \in \{1, \ldots, k-1\}$ is such that $e_i = e_{i+1}$ and $\mathbf{e}' = (e_1, \ldots, e_{i-1}, e_{i+2}, \ldots, e_k) \in \{\pm 1\}^{k-2}$ is obtained from $\mathbf{e}$ by deleting the ith and the $(i+1)$th entries.

For $\mathbf{e}, \mathbf{e}' \in V$, a path from $\mathbf{e}$ to $\mathbf{e}'$ is labeled by the sequence $\imath = (i_1, \ldots, i_t)$ and denoted by $\mathbf{e} \overset{\imath}{\curvearrowright} \mathbf{e}'$ if there exists $\mathbf{e}_1, \ldots, \mathbf{e}_t \in V$ and edges on the graph such that

$$\mathbf{e} = \mathbf{e}_1 \xrightarrow{i_1} \mathbf{e}_2 \xrightarrow{i_2} \cdots \xrightarrow{i_t} \mathbf{e}_t = \mathbf{e}'.$$

We write $\mathbf{e} \curvearrowright \mathbf{e}'$ if there exists a path in $\mathcal{E}$ from $\mathbf{e}$ to $\mathbf{e}'$. The t-*tuple* of natural numbers $\imath$ is the *pattern* of the path $\mathbf{e} \overset{\imath}{\curvearrowright} \mathbf{e}'$.

To make some of the basic arguments more formal it will also be convenient to introduce the coordinate *history* of the path $\mathbf{e} \overset{\imath}{\curvearrowright} \mathbf{e}'$. For $\mathbf{e} \in \{\pm 1\}^k$ and pattern $\imath = (i_1, \ldots, i_t)$ it is a sequence of t pairs of indices in $\{1, \ldots, k\}$ that keeps track of the coordinates of $\mathbf{e}$ that are deleted at each edge of the path. More explicitly, it is the sequence

$$((x_1, y_1), \ldots, (x_t, y_t))$$

where $x_i < y_i$ are the coordinates of $\mathbf{e}$ that are deleted in the jth edge, $j = 1, \ldots, t$. The coordinate history is defined recursively as follows. Set $(x_1, y_1) = (i_1, i_1 + 1)$. For $1 < s \leqslant t$ let $z_1 < \cdots < z_{k-2(s-1)}$ be such that

$$\{z_1, \ldots, z_{k-2(s-1)}\} = \{1, \ldots, k\} \setminus \{x_1, y_1, \ldots, x_{s-1}, y_{s-1}\}.$$

Then we define

$$x_s = z_{i_s} \quad \text{and} \quad y_s = z_{i_s+1}.$$

For future reference we record the following basic properties of the history of a path. They are straightforward from the definitions.

$$\text{For } 1 \leqslant i, j \leqslant t \text{ the inequalities } x_i < x_j < y_i < y_j \text{ cannot hold simultaneously.} \tag{19}$$

$$[x_i + 1, y_i - 1] \subseteq \{x_j, y_j : j = 1, \ldots, i-1\} \quad \text{for } i = 2, \ldots, t. \tag{20}$$

Here and henceforth, for integers $a \leqslant b$ we denote by $[a, b] = \{i \in \mathbb{Z} \mid a \leqslant i \leqslant b\}$ the associated interval of integers.

For $t \in \mathbb{Z}_{\geqslant 0}$ let $\mathbf{f}_t = (1, -1, \ldots, (-1)^{t-1}) \in \{\pm 1\}^t$ where $\mathbf{f}_0 \in V$ is the empty tuple of signs.

Definition 5. We define the subsets of vertices V_t and V_{-t} by

$$V_{\pm t} = \{\mathbf{e} \in V : \mathbf{e} \curvearrowright \pm \mathbf{f}_t\}$$

and let $\mathcal{E}_{\pm t}$ be the full subgraph of $\mathcal{E}$ with vertices $V_{\pm t}$.

https://doi.org/10.1017/S1474748019000161 Published online by Cambridge University Press

Clearly, $\mathbf{e} \in V$ is not the origin of any edge in $\mathcal{E}$ if and only if $\mathbf{e} = \pm\mathbf{f}_t$ for some $t \in \mathbb{Z}_{\geqslant 0}$. Since the number of coordinates decreases by two with every edge it follows that

$$V = \bigcup_{t \in \mathbb{Z}} V_t.$$

As we will soon see, this union is disjoint.

Lemma 8. *Let $\mathbf{e} \xrightarrow{i} \mathbf{e}'$ be an edge in $\mathcal{E}$ and $t \in \mathbb{Z}$. Then $\mathbf{e} \in V_t$ if and only if $\mathbf{e}' \in V_t$.*

Proof. If $\mathbf{e}' \overset{J}{\curvearrowright} \pm\mathbf{f}_t$ then clearly also $\mathbf{e} \overset{(i,J)}{\curvearrowright} \pm f_t$. The if part follows.

Let $((x_1, y_1), \ldots, (x_k, y_k))$ be the history of a path $\mathbf{e} = (e_1, \ldots, e_t) \curvearrowright \pm\mathbf{f}_t$. Note that $\{i, i+1\} \cap \{x_1, y_1, \ldots, x_k, y_k\}$ cannot be empty. We separate the proof into several cases.

Assume first that $(i, i+1) = (x_j, y_j)$ for some j. Then it is a simple consequence of (19) and (20) that

$$((x_j, y_j), (x_1, y_1), \ldots, (x_{j-1}, y_{j-1}), (x_{j+1}, y_{j+1}), \ldots, (x_k, y_k))$$

is the history of a path $\mathbf{e} \curvearrowright \pm\mathbf{f}_t$ that starts with the edge $\mathbf{e} \xrightarrow{i} \mathbf{e}'$. Truncating this edge gives a path $\mathbf{e}' \curvearrowright \pm\mathbf{f}_t$.

Next, assume that there exist $a \neq b$ such that $i \in \{x_a, y_a\}$ and $i+1 \in \{x_b, y_b\}$. It follows from (19) and (20) that there are three cases to consider:

(1) $y_a = i$ and $x_b = i+1$;

(2) $y_a = i$, $y_b = i+1$, $x_b < x_a$ and $a < b$;

(3) $x_a = i$, $x_b = i+1$, $y_b < y_a$ and $b < a$.

In case (1), let

- $A = \{j \in \{1, \ldots, k\} : x_a < x_j, y_j < i\}$,
- $B = \{j \in \{1, \ldots, k\} : i+1 < x_j, y_j < y_b\}$ and
- $C = \{j \in \{1, \ldots, k\} : x_j, y_j \notin [x_a, y_b]\}$.

In case (2) let

- $A = \{j \in \{1, \ldots, k\} : x_b < x_j, y_j < x_a\}$,
- $B = \{j \in \{1, \ldots, k\} : x_a < x_j, y_j < i\}$ and
- $C = \{j \in \{1, \ldots, k\} : x_j, y_j \notin [x_b, i+1]\}$.

In case (3) let

- $A = \{j \in \{1, \ldots, k\} : y_b < x_j, y_j < y_a\}$,
- $B = \{j \in \{1, \ldots, k\} : i+1 < x_j, y_j < y_b\}$ and
- $C = \{j \in \{1, \ldots, k\} : x_j, y_j \notin [i, y_a]\}$.

It then follows from (19) and (20) that with the standard linear order on A, on B, and on C the sequence $((x_j, y_j)_{j \in A}, (x_j, y_j)_{j \in B}, (i, i+1), (z, w), (x_j, y_j)_{j \in C})$ is the history of a path $\mathbf{e} \curvearrowright \pm\mathbf{f}_t$ where

$$(z, w) = \begin{cases} (x_a, y_b) & \text{in case (1)} \\ (x_b, x_a) & \text{in case (2)} \\ (y_b, y_a) & \text{in case (3).} \end{cases}$$

Since $(i, i+1)$ is in the history of this path, it follows from the case considered first that $\mathbf{e}' \curvearrowright \pm\mathbf{f}_t$.

The only case left is where $\{i, i+1\} \cap \{x_1, y_1, \ldots, x_k, y_k\}$ is a singleton set. It follows from (19) and (20) that if the intersection is $\{i\}$ then $i = y_a$ for some a, if the intersection is $\{i+1\}$ then $i = x_a$ for some a and that in either case, for this a the sequence

$$((x_1, y_1), \ldots, (x_{a-1}, y_{a-1}), (i, i+1), (x_{a+1}, y_{a+1}), \ldots, (x_k, y_k))$$

is the history of another path $\mathbf{e} \curvearrowright \pm\mathbf{f}_t$. We are reduced again to the first case and the lemma follows. □

As an immediate consequence of Lemma 8, by induction on the length of the path $\mathbf{e} \curvearrowright \mathbf{e}'$, we have the following.

Corollary 5. *For a path $\mathbf{e} \curvearrowright \mathbf{e}'$ in $\mathcal{E}$ and $t \in \mathbb{Z}$ we have $\mathbf{e} \in V_t$ if and only if $\mathbf{e}' \in V_t$. In particular, $\mathcal{E}_t$, $t \in \mathbb{Z}$ are the (non-directed) connected components of $\mathcal{E}$ and the union*

$$V = \bigsqcup_{t \in \mathbb{Z}} V_t$$

is disjoint.

Clearly, every $\mathbf{e} \in V \setminus \{\mathbf{f}_0\}$ is of the form $\mathbf{e} = \pm(\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, \ldots, (-1)^{m-1}\mathbf{1}_{s_m})$ for unique $m, s_1, \ldots, s_m \in \mathbb{N}$ where $\mathbf{1}_s = (1, \ldots, 1) \in \{\pm 1\}^s$, $s \in \mathbb{N}$. For our purposes it will be more convenient to represent elements of V differently, in terms of the $\mathbf{f}_t$.

For $m, t_1, \ldots, t_m \in \mathbb{N}$ let

$$\mathbf{f}_{t_1,\ldots,t_m} = (\mathbf{f}_{t_1}, (-1)^{\tau_2}\mathbf{f}_{t_2}, \ldots, (-1)^{\tau_m}\mathbf{f}_{t_m})$$

where $\tau_{2i} = t_1 + \cdots + t_{2i-1} + 1$ and $\tau_{2i+1} = t_1 + \cdots + t_{2i}$, $i = 1, \ldots, [m/2]$. For example,

$$\mathbf{f}_{3,1,1,2} = (1, -1, 1, 1, 1, 1, -1).$$

It is a simple observation that every $\mathbf{e} \in V \setminus \{\mathbf{f}_0\}$ is of the form $\mathbf{e} = \pm\mathbf{f}_{t_1,\ldots,t_m}$ for unique $m, t_1, \ldots, t_m \in \mathbb{N}$. Indeed, for $\mathbf{e} = (e_1, \ldots, e_k) \in V \setminus \{\mathbf{f}_0\}$ we have $\mathbf{e} = e_1\mathbf{f}_{t_1,\ldots,t_m}$ where

$$\{i \in \{1, \ldots, k\} : e_i = e_{i+1}\} = \{t_1 + \cdots + t_i : i = 1, \ldots, m-1\}$$

and $t_1 + \cdots + t_m = k$.

For $\mathbf{e} = \pm\mathbf{f}_{t_1,\ldots,t_m} \in V \setminus \{\mathbf{f}_0\}$ let

$$\tau(\mathbf{e}) = \pm(-1)^m \sum_{i=1}^{m} (-1)^i t_i.$$

We also set $\tau(\mathbf{f}_0) = 0$.

Lemma 9. *For $\mathbf{e} \in V$ we have $\mathbf{e} \in V_{\tau(\mathbf{e})}$.*

Proof. Let $m, t_1, \ldots, t_m \in \mathbb{N}$ and $\mathbf{e} = \pm\mathbf{f}_{t_1,\ldots,t_m}$. Since $\tau(\pm\mathbf{f}_{t_1}) = \pm t_1$ the case when $m = 1$ is straightforward. For $m > 1$ we proceed by induction on $t_1 + \cdots + t_m$. Note that $\mathbf{e} \xrightarrow{t_1} \mathbf{e}'$

https://doi.org/10.1017/S1474748019000161 Published online by Cambridge University Press

where

$$\mathbf{e}' = \begin{cases} \pm\mathbf{f}_{t_1-1,t_2-1,t_3,\ldots,t_m} & t_1,\ t_2 > 1 \\ \mp\mathbf{f}_{t_2-1,t_3,\ldots,t_m} & t_1 = 1 < t_2 \\ \pm f_{t_1-1+t_3,t_4,\ldots,t_m} & t_1 > t_2 = 1 \\ \pm\mathbf{f}_{t_3,\ldots,t_m} & t_1 = t_2 = 1. \end{cases} \quad (21)$$

In all cases we have $\tau(\mathbf{e}) = \tau(\mathbf{e}')$. By the induction hypothesis we have $\mathbf{e}' \in V_{\tau(\mathbf{e})}$ and by Lemma 8, we get that $\mathbf{e} \in V_{\tau(\mathbf{e})}$. □

The combinatorial description of discrete series representations that have trivial partial cuspidal support is based on the graphs $\mathcal{E}_0$ and $\mathcal{E}_1$. We finish this section with two simple lemmas concerning these subgraphs.

Lemma 10. *Let* $\mathbf{f}_0 \neq \mathbf{e} = \pm\mathbf{f}_{t_1,\ldots,t_m} \in V_0$. *Then there exists a path* $\mathbf{e} \overset{\imath}{\curvearrowright} \mathbf{f}_0$ *with pattern* $\imath$ *of the form* $(i_1, \ldots, i_T, t_1, \ldots, 2, 1)$ *where* $i_j > t_1 + 1$, $j = 1, \ldots, T$.

If in addition $\mathbf{e} = (e_1, \ldots, e_k)$ *with* $e_{t_1+2} = e_{t_1+1} (= e_{t_1})$ *then there exists a path* $\mathbf{e} \overset{\jmath}{\curvearrowright} \mathbf{f}_0$ *with pattern* $\jmath$ *of the form* $(i_1, \ldots, i_S, t_1 + 1, t_1, \ldots, 2, 1)$ *where* $i_j > t_1 + 2$, $j = 1, \ldots, S$.

Proof. We prove the first part by induction on $t_1 + \cdots + t_m$. Let $\mathbf{e} \xrightarrow{t_1} \mathbf{e}'$. It follows from Lemma 8 that $\mathbf{e}' \in V_0$. If $t_1 = 1$ it clearly follows that a path $\imath$ as required exists. This case includes the basis of induction. Assume now that $t_1 > 1$. By (21) we have that $\mathbf{e}' = \pm\mathbf{f}_{s_1,\ldots,s_k}$ with $s_1 \geqslant t_1 - 1$ (in particular $\mathbf{e}' \neq \mathbf{f}_0$). By induction there exists a path $\mathbf{e}' \overset{\imath'}{\curvearrowright} \mathbf{f}_0$ with pattern $\imath'$ of the form $(j_1, \ldots, j_S, s_1, \ldots, 2, 1)$ where $j_k > s_1 + 1$ for $k = 1, \ldots, S$. Thus, $\imath'$ is also of the form $(j_1, \ldots, j_{S+s_1-t_1+1}, t_1 - 1, \ldots, 2, 1)$ where $j_k > t_1 - 1$ for $j = 1, \ldots, S + s_1 - t_1 + 1$. It is then straightforward that $\mathbf{e} \overset{\imath}{\curvearrowright} \mathbf{f}_0$ with $\imath = (j_1 + 2, \ldots, j_{S+s_1-t_1+1} + 2, t_1, \ldots, 2, 1)$. The first part follows.

Assume now that $e_{t_1} = e_{t_1+2}$. Equivalently, assume that $m > 2$ and $t_2 = 1$. Applying Lemma 8 let $\mathbf{e}' \in V_0$ be such that $\mathbf{e} \xrightarrow{t_1+1} \mathbf{e}'$. Note that then $\mathbf{e}' = \pm\mathbf{f}_{s_1,\ldots,s_k}$ with $s_1 \geqslant t_1$. It therefore follows from the first part of the lemma that there exists a path $\mathbf{e}' \overset{\jmath'}{\curvearrowright} \mathbf{f}_0$ with pattern of the form $\jmath' = (j_1, \ldots, j_S, s_1, \ldots, 2, 1)$ and $j_k > s_1 + 1$ for $k = 1, \ldots, S$. Writing $\jmath' = (j_1, \ldots, j_{S+s_1-t_1}, t_1, \ldots, 2, 1)$ we have $j_k > t_1$ for $k = 1, \ldots, S + s_1 - t_1$. It is then straightforward that $\mathbf{e} \overset{\jmath}{\curvearrowright} \mathbf{f}_0$ with $\jmath = (j_1 + 2, \ldots, j_{S+s_1-t_1} + 2, t_1 + 1, t_1, \ldots, 2, 1)$. The lemma follows. □

Lemma 11. *Let* $\mathbf{f}_1 \neq \mathbf{e} = \pm\mathbf{f}_{t_1,\ldots,t_m} \in V_1$. *Then there exists a path* $\mathbf{e} \overset{\imath}{\curvearrowright} \mathbf{f}_1$ *with pattern* $\imath$ *of the form* $(i_1, \ldots, i_T, t_1, t_1 - 1, \ldots, x)$ *where* $x \in \{1, 2\}$, $x = 1$ *if* $t_1 = 1$ *and* $i_j > t_1 + 1$, $j = 1, \ldots, T$.

