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# SMALLEST WEAKLY CONTRACTIBLE NON-CONTRACTIBLE TOPOLOGICAL SPACES

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*Abstract* We characterize the topological spaces of minimum cardinality which are weakly contractible but not contractible. This is equivalent to finding the non-dismantlable posets of minimum cardinality such that the geometric realization of their order complexes are contractible. Specifically, we prove that all weakly contractible topological spaces with fewer than nine points are contractible. We also prove that there exist (up to homeomorphism) exactly two topological spaces of nine points which are weakly contractible but not contractible.

*Keywords:* finite topological space; weakly contractible topological space; partially ordered set; dismantlable poset

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Secondary 06A07

## 1. Introduction

Finite topological spaces have attracted increasing attention in recent years, principally from works by Barmak and Minian [**3**–**6**]. One of the main reasons for the interest in the theory of finite spaces is that they serve as models for weak homotopy types of compact polyhedra. More precisely, for every compact polyhedron $K$, there exists a finite $T_0$-space $\mathcal{X}(K)$ together with a weak homotopy equivalence $K \to \mathcal{X}(K)$ [**12**]. Moreover, there is a functorial correspondence between finite $T_0$-spaces and finite posets [**1**] which endows the theory of finite topological spaces with a natural combinatorial flavour. This allows the study of compact polyhedra (and often of general topological spaces) by means of combinatorial tools and gives a new insight into relevant topological questions [**2**].

A natural problem of this theory is to find finite topological spaces with the minimum number of points that have certain weak homotopy type. One of the first questions of this type was asked by May in [**10**], where he conjectures that, for all $n \in \mathbb{N}$, the $n$-fold non-Hausdorff suspension of the 0-sphere, denoted by $\mathbb{S}^n S^0$, is a *minimal finite model* of the $n$-sphere, that is, a finite topological space which is weak homotopy equivalent to the $n$-sphere with the minimum possible cardinality. This question was answered positively by

Barmak and Minian in [3]. Moreover, they also prove that $\mathbb{S}^n S^0$ is the only minimal finite model of the $n$-sphere. In the same article, they give a characterization of the minimal finite models of the finite graphs.

Further minimality questions were formulated in [9] and in [2] regarding finite models of the real projective plane and the torus. These problems were solved in [8], where a characterization of all the minimal finite models of these spaces is given.

Also, in [13] and in [2], a weakly contractible non-contractible space of nine points is given. Thus, another natural question to pose related to this theory is whether this is the minimum number of points that a weakly contractible non-contractible space can have [11, Problem 3.5.4]. In this article, we give an affirmative answer to this question and show that a weakly contractible non-contractible space must have at least nine points. Equivalently, any poset with fewer than nine points such that the geometric realization of its order complex is contractible must be dismantlable. Moreover, we prove that there exist (up to homeomorphism) exactly two topological spaces of nine points that are weakly contractible but not contractible.

## 2. Preliminaries

In this section, we will recall the basic notions of the theory of finite topological spaces and fix notation. For a comprehensive exposition on finite spaces, the reader may consult [2].

If $X$ is a finite topological space and $x \in X$, the intersection of all the open subsets of $X$ that contain $x$ is clearly an open subset and is denoted by $U_x$. Any finite $T_0$-space $X$ can be endowed with a partial order, which is defined as follows: $x_1 \leq x_2$ if and only if $U_{x_1} \subseteq U_{x_2}$. This defines a correspondence between finite $T_0$-spaces and finite posets which was first observed by Alexandroff [1]. Moreover, under this correspondence, continuous maps between finite $T_0$-spaces correspond to order-preserving morphisms between the respective posets. Hereafter, any finite $T_0$-space will be regarded also as a poset without further notice.

Let $X$ be a finite $T_0$-space and let $x \in X$. From the definition of the associated partial order it follows that $U_x = \{a \in X / a \leq x\}$. In a similar way, the smallest closed set that contains $x$ is $\overline{\{x\}} = \{a \in X / a \geq x\}$ and is denoted by $F_x$. It is also standard to define $\widehat{U}_x = \{a \in X / a < x\}$, $\widehat{F}_x = \{a \in X / a > x\}$, $C_x = U_x \cup F_x$ and $\widehat{C}_x = C_x - \{x\}$. We say that the point $x \in X$ is an *up beat point* (respectively, *down beat point*) of $X$ if the subposet $\widehat{F}_x$ has a minimum (respectively, if the subposet $\widehat{U}_x$ has a maximum) [2, 10, 14]. The notion of beat points in finite $T_0$-spaces corresponds to the notion of *irreducible points* in posets [13].

Stong proves, in [14], that if $x$ is a beat point of $X$ then $X - \{x\}$ is a strong deformation retract of $X$ and that two finite $T_0$-spaces are homotopy equivalent if and only if one obtains homeomorphic spaces after successively removing their beat points. It follows that a finite $T_0$-space is contractible if and only if its associated poset is dismantlable (by irreducibles). Also, using the results of Stong it is easy to prove that a finite $T_0$-space which has a maximum or a minimum is contractible.

If $X$ is a finite $T_0$-space, $\mathcal{K}(X)$ will denote the *order complex* of $X$, that is, the simplicial complex of the non-empty chains of $X$, and $|\mathcal{K}(X)|$ will denote its geometric realization. Also, $X^{\mathrm{op}}$ will denote the poset $X$ with the inverse order and will be called the *opposite*

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space of $X$. In addition, we define the *height* of a finite $T_0$-space $X$ as

$$h(X) = \max\{\#c - 1/c \text{ is a chain of } X\}.$$

Note that $h(X) = \dim \mathcal{K}(X)$.

McCord proves, in [12], that if $X$ is a finite $T_0$-space, then there exists a weak homotopy equivalence from the geometric realization of $\mathcal{K}(X)$ to $X$. In particular, any finite $T_0$-space is weak homotopy equivalent to its opposite space since their order complexes coincide. Note also that the aforementioned result of McCord implies that the singular homology groups of a finite $T_0$-space $X$ are isomorphic to the simplicial homology groups of $\mathcal{K}(X)$.

In the same article, McCord also proves that if $X$ is a finite topological space and $\mathbf{K}X$ denotes its Kolmogorov quotient, then the quotient map $X \to \mathbf{K}X$ is a homotopy equivalence. This implies that, given a finite space, one can obtain a homotopy equivalent finite $T_0$-space whose cardinality is not greater than that of the given space.

The *non-Hausdorff suspension* of a topological space $X$ is defined as the space $\mathbb{S}X$ whose underlying set is $X \amalg \{+, -\}$ and whose open sets are those of $X$ together with $X \cup \{+\}$, $X \cup \{-\}$ and $X \cup \{+, -\}$ [12]. Note that, if $X$ is a finite $T_0$-space, then the partial order in $\mathbb{S}X$ is induced by the partial order of $X$ together with the relations $x \leq +$ and $x \leq -$ for all $x \in X$. McCord proves that, for every topological space $X$, there exists a weak homotopy equivalence between the suspension of $X$ and $\mathbb{S}X$ [12]. As an example, he shows that, for all $n \in \mathbb{N}$, the $n$-sphere $S^n$ is weak homotopy equivalent to the $n$-fold non-Hausdorff suspension of the 0-sphere $S^0$. Observe that $\mathbb{S}^n S^0$ is a finite $T_0$-space of $2n + 2$ points.

May asked, in [10], if $\mathbb{S}^n S^0$ was the smallest space that is weak homotopy equivalent to the $n$-sphere. This question was answered by Barmak and Minian in [3]. More precisely, they proved the following theorem from which the affirmative answer to May's question follows.

**Theorem 2.1.** *Let $X \neq *$ be a finite topological space without beat points. Then $X$ has at least $2h(X) + 2$ points. Moreover, if $X$ has exactly $2h(X) + 2$ points, then it is homeomorphic to $\mathbb{S}^{h(X)} S^0$.*

In [7], we studied the homology groups of finite $T_0$-spaces obtaining several results and applications. Among them we mention the following proposition, which will be needed later. In what follows, homology will always mean homology with integer coefficients. Thus, the group of coefficients will be omitted from the notation.

**Proposition 2.2.** *Let $X$ be a finite $T_0$-space and let $D$ be an antichain in $X$. Then $H_n(X, X - D) \cong \bigoplus_{x \in D} \widetilde{H}_{n-1}(\widehat{C}_x)$ for every $n \in \mathbb{Z}$.*

If $X$ is a finite $T_0$-space, $\operatorname{mxl}(X)$ and $\operatorname{mnl}(X)$ will denote the subsets of maximal and minimal points of $X$, respectively. The following proposition states some simple facts concerning the maximal and minimal points of a finite $T_0$-space. Its proof will be omitted. The first two items appeared in [8].

**Proposition 2.3.**

(a) *Let $X$ be a connected and finite $T_0$-space with more than one point. Then* $\operatorname{mxl}(X) \cap \operatorname{mnl}(X) = \varnothing$.

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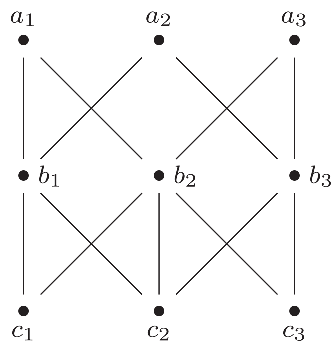


Figure 1. Weakly contractible non-contractible space of nine points.

(b) *Let $X$ be a finite $T_0$-space without beat points. If $a \in X - \mathrm{mxl}(X)$, then $\#(\widehat{F}_a \cap \mathrm{mxl}(X)) \geq 2$. Similarly, if $b \in X - \mathrm{mnl}(X)$ then $\#(\widehat{U}_b \cap \mathrm{mnl}(X)) \geq 2$.*

(c) *Let $X$ be a finite $T_0$-space without beat points. If $\#\mathrm{mxl}(X) = 2$, then $X$ is homeomorphic to $\mathbb{S}(X - \mathrm{mxl}(X))$.*

We end the preliminaries by giving a definition that will be used in the following section.

**Definition 2.4.** Let $X$ be a connected and finite $T_0$-space. We define

$$\mathcal{B}_X = X - (\mathrm{mxl}(X) \cup \mathrm{mnl}(X)).$$

## 3. Results

In this section, we will prove that a weakly contractible non-contractible space must have at least nine points. The equivalent formulation for posets states that if $P$ is a non-dismantlable poset such that $|\mathcal{K}(P)|$ is contractible, then $P$ must have at least nine points.

In addition, we will prove that there exist (up to homeomorphism) exactly two weakly contractible non-contractible spaces of nine points, or, equivalently, two non-dismantlable posets of nine points such that the geometric realizations of their order complexes are contractible.

In Figure 1, we exhibit a weakly contractible non-contractible $T_0$-space of nine points, which was also considered in [**13**, Figure 2] and in [**2**, Example 4.3.3]. Observe that this space is not contractible since it does not have beat points and it is weakly contractible since the geometric realization of its order complex is contractible.

We will give now several lemmas which will be useful for proving the main results of this article.

**Lemma 3.1.** *Let $X$ be a finite $T_0$-space without beat points. Let $a, b \in X$ with $a > b$. Then $\#\widehat{U}_a \geq \#\widehat{U}_b + 2$ and $\#\widehat{F}_b \geq \#\widehat{F}_a + 2$.*

**Proof.** Note that $\widehat{U}_a \supseteq U_b$ since $b < a$. And since $a$ is not a beat point of $X$, we obtain that $\widehat{U}_a \supsetneq U_b$. Therefore $\#\widehat{U}_a \geq \#U_b + 1 = \#\widehat{U}_b + 2$.