If in addition $\mathbf{e} = (e_1, \ldots, e_k)$ *with* $e_{t_1+2} = e_{t_1+1} (= e_{t_1})$ *then there exists a path* $\mathbf{e} \overset{\jmath}{\curvearrowright} \mathbf{f}_1$ *with pattern* $\jmath$ *of the form* $(i_1, \ldots, i_S, t_1 + 1, t_1, t_1 - 1, \ldots, x)$ *where* $x \in \{1, 2\}$ *and* $i_j > t_1 + 2$, $j = 1, \ldots, S$.

Proof. We prove the first part by induction on $t_1 + \cdots + t_m$. Let $\mathbf{e} \xrightarrow{t_1} \mathbf{e}'$. It follows from Lemma 8 that $\mathbf{e}' \in V_1$. If $t_1 = 1$ it clearly follows that a path $\imath$ as required exists. This case

https://doi.org/10.1017/S1474748019000161 Published online by Cambridge University Press

includes the basis of induction. Assume now that $t_1 > 1$. By (21) we have that $\mathbf{e}' = \pm\mathbf{f}_{s_1,\dots,s_k}$ with $s_1 \geqslant t_1 - 1$. If $\mathbf{e}' = \mathbf{f}_1$ then necessarily $\mathbf{e} = (1, -1, -1)$ and the path $\mathbf{e} \overset{(2)}{\curvearrowright} \mathbf{f}_1$ satisfies the required condition with $x = 2$. Assume now that $\mathbf{e}' \neq \mathbf{f}_1$. By induction there exists a path $\mathbf{e}' \overset{\imath'}{\curvearrowright} \mathbf{f}_1$ with pattern $\imath'$ of the form $(j_1, \dots, j_S, s_1, s_1 - 1, \dots, x)$ where $x \in \{1, 2\}$ and $j_k > s_1 + 1$ for $k = 1, \dots, S$. If we write $\jmath = (j_1, \dots, j_{S+s_1-t_1+1}, t_1 - 1, \dots, x)$ then $j_k > t_1 - 1$ for $j = 1, \dots, S + s_1 - t_1 + 1$. It is then straightforward that $\mathbf{e} \overset{\imath}{\curvearrowright} \mathbf{f}_1$ with $\imath = (j_1 + 2, \dots, j_{S+s_1-t_1+1} + 2, t_1, \dots, x)$. The first part follows.

Assume now that $e_{t_1} = e_{t_1+2}$. Applying Lemma 8 let $\mathbf{e}' \in V_0$ be such that $\mathbf{e} \xrightarrow{t_1+1} \mathbf{e}'$. Note that then $\mathbf{e}' = \pm\mathbf{f}_{s_1,\dots,s_k}$ with $s_1 \geqslant t_1$. It therefore follows from the first part of the lemma that there exists a path $\mathbf{e}' \overset{\jmath'}{\curvearrowright} \mathbf{f}_1$ with pattern of the form $\jmath' = (j_1, \dots, j_S, s_1, \dots, x)$ and $j_k > s_1 + 1$ for $k = 1, \dots, S$. Writing $\jmath' = (j_1, \dots, j_{S+s_1-t_1}, t_1, \dots, 2, 1)$ we have $j_k > t_1$ for $k = 1, \dots, S + s_1 - t_1$. It is then straightforward that $\mathbf{e} \overset{\jmath}{\curvearrowright} \mathbf{f}_0$ with $\jmath = (j_1 + 2, \dots, j_{S+s_1-t_1} + 2, t_1 + 1, t_1, \dots, x)$. The lemma follows. □

6. Vanishing in the discrete series class

Let

$$\mathrm{Irr}_{\mathrm{U}} = \bigsqcup_{n=0}^{\infty} \mathrm{Irr}(U_{2n})$$

where $\mathrm{Irr}(U_0) = \{\mathbf{1}_0\}$ consists of the trivial representation of the trivial group. In light of Proposition 2 we introduce the following class of representations. Let $\Pi^{\circ}_{\mathrm{disc}} \subseteq \mathrm{Irr}_{\mathrm{U}}$ be the set of discrete series representations with trivial partial cuspidal support. We first provide an interpretation of the Mœglin–Tadić classification for this particular class of irreducible discrete series representations.

6.1. Preliminaries on classification of discrete series representations

6.1.1. Let

$$\mathrm{Cusp}_{\mathrm{GL}} = \bigsqcup_{n=1}^{\infty} \mathrm{Cusp}(\mathrm{GL}_n(E)), \quad \mathrm{Irr}_{\mathrm{GL}} = \bigsqcup_{n=0}^{\infty} \mathrm{Irr}(\mathrm{GL}_n(E))$$

and $\nu(g) = |\det g|_E$ for any $g \in \mathrm{GL}_n(E)$ and $n \in \mathbb{N}$. A Zelevinsky segment is a subset of $\mathrm{Cusp}_{\mathrm{GL}}$ of the form

$$[a, b]_{(\rho)} = \{\nu^i \rho \mid i = a, a + 1, \dots, b\}$$

where $\rho \in \mathrm{Cusp}_{\mathrm{GL}}$, $a, b \in \mathbb{R}$ and $b - a + 2 \in \mathbb{N}$ (so that $[a, a - 1]_{(\rho)}$ is empty).

For a Zelevinsky segment $\Delta = [a, b]_{(\rho)}$ we associate the representation $L(\Delta) \in \mathrm{Irr}_{\mathrm{GL}}$, the unique irreducible quotient of $\nu^a \rho \times \nu^{a+1} \rho \times \cdots \times \nu^b \rho$. It is an essentially square-integrable representation and all essentially square-integrable representations in $\mathrm{Irr}_{\mathrm{GL}}$ are of this form.

For $\rho \in \mathrm{Cusp}_{\mathrm{GL}}$ and $a \in \mathbb{N}$ let

$$\delta(\rho, a) = L\left(\left[\frac{1-a}{2}, \frac{a-1}{2}\right]_{(\rho)}\right).$$

https://doi.org/10.1017/S1474748019000161 Published online by Cambridge University Press

For a Zelevinsky segment Δ we also set

$$\Delta^\vee = \{\rho^\vee \mid \rho \in \Delta\}, \quad \overline{\Delta} = \{\overline{\rho} \mid \rho \in \Delta\} \quad \text{and} \quad \nu^a \Delta = \{\nu^a \rho \mid \rho \in \Delta\}, \quad a \in \mathbb{R}.$$

6.1.2. Cuspidal reducibility points. Let $\rho \in \mathrm{Cusp}_{\mathrm{GL}}$. If ρ is unitary and $\rho^\vee \not\cong \overline{\rho}$ then $\nu^x \rho \rtimes \mathbf{1}_0$ is irreducible for all $x \in \mathbb{R}$. If ρ is conjugate self-dual, that is, such that $\rho^\vee \cong \overline{\rho}$ then there exists a unique $x \in \mathbb{R}_{\geqslant 0}$ such that $\nu^x \rho \rtimes \mathbf{1}_0$ is reducible (and so does $\nu^{-x} \rho \rtimes \mathbf{1}_0$ by Proposition 1(3)). In fact, it is proved in [12, Theorem 3.1] that this x is either 0 or $\frac{1}{2}$. This dichotomy for conjugate self-dual representations in $\mathrm{Cusp}_{\mathrm{GL}}$ is characterized in the literature in various different ways that we collect here together.

Theorem 6. *Let $\rho \in \mathrm{Cusp}_{\mathrm{GL}}$ be conjugate self-dual. The following are equivalent:*

(1) $\nu^{\frac{1}{2}} \rho \rtimes \mathbf{1}_0$ *is reducible;*

(2) *The Asai–Shahidi L-function associated with ρ (see e.g., [12, § 3]) has a pole at $s = 0$;*

(3) *The twisted tensor L-function associated to ρ by Flicker in the Appendix to [9] has a pole at $s = 0$;*

(4) ρ *is* $\mathrm{GL}(F)$*-distinguished.*

Proof. The equivalence of (1) and (2) follows from [12, Theorems 2.7–2.9]. The equivalence of (2) and (3) is immediate from [2, Theorem 1.6] where it is proved that the Asai–Shahidi and Flicker's twisted tensor L-functions coincide. The equivalence of (3) and (4) is [1, Corollary 1.5]. □

As we later observe, this dichotomy is also related to the parity of the L-parameter of ρ (for instance, see [13, Theorem 4.9]). With this in mind we make the following definition.

Definition 6. Let $\rho \in \mathrm{Cusp}_{\mathrm{GL}}$ be conjugate self-dual. We say that ρ is *even* if the equivalent conditions (1)–(4) of Theorem 6 hold and *odd* otherwise.

We will later see that the L-parameter of an even ρ is conjugate orthogonal and of an odd ρ is conjugate symplectic.

6.1.3. Define an admissible datum to be a pair of the form $(\mathcal{J}, (\mathbf{a}_\rho, \epsilon_\rho)_{\rho \in \mathcal{J}})$ where

- $\mathcal{J}$ is a finite set of conjugate self-dual representations in $\mathrm{Cusp}_{\mathrm{GL}}$.
- For each $\rho \in \mathcal{J}$ there is a natural number $k_\rho \in \mathbb{N}$, that is necessarily even if ρ is odd, such that $\mathbf{a}_\rho = (a_1, \ldots, a_{k_\rho})$ with $a_1 > \cdots > a_{k_\rho} \geqslant 0$ and such that $a_1, \ldots, a_{k_\rho}$ are all in $\mathbb{Z}$ if ρ is odd and all in $\frac{1}{2} + \mathbb{Z}$ if ρ is even.
- ϵ_ρ is a k_ρ-tuple of signs in $V_0 \sqcup V_1$ (see Definition 5 for the notation).

6.1.4. For $\pi \in \Pi^\circ_{\mathrm{disc}}$ and any $\rho \in \mathrm{Cusp}_{\mathrm{GL}}$ conjugate self-dual, let $\mathrm{Jord}_\rho(\pi)$ be the set of integers $a \in \mathbb{N}$ such that:

- a is even if ρ is even and odd if ρ is odd,
- $\delta(\rho, a) \rtimes \pi$ is irreducible.

https://doi.org/10.1017/S1474748019000161 Published online by Cambridge University Press

Let $k_\rho(\pi) = |\mathrm{Jord}_\rho(\pi)|$. Let

$$\mathcal{J}_\pi = \{\rho : k_\rho(\pi) > 0\}$$

and for $\rho \in \mathcal{J}_\pi$ let $\mathbf{a}_\rho(\pi) = (a_1, \dots, a_{k_\rho(\pi)})$ where $a_1 > \cdots > a_{k_\rho(\pi)}$ and

$$\mathrm{Jord}_\rho(\pi) = \{2a_i + 1 : i = 1, \dots, k_\rho(\pi)\}.$$

Set $\mathrm{Jord}(\pi) = \{(\rho, a) \mid \rho \in \mathcal{J}_\pi, a \in \mathrm{Jord}_\rho(\pi)\}$.

Mœglin defined the function $\epsilon_\pi : \mathrm{Jord}(\pi) \to \{\pm 1\}$ for $\pi \in \Pi^\circ_{\mathrm{disc}}$ (see [28, § 2]) which we now recall.

First suppose that ρ is odd so that $\rho \rtimes \mathbf{1}_0$ is reducible. It is also known that $\rho \rtimes \mathbf{1}_0$ is a direct sum of two inequivalent tempered representations of which exactly one is generic. Call the generic component τ_1. Now define $\epsilon_\pi((\rho, 2a_1 + 1)) = 1$ if and only if there exists a representation $\tau \in \mathrm{Irr}_{\mathrm{GL}}$ such that $\pi \hookrightarrow \tau \times L([1, a_1]_{(\rho)}) \rtimes \tau_1$.

Suppose now that ρ is even. In this case define $\epsilon_\pi((\rho, 2a_{k_\rho(\pi)} + 1)) = 1$ if and only if there exists a representation $\sigma \in \mathrm{Irr}_{\mathrm{U}}$ such that $\pi \hookrightarrow L([\frac{1}{2}, a_{k_\rho(\pi)}]_{(\rho)}) \rtimes \sigma$.

In either of the two cases we now define ϵ_π on $\mathrm{Jord}_\rho(\pi)$ in the following recursive manner. For any $i \in \{1, \dots, k_\rho(\pi) - 1\}$, we require that $\epsilon_\pi((\rho, 2a_i + 1)) = \epsilon_\pi((\rho, 2a_{i+1} + 1))$ if and only if there exists a representation $\sigma \in \mathrm{Irr}_{\mathrm{U}}$ such that $\pi \hookrightarrow L([a_{i+1} + 1, a_i]_{(\rho)}) \rtimes \sigma$.

Finally, we define $\epsilon_\rho(\pi) = (e_1, \dots, e_{k_\rho(\pi)})$ by setting $e_i = \epsilon_\pi((\rho, 2a_i + 1))$.

6.1.5. The Mœglin–Tadić classification implies that $\pi \mapsto (\mathcal{J}_\pi, (\mathbf{a}_\rho(\pi), \epsilon_\rho(\pi))_{\rho \in \mathcal{J}_\pi})$ is a bijection from $\Pi^\circ_{\mathrm{disc}}$ to the set of admissible data (see [28, Theorem 6.1]).

6.1.6. Next we explain how to extract from an admissible datum corresponding to a representation $\pi \in \Pi^\circ_{\mathrm{disc}}$, ways to realize π as a quotient of a representation induced from an essentially square-integrable representation of a standard Levi subgroup of the Siegel Levi. We apply the results in [28, §§ 7 and 9]. We freely use below notation introduced in § 5.

Mœglin and Tadić realized the discrete series representations as subrepresentations of certain induced representations. For distinction problems it is more convenient to realize representations as quotients. We translate their results via contragredient (recall (1)). Note that $\Pi^\circ_{\mathrm{disc}}$ is preserved by $\pi \mapsto \pi^\vee$ and $\mathcal{J}_{\pi^\vee} = \{\rho^\vee \mid \rho \in \mathcal{J}_\pi\}$, $\pi \in \Pi^\circ_{\mathrm{disc}}$. In fact, we have

$$\mathrm{Jord}(\pi^\vee) = \{(\rho^\vee, a) : (\rho, a) \in \mathrm{Jord}(\pi)\}. \tag{22}$$

Indeed, ρ and $\rho^\vee$ have the same parity (for example, since one is $\mathrm{GL}(F)$-distinguished if and only if the other is) and since $\delta(\rho^\vee, a) = \delta(\rho, a)^\vee$ it follows from (1) that $\delta(\rho^\vee, a) \rtimes \pi^\vee = (\delta(\rho, a) \rtimes \pi)^\vee$. Therefore $\mathbf{a}_{\rho^\vee}(\pi^\vee) = \mathbf{a}_\rho(\pi)$.

Fix $\pi \in \Pi^\circ_{\mathrm{disc}}$ and $\rho \in \mathcal{J}_\pi$ and let $\mathbf{a} = \mathbf{a}_{\rho^\vee}(\pi^\vee) = (a_1, \dots, a_k)$ and $\epsilon = \epsilon_{\rho^\vee}(\pi^\vee) \in V_x$ with $x \in \{0, 1\}$. For any path $\epsilon \overset{\iota}{\curvearrowright} \mathbf{f}_x$ we associate a representation $I_\rho(\mathbf{a}, \epsilon, \iota)$ as follows. Let $((x_1, y_1), \dots, (x_m, y_m))$ be the history of the path ι (in particular $m = [k/2]$). If $k = 2m$ (i.e., $x = 0$) we set

$$I_\rho(\mathbf{a}, \epsilon, \iota) = L([-a_{x_1}, a_{y_1}]_{(\rho)}) \times \cdots \times L([-a_{x_m}, a_{y_m}]_{(\rho)})$$

https://doi.org/10.1017/S1474748019000161 Published online by Cambridge University Press

and if $k = 2m + 1$ (i.e., $x = 1$) and $\{z\} = \{1, \ldots, k\} \setminus \{x_1, \ldots, x_m, y_1, \ldots, y_m\}$ then

$$I_\rho(\mathbf{a}, \epsilon, \iota) = L([-a_{x_1}, a_{y_1}]_{(\rho)}) \times \cdots \times L([-a_{x_m}, a_{y_m}]_{(\rho)}) \times L([-a_z, -1/2]_{(\rho)}).$$

Now let $\pi \in \Pi^\circ_{\mathrm{disc}}$ and for any $\rho \in \mathcal{J}_\pi$ choose a path $\epsilon_{\rho^\vee}(\pi^\vee) \overset{\iota_\rho}{\curvearrowright} \mathbf{f}_{x(\rho)}$ where $x(\rho) = x(\rho, \pi)$ is either zero or one. Let $\mathcal{J}_\pi = \{\rho_1, \ldots, \rho_n\}$ be ordered arbitrarily. Then π is a quotient of the representation

$$I_\pi(\iota_{\rho_1}, \ldots, \iota_{\rho_n}) = I_{\rho_1}(\mathbf{a}_{\rho_1^\vee}(\pi^\vee), \epsilon_{\rho_1^\vee}(\pi^\vee), \iota_{\rho_1}) \times \cdots \times I_{\rho_n}(\mathbf{a}_{\rho_n^\vee}(\pi^\vee), \epsilon_{\rho_n^\vee}(\pi^\vee), \iota_{\rho_n}) \rtimes \mathbf{1}_0.$$

6.2. Statement and proof of the vanishing result

6.2.1. One of our main applications of the geometric lemma is the following.

Proposition 4. *Let $\Delta_1, \ldots, \Delta_m$ be segments satisfying the following property. There exists $k \leqslant m$, such that*

- *for some $\rho \in \mathrm{Cusp}_{\mathrm{GL}}$ conjugate self-dual we have $\Delta_{m+1-i} = [-a_i, b_i]_{(\rho)}$ for all $i = 1, \ldots, k$ where the difference between any two elements in $\{a_1, \ldots, a_k, b_1, \ldots, b_k\}$ is an integer, and $a_1 > \cdots > a_k > b_k > \cdots > b_1$;*
- *for $i \leqslant m - k$, Δ_i is of the form $[-a, b]_{(\rho')}$ where $\rho' \in \mathrm{Cusp}_{\mathrm{GL}}$ is unitary and either $\rho' \not\cong \rho$ or $b < a < b_1$.*

If the representation $\sigma = L(\Delta_1) \otimes \cdots \otimes L(\Delta_m) \otimes \mathbf{1}_0$ (of the relevant Levi subgroup M of U_{2n}) admits a relevant orbit (see Definition 3) then k is even and $a_{2i} = a_{2i-1} - 1$, $i = 1, \ldots, k/2$.

Proof. We apply the notation of § 4.4 for the relevant orbit. Applying [43, § 9.5] write

$$r_{L,M}(\sigma) = \bigotimes_{\iota \in \mathcal{I}} L(\Delta_\iota) \otimes \mathbf{1}_0$$

and recall that

$$\Delta_{(m+1-i, j_{m+1-i})} = [-a_i, x_i]_{(\rho)}$$

for some $x_i \leqslant b_i$, $i = 1, \ldots, k$. (Indeed, if ρ is a representation of $\mathrm{GL}_d(E)$ then $d(x_i + a_i + 1) = a_{(m+1-i, j_{m+1-i})}$ is the size of the $(m + 1 - i, j_{m+1-i})$-block of L.) Since the representation $L(\Delta)$ is generic it is not Sp-distinguished for any Zelevinsky segment Δ (see [16, Theorem 3.2.2]). It therefore follows from Corollary 4 that $\mathfrak{c}_+(w)$ is empty. By assumption

$$\nu L(\overline{\Delta}^\vee_{(m+1-i, j_{m+1-i})}) = L([1 - x_i, a_i + 1]_{(\rho)}) \neq L(\Delta_\iota)$$

for any $\iota \in \mathcal{I}$. It therefore further follows from Corollary 4 that

$$(m + 1 - i, j_{m+1-i}) \in \mathfrak{c}, \quad i = 1, \ldots, k.$$

To complete the proof of this proposition we prove by induction that if $2i - 1 \leqslant k$ then also $2i \leqslant k$, $\tau(m + 1 - 2i, j_{m+1-2i}) = (m + 2 - 2i, j_{m+2-2i})$ and $a_{2i} = a_{2i-1} - 1$.

For $i = 1$ clearly $\tau(m, j_m) \prec (m, j_m)$ (since (m, j_m) is the maximal element of $\mathcal{I}$ and it is not a fixed point of τ). Corollary 4 now further implies that

$$L(\Delta_{\tau(m,j_m)}) = \nu L(\Delta_{(m,j_m)}) = L([1 - a_1, 1 + x_1]_{(\rho)}).$$

https://doi.org/10.1017/S1474748019000161 Published online by Cambridge University Press

It follows from (17) and (18) that $\tau(m, j_m) = (a, b)$ with $a < m$. The assumptions of the proposition now imply that $k \geqslant 2$, $\tau(m, j_m) = (m-1, j_{m-1})$ and $a_2 = a_1 - 1$.

We proceed by induction on i. It follows from the order constraints (17) and (18) on τ that $(n, j) \in \mathfrak{c}$ if and only if $j = j_n$ for $n \in \{m-1, m\}$. Now assume that $\tau(m+1-2i, j_{m+1-2i}) = (m+2-2i, j_{m+2-2i})$ and $a_{2i} = a_{2i-1} - 1$ for $i = 1, \ldots, s-1$, that $(n, j) \in \mathfrak{c}$ if and only if $j = j_n$ for $m+3-2s \leqslant n \leqslant m$ and that $2s-1 \leqslant k$. Since $\tau(m+2-2s, j_{m+2-2s}) \in \mathfrak{c}$ it follows from the induction hypothesis that $\tau(m+2-2s, j_{m+2-2s}) = (i, j)$ with $i \leqslant m+2-2s$ and from (17) and (18) that in fact $i < m+2-2s$. It therefore follows from Corollary 4 that

$$L(\Delta_{\tau(m+2-2s, j_{m+2-2s})}) = \nu L(\Delta_{(m+2-2s, j_{m+2-2s})}) = L([1-a_{2s-1}, 1+x_{2s-1}]_{(\rho)}).$$

The assumptions of the proposition now imply that $2s \leqslant k$, $\tau(m+1-2s, j_{m+1-2s}) = (m+2-2s, j_{m+2-2s})$ and $a_{2s} = a_{2s-1} - 1$. The constraints (17) and (18) on τ also imply that $(n, j) \in \mathfrak{c}$ if and only if $j = j_n$ for $m+1-2s \leqslant n \leqslant m$. The proposition follows. □

6.2.2. We recall that the graph of signs is defined in § 5. We freely use the notation introduced there.

Theorem 7. *Let $\pi \in \Pi^{\circ}_{\mathrm{disc}}$. Suppose that there exists $\rho \in \mathcal{J}_\pi$, with $\epsilon_{\pi^\vee}(\rho^\vee) = (e_1, \ldots, e_k) = \pm\mathbf{f}_{t_1,\ldots,t_m}$ and $\mathbf{a}_{\pi^\vee}(\rho^\vee) = (a_1, \ldots, a_k)$, that satisfies at least one of the following three conditions:*

(1) *t_1 is odd,*

(2) *$a_{2i-1} > a_{2i} + 1$ for some $i \leqslant t_1/2$,*

(3) *$e_{t_1+2} = e_{t_1+1} (= e_{t_1})$.*

Then π is not Sp*-distinguished.*

Proof. In order to show that π is not Sp-distinguished we realize it as a quotient of an induced representation I that is not Sp-distinguished. We separate into several cases, and in each case, writing $I = \mathbf{i}_{G,M}(\sigma)$, we apply Theorem 5 and show that I is not Sp-distinguished by showing that σ has no relevant orbit.

Write $\mathcal{J}_\pi = \{\rho_1, \ldots, \rho_n\}$ with $\rho = \rho_n$ and choose an arbitrary path $\epsilon_{\pi^\vee}(\rho_k^\vee) \overset{\iota_k}{\curvearrowright} \mathbf{f}_{x_k}$ with $x_k \in \{0, 1\}$ for $k = 1, \ldots, n-1$. Let $\mathbf{e} = \epsilon_{\pi^\vee}(\rho^\vee)$. If $\mathbf{e} \in V_0$ then choose a path $\mathbf{e} \overset{\iota_n}{\curvearrowright} \mathbf{f}_0$ to satisfy the requirement of $\imath$ in Lemma 10 if either (1) or (2) above hold and $\jmath$ if (3) holds. Then $I_\pi(\iota_1, \ldots, \iota_n)$ is of the form $\mathbf{i}_{G,M}(\sigma)$ where σ satisfies the assumptions of Proposition 4 with $k = t_1$ in Cases (1) or (2) and with $k = t_1 + 1$ in Case (3). It therefore follows from the proposition that $I_\pi(\iota_1, \ldots, \iota_n)$ and therefore also π is not Sp-distinguished.

Assume now that $\mathbf{e} \in V_1$. Choose a path $\mathbf{e} \overset{\iota_n}{\curvearrowright} \mathbf{f}_1$ to satisfy the requirement of $\imath$ in Lemma 11 if either (1) or (2) above hold and $\jmath$ if (3) holds. If the pattern of the path ι_n ends at $x = 2$ then as in the previous case it follows that $I_\pi(\iota_1, \ldots, \iota_n)$ is of the form $\mathbf{i}_{G,M}(\sigma)$ where σ satisfies the assumptions of Proposition 4 with $k = t_1$ in Cases (1) or (2) and with $k = t_1 + 1$ in Case (3). It follows from the proposition that σ has no contributing orbit and hence π is not Sp-distinguished. Assume now that ι_n ends at $x = 1$. Then

$$I_\pi(\iota_1, \ldots, \iota_n) = L(\Delta_1) \times \cdots \times L(\Delta_m) \times L(\Delta) \rtimes \mathbf{1}_0$$

https://doi.org/10.1017/S1474748019000161 Published online by Cambridge University Press

where $\sigma = L(\Delta_1) \otimes \cdots \otimes L(\Delta_m) \otimes \mathbf{1}_0$ satisfies the assumptions of Proposition 4 with $k = t_1$ in Cases (1) or (2) and with $k = t_1 + 1$ in Case (3). Furthermore $L(\Delta) = L([-a, -1/2]_{(\rho)})$ where in the notation of Proposition 4 we have $0 < a < b_1$ and $-1/2 < b_1$. In other words, $[-a_i, b_i]_{(\rho)}$ contains $[-a, -1/2]_{(\rho)}$ and therefore $L(\Delta) \times L(\Delta_{m+1-i}) \cong L(\Delta_{m+1-i}) \times L(\Delta)$ for $i = 1, \ldots, k$. It follows that $I_\pi(\iota_1, \ldots, \iota_n)$ is isomorphic to

$$L(\Delta_1) \times \cdots \times L(\Delta_{m-k}) \times L(\Delta) \times L(\Delta_{m+1-k}) \times \cdots \times L(\Delta_m) \rtimes \mathbf{1}_0$$

which is of the form $\mathbf{i}_{G,M}(\sigma)$ where σ satisfies the assumptions of Proposition 4 with $k = t_1$ in Cases (1) or (2) and with $k = t_1 + 1$ in Case (3). Again it follows from the proposition that σ has no contributing orbit and hence π is not Sp-distinguished. □

7. Vanishing in the tempered class

In this section we obtain a sufficient condition for a tempered representation to be not Sp-distinguished.

7.1. Tempered representations of U_{2n}

Given an irreducible tempered representation π of U_{2n}, it is known due to Harish-Chandra that there exist irreducible discrete series representations $\pi_{\mathrm{ds}} \in \mathrm{Irr}(U_{2m})$ (for some $m \leqslant n$) and $\delta_1, \ldots, \delta_t \in \mathrm{Irr}_{\mathrm{GL}}$ such that π is a quotient (in fact a direct summand) of the unitary representation $\delta_1 \times \cdots \times \delta_t \rtimes \pi_{\mathrm{ds}}$. Furthermore, π_{ds} and the multi-set $\{\delta_1, \overline{\delta_1}^\vee, \ldots, \delta_t, \overline{\delta_t}^\vee\}$ are uniquely determined by π (see [42, Proposition III.4.1]). We remark further that $\delta_1, \ldots, \delta_t$ can be reordered arbitrarily and each δ_i can be replaced by $\overline{\delta_i}^\vee$.

Following [41, Definition 5.4] we define the Jordan set of the tempered representation π in the following manner. Let $\delta_i = \delta(\rho_i, a_i)$ where $\rho_i \in \mathrm{Cusp}_{\mathrm{GL}}$ is unitary $i = 1, \ldots, t$. Then define $\mathrm{Jord}(\pi)$ to be the multi-set

$$\{(\rho_1, a_1), \ldots, (\rho_t, a_t), (\overline{\rho_1}^\vee, a_1), \ldots, (\overline{\rho_t}^\vee, a_t)\} \cup \mathrm{Jord}(\pi_{\mathrm{ds}}).$$

We also write

$$\{(\rho_1, a_1), \ldots, (\rho_t, a_t), (\overline{\rho_1}^\vee, a_1), \ldots, (\overline{\rho_t}^\vee, a_t)\} = \mathrm{Jord}(\pi) - \mathrm{Jord}(\pi_{\mathrm{ds}}).$$

Note that, unlike in the case of discrete series representations, $\mathrm{Jord}(\pi)$ can have multiplicities and may contain (ρ, a) and (ρ, a') for a and a' of different parity.

Define $\Pi^\circ_{\mathrm{temp}} \subseteq \mathrm{Irr}_{\mathrm{U}}$ to be the class of tempered representations with trivial partial cuspidal support. Thus, $\pi \in \Pi^\circ_{\mathrm{temp}}$ if and only if $\pi_{\mathrm{ds}} \in \Pi^\circ_{\mathrm{disc}}$.

If $\pi \in \mathrm{Irr}_{\mathrm{U}}$ is tempered and not in $\Pi^\circ_{\mathrm{temp}}$ then π is not Sp-distinguished by Proposition 2. We can also exclude distinction for many representations in $\Pi^\circ_{\mathrm{temp}}$.

7.2. A vanishing result for tempered representations

In the following vanishing result for certain tempered representations the condition depends only on the Jordan blocks.

Proposition 5. *Let $\pi \in \Pi^\circ_{\mathrm{temp}}$ be such that there exists $(\rho, a) \in \mathrm{Jord}(\pi) - \mathrm{Jord}(\pi_{\mathrm{ds}})$ which satisfies at least one of the following conditions:*

https://doi.org/10.1017/S1474748019000161 Published online by Cambridge University Press

(1) *ρ is not conjugate self-dual,*

(2) *ρ is conjugate self-dual, and even if and only if a is odd,*

(3) *ρ is conjugate self-dual, and for $(\rho, b) \in \mathrm{Jord}(\pi_{\mathrm{ds}})$ we have that $b \leqslant a$.*

Then π is not Sp*-distinguished.*

Proof. Let $(\rho, a) \in \mathrm{Jord}(\pi) - \mathrm{Jord}(\pi_{\mathrm{ds}})$ satisfy one of the conditions (1)–(3) of the proposition with a maximal. That is, if $(\rho', a') \in \mathrm{Jord}(\pi) - \mathrm{Jord}(\pi_{\mathrm{ds}})$ also satisfies one of the three conditions then $a' \leqslant a$. Let

$$\Delta_1 = \left[\frac{1-a}{2}, \frac{a-1}{2}\right]_{(\rho)}.$$

Let D be the set of segments of the form $[\frac{1-a'}{2}, \frac{a'-1}{2}]_{(\rho')}$ for all pairs $(\rho', a') \in \mathrm{Jord}(\pi) - \mathrm{Jord}(\pi_{\mathrm{ds}})$ such that if ρ is not conjugate self-dual then $\rho' \neq \bar{\rho}^{\vee}$ and let D' be the set of segments of the form $[\frac{1-a'}{2}, \frac{b'-1}{2}]_{(\rho')}$ where $(\rho', a') \in \mathrm{Jord}(\pi_{\mathrm{ds}})$ and either $(\rho', b') \in \mathrm{Jord}(\pi_{\mathrm{ds}})$ or (only if ρ' is even) $b' = 0$. We claim that every $\Delta \in D \cup D'$ is either contained in Δ_1 or disjoint from the set $\{\nu^{j+(a-1)/2}\rho, \nu^{j+(a-1)/2}\bar{\rho}^{\vee} : j \in \mathbb{Z}\}$.