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Applying this result to $X^{\mathrm{op}}$, we obtain that $\#\widehat{F}_b \geq \#\widehat{F}_a + 2$. □

**Lemma 3.2.** *Let $X$ be a finite $T_0$-space such that $h(X) = 2$. If $X$ is weakly contractible, then $H_2(A) = 0$ for all subspaces $A \subseteq X$.*

**Proof.** Let $A$ be a subspace of $X$. Note that $H_3(X, A) = 0$ since $h(X) = 2$. The result then follows from the exact sequence $H_3(X, A) \longrightarrow H_2(A) \longrightarrow H_2(X)$. □

**Lemma 3.3.** *Let $X$ be a weakly contractible finite $T_0$-space such that $h(X) = 2$. Let $b, b' \in X - (\mathrm{mxl}(X) \cup \mathrm{mnl}(X))$ such that $b \neq b'$. If $\#(\widehat{F}_b \cap \widehat{F}_{b'}) \geq 2$, then $\#(\widehat{U}_b \cap \widehat{U}_{b'}) \leq 1$.*

**Proof.** Note that $\{b, b'\}$ must be an antichain. Suppose that $\#(\widehat{U}_b \cap \widehat{U}_{b'}) \geq 2$. Then there exist distinct elements $a, a' \in \widehat{F}_b \cap \widehat{F}_{b'}$ and distinct elements $c, c' \in \widehat{U}_b \cap \widehat{U}_{b'}$. Note that $\{a, a'\} \subseteq \mathrm{mxl}(X)$ and that $\{c, c'\} \subseteq \mathrm{mnl}(X)$. Hence the subspace $A = \{a, a', b, b', c, c'\}$ is homeomorphic to $\mathbb{S}^2 S^0$ and then $H_2(A) \neq 0$, which contradicts Lemma 3.2. Thus $\#(\widehat{U}_b \cap \widehat{U}_{b'}) \leq 1$. □

**Lemma 3.4.** *Let $X$ be a topological space such that, for some $x_0 \in X$, $\pi_1(X, x_0)$ is not a non-trivial perfect group. If $\mathbb{S}X$ is weakly contractible, then $X$ is weakly contractible.*

**Proof.** Let $\Sigma X$ be the suspension of $X$. Since $\Sigma X$ is weak homotopy equivalent to $\mathbb{S}X$, we obtain that $\Sigma X$ is weakly contractible. Thus $H_n(\Sigma X) = 0$ for all $n \in \mathbb{N}$. Hence $X$ is path-connected and $H_n(X) = 0$ for all $n \in \mathbb{N}$. Thus $\pi_1(X, x_0)$ is a perfect group. Therefore $\pi_1(X, x_0)$ must be the trivial group. The result then follows from Hurewicz's theorem. □

The following proposition shows that the height of a weakly contractible non-contractible finite $T_0$-space must be greater than one.

**Proposition 3.5.** *Let $X$ be a weakly contractible finite $T_0$-space with $h(X) \leq 1$. Then $X$ is contractible.*

**Proof.** Suppose that $X$ is not contractible. Without loss of generality, we may assume that $X$ does not have beat points. Let $E$ denote the set of edges of the Hasse diagram of $X$. Let $\mathcal{R} \subseteq X \times E$ be the relation defined by $x\mathcal{R}a$ if and only if the point $x$ belongs to the edge $a$. Note that $\#\mathcal{R} = 2\#E$.

On the other hand, since $X$ is a path-connected space and $\#X \geq 2$ (as we are assuming that $X$ is not contractible), from the first item of Proposition 2.3, we obtain that $\mathrm{mxl}(X) \cap \mathrm{mnl}(X) = \varnothing$. Thus, for each $x \in X$, we obtain that $x \notin \mathrm{mxl}(X)$ or $x \notin \mathrm{mnl}(X)$. And since $X$ does not have beat points, it follows that $\mathcal{R}(x) \geq 2$ for all $x \in X$. Hence $\#\mathcal{R} \geq 2\#X$.

Therefore $\#E \geq \#X$. Thus $1 = \chi(X) = \#X - \#E \leq 0$, which entails a contradiction. □

We will now prove one of the main results of this article.

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**Theorem 3.6.** *Let $X$ be a non-contractible topological space that is weakly contractible. Then $\#X \geq 9$.*

*Equivalently, if $P$ is a non-dismantlable poset such that $|\mathcal{K}(P)|$ is contractible, then $\#P \geq 9$.*

**Proof.** Observe that we may assume that $X$ is a finite $T_0$-space without beat points. By Proposition 3.5, $h(X) \geq 2$. And by Theorem 2.1, if $h(X) \geq 4$, then $\#X \geq 10$. Thus, we may assume that $h(X) = 2$ or $h(X) = 3$.

*Case 1*: $h(X) = 3$. By Theorem 2.1, $\#X \geq 8$. If $\#X = 8$, then, again by Theorem 2.1, $X$ is homeomorphic to $\mathbb{S}^3 S^0$ which is weak equivalent to $S^3$. But this is a contradiction since $X$ is weakly contractible. Thus $\#X \geq 9$.

*Case 2*: $h(X) = 2$. Applying items $(a)$ and $(b)$ of Proposition 2.3, we obtain that $\operatorname{mxl}(X) \cap \operatorname{mnl}(X) = \varnothing$, $\#\operatorname{mxl}(X) \geq 2$ and $\#\operatorname{mnl}(X) \geq 2$.

We will prove now that $\#\operatorname{mxl}(X) \geq 3$. Suppose that $\#\operatorname{mxl}(X) = 2$. Then $X$ is homeomorphic to $\mathbb{S}(X - \operatorname{mxl}(X))$ by item $(c)$ of Proposition 2.3. Let $Y = X - \operatorname{mxl}(X)$. Note that $h(Y) = 1$ and thus $\pi_1(Y, y_0)$ is a free group for all $y_0 \in Y$. By Lemma 3.4, $Y$ is weakly contractible. Since $X$ does not have beat points, the same holds for $Y$. And since $\operatorname{mnl}(X) \subseteq Y$, we obtain that $\#Y \geq 2$ and hence $Y$ is not contractible, which contradicts Proposition 3.5. Thus $\#\operatorname{mxl}(X) \geq 3$.

Working with $X^{\mathrm{op}}$ in a similar way, we obtain that $\#\operatorname{mnl}(X) \geq 3$.

Since $h(X) = 2$, we obtain that $\mathcal{B}_X$ is a non-empty antichain of $X$. If $\#\mathcal{B}_X \geq 3$, then $\#X \geq 9$. Thus we may assume that $\#\mathcal{B}_X \leq 2$.

*Case 2.1*: $\#\mathcal{B}_X = 1$. Suppose that $\mathcal{B}_X = \{b\}$. Note that $\widehat{F}_b \subseteq \operatorname{mxl}(X)$ and $\widehat{U}_b \subseteq \operatorname{mnl}(X)$. Let $\alpha_b = \#\widehat{F}_b$ and $\beta_b = \#\widehat{U}_b$. By Proposition 2.3, $\alpha_b \geq 2$ and $\beta_b \geq 2$.