Indeed, assume first that (ρ, a) satisfies either assumption (1) or (2). Then every segment in D' is disjoint from $\{\nu^{j+(a-1)/2}\rho, \nu^{j+(a-1)/2}\bar{\rho}^{\vee} : j \in \mathbb{Z}\}$ (see § 6.1.6). Let $(\rho', a') \in \mathrm{Jord}(\pi) - \mathrm{Jord}(\pi_{\mathrm{ds}})$ be such that $\Delta = [\frac{1-a'}{2}, \frac{a'-1}{2}]_{(\rho')} \in D$. If either $\rho' \neq \rho$ or $a \not\equiv a' (\mathrm{mod}\, 2)$ then Δ is disjoint from $\{\nu^{j+(a-1)/2}\rho, \nu^{j+(a-1)/2}\bar{\rho}^{\vee} : j \in \mathbb{Z}\}$. Otherwise, $(\rho, a') = (\rho', a')$ satisfies the same assumption (either (1) or (2)) as (ρ, a) and by maximality of a we have $a' \leqslant a$. Therefore $\Delta \subseteq \Delta_1$. Assume now that ρ is conjugate self-dual and has the same parity as a. Then (ρ, a) satisfies assumption (3). Let $\Delta' = [\frac{1-a'}{2}, \frac{b'-1}{2}]_{(\rho')} \in D'$ with $(\rho', a') \in \mathrm{Jord}(\pi_{\mathrm{ds}})$. If $\rho' = \rho$ then $a', b' \leqslant a$ and therefore $\Delta' \subseteq \Delta_1$. Otherwise, clearly Δ' is disjoint from $\{\nu^{j+(a-1)/2}\rho, \nu^{j+(a-1)/2}\bar{\rho}^{\vee} : j \in \mathbb{Z}\} = \{\nu^{j+(a-1)/2}\rho : j \in \mathbb{Z}\}$. Let $(\rho', a') \in \mathrm{Jord}(\pi) - \mathrm{Jord}(\pi_{\mathrm{ds}})$ be such that $\Delta = [\frac{1-a'}{2}, \frac{a'-1}{2}]_{(\rho')} \in D$. As above, if either $\rho \neq \rho'$ or $a \not\equiv a' (\mathrm{mod}\, 2)$ then Δ is disjoint from $\{\nu^{j+(a-1)/2}\rho : j \in \mathbb{Z}\}$. Assume that $\rho' = \rho$ and $a \equiv a' (\mathrm{mod}\, 2)$. If (ρ, a') fails to satisfy condition (3) then clearly $a' < a$. Otherwise, by maximality of a we have $a' \leqslant a$. Either way this gives that $\Delta \subseteq \Delta_1$. The claim follows.

Taking (22) into account, it follows from the discussion in §§ 7.1 and 6.1.6 that π can be realized as an irreducible quotient of an induced representation of the form

$$I_\pi = L(\Delta_1) \times \cdots \times L(\Delta_k) \rtimes \mathbf{1}_0$$

where for all $i = 1, \ldots, k$ either Δ_i belongs to D or to D'. As a consequence, we have that either $\Delta_i \subseteq \Delta_1$ or Δ_i is disjoint from $\{\nu^{j+(a-1)/2}\rho, \nu^{j+(a-1)/2}\bar{\rho}^{\vee} : j \in \mathbb{Z}\}$ for $i = 2, \ldots, k$.

Assume by contradiction that π is Sp-distinguished. Then I_π is also Sp-distinguished and by Theorem 5 the representation $\sigma = L(\Delta_1) \otimes \cdots \otimes L(\Delta_k) \otimes \mathbf{1}_0$ has a relevant orbit.

We apply the notation of § 4.4 for the relevant orbit. Let $x = (a-1)/2$. By [43, § 9.5] write

$$r_{L,M}(L(\Delta_1) \otimes \cdots \otimes L(\Delta_k) \otimes \mathbf{1}_0) = \bigotimes_{\iota \in \mathcal{I}} L(\Delta_\iota) \otimes \mathbf{1}_0$$

and recall that

$$\Delta_{(1,j_1)} = [-x, y]_{(\rho)}$$

https://doi.org/10.1017/S1474748019000161 Published online by Cambridge University Press

for some $y \in \{-x, -x+1, \dots, x\}$ and that by assumption

$$\nu^{-x-1}\rho, \nu^{1+x}\overline{\rho^\vee} \notin \bigcup_{i=1}^{k} \Delta_i.$$

By [16, Theorem 3.2.2] and Corollary 4 the set $\mathfrak{c}_+(w)$ is empty. But we now have

$$\nu^{-1}\Delta_{(1,j_1)} = [-x-1, y-1]_{(\rho)} \neq \Delta_\iota, \quad \iota \in \mathcal{I}$$

and

$$\nu\overline{\Delta_{(1,j_1)}}^\vee = [1-y, 1+x]_{(\overline{\rho}^\vee)} \neq \Delta_\iota, \quad \iota \in \mathcal{I}$$

which contradicts Corollary 4. The proposition follows. □

8. Representation theory of general linear groups

Before we continue with further applications of the geometric lemma to Sp-distinction, we need to introduce some further notation and recall the Langlands and Zelevinsky classifications of $\mathrm{Irr}_{\mathrm{GL}}$.

Zelevinsky segments (henceforth, simply segments) were defined in § 6.1.1. For a segment $\Delta = [a, b]_{(\rho)}$, we denote by $b(\Delta) = \nu^a\rho$ its beginning, by $e(\Delta) = \nu^b\rho$ its end and by $\ell(\Delta) = b - a + 1$ its length.

8.1. Classification of $\mathrm{Irr}_{\mathrm{GL}}$

We refer to [43] for the results stated in this section.

8.1.1. Segment representations. Let $\Delta = [a, b]_{(\rho)}$ be a Zelevinsky segment. The representation $\nu^a\rho \times \nu^{a+1}\rho \times \cdots \times \nu^b\rho$ has a unique irreducible subrepresentation which we denote by $Z(\Delta)$. By convention, if the segment Δ is empty then $Z(\Delta)$ is the trivial representation of the trivial group.

8.1.2. For a segment Δ the representation $L(\Delta)$ is defined in § 6.1.1. We remark that $\Delta \mapsto L(\Delta)$ is a bijection between the set of segments and the subset of essentially square-integrable representations in $\mathrm{Irr}_{\mathrm{GL}}$.

8.1.3. For two segments Δ_1 and Δ_2, we write $\Delta_1 \prec \Delta_2$ to denote that Δ_1 precedes Δ_2 (see [43, § 4.1] for the definition).

8.1.4. We denote the set of finite multi-sets of segments by $\mathcal{O}$. For $\mathfrak{m} \in \mathcal{O}$ we write $\mathfrak{m} = \{\Delta_1, \dots, \Delta_t\}$ as an unordered t-tuple. We also let $\mathfrak{m}^\vee = \{\Delta_1^\vee, \dots, \Delta_t^\vee\}$ and similarly define the multi-sets $\overline{\mathfrak{m}}$ and $\nu^a\mathfrak{m}$, $a \in \mathbb{R}$. When choosing a specific order, by abuse of notation we write $\mathfrak{m} = (\Delta_1, \dots, \Delta_t)$ as an ordered t-tuple.

8.1.5. The Zelevinsky classification and the Langlands classification provide bijections between the sets $\mathcal{O}$ and $\mathrm{Irr}_{\mathrm{GL}}$. We denote by $\mathfrak{m} \mapsto Z(\mathfrak{m})$ the bijection defined by [43, Theorem 6.1] and by $\mathfrak{m} \mapsto L(\mathfrak{m})$ the bijection defined by the Langlands classification (see e.g., [37]). We remark that if $\mathfrak{m} = (\Delta_1, \dots, \Delta_t)$ is ordered so that Δ_i does not precede Δ_j for all $1 \leqslant i < j \leqslant t$ then $Z(\mathfrak{m})$ is the unique irreducible subrepresentation of $Z(\Delta_1) \times \cdots \times Z(\Delta_t)$ and $L(\mathfrak{m})$ is the unique irreducible quotient of $L(\Delta_1) \times \cdots \times L(\Delta_t)$.

https://doi.org/10.1017/S1474748019000161 Published online by Cambridge University Press

8.1.6. It follows from the two classifications above that for any $\mathfrak{m} \in \mathcal{O}$ there exists a unique $\mathfrak{m}^t \in \mathcal{O}$ such that $Z(\mathfrak{m}) = L(\mathfrak{m}^t)$. The function $\mathfrak{m} \mapsto \mathfrak{m}^t$ is an involution on $\mathcal{O}$. Given a multi-set $\mathfrak{m}$, an algorithm to compute $\mathfrak{m}^t$ is provided in [30].

For $\pi = Z(\mathfrak{m}) \in \mathrm{Irr}_{\mathrm{GL}}$, let $\pi^t = L(\mathfrak{m})$. The map $\pi \mapsto \pi^t$ is the Zelevinsky involution on $\mathrm{Irr}_{\mathrm{GL}}$.

8.1.7. For an irreducible cuspidal $\rho \in \mathrm{Cusp}_{\mathrm{GL}}$ define its *cuspidal line*

$$\rho^{\mathbb{Z}} = \{\nu^m \rho \mid m \in \mathbb{Z}\}.$$

To $\rho^{\mathbb{Z}}$ we transfer the standard order $\leqslant$ on $\mathbb{Z}$. That is, for $\rho, \rho' \in \mathrm{Cusp}_{\mathrm{GL}}$ we write $\rho \leqslant \rho'$ if $\rho' = \nu^n \rho$ for some $n \in \mathbb{Z}_{\geqslant 0}$.

8.1.8. For every $\pi \in \mathrm{Irr}_{\mathrm{GL}}$ there exist $\rho_1, \dots, \rho_k \in \mathrm{Cusp}_{\mathrm{GL}}$, unique up to rearrangement, so that π is isomorphic to a subrepresentation of $\rho_1 \times \cdots \times \rho_k$. Let the multi-set of cuspidal representations

$$\mathrm{Supp}(\pi) = \{\rho_1, \dots, \rho_k\}$$

be the cuspidal support of π.

8.1.9. For $\mathfrak{m} \in \mathcal{O}$ let $\mathrm{Supp}(\mathfrak{m})$ be the multi-set of cuspidal representations consisting of the union (with multiplicities) of all the segments in $\mathfrak{m}$. We then have $\mathrm{Supp}(\mathfrak{m}) = \mathrm{Supp}(Z(\mathfrak{m})) = \mathrm{Supp}(L(\mathfrak{m}))$.

Definition 7. We say that a representation $\pi \in \mathrm{Irr}_{\mathrm{GL}}$ (respectively multi-set $\mathfrak{m} \in \mathcal{O}$) is *rigid* if $\mathrm{Supp}(\pi) \subseteq \rho^{\mathbb{Z}}$ (respectively $\mathrm{Supp}(\mathfrak{m}) \subseteq \rho^{\mathbb{Z}}$) for some $\rho \in \mathrm{Cusp}_{\mathrm{GL}}$. We then also say that π (respectively $\mathfrak{m}$) is supported on $\rho^{\mathbb{Z}}$.

8.1.10. Exponent of a representation. For a representation $\pi \in \mathrm{Irr}_{\mathrm{GL}}$ with central character ω_π let $\alpha = \exp(\pi) \in \mathbb{R}$ be the exponent of π. It is the unique real number such that $\nu^{-\alpha}\omega_\pi$ is a unitary character.

For a segment Δ define its exponent by

$$\exp(\Delta) = \exp(Z(\Delta)) = \exp(L(\Delta)).$$

8.2. Langlands parameters

Let W_E denote the Weil group of E and

$$W'_E = W_E \times \mathrm{SL}_2(\mathbb{C})$$

the Weil–Deligne group. The Langlands parameter (henceforth, L-parameter) of an irreducible representation of $\mathrm{GL}_n(E)$ is an n-dimensional continuous semi-simple complex representation of the group W'_E. Let $\Phi(\mathrm{GL}_n(E))$ be the set of all equivalence classes of L-parameters for the group $\mathrm{GL}_n(E)$. The Langlands reciprocity map, is the bijection

$$\mathrm{rec}_{\mathrm{n}} : \mathrm{Irr}(\mathrm{GL}_n(E)) \to \Phi(\mathrm{GL}_n(E))$$

established in [15]. We denote by $\mathrm{rec}_{\mathrm{GL}} : \bigsqcup_{n=1}^{\infty} \mathrm{Irr}(\mathrm{GL}_n(E)) \to \bigsqcup_{n=1}^{\infty} \Phi(\mathrm{GL}_n(E))$ the union of these bijections for all n.

https://doi.org/10.1017/S1474748019000161 Published online by Cambridge University Press

For an L-parameter $\phi \in \Phi(\mathrm{GL}_n(E))$, denote by ${}^c\phi^\vee$ its conjugate-dual parameter (see, for instance, [32, § 2.2] for the precise definition). The following fact [15, Lemma VII.1.6] will be used without further mention:

$$\mathrm{rec}_{\mathrm{GL}}(\overline{\pi}^\vee) = {}^c(\mathrm{rec}_{\mathrm{GL}}(\pi))^\vee.$$

We say that a parameter $\phi \in \Phi(\mathrm{GL}_n(E))$ is conjugate self-dual if $\phi \cong {}^c\phi^\vee$. We refer to [32, § 2.2] for the notion of a conjugate self-dual parameter with parity ± 1 and we also apply the terminology of [10, § 3] to say that $\phi \in \Phi(\mathrm{GL}_n(E))$ conjugate self-dual is conjugate orthogonal if it has parity 1 and conjugate symplectic if it has parity -1.

The results of Mok [32, Lemma 2.2.1 and Theorem 2.5.1] combined with [12, Theorems 2.7 and 2.8] imply the following.

Theorem 8. *Let* $\rho \in \mathrm{Cusp}_{\mathrm{GL}}$ *be conjugate self-dual. Then* ρ *is even (see Definition 6) if and only if* $\mathrm{rec}_{\mathrm{GL}}(\rho)$ *is conjugate orthogonal.*

Remark 5. As a consequence, a conjugate self-dual discrete series representation in $\mathrm{Irr}_{\mathrm{GL}}$ always has parity determined as follows. Let $\rho \in \mathrm{Cusp}_{\mathrm{GL}}$ be conjugate self-dual and $a \in \mathbb{N}$. Then $\delta(\rho, a)$ is conjugate self-dual with parity $(-1)^{a-1}\eta_\rho$ where η_ρ is the parity of ρ.

8.3. Ladder representations

The class of ladder representations was introduced in [19]. The Jacquet modules of a ladder representation are calculated explicitly in [17, Corollary 2.2]. Moreover, this class is preserved by the Zelevinsky involution and the algorithm provided in [30] to compute the Zelevinsky involution of an irreducible representation takes a much simpler form when the representation is a ladder (see [19, § 3]). Some of these results and structural properties make this class more approachable in comparison to the entire admissible dual for the purpose of distinction problems (for instance, see [26]). We will now recall their definition.

8.3.1. Definition of ladder representations.

Definition 8. An (ordered) multi-set $\mathfrak{m} \in \mathcal{O}$ is called a *ladder* if there exist $\rho \in \mathrm{Cusp}_{\mathrm{GL}}$ and integers

$$a_1 > \cdots > a_k \quad \text{and} \quad b_1 > \cdots > b_k$$

such that $\mathfrak{m} = (\Delta_1, \ldots, \Delta_k)$ where $\Delta_i = [a_i, b_i]_{(\rho)}$. A representation $\pi \in \mathrm{Irr}_{\mathrm{GL}}$ is called a ladder representation if $\pi = L(\mathfrak{m})$ (or equivalently, if $\pi = Z(\mathfrak{m})$) where $\mathfrak{m}$ is a ladder.

8.3.2. In the above notation, the ladder representation $\pi = L(\mathfrak{m})$ is called a Speh representation if

$$a_i = a_{i+1} + 1 \quad \text{and} \quad b_i = b_{i+1} + 1 \quad \text{for } i = 1, \ldots, k-1. \tag{23}$$

We recall the fact that any unitary representation in $\mathrm{Irr}_{\mathrm{GL}}$ can be obtained as a parabolic induction of Speh representations (see [39, Theorem D]).

https://doi.org/10.1017/S1474748019000161 Published online by Cambridge University Press

9. Further applications of the geometrical lemma to distinction

Let $P = M \ltimes U$ be a standard parabolic subgroup of U_{2n}, contained in the Siegel parabolic, with its standard Levi decomposition and fix a double coset in $P\backslash U_{2n}/\mathrm{Sp}_{2n}$. We will now collect some simple consequences of the geometric lemma (described in § 4) that we use later in the article. Throughout this section we freely apply the notation of § 4.4 for the chosen orbit.

Lemma 12. *Let $\iota = (c, d) \in \mathcal{I}$ be the minimal index in $\mathfrak{c}$ such that $\tau(\iota) \neq \iota$. Then $\tau(\iota) = (a, j_a)$ for some $a > c$.*

Proof. Write $\tau(\iota) = (a, b)$. By the minimality of ι and (18), it follows that $c < a$. Assume if possible that $b < j_a$. By (17) we have $(a, j_a) \in \mathfrak{c}$ and by (18) we have $\tau(a, j_a) \prec \tau(a, b) = \iota$ which is contradicting the minimality of ι. □

Lemma 13. *Let $\rho \in \mathrm{Cusp}_{\mathrm{GL}}$ and $\mathfrak{m} = (\Delta_1, \dots, \Delta_t) \in \mathcal{O}$ be an ordered multi-set with $\Delta_i = [a_i, b_i]_{(\rho)}$, $i = 1, \dots, t$ and $b_1 \leqslant \cdots \leqslant b_t$. Suppose that $w \in {}_MW_M \cap W[2]$ is relevant to $Z(\Delta_1) \otimes \cdots \otimes Z(\Delta_t) \otimes \mathbf{1}_0$. For the orbit corresponding to w we have $\mathfrak{c} = \mathfrak{c}_+(w)$. In particular, $\mathfrak{c}$ is of the form*

$$\mathfrak{c} = \{(a, j_a) : a \in A\}$$

for some $A \subseteq \{1, \dots, t\}$.

Proof. Let $L = M \cap w^{-1}Mw$ and applying [43, Proposition 3.4], in the above notation write

$$r_{L,M}(Z(\Delta_1) \otimes \cdots \otimes Z(\Delta_t) \otimes \mathbf{1}_0) = \bigotimes_{\iota \in \mathcal{I}} Z(\Delta_\iota) \otimes \mathbf{1}_0.$$

Then $\Delta_{(i,j_i)} = [y_i, b_i]_{(\rho)}$ for some $y_i \geqslant a_i$ for $i = 1, \dots, t$. Assume by contradiction that there exists an index in $\mathfrak{c}$ not fixed by the involution τ and denote the minimal such index by $\iota = (c, d)$. By Lemma 12, there exists $a \in \{c + 1, \dots, t\}$ such that $\tau(\iota) = (a, j_a)$. Thus, by Corollary 4 we have $\nu^{-1}Z(\Delta_\iota) = Z(\Delta_{(a,j_a)})$. This implies that $\Delta_\iota = [y_a + 1, b_a + 1]_{(\rho)}$ which contradicts the hypothesis on $\mathfrak{m}$ and the fact that $c < a$. Thus, $\mathfrak{c} = \mathfrak{c}_+(w)$. The last part of the lemma follows from (18). □

Lemma 14. *Let $\pi_1 \in \Pi(\mathrm{GL}_{p_1}(E))$ and $\pi_2 \in \Pi(\mathrm{GL}_{2p_2}(E))$ be such that $\nu^{-1/2}\pi_1$ is $\mathrm{GL}(F)$-distinguished and π_2 is Sp-distinguished ($p_1, p_2 \geqslant 0$). Then $\pi_1 \times \pi_2 \rtimes 1$ is Sp-distinguished.*

Proof. This follows from a combination of an open orbit and a closed orbit argument for lifting distinguished representations via parabolic induction. Indeed, it follows from [34, Proposition 7.2] that the representation $\pi_2 \rtimes 1$ is Sp-distinguished and now the lemma follows from [34, Proposition 7.1]. □

Lemma 15. *Let $\rho_1, \dots, \rho_k \in \mathrm{Cusp}_{\mathrm{GL}}$ be unitary representations such that $\rho_i \not\cong \rho_j, \overline{\rho_j}^\vee$ if $i \neq j$. Let*

$$\tau = L([a_1, b_1]_{(\rho_1)}) \times \cdots \times L([a_k, b_k]_{(\rho_k)})$$

https://doi.org/10.1017/S1474748019000161 Published online by Cambridge University Press

where $a_i, b_i \in \mathbb{R}$ *are such that* $b_i - a_i \in \mathbb{Z}_{\geqslant 0}$, $i = 1, \dots, k$. *Then the representation* $\tau \rtimes \mathbf{1}_0 \in \Pi(U_{2n})$ *is* Sp-*distinguished if and only if* $\nu^{-1/2}\tau$ *is* GL(F)-*distinguished. In particular, if* $\tau \rtimes \mathbf{1}_0$ *is* Sp-*distinguished then* $\exp([a_i, b_i]_{(\rho_i)}) = \frac{1}{2}$ *for all* $i = 1, \dots, k$.

Proof. Let $\Delta_i = [a_i, b_i]_{(\rho_i)}$ for $i = 1, \dots, k$. If $\nu^{-1/2}\tau$ is GL(F)-distinguished, then the representation $\tau \rtimes \mathbf{1}_0$ is Sp-distinguished by Lemma 14. To prove the 'only if' part, suppose that $\tau \rtimes \mathbf{1}_0$ is Sp-distinguished. Let $\sigma = L(\Delta_1) \otimes \cdots \otimes L(\Delta_k) \otimes \mathbf{1}_0$ and M be the standard Levi subgroup of U_{2n} corresponding to σ. Now Sp-distinction of $\tau \rtimes \mathbf{1}_0$ implies that there exists a $w \in {}_M W_M \cap W[2]$ which is relevant to σ. Let $L = M \cap w^{-1} M w$ and applying [43, § 9.5] write

$$r_{L,M}(L(\Delta_1) \otimes \cdots \otimes L(\Delta_k) \otimes \mathbf{1}_0) = \bigotimes_{\iota \in \mathcal{I}} L(\Delta_\iota) \otimes \mathbf{1}_0$$

(in the notation of § 4.4). By the hypothesis, Corollary 4 and the fact that no essentially square-integrable representation in $\mathrm{Irr}_{\mathrm{GL}}$ is Sp-distinguished (by [16, Theorem 3.2.2]), we obtain that $j_i = 1, i = 1, \dots, k$, $\mathfrak{c} = \emptyset$ and the involution τ on $\mathcal{I}$ is trivial. Thus $\nu^{-1/2} L(\Delta_i)$ is GL(F)-distinguished for each $i \in \{1, \dots, k\}$. By [7, Proposition 12] $\exp(\Delta_i) = \frac{1}{2}$. The representation $\nu^{-1/2}\tau$ is GL(F)-distinguished by [8, Proposition 26]. □

10. L-parameters for the unitary groups and the base change map

We now collect some preliminaries relevant to this article on the L-parameters of quasi-split unitary groups and the base change map. We begin by recalling the Langlands quotient theorem for these groups.

10.1. Langlands quotient theorem

10.1.1. A standard module $\lambda \in \Pi(U_{2n})$ is a representation of the form

$$\nu^{x_1}\tau_1 \times \cdots \times \nu^{x_k}\tau_k \rtimes \tau_{\mathrm{temp}}$$

where $k \geqslant 0$, τ_i $(1 \leqslant i \leqslant k)$ are tempered representations in $\mathrm{Irr}_{\mathrm{GL}}$, τ_{temp} is a tempered representation in $\mathrm{Irr}_{\mathrm{U}}$ and $x_1 > \cdots > x_k > 0$. The representation $\nu^{x_1}\tau_1 \times \cdots \times \nu^{x_k}\tau_k$ is the GL-part of λ.

10.1.2. The Langlands quotient theorem for U_{2n} (see [37]) says that a standard module $\lambda \in \Pi(U_{2n})$ has a unique irreducible (Langlands) quotient that we denote by $LQ(\lambda)$ and the map $\lambda \mapsto LQ(\lambda)$ from standard modules in $\Pi(U_{2n})$ to $\mathrm{Irr}(U_{2n})$ is a bijection.

10.2. Base change

We will now recall the definition and some preliminary facts on the base change map from $\mathrm{Irr}(U_{2n})$ to $\mathrm{Irr}(\mathrm{GL}_{2n}(E))$.

10.2.1. L-parameter for U_{2n}. Let W_F denote the Weil group and $W'_F := W_F \times \mathrm{SL}_2(\mathbb{C})$ the Weil–Deligne group of F. The Langlands dual group of U_{2n} is the semi-direct product ${}^L U_{2n} = \mathrm{GL}_{2n}(\mathbb{C}) \rtimes W_F$ where the action of W_F factors through the Galois group $\mathrm{Gal}(E/F)$. An L-parameter of U_{2n} is a map $W'_F \to {}^L U_{2n}$ satisfying several properties (see, for instance, [10, § 8] or [32, § 2.2]). Let $\Phi(U_{2n})$ denote the set of all equivalence

https://doi.org/10.1017/S1474748019000161 Published online by Cambridge University Press

classes of L-parameters of the group U_{2n}. The Langlands reciprocity map for this case was established by Mok (see [32, Theorem 2.5.1]). We denote by $\mathrm{rec}_{\mathrm{U}}$ the union over all $n \in \mathbb{N}$ of the finite to one surjective maps from $\mathrm{Irr}(U_{2n})$ to $\Phi(U_{2n})$ defined by Mok.

Given $\pi \in \mathrm{Irr}(U_{2n})$, the fiber of the map $\mathrm{rec}_{\mathrm{U}}$ containing π is the L-packet of π.

10.2.2. The restriction map ξ'_{bc}. For $\phi \in \Phi(U_{2n})$ the restriction $\phi|_{W'_E}$ lies in $\Phi(\mathrm{GL}_{2n}(E))$. Denote this restriction map $(\phi \mapsto \phi|_{W'_E}) : \bigsqcup_{n=1}^{\infty} \Phi(U_{2n}) \to \bigsqcup_{n=1}^{\infty} \Phi(\mathrm{GL}_{2n}(E))$ by ξ'_{bc}. The map ξ'_{bc} is injective (see [10, Theorem 8.1(ii)] or [32, § 2.2]).

10.2.3. Base change, denoted by ξ_{bc}, is the functorial transfer from $\mathrm{Irr}(U_{2n})$ to $\mathrm{Irr}(\mathrm{GL}_{2n}(E))$ defined by

$$\xi'_{\mathrm{bc}}(\mathrm{rec}_{\mathrm{U}}(\pi)) = \mathrm{rec}_{\mathrm{GL}}(\xi_{\mathrm{bc}}(\pi)).$$

10.2.4. Image of ξ_{bc}. The next result follows from [32, Lemma 2.2.1 and Theorem 2.5.1] (see also [10, Theorem 8.1]).

Proposition 6. *A representation $\pi \in \mathrm{Irr}(\mathrm{GL}_{2n}(E))$ is in the image of the map ξ_{bc} if and only if $\pi^\vee \cong \overline{\pi}$ and $\mathrm{rec}_{\mathrm{GL}}(\pi)$ is conjugate symplectic.*

In what follows we provide some information about the base change fibers (L-packets). First, we remark that this task is reduced to tempered L-packets as follows.

10.2.5. The map ξ_{bc} takes tempered representations to tempered representations. If $\pi \in \mathrm{Irr}_{\mathrm{GL}}$ lies in the image of the base change map then it is the Langlands quotient of a standard module of the form

$$\nu^{x_1}\tau_1 \times \cdots \times \nu^{x_k}\tau_k \times \pi_{\mathrm{temp}} \times \nu^{-x_k}\overline{\tau_k}^\vee \times \cdots \times \nu^{-x_1}\overline{\tau_1}^\vee$$

for unique real numbers $x_1 > \cdots > x_k > 0$ and tempered representations $\tau_1, \ldots, \tau_k$ and π_{temp} in $\mathrm{Irr}_{\mathrm{GL}}$ so that $\mathrm{rec}_{\mathrm{GL}}(\pi_{\mathrm{temp}})$ is conjugate symplectic. We then have

$$\xi_{\mathrm{bc}}^{-1}(\pi) = \{\mathrm{LQ}(\nu^{x_1}\tau_1 \times \cdots \times \nu^{x_k}\tau_k \rtimes \tau_{\mathrm{temp}}) \mid \tau_{\mathrm{temp}} \in \xi_{\mathrm{bc}}^{-1}(\pi_{\mathrm{temp}})\}.$$

10.2.6. An L-packet of tempered representations in $\mathrm{Irr}_{\mathrm{U}}$ either consists entirely of discrete series representations or contains none (see [27, Theorem 5.7]). Mœglin further explicated in [27, § 5] the L-packets consisting of discrete series representations.

A tempered representation $\pi \in \mathrm{Irr}_{\mathrm{GL}}$ is called *F-discrete* if it has the form

$$\pi \cong \delta_1 \times \cdots \times \delta_k \tag{24}$$

where $\delta_1, \ldots, \delta_k \in \mathrm{Irr}_{\mathrm{GL}}$ are discrete series representations such that $\mathrm{rec}_{\mathrm{GL}}(\delta_i)$ is conjugate symplectic for $i = 1, \ldots, k$ and $\delta_i \ncong \delta_j$ for $i \neq j$.

Recall from Remark 5 that for $\rho \in \mathrm{Cusp}_{\mathrm{GL}}$ and $a \in \mathbb{N}$ we have that $\mathrm{rec}_{\mathrm{GL}}(\delta(\rho, a))$ is conjugate symplectic if and only if ρ is conjugate self-dual of parity $(-1)^a$.

For $\pi \in \mathrm{Irr}_{\mathrm{GL}}$ tempered and in the image of base change we have that $\xi_{\mathrm{bc}}^{-1}(\pi)$ consists of discrete series representations if and only if π is stable F-discrete.

Let π be stable F-discrete as in (24) and write $\delta_i = \delta(\rho_i, a_i)$, $i = 1, \ldots, k$. Then the L-packet $\xi_{\mathrm{bc}}^{-1}(\pi)$ consists of discrete series representation $\pi' \in \mathrm{Irr}_{\mathrm{U}}$ such that $\mathrm{Jord}(\pi') = \{(\rho_i, a_i), \mid i = 1, \ldots, k\}$. It further follows from [27, Theorem 7.1] that the base change fiber is of cardinality

$$|\xi_{\mathrm{bc}}^{-1}(\pi)| = 2^{k-1}.$$

This observation, that a discrete series L-packet is determined by the Jordan set, is a consequence of the fact that the extended cuspidal support of a discrete series $\tau \in \mathrm{Irr}_{\mathrm{U}}$ as defined in [27, § 5.4] is, in fact, the cuspidal support of $\times_{(\rho,a)\in\mathrm{Jord}(\tau)}\delta(\rho, a)$. This, in turn, follows from the fact that the basic assumption (BA) of [28] (and hence also [28, (2-3)]) is now a theorem (see [27, Proposition 3.1] and [28, § 12]).

10.2.7. Let $\pi \in \mathrm{Irr}_{\mathrm{GL}}$ be tempered and in the image of base change. Then there is a unique stable F-discrete representation $\pi_0 \in \mathrm{Irr}_{\mathrm{GL}}$ and a tempered representation $\tau \in \mathrm{Irr}_{\mathrm{GL}}$ such that

$$\pi \cong \tau \times \pi_0 \times \overline{\tau}^\vee.$$

Furthermore, $\tau \times \overline{\tau}^\vee$ is uniquely determined by π. We have

$$\xi_{\mathrm{bc}}^{-1}(\pi) = \{\tau' \in \mathrm{Irr}_{\mathrm{U}} \mid \tau' \hookrightarrow \tau \rtimes \pi_0' \text{ for some } \pi_0' \in \xi_{\mathrm{bc}}^{-1}(\pi_0)\}.$$

Write $\tau = \delta_1 \times \cdots \times \delta_t$ where $\delta_i \in \mathrm{Irr}_{\mathrm{GL}}$ is a discrete series representation $i = 1, \ldots, t$. In fact, it follows from [28, Theorem 13.1] (and the fact that $\mathrm{Jord}(\pi_0')$ is independent of $\pi_0' \in \xi_{\mathrm{bc}}^{-1}(\pi_0)$) that

$$|\xi_{\mathrm{bc}}^{-1}(\pi)| = 2^m |\xi_{\mathrm{bc}}^{-1}(\pi_0)|$$

where m is the number of pairs (ρ, a) such that $\rho \in \mathrm{Cusp}_{\mathrm{GL}}$ is conjugate self-dual with parity $(-1)^a$, $\delta(\rho, a) \rtimes \pi_0'$ is reducible for one (and hence all) $\pi_0' \in \xi_{\mathrm{bc}}^{-1}(\pi_0)$ and $\delta(\rho, a) \cong \delta_i$ for some $i \in \{1, \ldots, t\}$. Furthermore, in the above notation, write $\delta_i = \delta(\sigma_i, b_i)$, with $\sigma_i \in \mathrm{Cusp}_{\mathrm{GL}}$ and $b_i \in \mathbb{N}$, $i = 1, \ldots, t$ and $\pi_0 = \delta(\rho_1, a_1) \times \cdots \times \delta(\rho_k, a_k)$ as in (24). Then $\xi_{\mathrm{bc}}^{-1}(\pi)$ consists of the tempered representations $\pi' \in \mathrm{Irr}_{\mathrm{U}}$ such that $\mathrm{Jord}(\pi')$ is the multi-set

$$\{(\sigma_i, b_i), (\overline{\sigma_i}^\vee, b_i) \mid i = 1, \ldots, t\} \cup \{(\rho_i, a_i) \mid i = 1, \ldots, k\}.$$

10.2.8. Next we make more explicit the L-packets of discrete series representations that contain a single member.

For $\rho \in \mathrm{Cusp}_{\mathrm{GL}}$ even and $a \in 2\mathbb{N}$ let $\tau^+(\rho, a) \in \Pi^\circ_{\mathrm{disc}}$ be the representation associated in § 6.1.5 to the admissible data $(\mathcal{J}, (\mathbf{a}, \epsilon))$ where $\mathcal{J} = \{\rho\}$, $\mathbf{a} = ((a-1)/2)$ and $\epsilon = (1)$ (so that $k_\rho = 1$). In other words, $\tau^+(\rho, a)$ is the unique irreducible quotient of

$$L\left(\left[\frac{1-a}{2}, -\frac{1}{2}\right]_{(\rho)}\right) \rtimes \mathbf{1}_0$$

and is a strongly positive discrete series representation (see, for example, [28, § 7]).