Since $\operatorname{mxl}(X) \cap \operatorname{mnl}(X) = \varnothing$, it follows that $\#\widehat{U}_a \geq 2$ for all $a \in \operatorname{mxl}(X)$ by the second item of Proposition 2.3. And from Lemma 3.1, we obtain that $\#\widehat{U}_a \geq \beta_b + 2$ for all $a \in \widehat{F}_b$. Let $l$ denote the number of 1-chains of $X$ and let $m = \#\operatorname{mxl}(X)$. From the previous inequalities it follows that

$$l = \#\widehat{U}_b + \sum_{a \in \operatorname{mxl}(X)} \#\widehat{U}_a = \widehat{U}_b + \sum_{a \in \widehat{F}_b} \#\widehat{U}_a + \sum_{a \in \operatorname{mxl}(X) - \widehat{F}_b} \#\widehat{U}_a$$
$$\geq \beta_b + \alpha_b(\beta_b + 2) + 2(m - \alpha_b) = \beta_b + \alpha_b\beta_b + 2m \geq \alpha_b\beta_b + 8.$$

On the other hand, note that the number of 2-chains of $X$ is $\alpha_b\beta_b$.

Hence

$$1 = \chi(X) = \#X - l + \alpha_b\beta_b \leq \#X - (\alpha_b\beta_b + 8) + \alpha_b\beta_b = \#X - 8.$$

Therefore $\#X \geq 9$.

*Case 2.2*: $\#\mathcal{B}_X = 2$. If $\#\operatorname{mxl}(X) \geq 4$ or $\#\operatorname{mnl}(X) \geq 4$, then $\#X \geq 9$. Thus we may assume that $\#\operatorname{mxl}(X) = \#\operatorname{mnl}(X) = 3$. As above, note that $\widehat{F}_b \subseteq \operatorname{mxl}(X)$ and $\widehat{U}_b \subseteq \operatorname{mnl}(X)$ for all $b \in \mathcal{B}_X$ since $\mathcal{B}_X$ is an antichain.

First, we will prove that, for all $b \in \mathcal{B}_X$, $\#\widehat{U}_b = 2$ and $\#\widehat{F}_b = 2$. Let $b \in \mathcal{B}_X$. Clearly, $2 \leq \#\widehat{U}_b \leq 3$, where the first inequality follows from Proposition 2.3. Suppose that $\#\widehat{U}_b = 3$.

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Then $\widehat{U}_b = \mathrm{mnl}(X)$. Let $b'$ be the element of $\mathcal{B}_X - \{b\}$. We claim that $\widehat{F}_b \subseteq \widehat{F}_{b'}$. Indeed, let $a \in \widehat{F}_b$. Then $\widehat{U}_a \supseteq U_b$, and since $a$ is not a beat point of $X$, we obtain that $\widehat{U}_a \supsetneq U_b$. But since $U_b \supseteq \mathrm{mnl}(X)$, it follows that $b' \in \widehat{U}_a$ and hence $a \in \widehat{F}_{b'}$. By Proposition 2.3, $\#\widehat{F}_b \geq 2$ and $\#\widehat{U}_{b'} \geq 2$. Thus $\#(\widehat{F}_b \cap \widehat{F}_{b'}) \geq 2$ and $\#(\widehat{U}_b \cap \widehat{U}_{b'}) \geq 2$, which contradicts Lemma 3.3. Therefore $\#\widehat{U}_b \neq 3$ and hence $\#\widehat{U}_b = 2$. In a similar way, we obtain that $\#\widehat{F}_b = 2$.

Now, let $\mathrm{mxl}(X) = \{a_1, a_2, a_3\}$, $\mathcal{B}_X = \{b_1, b_2\}$ and $\mathrm{mnl}(X) = \{c_1, c_2, c_3\}$. Without loss of generality, we may assume that $\widehat{F}_{b_1} = \{a_1, a_2\}$ and $\widehat{U}_{b_1} = \{c_1, c_2\}$. We will prove that $\#\widehat{U}_{a_1} + \#\widehat{U}_{a_2} + \#\widehat{U}_{a_3} \geq 12$ by analyzing two cases: $b_2 < a_3$ and $b_2 \nless a_3$.

If $b_2 < a_3$, then $\#\widehat{U}_{a_3} \geq \#\widehat{U}_{b_2} + 2 = 4$ by Lemma 3.1. And since $b_1 < a_1$ and $b_1 < a_2$, we also obtain that $\#\widehat{U}_{a_1} \geq 4$ and $\#\widehat{U}_{a_2} \geq 4$ by Lemma 3.1. Thus $\#\widehat{U}_{a_1} + \#\widehat{U}_{a_2} + \#\widehat{U}_{a_3} \geq 12$.

If $b_2 \nless a_3$, then $\widehat{F}_{b_2} = \{a_1, a_2\} = \widehat{F}_{b_1}$. By Lemma 3.3, $\widehat{U}_{b_2} \neq \widehat{U}_{b_1} = \{c_1, c_2\}$. Hence $c_3 < b_2$. Thus $\widehat{U}_{a_1} = \widehat{U}_{a_2} = \{b_1, b_2, c_1, c_2, c_3\}$, and since $\#\widehat{U}_{a_3} \geq 2$ by Proposition 2.3, we obtain that $\#\widehat{U}_{a_1} + \#\widehat{U}_{a_2} + \#\widehat{U}_{a_3} \geq 12$.

Therefore $\#\widehat{U}_{a_1} + \#\widehat{U}_{a_2} + \#\widehat{U}_{a_3} \geq 12$ in any case.

As above, let $l$ denote the number of 1-chains of $X$. Then

$$l = \#\widehat{U}_{a_1} + \#\widehat{U}_{a_2} + \#\widehat{U}_{a_3} + \#\widehat{U}_{b_1} + \#\widehat{U}_{b_2} \geq 12 + 2 + 2 = 16.$$

Now, note that the number of 2-chains of $X$ is $\#\widehat{F}_{b_1}\#\widehat{U}_{b_1} + \#\widehat{F}_{b_2}\#\widehat{U}_{b_2} = 8$. Thus

$$\chi(X) = \#X - l + 8 \leq 8 - 16 + 8 = 0$$

and hence the space $X$ is not weakly contractible. □

As a corollary of the previous theorem, we obtain that the space of Figure 1 is a weakly contractible non-contractible space with the minimum possible number of points. In the following theorem, we find all the weakly contractible non-contractible spaces of this minimum number of points.

**Theorem 3.7.** *Let $X$ be a weakly contractible non-contractible topological space such that $\#X = 9$. Then $X$ is homeomorphic to either the space of Figure 1 or its opposite.*

**Proof.** By Theorem 3.6, we may assume that $X$ is a finite $T_0$-space without beat points. By Proposition 2.3, $\#\mathrm{mxl}(X) \geq 2$. If $\#\mathrm{mxl}(X) = 2$, $X$ is homeomorphic to $\mathbb{S}(X - \mathrm{mxl}(X))$ by Proposition 2.3. Note that $X - \mathrm{mxl}(X)$ does not have beat points. Since $\#(X - \mathrm{mxl}(X)) = 7$, from [**8**, Theorem 5.7] we obtain that $\pi_1(X - \mathrm{mxl}(X), x_0)$ is a free group for all $x_0 \in X - \mathrm{mxl}(X)$. Thus $X - \mathrm{mxl}(X)$ is weakly contractible by Lemma 3.4, which contradicts Theorem 3.6. Therefore $\#\mathrm{mxl}(X) \geq 3$. Applying this argument to $X^{\mathrm{op}}$, we obtain that $\#\mathrm{mnl}(X) \geq 3$.

By Proposition 3.5, $h(X) \geq 2$. Thus $\mathcal{B}_X \neq \varnothing$. On the other hand, $\mathrm{mxl}(X) \cap \mathrm{mnl}(X) = \varnothing$ by Proposition 2.3 and hence $\#(\mathrm{mxl}(X) \cup \mathrm{mnl}(X)) = \#\mathrm{mxl}(X) + \#\mathrm{mnl}(X) \geq 6$. Therefore $\#\mathcal{B}_X \leq 3$.

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We will analyze three cases which correspond to the possible cardinalities of the subset $\mathcal{B}_X$.

*Case 1*: $\#\mathcal{B}_X = 1$. Let $b$ be the only element of $\mathcal{B}_X$. Let $n = \#\text{mnl}(X)$, $\alpha = \#\widehat{F}_b$ and $\beta = \#\widehat{U}_b$. By Proposition 2.3, $\alpha \geq 2$ and $\beta \geq 2$. Let $\mathcal{R} \subseteq X \times X$ be the order relation of $X$ and let $\mathcal{S} = \mathcal{R} \cap (\text{mnl}(X) \times \text{mxl}(X))$. Let $\mathcal{S}_1 = \mathcal{R} \cap (\widehat{U}_b \times \widehat{F}_b)$, $\mathcal{S}_2 = \mathcal{R} \cap (\widehat{U}_b \times (\text{mxl}(X) - \widehat{F}_b))$ and $\mathcal{S}_3 = \mathcal{R} \cap ((\text{mnl}(X) - \widehat{U}_b) \times \text{mxl}(X))$. Clearly, $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ are pairwise disjoint and $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$.

Note that $\#\mathcal{S}_1 = \alpha\beta$. Also, if $z \in \widehat{U}_b$, then $\widehat{F}_z \supsetneq F_b$ since $z$ is not a beat point of $X$. Hence $\widehat{F}_z \cap (\text{mxl}(X) - \widehat{F}_b) \neq \varnothing$ for all $z \in \widehat{U}_b$. Thus $\#\mathcal{S}_2 \geq \beta$. On the other hand, from Proposition 2.3, we obtain that $\#\mathcal{S}_3 \geq 2\#(\text{mnl}(X) - \widehat{U}_b) = 2(n - \beta)$. Therefore

$$\#\mathcal{S} \geq \alpha\beta + \beta + 2(n-\beta) \geq \alpha\beta + 2 + 2(n-\beta).$$

Proceeding in a similar way, we also obtain that

$$\#\mathcal{S} \geq \alpha\beta + 2 + 2\#(\text{mxl}(X) - \widehat{F}_b) = \alpha\beta + 2 + 2(8 - n - \alpha).$$

Hence

$$\begin{aligned}\#\mathcal{S} &\geq \alpha\beta + 2 + 2\max\{n-\beta, 8-n-\alpha\} \\ &\geq \alpha\beta + 2 + (n-\beta) + (8-n-\alpha) = \alpha\beta + 10 - \alpha - \beta.\end{aligned}$$

Thus we obtain that

$$\chi(X) = 9 - (\#\mathcal{S} + \alpha + \beta) + \alpha\beta \leq 9 - \alpha\beta - 10 + \alpha\beta = -1.$$

Hence $X$ is not weakly contractible. Therefore this case is not possible.

*Case 2*: $\#\mathcal{B}_X = 2$. Without loss of generality, we may assume that $\#\text{mxl}(X) = 3$ and $\#\text{mnl}(X) = 4$. Let $b_1$ and $b_2$ be the elements of $\mathcal{B}_X$.

We will prove that $\mathcal{B}_X$ is an antichain. Indeed, if $b_1 < b_2$, then $\#\widehat{F}_{b_1} \geq \#\widehat{F}_{b_2} + 2 \geq 4$ by Lemma 3.1 and Proposition 2.3. Thus $\widehat{F}_{b_1} = \{b_2\} \cup \text{mxl}(X)$. Let $c \in \text{mnl}(X) \cap U_{b_1}$. Then $F_{b_1} \subseteq \widehat{F}_c \subseteq X - \text{mnl}(X) = \mathcal{B}_X \cup \text{mxl}(X) = F_{b_1}$. Hence $\widehat{F}_c = F_{b_1}$ and then $c$ is a beat point of $X$, which contradicts our assumptions. Therefore $\mathcal{B}_X$ must be an antichain.