Lemma 16. *Let $\delta \in \mathrm{Irr}(\mathrm{GL}_{2n}(E))$ be a discrete series representation in the image of base change. Write $\delta = \delta(\rho, a)$ where $\rho \in \mathrm{Cusp}_{\mathrm{GL}}$ is conjugate self-dual with parity $(-1)^a$ and let $\tau \in \mathrm{Irr}_{\mathrm{U}}$ be the discrete series representation such that $\{\tau\} = \xi_{\mathrm{bc}}^{-1}(\delta)$.*

(1) *If a (and hence ρ) is odd then $\tau \notin \Pi^\circ_{\mathrm{disc}}$.*

(2) *If a (and hence ρ) is even then $\tau = \tau^+(\rho, a)$.*

Proof. This follows from [27, § 5]. In particular, when a is odd the extended cuspidal support of τ in the sense of loc. cit. contains ρ with multiplicity one while when a

https://doi.org/10.1017/S1474748019000161 Published online by Cambridge University Press

is even, the extended cuspidal support of $\tau^+(\rho, a)$ is the cuspidal support of δ. The lemma therefore follows from [27, Theorem 5.7]. □

10.2.9. It is observed in [6, Proposition 12] that if an L-packet in $\mathrm{Irr}_{\mathrm{U}}$ is associated to an Arthur packet and its base change is Sp-distinguished then the L-packet consists of a single representation. We notice here that this observation holds for all unitary representations in $\mathrm{Irr}_{\mathrm{GL}}$.

Proposition 7. *Let $\pi \in \mathrm{Irr}_{\mathrm{GL}}$ be unitary, conjugate self-dual and* Sp*-distinguished. Then π is in the image of base change and its base change fiber is a singleton, i.e., $|\xi_{\mathrm{bc}}^{-1}(\pi)| = 1$.*

Proof. Write $\pi = L(\mathfrak{m})$. In light of § 10.2.5 it is enough to show that no segment in $\mathfrak{m}$ has exponent zero. For a segment Δ and $a \in \mathbb{N}$ let $\mathfrak{m}(\Delta, a) = \{\nu^{\frac{1-a}{2}+i}\Delta \mid i = 0, 1, \ldots, a-1\}$. It follows from [35, Corollary 2] that $\pi = \pi_1 \times \cdots \times \pi_k$ where each $\pi_i = L(\mathfrak{m}_i)$ is such that $\mathfrak{m}_i$ is either of the form $\mathfrak{m}(\Delta, 2a)$ or of the form $\nu^{\alpha}\mathfrak{m}(\Delta, 2a) + \nu^{-\alpha}\mathfrak{m}(\Delta, 2a)$ where $\exp(\Delta) = 0$, $a \in \mathbb{N}$ and $0 < \alpha < \frac{1}{2}$. It easily follows that no segment in $\mathfrak{m}_i$ has exponent zero and since $\mathfrak{m} = \mathfrak{m}_1 + \cdots + \mathfrak{m}_k$ the proposition follows. (Here, we view multi-sets as functions of finite support, hence addition is defined.) □

10.2.10. Note that Proposition 6 implies that for any rigid π in the image of the map ξ_{bc}, there exists a unique conjugate self-dual representation $\rho \in \mathrm{Cusp}_{\mathrm{GL}}$ such that either $\mathrm{Supp}(\pi) \subseteq \rho^{\mathbb{Z}}$ or $\mathrm{Supp}(\pi) \subseteq (\nu^{\frac{1}{2}}\rho)^{\mathbb{Z}}$. We make the following definition.

Definition 9. We say that a conjugate self-dual rigid representation $\pi \in \mathrm{Irr}_{\mathrm{GL}}$ is *centered* at $\rho \in \mathrm{Cusp}_{\mathrm{GL}}$, if $\overline{\rho} \cong \rho^{\vee}$ and $\mathrm{Supp}(\pi) \subseteq \rho^{\mathbb{Z}} \sqcup (\nu^{\frac{1}{2}}\rho)^{\mathbb{Z}}$.

11. Distinction and the base change map

Dijols and Prasad conjectured in [6, Conjecture 3] that an L-packet of Arthur type in Irr_U contains an Sp-distinguished member if and only if the base change of the L-packet is Sp-distinguished. In this section we study a family of representations that indicate that the Arthur type assumption is necessary. Namely, we show that for some Sp-distinguished ladder representations in $\mathrm{Irr}_{\mathrm{GL}}$ the unique element of its base change fiber is not Sp-distinguished.

11.1. Reducibility, distinction and L-packets associated to ladders

Let $\pi = L(\mathfrak{m})$ be a ladder representation. We recall the equivalent conditions for π to be Sp-distinguished and for π to be in the image of base change.

11.1.1. It follows from [26, Theorem 10.3] that the representation π is Sp-distinguished if and only if the ladder $\mathfrak{m}$ is of the form $(\Delta_1, \Delta_2, \ldots, \Delta_{2s-1}, \Delta_{2s})$ where $\Delta_{2i-1} = \nu\Delta_{2i}$, $i = 1, \ldots, s$.

11.1.2. Assume that π is conjugate self-dual and write $\mathfrak{m} = (\Delta_1, \ldots, \Delta_t)$ so that $\Delta_{t+1-i} = \overline{\Delta_i}^{\vee}$, $i = 1, \ldots, t$. Then according to § 10.2 (and in particular Lemma 16) π is in the image of base change if and only if either t is even or $\mathrm{rec}_{\mathrm{GL}}(L(\Delta_{[(t+1)/2]}))$ is

https://doi.org/10.1017/S1474748019000161 Published online by Cambridge University Press

conjugate symplectic. When this is the case, $\pi = \xi_{\mathrm{bc}}(\tau)$ for a unique $\tau \in \mathrm{Irr}_{\mathrm{U}}$ that is determined as follows. We have

$$\tau = \begin{cases} \mathrm{LQ}(L(\Delta_1) \times \cdots \times L(\Delta_{t/2}) \rtimes \mathbf{1}_0) & t \text{ is even} \\ \mathrm{LQ}(L(\Delta_1) \times \cdots \times L(\Delta_{(t-1)/2}) \rtimes \sigma) & t \text{ is odd} \end{cases} \tag{25}$$

where for t odd we have $\{\sigma\} = \xi_{\mathrm{bc}}^{-1}(L(\Delta_{[(t+1)/2]}))$. Recall that by Lemma 16 if $\ell(\Delta_{[(t+1)/2]})$ is odd then σ has a non-trivial partial cuspidal support while if $\ell(\Delta_{[(t+1)/2]})$ is even then $\sigma = \tau^+(\rho, a)$ where $\Delta_{[(t+1)/2]} = [(1-a)/2, (a-1)/2]_{(\rho)}$.

11.2. The case of an even ladder

In this section we study the relation between Sp-distinction of a ladder representation and of its base change fiber in the case that the ladder has even number of segments.

11.2.1. An auxiliary result on irreducibility. Define

$$\mathcal{S} = \{\rho \in \mathrm{Cusp}_{\mathrm{GL}} \mid \rho \rtimes \mathbf{1}_0 \text{ is reducible}\}.$$

We state below a special case of [21, Theorems 1.1 and 1.2] that we are going to use. Here and otherwise, for a multi-set $\mathfrak{n} = \{\Delta_1, \dots, \Delta_t\}$ of segments let $\mathfrak{n}_{>0}$ be the multi-set defined by

$$\mathfrak{n}_{>0} = \{\Delta_i \mid 1 \leqslant i \leqslant t, \exp(\Delta_i) > 0\}.$$

Theorem 9. *Let $\pi = L(\mathfrak{m})$ be a ladder representation. Then $\pi \rtimes \mathbf{1}_0$ is irreducible if and only if $\mathrm{Supp}(\pi) \cap \mathcal{S} = \emptyset$ and $L(\mathfrak{m}_{>0}) \times L((\overline{\mathfrak{m}}^\vee)_{>0})$ is irreducible.*

11.2.2. In (25) (and in the notation of § 11.1.2) in the case that t is even, we express τ, the unique member of the base change fiber of the ladder representation π as the Langlands quotient of a representation induced from a ladder on the Siegel parabolic. The following corollary tells us when this induced representation is irreducible.

Corollary 6. *Let $\mathfrak{m} = (\Delta_1, \dots, \Delta_t)$ be a ladder and let $\pi = L(\mathfrak{m})$. Moreover, suppose that $\pi^\vee = \overline{\pi}$. Let $k = [\frac{t+1}{2}]$, $1 \leqslant k' \leqslant k$, $\mathfrak{n} = (\Delta_{k'}, \dots, \Delta_k)$ and $\pi' = L(\mathfrak{n})$. Then the following are equivalent*

(1) *$\pi \rtimes \mathbf{1}_0$ is irreducible*

(2) *$\Delta_k \cap \mathcal{S}$ is empty*

(3) *$\pi' \rtimes \mathbf{1}_0$ is irreducible.*

Proof. By assumption $\mathfrak{m} = \overline{\mathfrak{m}}^\vee$. Since $\mathfrak{m}$ is a ladder this means that $\Delta_i^\vee = \overline{\Delta_{t+1-i}}$, $i = 1, \dots, t$. It therefore follows from [12, Theorem 3.1] (see § 6.1.2 above) that the following conditions are equivalent

- $\mathrm{Supp}(\pi) \cap \mathcal{S}$ is empty
- $\Delta_k \cap \mathcal{S}$ is empty
- $\mathrm{Supp}(\pi') \cap \mathcal{S}$ is empty.

https://doi.org/10.1017/S1474748019000161 Published online by Cambridge University Press

It follows from [20, Proposition 6.20 and Lemma 6.21] that $L(\mathfrak{m}_{>0}) \times L(\mathfrak{m}_{>0})$ is irreducible. Thus, by Theorem 9 we have that $\pi \rtimes \mathbf{1}_0$ is irreducible if and only if $\mathrm{Supp}(\pi) \cap \mathcal{S}$ is empty.

Note further that $\mathfrak{n}$ is a ladder and $(\overline{\mathfrak{n}}^\vee)_{>0}$ is the empty set. It therefore follows from Theorem 9 that $\pi' \rtimes \mathbf{1}_0$ is irreducible if and only if $\mathrm{Supp}(\pi') \cap \mathcal{S}$ is empty. The corollary follows. □

11.2.3. In the case of a ladder with even number of segments, Sp-distinction is preserved by the base change map as the following result shows.

Proposition 8. *Let $\pi \in \mathrm{Irr}_{\mathrm{GL}}$ be a ladder representation that is conjugate self-dual. Suppose that $\pi = L(\mathfrak{m})$ where $|\mathfrak{m}|$ is an even integer. Let $\tau \in \mathrm{Irr}_{\mathrm{U}}$ be the unique representation such that $\xi_{\mathrm{bc}}(\tau) = \pi$ (see (25)). If τ is* Sp*-distinguished then π is* Sp*-distinguished.*

Proof. Write $\pi = L(\mathfrak{m})$ where $\mathfrak{m} = (\Delta_1, \dots, \Delta_{2s})$ such that $\Delta_i^\vee = \overline{\Delta_{2s-i+1}}$, $i = 1, \dots, 2s$. By (25) τ is the unique irreducible quotient of the representation

$$L(\Delta_1) \times \cdots \times L(\Delta_s) \rtimes \mathbf{1}_0.$$

This induced representation is Sp-distinguished since τ is. Thus it follows from Theorem 5 that $\sigma = L(\Delta_1) \otimes \cdots \otimes L(\Delta_s) \otimes \mathbf{1}_0$ admits a relevant orbit. We use the notation of § 4.4 for this orbit except that (since the symbol τ is already taken) we denote by η the involution on $\mathcal{I}$ associated with the contributing orbit. Applying [43, § 9.5] write

$$r_{L,M}(\sigma) = \bigotimes_{\iota \in \mathcal{I}} L(\Delta_\iota) \otimes \mathbf{1}_0.$$

Let $\Delta_i = [a_i, b_i]_{(\rho)}$, $i = 1, \dots, 2s$ for some $\rho \in \mathrm{Cusp}_{\mathrm{GL}}$ conjugate self-dual. By assumption, we have $(a_i + b_i)/2 = \exp(\Delta_i) > 0$, $i = 1, \dots, s$. Therefore, $0 < a_s + b_s \leqslant a_s + b_{s-1} - 1$ (the second inequality follows from the definition of a ladder). That is, $1 - b_{s-1} < a_s$, and therefore

$$\nu^{1-b_i}\rho \notin \bigcup_{j=1}^{s} \Delta_j$$

for any $i = 1, \dots, s-1$. Recalling that

$$\Delta_{(i,1)} = [x_i, b_i]_{(\rho)}$$

for some $x_i \leqslant b_i$, $i = 1, \dots, s$, by Corollary 4 and (17) it follows that

$$\{(i, j) \in \mathcal{I} \mid i = 1, \dots, s-1, \; j = 1, \dots, j_i\} \subseteq \mathfrak{c}. \tag{26}$$

Claim 1. *If $2i - 1 < s$ then $j_{2i-1} = j_{2i} = 1$ and $\eta(2i-1, 1) = (2i, 1)$.*

We prove the claim by induction on i. Recall first that since the representation $L(\Delta)$ is generic it is not Sp-distinguished for any segment Δ (see [16, Theorem 3.2.2]). Therefore $\mathfrak{c}_+(w)$ is empty. For the basis of induction assume that $1 < s$ and let $\eta(1, 1) = (i', j')$. It follows from (18) and (26) that $i' > 1$ and from (26) and Corollary 4 that

https://doi.org/10.1017/S1474748019000161 Published online by Cambridge University Press

$\Delta_{(i',j')} = \nu^{-1}\Delta_{(1,1)}$. In particular, $\nu^{b_1-1}\rho \in \Delta_{(i',j')} \subseteq \Delta_{i'}$. Since $\mathfrak{m}$ is a ladder this implies that $(i', j') = (2, 1)$. If either $j_1 > 1$ or $j_2 > 1$ we get a contradiction to (17), (18) and (26).

Suppose that the claim is true for $1, 2, \ldots, i-1$. For the induction step, let $\eta(2i-1, 1) = (i', j')$. By the induction hypothesis $i' \geqslant 2i-1$. It further follows from Corollary 4, (18) and (26) that in fact $i' > 2i-1$ and we have $\Delta_{(i',j')} = \nu^{-1}\Delta_{(2i-1,1)}$. In particular, $\nu^{b_{2i-1}-1}\rho \in \Delta_{(i',j')} \subseteq \Delta_{i'}$. Since, $\mathfrak{m}$ is a ladder this implies that $(i', j') = (2i, 1)$. By the induction hypothesis if $j_k > 1$ for $k \in \{2i-1, 2i\}$ then $\tau(k, 2) = (a, b)$ with $a \geqslant 2i-1$. As in the case $i = 1$, now this contradicts (17), (18) and (26). The claim follows.

If s is even then by (26), Claim 1 and Corollary 4 the ladder $(\Delta_1, \ldots, \Delta_s)$, and hence $\mathfrak{m}$, is of the form described in § 11.1.1 and thus the proposition follows. So suppose now that s is odd. As above the ladder $(\Delta_1, \ldots, \Delta_{s-1})$ is of the form described in § 11.1.1. By Claim 1, (17) and (18) we get that $j_s \leqslant 2$ and $\eta(s, i) = (s, i)$, $i = 1, \ldots, j_s$. By Corollary 4 and [16, Theorem 3.2.2] we further get that $j_s = 1$, $(s, 1) \in \mathcal{I} \setminus \mathfrak{c}$ and $\nu^{-1/2}L(\Delta_s)$ is $\mathrm{GL}(F)$-distinguished. By [7, Proposition 12] $\exp(\Delta_s) = \frac{1}{2}$. Coupled with the fact that $\Delta_s^\vee \cong \overline{\Delta_{s+1}}$, we get that $\Delta_s = \nu\Delta_{s+1}$, and the multi-set $\mathfrak{m}$ is again of the form described in § 11.1.1. This proves the proposition. □

11.2.4. The converse of Proposition 8 is not true and we have the following result in that direction.

Proposition 9. *Let $\pi \in \mathrm{Irr}_{\mathrm{GL}}$ be a ladder representation that is conjugate self-dual and* Sp*-distinguished. Write $\pi = L(\mathfrak{m})$ and $\mathfrak{m} = (\Delta_1, \Delta_2, \ldots, \Delta_{2s-1}, \Delta_{2s})$ as in § 11.1.1. Let $\tau' = L(\Delta_1, \ldots, \Delta_s) \rtimes \mathbf{1}_0$ and let $\tau \in \mathrm{Irr}_{\mathrm{U}}$ be the unique representation such that $\xi_{\mathrm{bc}}(\tau) = \pi$ (see (25)). We have*

(1) *τ is the unique irreducible quotient of τ' and $\tau = \tau'$ if and only if $\Delta_s \cap \mathcal{S}$ is empty.*

(2) *If s is even then τ' is* Sp*-distinguished.*

(3) *If s is odd and $\Delta_s \cap \mathcal{S}$ is empty then τ is not* Sp*-distinguished.*

Proof. The first part is immediate from § 10.2 and Corollary 6. For the second part, it follows from § 11.1.1 that if s is even then $L(\Delta_1, \ldots, \Delta_s)$ is Sp-distinguished and therefore from Lemma 14 that τ' is Sp-distinguished.