Since $\mathcal{B}_X$ is an antichain, we obtain that $h(X) = 2$ and that $\widehat{F}_{b_1} \cup \widehat{F}_{b_2} \subseteq \text{mxl}(X)$ and $\widehat{U}_{b_1} \cup \widehat{U}_{b_2} \subseteq \text{mnl}(X)$. For $j \in \{1, 2\}$, let $\alpha_j = \#\widehat{F}_{b_j}$ and $\beta_j = \#\widehat{U}_{b_j}$. Note that $\alpha_j \geq 2$ and $\beta_j \geq 2$ for all $j \in \{1, 2\}$ by Proposition 2.3. As in the previous case, let $\mathcal{R} \subseteq X \times X$ be the order relation of $X$ and let $\mathcal{S} = \mathcal{R} \cap (\text{mnl}(X) \times \text{mxl}(X))$. Note that the number of 1-chains of $X$ that contain $b_1$ is $\alpha_1 + \beta_1$ and that the number of 1-chains of $X$ that contain $b_2$ is $\alpha_2 + \beta_2$. Thus

$$\chi(X) = 9 - (\alpha_1 + \beta_1 + \alpha_2 + \beta_2 + \#\mathcal{S}) + \alpha_1\beta_1 + \alpha_2\beta_2 = 7 - \#\mathcal{S} + \sum_{j=1}^{2}(\alpha_j - 1)(\beta_j - 1).$$

We will analyze two subcases.

*Case 2.1*: $\#(\widehat{F}_{b_1} \cap \widehat{F}_{b_2}) = 1$. Since $\#\text{mxl}(X) = 3$, we obtain that $\alpha_1 = \alpha_2 = 2$ and $\widehat{F}_{b_1} \cup \widehat{F}_{b_2} = \text{mxl}(X)$. Let $a$ be the only point of $\widehat{F}_{b_2} - \widehat{F}_{b_1}$.

We will now prove that $\beta_2 \leq 3$. Suppose that $\beta_2 > 3$. Since $\#\mathrm{mnl}(X) = 4$, we obtain that $\beta_2 = 4$ and $\widehat{U}_{b_2} = \mathrm{mnl}(X)$. Thus $\widehat{U}_a = U_{b_2}$ and hence $a$ is a beat point of $X$, which entails a contradiction. Therefore $\beta_2 \leq 3$.

We will now prove that $\mathrm{mxl}(X) \subseteq F_c$ for all $c \in \widehat{U}_{b_1}$. Let $c \in \widehat{U}_{b_1}$. If $b_2 > c$, then $F_c \supseteq F_{b_1} \cup F_{b_2} \supseteq \mathrm{mxl}(X)$. If $b_2 \not> c$, then $F_{b_1} \subsetneq \widehat{F}_c \subseteq \{b_1\} \cup \mathrm{mxl}(X) = F_{b_1} \cup \{a\}$. And since $c$ is not a beat point of $X$, we obtain that $\widehat{F}_c = F_{b_1} \cup \{a\}$ and hence $\mathrm{mxl}(X) \subseteq F_c$.

Since $\mathrm{mxl}(X) \subseteq F_c$ for all $c \in \widehat{U}_{b_1}$ and applying Proposition 2.3, we obtain that

$$\#\mathcal{S} \geq 3\#\widehat{U}_{b_1} + 2\#(\mathrm{mnl}(X) - \widehat{U}_{b_1}) = 3\beta_1 + 2(4 - \beta_1) = \beta_1 + 8.$$

Thus

$$\chi(X) = 7 - \#\mathcal{S} + \sum_{j=1}^{2}(\alpha_j - 1)(\beta_j - 1) \leq 7 - \beta_1 - 8 + \beta_1 - 1 + \beta_2 - 1 = \beta_2 - 3 \leq 0$$

and hence $X$ is not weakly contractible.

*Case 2.2*: $\#(\widehat{F}_{b_1} \cap \widehat{F}_{b_2}) \geq 2$. By Lemma 3.3, $\#(\widehat{U}_{b_1} \cap \widehat{U}_{b_2}) \leq 1$. Since $\beta_1 \geq 2$ and $\beta_2 \geq 2$, we obtain that $\widehat{U}_{b_1} - \widehat{U}_{b_2} \neq \varnothing$ and $\widehat{U}_{b_2} - \widehat{U}_{b_1} \neq \varnothing$.

We will now prove that $\alpha_1 = \alpha_2 = 2$ and that $\mathrm{mxl}(X) \subseteq F_c$ for all $c \in (\widehat{U}_{b_1} - \widehat{U}_{b_2}) \cup (\widehat{U}_{b_2} - \widehat{U}_{b_1})$. Let $c \in \widehat{U}_{b_1} - \widehat{U}_{b_2}$. We have that $F_{b_1} \subseteq \widehat{F}_c \subseteq \{b_1\} \cup \mathrm{mxl}(X)$. Since $c$ is not a beat point of $X$, we obtain that $\widehat{F}_c \neq F_{b_1}$ and thus $\mathrm{mxl}(X) \nsubseteq F_{b_1}$, which implies that $\alpha_1 = 2$ since $\#\mathrm{mxl}(X) = 3$. Then $\#(\mathrm{mxl}(X) - F_{b_1}) = 1$, and since $\widehat{F}_c \neq F_{b_1}$, we obtain that $\mathrm{mxl}(X) \subseteq F_c$. In a similar way, we obtain that $\alpha_2 = 2$ and that $\mathrm{mxl}(X) \subseteq F_c$ for all $c \in \widehat{U}_{b_2} - \widehat{U}_{b_1}$.

Thus, applying Proposition 2.3, we obtain that $\#\mathcal{S} \geq 3 + 3 + 2 + 2 = 10$. On the other hand, note that $\beta_1 + \beta_2 \leq 5$ since $\#(\widehat{U}_{b_1} \cap \widehat{U}_{b_2}) \leq 1$. Hence

$$\chi(X) = 7 - \#\mathcal{S} + \sum_{j=1}^{2}(\alpha_j - 1)(\beta_j - 1) \leq 7 - 10 + \beta_1 - 1 + \beta_2 - 1 = \beta_1 + \beta_2 - 5 \leq 0.$$

Therefore $X$ is not weakly contractible.

*Case 3*: $\#\mathcal{B}_X = 3$. Note that $\#\mathrm{mxl}(X) = \#\mathrm{mnl}(X) = 3$. Let $b_1$, $b_2$ and $b_3$ be the elements of $\mathcal{B}_X$.

Suppose that $\mathcal{B}_X$ is a chain. Without loss of generality, we may assume that $b_1 < b_2 < b_3$. Applying Lemma 3.1 and Proposition 2.3, we obtain that

$$\#\widehat{F}_{b_1} \geq \#\widehat{F}_{b_2} + 2 \geq \#\widehat{F}_{b_3} + 4 \geq 6,$$

but this cannot be possible since $\widehat{F}_{b_1} \subseteq \{b_2, b_3\} \cup \mathrm{mxl}(X)$. Thus $\mathcal{B}_X$ is not a chain.