Suppose now that $s = 2k+1$ for some $k \in \mathbb{Z}_{\geqslant 0}$ and $\Delta_s \cap \mathcal{S}$ is empty, and assume by contradiction that τ is Sp-distinguished. By Corollary 6 (with $k' = s$) we have that $L(\Delta_{2k+1}) \rtimes \mathbf{1}_0$ is irreducible and by Proposition 1 we have that

$$L(\Delta_{2k+1}) \rtimes \mathbf{1}_0 \cong L(\overline{\Delta_{2k+1}})^\vee \rtimes \mathbf{1}_0.$$

Since π is conjugate self-dual we have $\overline{\Delta_{2k+1}}^\vee = \Delta_{2k+2}$ and therefore

$$L(\Delta_{2k+1}) \rtimes \mathbf{1}_0 \cong L(\Delta_{2k+2}) \rtimes \mathbf{1}_0.$$

Thus τ is the unique irreducible quotient of

$$L(\Delta_1) \times \cdots \times L(\Delta_{2k}) \times L(\Delta_{2k+2}) \rtimes \mathbf{1}_0.$$

https://doi.org/10.1017/S1474748019000161 Published online by Cambridge University Press

Since $L(\Delta_1, \dots, \Delta_{2k}, \Delta_{2k+2})$ is the unique irreducible quotient of the representation $L(\Delta_1) \times \cdots \times L(\Delta_{2k}) \times L(\Delta_{2k+2})$, we have that τ is the unique irreducible quotient of

$$L(\Delta_1, \dots, \Delta_{2k}, \Delta_{2k+2}) \rtimes \mathbf{1}_0.$$

Since by assumption we have that $\Delta_{2k+1} = \nu\Delta_{2k+2}$ as well as $\Delta_{2k+1} = \overline{\Delta_{2k+2}}^\vee$ we deduce that $\exp(\Delta_{2k+1}) = \frac{1}{2}$ and $\exp(\Delta_{2k+2}) = -\frac{1}{2}$. Let $\rho \in \mathrm{Cusp}_{\mathrm{GL}}$ be such that π is centered at ρ (see Definition 9) and $a \in \frac{1}{2}\mathbb{Z}$ such that $\Delta_{2k+1} = [-a, 1+a]_{(\rho)}$. Then $\Delta_{2k+2} = [-(1+a), a]_{(\rho)}$. Note that $a \geqslant -1/2$ (otherwise Δ_{2k+1} would be empty).

Let us first treat the case when $k > 0$. Let $\mathfrak{m}' = (\Delta'_1, \dots, \Delta'_l)$ denote the multi-set $\{\Delta_1, \dots, \Delta_{2k}, \Delta_{2k+2}\}^t$ ordered such that $e(\Delta'_1) \leqslant \cdots \leqslant e(\Delta'_l)$. Set

$$\tilde{\zeta}(\mathfrak{m}') \cong Z(\Delta'_1) \times \cdots \times Z(\Delta'_l).$$

It follows from the Zelevinsky classification [43] that $Z(\mathfrak{m}')$ is the unique irreducible quotient of $\tilde{\zeta}(\mathfrak{m}')$. Since $L(\Delta_1, \dots, \Delta_{2k}, \Delta_{2k+2}) = Z(\mathfrak{m}')$ we deduce that τ is an irreducible quotient of $\tilde{\zeta}(\mathfrak{m}') \rtimes \mathbf{1}_0$.

By the Mœglin–Waldspurger algorithm for ladder representations (provided in [19, § 3]) it follows that $\Delta'_1 = \{\nu^{-(1+a)}\rho\}$ is a segment of length one. Since the representation $\tilde{\zeta}(\mathfrak{m}') \rtimes \mathbf{1}_0$ has τ as an irreducible quotient, it is Sp-distinguished.

We now apply the geometric lemma for $\tilde{\zeta}(\mathfrak{m}') \rtimes \mathbf{1}_0$ viewed as induced from

$$\sigma = Z(\Delta'_1) \otimes \cdots \otimes Z(\Delta'_l) \otimes \mathbf{1}_0.$$

It follows from Theorem 5 that σ admits a relevant orbit. We use the notation of § 4.4 for this orbit except that (since the symbol τ is already taken) we denote by η the involution on $\mathcal{I}$ associated with the contributing orbit.

Applying [43, Proposition 3.4] write $r_{L,M}(\sigma) = \bigotimes_{\iota\in\mathcal{I}} Z(\Delta'_\iota) \otimes \mathbf{1}_0$ and note that $j_1 = 1$ and $\Delta'_{(1,1)} = \Delta'_1 = \{\nu^{-(1+a)}\rho\}$. If $(1, 1) \in \mathfrak{c}$, then by Lemma 13 we have that $\eta(1, 1) = (1, 1)$ and by Corollary 4 we deduce that $Z(\Delta'_{(1,1)}) = \nu^{-(1+a)}\rho \in \mathrm{Cusp}_{\mathrm{GL}}$ is Sp-distinguished. This contradicts [16, Theorem 3.2.2].

Thus, $(1, 1) \in \mathcal{I} \setminus \mathfrak{c}$. Since $\exp(Z(\Delta'_{(1,1)})) = -(1+a) \leqslant -1/2$ the representation $\nu^{-1/2} Z(\Delta'_{(1,1)})$ is not $\mathrm{GL}(F)$-distinguished. It therefore follows from Corollary 4 that $\eta(1, 1) \neq (1, 1)$ and that $Z(\Delta_{\eta(1,1)}) = \nu^{2+a}\rho$, and from (18) that $\eta(1, 1) = (\alpha, 1)$ for some $\alpha \in \{2, \dots, l\}$. Let $e(\Delta_{2k}) = \nu^b\rho$. Since $(\Delta_1, \dots, \Delta_{2k+1})$ is a ladder we have that $b \geqslant 2+a$. Thus from the Mœglin–Waldspurger algorithm it follows that $\Delta'_\alpha \in \{\Delta_1, \dots, \Delta_{2k}\}^t$. By § 11.1.1, the representation $L(\Delta_1, \dots, \Delta_{2k})$ is Sp-distinguished. By [26, Proposition 7.5] we get that $\ell(\Delta'_\alpha)$ is even and therefore $j_\alpha \geqslant 2$. If $(\alpha, 2) \in \mathfrak{c}$ then it follows from Lemma 13 that $j_\alpha = 2$ and that $Z(\Delta'_{(\alpha,2)})$ is Sp-distinguished. Since in this case $\ell(\Delta'_{(\alpha,2)})$ is odd, this contradicts [26, Proposition 7.5]. Therefore, we get that $(\alpha, 2) \in \mathcal{I} \setminus \mathfrak{c}$. Note that $b(\Delta'_{(\alpha,2)}) = \nu^{a+3}\rho$. Since $\nu^{-(2+a)}\rho \notin \mathrm{Supp}(\mathfrak{m}')$ this contradicts Corollary 4. Thus we get that when $k > 0$, $\tilde{\zeta}(\mathfrak{m}') \rtimes \mathbf{1}_0$ and hence τ cannot be Sp-distinguished.

Assume now that $k = 0$. In this case τ is a quotient of $L(\Delta_2) \rtimes \mathbf{1}_0$ which is then Sp-distinguished. As observed above, we have $\exp(\Delta_2) = -\frac{1}{2}$. On the other hand, by Lemma 15 we get that $\exp(\Delta_2) = \frac{1}{2}$ which gives us a contradiction. Thus τ cannot be Sp-distinguished. This completes the proof of the proposition. □

11.3. The case of an odd ladder

Next we consider a ladder representation π of the form $L(\Delta_1, \ldots, \Delta_{2k+1})$ in the image of the map ξ_{bc}. In many cases we show below that the unique $\tau \in \mathrm{Irr}_{\mathrm{U}}$ such that $\xi_{\mathrm{bc}}(\tau) = \pi$ is not Sp-distinguished.

Recall from § 11.1.1 that π is not Sp-distinguished, and from § 10.2 that $\Delta_{2k+2-i} = \overline{\Delta_i}^\vee$, $i = 1, \ldots, 2k+1$ and $L(\Delta_{k+1}) = \delta(\rho, a)$ for some $\rho \in \mathrm{Cusp}_{\mathrm{GL}}$ and $a \in \mathbb{N}$ so that ρ is conjugate self-dual of parity $(-1)^a$.

Proposition 10. *Let $\pi = L(\Delta_1, \ldots, \Delta_{2k+1}) \in \mathrm{Irr}_{\mathrm{GL}}$ be a ladder representation in the image of ξ_{bc}. Write $L(\Delta_{k+1}) = \delta(\rho, a)$ as above and let $\tau \in \mathrm{Irr}_{\mathrm{U}}$ be such that $\{\tau\} = \xi_{\mathrm{bc}}^{-1}(\pi)$. If either a is odd, $k = 0$ or $b(\Delta_k) \neq \nu^{3/2}\rho$ (this inequality holds, in particular, if $b(\Delta_k) = \nu\, b(\Delta_{k+1})$) then τ is not* Sp*-distinguished.*

Proof. If a is odd then it follows from (25) that τ has non-trivial partial cuspidal support. Thus, τ is not Sp-distinguished by Proposition 2.

Assume that a is even. It follows from (25) that τ is the unique irreducible quotient of $L(\Delta_1, \ldots, \Delta_k) \rtimes \tau^+(\rho, a)$. Let $b = (a-1)/2$ and recall further that $\tau^+(\rho, a)$ is the unique irreducible quotient of $L([-b, -\frac{1}{2}]_{(\rho)}) \rtimes \mathbf{1}_0$ (see § 10.2.8). This implies that τ is an irreducible quotient of the representation

$$L(\Delta_1, \ldots, \Delta_k) \times L([-b, -1/2]_{(\rho)}) \rtimes \mathbf{1}_0. \tag{27}$$

To complete the proof of the proposition it is enough to prove that this induced representation is not Sp-distinguished. If $k = 0$ this is immediate from Lemma 15. Assume that $k > 0$ and $b(\Delta_k) \neq \nu^{3/2}\rho$ and assume by contradiction that the representation in (27) is Sp-distinguished.

It follows from Theorem 5 that $\sigma = L(\Delta_1, \ldots, \Delta_k) \otimes L([-b, -1/2]_{(\rho)}) \otimes \mathbf{1}_0$ admits a relevant orbit. We use the notation of § 4.4 for this orbit except that (since the symbol τ is already taken) we denote by η the involution on $\mathcal{I}$ associated with the contributing orbit. Let $\sigma' = \bigotimes_{(i,j)\in\mathcal{I}} \pi'_{i,j} \otimes \mathbf{1}_0$ be an irreducible subquotient of $r_{L,M}(\sigma)$ that admits the corresponding non-trivial invariant functional (i.e., σ' is (L_x, δ_x)-distinguished).

Claim 2. *If $(1, 1) \notin \mathfrak{c}$, then $\eta(1, 1) \neq (1, 1)$.*

Observe that $\exp(\Delta_i) \geqslant 1$, $i = 1, \ldots, k$. By [17, Theorem 2.1], we get that $\pi_{1,1}$ is a (ladder) representation of the form $L(\Delta'_1, \ldots, \Delta'_l)$ where $\{d_1, \ldots, d_l\}$ is a subset of $\{1, \ldots, k\}$ such that $\Delta'_i \subseteq \Delta_{d_i}$ and $e(\Delta'_i) = e(\Delta_{d_i})$, $i = 1, \ldots, l$. In particular, $\exp(\Delta'_i) \geqslant 1$, $i = 1, \ldots, l$ and therefore also $\exp(\pi_{1,1}) \geqslant 1$. Thus $\nu^{-1/2}\pi_{1,1}$ cannot be $\mathrm{GL}(F)$-distinguished (by [7, Proposition 12]) and Claim 2 follows from Corollary 4.

Claim 3. *We have $(2, j) \in \mathfrak{c}$ for all $j = 1, \ldots, j_2$.*

By (17) it is enough to show that $(2, 1) \in \mathfrak{c}$. Assume by contradiction that $(2, 1) \in \mathcal{I} \setminus \mathfrak{c}$. It follows from [43, Proposition 9.5] that $\pi_{2,1}$ is of the form $L([x, -\frac{1}{2}]_{(\rho)})$ where $-b \leqslant x \leqslant -\frac{1}{2}$ and in particular $\exp(\pi_{2,j}) < 0$. As in Claim 2, it therefore follows from Corollary 4 and [7, Proposition 12] that $\eta(2, 1) \neq (2, 1)$.

By (18) and Claim 2 we have $\eta(2,1)=(1,1)$. It therefore follows from Corollary 4 that $\pi_{1,1}=L([\frac{3}{2},1-x]_{(\rho)})$. Since by assumption $b(\Delta_k)\neq\frac{3}{2}$ we must have $j_1>1$.

In the notation of the proof of Claim 2 we now have $l=1$ and $\exp(e(\Delta_{d_1}))=1-x\leqslant 1+b$. Since $\exp(e(\Delta_i))>1+b$ for $i=1,\dots,k-1$ and $\exp(e(\Delta_k))\geqslant 1+b$ it follows that $d_1=k$ and $x=-b$. This forces $j_2=1$ and $\pi_{2,1}=L([-b,-\frac{1}{2}]_{(\rho)})$.

Moreover, from (18) we get that $j_1=2$, $(1,2)\in\mathfrak{c}$ and $\eta(1,2)=(1,2)$. From [17, Theorem 2.1] we get that $\pi_{1,2}=L(\Delta_1,\dots,\Delta_{k-1},[y,\frac{1}{2}]_{(\rho)})$ where $b(\Delta_k)=\nu^y\rho$. It further follows from Corollary 4 that $\pi_{1,2}$ is Sp-distinguished. By § 11.1.1 we get that $e(\Delta_{k-1})=\nu^{3/2}\rho$. This contradicts the facts that $(\Delta_{k-1},\Delta_k,\Delta_{k+1})$ is a ladder, $e(\Delta_{k+1})=\nu^b\rho$ and $b\geqslant 1/2$. This proves Claim 3.

Note that Claims 2 and 3 together with (18) imply that $\mathcal{I}=\mathfrak{c}$. Since no essentially square-integrable representation in $\mathrm{Irr}_{\mathrm{GL}}$ is Sp-distinguished [16, Theorem 3.2.2], it further follows that $j_2=1$, $(2,1)\in\mathfrak{c}$, and $\eta(2,1)=(1,1)$. Thus, by Corollary 4 we have $\pi_{1,1}=L([1-b,\frac{1}{2}]_{(\rho)})$. This means again that $l=1$ and $e(\Delta_{d_1})=\nu^{1/2}\rho$ which is a contradiction (since $e(\Delta_i)\geqslant\nu^{1+b}\rho$, $i=1,\dots,k$). The proposition is now proved. □

11.3.1. We summarize the results obtained in this section for the class of Speh representations in the image of the map ξ_{bc}. By § 11.1.1 a Speh representation $\pi=L(\mathfrak{m})$ is Sp-distinguished if and only if $|\mathfrak{m}|$ is an even integer. The following is a consequence of Propositions 9 and 10.

Theorem 10. *Let $\pi=L(\mathfrak{m})\in\mathrm{Irr}_{\mathrm{GL}}$ be a Speh representation contained in the image of the base change map ξ_{bc} and let $\tau\in\mathrm{Irr}_{\mathrm{U}}$ be such that $\{\tau\}=\xi_{\mathrm{bc}}^{-1}(\pi)$. Then we have the following:*

(1) *Suppose that π is not* Sp*-distinguished. Then τ is not* Sp*-distinguished.*

(2) *Suppose that π is* Sp*-distinguished and write $\mathfrak{m}=(\nu^{2k-1}\Delta,\nu^{2k-2}\Delta,\dots,\Delta)$ for a segment Δ and $k\in\mathbb{N}$. Suppose further that $\nu^k\Delta\cap\mathcal{S}$ is empty. Then τ is* Sp*-distinguished if and only if k is even.*

Remark 6. While part (1) of the theorem supports [6, Conjecture 3], part (2) is not expected to contradict it. In the notation of part (2) of the theorem, if k is odd and $\nu^k\Delta\cap\mathcal{S}$ is empty then τ is not expected to have an Arthur parameter. (Note that τ is fully induced from a Speh representation with positive exponent.) We thank Dipendra Prasad for this insight.

12. Distinction for standard modules with an irreducible GL-part

In this section we characterize distinction of standard modules with a generic GL-part.

12.1.

Let $\pi=L(\Delta_1,\dots,\Delta_t)\in\mathrm{Irr}_{\mathrm{GL}}$. By [43, Theorem 9.7] the representation π is generic if and only if π is a standard module. That is,

$$\pi\cong L(\Delta_1)\times\cdots\times L(\Delta_t)$$

(for some and therefore any order on the segments).