We will now prove that $\mathcal{B}_X$ is an antichain. Assume the contrary. Since $\mathcal{B}_X$ is neither a chain nor an antichain, without loss of generality, we may assume that $b_1$ and $b_3$ are incomparable and that $b_1$ and $b_2$ are comparable. In addition, considering $X^{\mathrm{op}}$, if necessary, we may suppose that $b_1 < b_2$.

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Note that $b_2 \not< b_3$ since $\mathcal{B}_X$ is not a chain. Hence $\widehat{F}_{b_2} \subseteq \mathrm{mxl}(X)$. By Proposition 2.3, $\#\widehat{F}_{b_2} \geq 2$. Let $a_1$ and $a_2$ be distinct elements of $\widehat{F}_{b_2}$ and let $a_3$ be the remaining maximal element of $X$. By Lemma 3.1, $\#\widehat{F}_{b_1} \geq \#\widehat{F}_{b_2} + 2 \geq 4$, and since $b_1$ and $b_3$ are incomparable, it follows that $\widehat{F}_{b_1} = \{b_2, a_1, a_2, a_3\}$ and $\widehat{F}_{b_2} = \{a_1, a_2\}$.

Now observe that $\widehat{F}_{b_3} \subseteq \{b_2, a_1, a_2, a_3\}$ since $b_1$ and $b_3$ are incomparable. We claim that $\widehat{F}_{b_3}$ is not path-connected. Indeed, if $b_3 < b_2$, then, proceeding as in the previous paragraph, we obtain that $\widehat{F}_{b_3} = \{b_2, a_1, a_2, a_3\}$, which is not path-connected. And if $b_3 \not< b_2$, then $\widehat{F}_{b_3} \subseteq \mathrm{mxl}(X)$ and hence $\widehat{F}_{b_3}$ is a discrete subspace and $\#\widehat{F}_{b_3} \geq 2$ by Proposition 2.3. Thus $\widehat{F}_{b_3}$ is not path-connected.

By Proposition 2.3, $\#(\widehat{U}_{b_1} \cap \mathrm{mnl}(X)) \geq 2$. Let $c_1$ and $c_2$ be distinct elements of $\widehat{U}_{b_1} \cap \mathrm{mnl}(X)$ and let $c_3$ be the remaining minimal element of $X$. Since $c_1 < b_1$ and $c_1$ is not a beat point of $X$, we obtain that $\{b_1, b_2, a_1, a_2, a_3\} = F_{b_1} \subsetneq \widehat{F}_{c_1} \subseteq X - \mathrm{mnl}(X)$. Hence $\widehat{F}_{c_1} = \{b_1, b_2, b_3, a_1, a_2, a_3\}$. In a similar way, $\widehat{F}_{c_2} = \{b_1, b_2, b_3, a_1, a_2, a_3\}$.

From Proposition 2.2 we obtain that $H_1(\widehat{F}_{c_1}) \cong H_1(\widehat{F}_{c_1}, F_{b_1}) \cong \widetilde{H}_0(\widehat{F}_{b_3}) \neq 0$ since $\widehat{F}_{b_3}$ is not path-connected. Applying Proposition 2.2 again, we obtain that $H_2(X) \cong H_2(X, F_{c_2}) \cong H_1(\widehat{F}_{c_1}) \oplus H_1(\widehat{F}_{c_3}) \neq 0$ and hence $X$ is not weakly contractible. Therefore $\mathcal{B}_X$ must be an antichain.

Since $\mathcal{B}_X$ is an antichain, we obtain that $h(X) = 2$. For $j \in \{1, 2, 3\}$, let $\alpha_j = \#\widehat{F}_{b_j}$ and $\beta_j = \#\widehat{U}_{b_j}$. Note that $\alpha_j \geq 2$ and $\beta_j \geq 2$ for all $j \in \{1, 2, 3\}$ by Proposition 2.3. As in the previous cases, let $\mathcal{R} \subseteq X \times X$ be the order relation of $X$ and let $\mathcal{S} = \mathcal{R} \cap (\mathrm{mnl}(X) \times \mathrm{mxl}(X))$. Thus

$$1 = \chi(X) = 9 - \left(\sum_{j=1}^{3} \alpha_j + \sum_{j=1}^{3} \beta_j + \#\mathcal{S}\right) + \sum_{j=1}^{3} \alpha_j \beta_j = 6 - \#\mathcal{S} + \sum_{j=1}^{3} (\alpha_j - 1)(\beta_j - 1).$$

Hence

$$\#\mathcal{S} = 5 + \sum_{j=1}^{3} (\alpha_j - 1)(\beta_j - 1) \geq 8.$$

Thus $\#\mathcal{S} = 8$ or $\#\mathcal{S} = 9$ and hence $\sum_{j=1}^{3} (\alpha_j - 1)(\beta_j - 1) \in \{3, 4\}$. Therefore at least five of the numbers $\alpha_1$, $\alpha_2$, $\alpha_3$, $\beta_1$, $\beta_2$ and $\beta_3$ are equal to 2 and the remaining one might be 2 or 3. Without loss of generality and considering $X^{\mathrm{op}}$, if necessary, we may assume that $\alpha_1 = \alpha_2 = \alpha_3 = 2$.

**Claim 1.** *$\#(\widehat{F}_{b_k} \cap \widehat{F}_{b_l}) = 1$ for all $k, l \in \{1, 2, 3\}$ with $k \neq l$.*

Suppose that there exist $k, l \in \{1, 2, 3\}$ with $k \neq l$ such that $\#(\widehat{F}_{b_k} \cap \widehat{F}_{b_l}) = 2$. Without loss of generality, we may assume that $\#(\widehat{F}_{b_1} \cap \widehat{F}_{b_2}) = 2$. Thus $\#(\widehat{U}_{b_1} \cap \widehat{U}_{b_2}) \leq 1$ by Lemma 3.3. And since $\#\mathrm{mnl}(X) = 3$, we obtain that $\#(\widehat{U}_{b_1} \cap \widehat{U}_{b_2}) = 1$. Hence $\beta_1 = \beta_2 = 2$ and $\widehat{U}_{b_1} \cup \widehat{U}_{b_2} = \mathrm{mnl}(X)$.