12.2.

Let $\pi = L(\Delta_1, \ldots, \Delta_t) \in \mathrm{Irr}_{\mathrm{GL}}$ be generic. By [24, Theorem 5.2] the representation π is $\mathrm{GL}(F)$-distinguished if and only if there exists an involution $w \in S_t$ such that

- $\Delta_{w(i)} = \overline{\Delta_i}^{\vee}$, $i = 1, \ldots, t$ and
- $L(\Delta_i)$ is $\mathrm{GL}(F)$-distinguished if $w(i) = i$.

In fact, we will only apply the 'if' part of this statement, which is a consequence of [23, Theorem 4.2] and [8, Proposition 26].

12.3.

The main result of the section is as follows.

Theorem 11. *Let $\Delta_1, \ldots, \Delta_t$ be segments such that $\exp(\Delta_1) \geqslant \cdots \geqslant \exp(\Delta_t) > 0$ and $\pi = L(\Delta_1, \ldots, \Delta_t) \in \mathrm{Irr}_{\mathrm{GL}}$ is generic. Then, the standard module $\pi \rtimes \mathbf{1}_0$ is* Sp*-distinguished if and only if $\nu^{-1/2}\pi$ is $\mathrm{GL}(F)$-distinguished.*

Proof. If $\nu^{-1/2}\pi$ is a $\mathrm{GL}(F)$-distinguished representation, then $\pi \rtimes \mathbf{1}_0$ is Sp-distinguished by Lemma 14. Suppose that $\pi \rtimes \mathbf{1}_0$ is Sp-distinguished. As remarked in § 12.1, we may rearrange the segments $\Delta_1, \ldots, \Delta_t$ in any order. Writing $\Delta_i = [a_i, b_i]_{(\rho_i)}$ where $\rho_i \in \mathrm{Cusp}_{\mathrm{GL}}$ is unitary we may assume that the following conditions are satisfied:

$$a_1 \leqslant \cdots \leqslant a_t \quad \text{and if } a_i = a_{i+1} \text{ then } b_i \geqslant b_{i+1}. \tag{28}$$

It follows from Theorem 5 that $\sigma = L(\Delta_1) \otimes \cdots \otimes L(\Delta_t) \otimes \mathbf{1}_0$ admits a relevant orbit. We use the notation of § 4.4 for this orbit. Applying [43, § 9.5] write

$$r_{L,M}(\sigma) = \bigotimes_{\iota \in \mathcal{I}} L(\Delta_\iota) \otimes \mathbf{1}_0$$

and recall that

$$\Delta_{(i,1)} = [x_i, b_i]_{(\rho_i)} \quad \text{and} \quad \Delta_{(i,j_i)} = [a_i, y_i]_{(\rho_i)}$$

for some $a_i \leqslant x_i \leqslant y_i \leqslant b_i$, $i = 1, \ldots, t$.

Claim 4. *The set $\mathfrak{c}$ is empty.*

Assume by contradiction that $\mathfrak{c}$ is not empty and let $i \in \{1, \ldots, t\}$ be minimal such that $(i, j) \in \mathfrak{c}$ for some j. By (17) we have $(i, j_i) \in \mathfrak{c}$. Let $\tau(i, j_i) = (i', j')$. Since no essentially square-integrable representation is Sp-distinguished [16, Theorem 3.2.2], it follows from Corollary 4 that $(i', j') \neq (i, j_i)$. By minimality of i and (18) we have $i' > i$ and it further follows from Corollary 4 that

$$\nu^{-1} L(\Delta_{(i,j_i)}) = L(\Delta_{(i',j')}).$$

In particular, $a_{i'} \leqslant a_i - 1$ which contradicts (28). This proves Claim 4.

Claim 5. *We have $j_i = 1$ for every $i \in \{1, \ldots, t\}$.*

To prove the claim we show, by induction on k, that for all $1 \leqslant i \leqslant k$, we have:

(1) $j_i = 1$;

(2) if $\tau(i, 1) = (i', j')$ then $j_{i'} = 1$.

https://doi.org/10.1017/S1474748019000161 Published online by Cambridge University Press

Suppose that the induction hypothesis holds for $k-1$ and let $\tau(k,1)=(k',j')$. If $k'<k$, then the induction hypothesis implies the statement for k. Assume now that $k'\geqslant k$. It follows from (18), Claim 4 and the induction hypothesis that $j'=1$ and further from Corollary 4 that $\Delta_{(k',1)}=\nu\overline{\Delta_{(k,1)}}^{\vee}$. Since $\exp(\Delta_i)>0$ we get that $\exp(\Delta_{(i,1)})>0$ for every $i=1,\dots,t$. Since $\exp(\Delta_{(k,1)})=1-\exp(\Delta_{(k',1)})$, we have $0<\exp(\Delta_{(k,1)}),\exp(\Delta_{(k',1)})<1$.

If $\exp(\Delta_{(k,1)})=\exp(\Delta_{(k',1)})=\frac{1}{2}$ then we must have $j_k=j_{k'}=1$ since $\exp(\Delta_i)>0$ for all i. This proves the induction step in this case. Otherwise, i.e., if $\exp(\Delta_{(k,1)})\neq\exp(\Delta_{(k',1)})$, then the same argument shows that $j_k,\ j_{k'}\leqslant 2$, if one of them equals 2 the other must equal 1, and $\ell(\Delta_i)\leqslant\ell(\Delta_{(i,1)})+1$ for $i\in\{k,k'\}$.

By assumption, since $\exp(\Delta_{(k,1)})\neq\frac{1}{2}$, the representation $\nu^{-1/2}L(\Delta_{(k,1)})$ cannot be $\mathrm{GL}(F)$-distinguished and therefore, by Corollary 4 we have $k'>k$, $\Delta_{(k',1)}=[1-b_k,b_{k'}]_{(\rho_{k'})}$ and $\overline{\rho}_k^{\vee}\cong\rho_{k'}$.

If $j_{k'}=2$, then $\Delta_{(k',2)}=\{\nu^{-b_k}\rho_{k'}\}$. Let $\tau(k',2)=(l,l')$. By (18) and Claim 4 we have $l>k$ and further applying Corollary 4 we have $\Delta_{(l,l')}=\{\nu^{b_k+1}\rho_k\}$. But now (28) implies that Δ_k and Δ_l are linked which contradicts the fact that π is generic (see [43, Theorem 9.7]). Thus we conclude that $j_{k'}=1$.

If $j_k=2$ then similarly $\Delta_{(k,2)}=\{\nu^{-b_{k'}}\rho_k\}$ while (since $j_{k'}=1$)

$$\Delta_{k'}=\Delta_{(k',1)}=[1-b_k,b_{k'}]_{(\rho_{k'})}.$$

Let $\tau(k,2)=(l,l')$. By (18) we have that $l>k'$ and further by the induction hypothesis, that $l'=1$. By Claim 4 and Corollary 4 we have $\Delta_{(l,1)}=\{\nu^{b_{k'}+1}\rho_{k'}\}$. Again, (28) implies that Δ_l and $\Delta_{k'}$ are linked which contradicts the fact that π is generic. Thus we conclude that $j_k=1$ which finishes the proof of the induction. This completes the proof of Claim 5.

The theorem is now immediate from Claims 4 and 5, Corollary 4 and § 12.2. □

Remark 7. In the notation of Theorem 11, if $\nu^{-\frac{1}{2}}\pi$ is $\mathrm{GL}(F)$-distinguished and furthermore $\exp(\Delta_i)\in\frac{1}{2}\mathbb{Z}$, $i=1,\dots,t$ then it follows from § 12.2, in its notation, that w is trivial and therefore $\nu^{-\frac{1}{2}}\pi$ is tempered.

When $\pi\in\mathrm{Irr}(\mathrm{GL}_{2n}(E))$ for some $n\in\mathbb{N}$ is such that $\nu^{-\frac{1}{2}}\pi$ is tempered and $\mathrm{GL}(F)$-distinguished, Morimoto proved in [33] the stronger result that $\mathrm{LQ}(\pi\rtimes\mathbf{1}_0)$ is Sp-distinguished. We expect this to be true more generally, for the setting of Theorem 11. However, this will require methods different from the ones employed in this work.

Acknowledgements. The authors wish to thank K. Morimoto for sharing his preprint [33] with them and Dipendra Prasad for answering their many questions. The first named author also wishes to thank Sandeep Varma for several helpful conversations and the Tata Institute of Fundamental Research, Mumbai, for providing an effective environment in which a part of this work was done. Finally, we thank the referee for a careful reading of this work and several corrections and insightful remarks.

References

1. U. K. Anandavardhanan, A. C. Kable and R. Tandon, Distinguished representations and poles of twisted tensor L-functions, *Proc. Amer. Math. Soc.* **132**(10) (2004), 2875–2883, MR 2063106.

2. U. K. Anandavardhanan and C. S. Rajan, Distinguished representations, base change, and reducibility for unitary groups, *Int. Math. Res. Not. IMRN* (14) (2005), 841–854, MR 2146859.
3. I. N. Bernšteĭn and A. V. Zelevinskiĭ, Representations of the group GL(n, F), where F is a local non-Archimedean field, *Uspekhi Mat. Nauk* **31**(3(189)) (1976), 5–70, MR 0425030 (54 #12988).
4. I. N. Bernstein and A. V. Zelevinsky, Induced representations of reductive $\mathfrak{p}$-adic groups. I, *Ann. Sci. Éc. Norm. Supér. (4)* **10**(4) (1977), 441–472, MR 0579172 (58 #28310).
5. J. N. Bernstein, On the support of Plancherel measure, *J. Geom. Phys.* **5**(4) (1988), 663–710 (1989), MR 1075727 (91k:22027).
6. S. Dijols and D. Prasad, Symplectic models for unitary groups, *Trans. Amer. Math. Soc.*, doi:10.1090/tran/7651.
7. Y. Z. Flicker, On distinguished representations, *J. Reine Angew. Math.* **418** (1991), 139–172, MR 1111204.
8. Y. Z. Flicker, Distinguished representations and a Fourier summation formula, *Bull. Soc. Math. France* **120**(4) (1992), 413–465, MR 1194271.
9. Y. Z. Flicker, On zeroes of the twisted tensor L-function, *Math. Ann.* **297**(2) (1993), 199–219, MR 1241802.
10. W. T. Gan, B. H. Gross and D. Prasad, Symplectic local root numbers, central critical L values, and restriction problems in the representation theory of classical groups, *Astérisque* (346) (2012), 1–109. Sur les conjectures de Gross et Prasad. I. MR 3202556.
11. I. M. Gel'fand and D. A. Kajdan, Representations of the group GL(n, K) where K is a local field, in *Lie Groups and their Representations (Proc. Summer School, Bolyai János Math. Soc., Budapest, 1971)*, pp. 95–118 (Halsted, New York, 1975), MR 0404534 (53 #8334).
12. D. Goldberg, Some results on reducibility for unitary groups and local Asai L-functions, *J. Reine Angew. Math.* **448** (1994), 65–95, MR 1266747.
13. M. Gurevich, J.-J. Ma and A. Mitra, On two questions concerning representations distinguished by the Galois involution, *Forum Math.* **30**(1) (2018), 141–157, MR 3739332.
14. M. Hanzer, The generalized injectivity conjecture for classical p-adic groups, *Int. Math. Res. Not. IMRN* (2) (2010), 195–237, MR 2581039 (2011b:22025).
15. M. Harris and R. Taylor, *The Geometry and Cohomology of Some Simple Shimura Varieties*, Annals of Mathematics Studies, 151, (Princeton University Press, Princeton, NJ, 2001). With an appendix by Vladimir G. Berkovich, MR MR1876802 (2002m:11050).
16. M. J. Heumos and S. Rallis, Symplectic-Whittaker models for Gl$_n$, *Pacific J. Math.* **146**(2) (1990), 247–279, MR 1078382 (91k:22036).
17. A. Kret and E. Lapid, Jacquet modules of ladder representations, *C. R. Math. Acad. Sci. Paris* **350**(21–22) (2012), 937–940, MR 2996769.
18. R. Kumanduri, Distinguished representations for unitary groups, *Pacific J. Math.* **178**(2) (1997), 293–306, MR 1447416.
19. E. Lapid and A. Mínguez, On a determinantal formula of Tadić, *Amer. J. Math.* **136**(1) (2014), 111–142, MR 3163355.
20. E. Lapid and A. Mínguez, On parabolic induction on inner forms of the general linear group over a non-archimedean local field, *Selecta Math. (N.S.)* **22**(4) (2016), 2347–2400, MR 3573961.
21. E. Lapid and M. Tadić, Some results on reducibility of parabolic induction for classical groups, *Amer. J. Math.* (2017) (to appear) arXiv:1703.09475.
22. E. M. Lapid and J. D. Rogawski, Periods of Eisenstein series: the Galois case, *Duke Math. J.* **120**(1) (2003), 153–226, MR 2010737.

23. N. Matringe, Distinction of some induced representations, *Math. Res. Lett.* **17**(1) (2010), 77–97, MR 2592729 (2011a:22021).
24. N. Matringe, Distinguished generic representations of GL(n) over p-adic fields, *Int. Math. Res. Not. IMRN* (1) (2011), 74–95, MR 2755483 (2012f:22032).
25. A. Mitra and O. Offen, Vanishing of local symplectic periods for cuspidal representations of the unitary group, *C. R. Math. Acad. Sci. Paris* **355**(1) (2017), 15–19, MR 3590280.
26. A. Mitra, O. Offen and E. Sayag, Klyachko models for ladder representations, *Doc. Math.* **22** (2017), 611–657, MR 3628792.
27. C. Mœglin, Classification et changement de base pour les séries discrètes des groupes unitaires p-adiques, *Pacific J. Math.* **233**(1) (2007), 159–204, MR 2366373.
28. C. Mœglin and M. Tadić, Construction of discrete series for classical p-adic groups, *J. Amer. Math. Soc.* **15**(3) (2002), 715–786 (electronic), MR 1896238 (2003g:22020).
29. C. Mœglin, M.-F. Vignéras and J.-L. Waldspurger, *Correspondances de Howe Sur un Corps p-Adique*, Lecture Notes in Mathematics, 1291, (Springer, Berlin, 1987), MR 1041060 (91f:11040).
30. C. Mœglin and J.-L. Waldspurger, Sur l'involution de Zelevinski, *J. Reine Angew. Math.* **372** (1986), 136–177, MR 863522 (88c:22019).
31. C. Mœglin and J.-L. Waldspurger, *Spectral Decomposition and Eisenstein Series*, Cambridge Tracts in Mathematics, 113, (Cambridge University Press, Cambridge, 1995). Une paraphrase de l'Écriture [A paraphrase of Scripture], MR 1361168 (97d:11083).
32. C. P. Mok, Endoscopic classification of representations of quasi-split unitary groups, *Mem. Amer. Math. Soc.* **235**(1108) (2015), vi + 248, MR 3338302.
33. K. Morimoto, Model transitions for representations of unitary type, *Int. Math. Res. Not. IMRN* (to appear) doi:10.1093/imrn/rny048.
34. O. Offen, On parabolic induction associated with a p-adic symmetric space, *J. Number Theory* **170** (2017), 211–227, MR 3541705.
35. O. Offen and E. Sayag, Uniqueness and disjointness of Klyachko models, *J. Funct. Anal.* **254**(11) (2008), 2846–2865, MR 2414223.
36. Y. Sakellaridis and A. Venkatesh, Periods and harmonic analysis on spherical varieties, *Astérisque* (396) (2017), viii + 360, MR 3764130.
37. A. J. Silberger, The Langlands quotient theorem for p-adic groups, *Math. Ann.* **236**(2) (1978), 95–104, MR 0507262 (58 #22413).
38. T. A. Springer, Some results on algebraic groups with involutions, in *Algebraic Groups and Related Topics (Kyoto/Nagoya, 1983)*, Adv. Stud. Pure Math., 6, pp. 525–543 (North-Holland, Amsterdam, 1985), MR 803346 (86m:20050).
39. M. Tadić, Classification of unitary representations in irreducible representations of general linear group (non-Archimedean case), *Ann. Sci. Éc. Norm. Supér. (4)* **19**(3) (1986), 335–382, MR 870688 (88b:22021).
40. M. Tadić, Representations of p-adic symplectic groups, *Compos. Math.* **90**(2) (1994), 123–181, MR 1266251 (95a:22025).
41. M. Tadić, On tempered and square integrable representations of classical p-adic groups, *Sci. China Math.* **56**(11) (2013), 2273–2313, MR 3123571.
42. J.-L. Waldspurger, La formule de Plancherel pour les groupes p-adiques (d'après Harish-Chandra), *J. Inst. Math. Jussieu* **2**(2) (2003), 235–333, MR 1989693 (2004d:22009).
43. A. V. Zelevinsky, Induced representations of reductive $\mathfrak{p}$-adic groups. II. On irreducible representations of GL(n), *Ann. Sci. Éc. Norm. Supér. (4)* **13**(2) (1980), 165–210, MR 58 4084 (83g:22012).