Let $a_1$ and $a_2$ be the elements of $\widehat{F}_{b_1} \cap \widehat{F}_{b_2}$ and let $a_3$ be the remaining maximal element of $X$. Note that $U_{a_1} \supseteq \mathrm{mnl}(X)$ and $U_{a_2} \supseteq \mathrm{mnl}(X)$.

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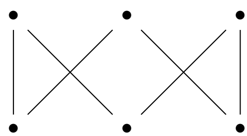
Suppose that $\{a_1, a_2\} \subseteq F_{b_3}$. Then $\#(\widehat{U}_{b_1} \cap \widehat{U}_{b_3}) \leq 1$ and $\#(\widehat{U}_{b_2} \cap \widehat{U}_{b_3}) \leq 1$ by Lemma 3.3. And since $\#\mathrm{mnl}(X) = 3$, we obtain that $\beta_3 = 2$, $\#(\widehat{U}_{b_1} \cap \widehat{U}_{b_3}) = 1$ and $\#(\widehat{U}_{b_2} \cap \widehat{U}_{b_3}) = 1$. Hence $|\mathcal{K}(X - \{a_3\})|$ is homeomorphic to $S^2$, which contradicts Lemma 3.2. Thus $\{a_1, a_2\} \nsubseteq F_{b_3}$.

Hence, $a_3 \in \widehat{F}_{b_3}$. Since $\alpha_1 = \alpha_2 = 2$, we obtain that $b_1 \nless a_3$ and $b_2 \nless a_3$. And since $a_3$ is not a beat point of $X$, we obtain that $U_{b_3} \subsetneq \widehat{U}_{a_3} \subseteq \{b_3\} \cup \mathrm{mnl}(X)$. But $\#U_{b_3} \geq 3$. Thus $\#U_{b_3} = 3$, $\beta_3 = 2$ and $\widehat{U}_{a_3} = \{b_3\} \cup \mathrm{mnl}(X)$. Hence $\mathcal{S} = \mathrm{mnl}(X) \times \mathrm{mxl}(X)$. Thus

$$\chi(X) = 6 - \#\mathcal{S} + \sum_{j=1}^{3} (\alpha_j - 1)(\beta_j - 1) = 6 - 9 + 3 = 0,$$

which entails a contradiction. This proves Claim 1.

Observe that, from Claim 1, we obtain that $X - \mathrm{mnl}(X)$ is homeomorphic to the following space.



**Claim 2.** *If $k, l \in \{1, 2, 3\}$ are distinct elements such that $\beta_k = \beta_l = 2$, then $\#(\widehat{U}_{b_k} \cap \widehat{U}_{b_l}) = 1$.*

Suppose that there exist distinct elements $k, l \in \{1, 2, 3\}$ such that $\beta_k = \beta_l = 2$ and $\#(\widehat{U}_{b_k} \cap \widehat{U}_{b_l}) = 2$. Let $m$ be the remaining element of $\{1, 2, 3\}$, let $c_1$ and $c_2$ be the elements of $\widehat{U}_{b_k} \cap \widehat{U}_{b_l}$ and let $c_3$ be the remaining element of $\mathrm{mnl}(X)$. Note that $\mathrm{mxl}(X) \subseteq F_{c_1} \cap F_{c_2}$. If $\{c_1, c_2\} \subseteq \widehat{U}_{b_m}$, then $|\mathcal{K}(X - \{c_3\})|$ is homeomorphic to $S^2$, which contradicts Lemma 3.2. Thus $\{c_1, c_2\} \nsubseteq \widehat{U}_{b_m}$. Hence $\beta_m = 2$ and $c_3 \in U_{b_m}$. If either $c_3 < b_k$ or $c_3 < b_l$, then $\mathrm{mxl}(X) \subseteq F_{c_3}$. Otherwise, since $c_3$ is not a beat point of $X$, we obtain that $F_{b_m} \subsetneq \widehat{F}_{c_3} \subseteq \{b_m\} \cup \mathrm{mxl}(X)$. As $\#F_{b_m} = 3$, it follows that $\widehat{F}_{c_3} = \{b_m\} \cup \mathrm{mxl}(X)$ and hence $\mathrm{mxl}(X) \subseteq F_{c_3}$. Thus $\mathrm{mxl}(X) \subseteq F_{c_3}$ in any case.

Hence $\mathcal{S} = \mathrm{mnl}(X) \times \mathrm{mxl}(X)$. Thus

$$\chi(X) = 6 - \#\mathcal{S} + \sum_{j=1}^{3} (\alpha_j - 1)(\beta_j - 1) = 6 - 9 + 3 = 0,$$

which entails a contradiction. This proves Claim 2.

Now, let $a_1$, $a_2$ and $a_3$ be the only elements of $\widehat{F}_{b_1} \cap \widehat{F}_{b_2}$, $\widehat{F}_{b_1} \cap \widehat{F}_{b_3}$ and $\widehat{F}_{b_2} \cap \widehat{F}_{b_3}$, respectively. Without loss of generality, we may assume that $\beta_1 = \beta_3 = 2$. Thus $\#(\widehat{U}_{b_1} \cap \widehat{U}_{b_3}) = 1$ by Claim 2. Let $c_1$, $c_2$ and $c_3$ be the only elements of $\widehat{U}_{b_1} - \widehat{U}_{b_3}$, $\widehat{U}_{b_1} \cap \widehat{U}_{b_3}$ and $\widehat{U}_{b_3} - \widehat{U}_{b_1}$, respectively.

If $\beta_2 = 2$, then $\#(\widehat{U}_{b_2} \cap \widehat{U}_{b_1}) = 1$ and $\#(\widehat{U}_{b_2} \cap \widehat{U}_{b_3}) = 1$ by Claim 2. Then $X$ is homeomorphic to the following space.

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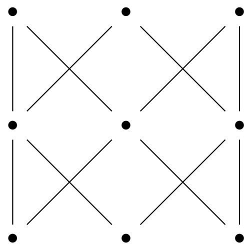

Hence $|\mathcal{K}(X)|$ is homotopy equivalent to $S^1$ and then $X$ is not weakly contractible. Thus $\beta_2 = 3$ and hence $X$ is homeomorphic to the space of Figure 1. □

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