J. Inst. Math. Jussieu (2020) **19**(5), 1573–1600 1573
doi:10.1017/S1474748018000488

RECOVERY OF NON-COMPACTLY SUPPORTED COEFFICIENTS OF ELLIPTIC EQUATIONS ON AN INFINITE WAVEGUIDE

YAVAR KIAN

Aix Marseille Univ, Université de Toulon, CNRS, CPT, Marseille, France
(yavar.kian@univ-amu.fr)

(Received 19 October 2017; revised 2 October 2018; accepted 11 October 2018; first published online 5 November 2018)

Abstract We consider the unique recovery of a non-compactly supported and non-periodic perturbation of a Schrödinger operator in an unbounded cylindrical domain, also called waveguide, from boundary measurements. More precisely, we prove recovery of a general class of electric potentials from the partial Dirichlet-to-Neumann map, where the Dirichlet data is supported on slightly more than half of the boundary and the Neumann data is taken on the other half of the boundary. We apply this result in different contexts including recovery of some general class of non-compactly supported coefficients from measurements on a bounded subset and recovery of an electric potential, supported on an unbounded cylinder, of a Schrödinger operator in a slab.

Keywords: inverse problems; elliptic equations; scalar potential; unbounded domain; infinite cylindrical waveguide; slab; partial data; Carleman estimate

2010 *Mathematics subject classification*: Primary 35R30; 35J15

1. Introduction

Let Ω be an unbounded open set of $\mathbb{R}^3$ taking the form $\Omega := \omega \times \mathbb{R}$, with ω a $\mathcal{C}^2$ bounded open set of $\mathbb{R}^2$. We associate with every point $x \in \Omega$ the coordinate $x = (x', x_3)$, where $x_3 \in \mathbb{R}$ and $x' := (x_1, x_2) \in \omega$. For $q \in L^\infty(\Omega)$ such that 0 is not in the spectrum of $-\Delta + q$ with Dirichlet boundary condition, we introduce the following boundary value problem (BVP):

$$\begin{cases} (-\Delta + q)v = 0, & \text{in } \Omega, \\ v = f, & \text{on } \Gamma := \partial\Omega. \end{cases} \tag{1.1}$$

Recall that $\Gamma = \partial\omega \times \mathbb{R}$ and that the outward unit normal vector ν to Γ takes the form

$$\nu(x', x_3) = (\nu'(x'), 0), \quad x = (x', x_3) \in \Gamma,$$

where ν' is the outward unit normal vector of $\partial\omega$. From now on, we denote by ν both exterior unit vectors normal to $\partial\omega$ and to Γ. We fix $\theta_0 \in \mathbb{S}^1 := \{y \in \mathbb{R}^2;\ |y| = 1\}$ and we introduce the θ_0-illuminated (respectively, θ_0-shadowed) face of $\partial\omega$ as

$$\partial\omega_{\theta_0}^- := \{x \in \partial\omega;\ \theta_0 \cdot \nu(x) \leqslant 0\} \text{ (respectively, } \partial\omega_{\theta_0}^+ = \{x \in \partial\omega;\ \theta_0 \cdot \nu(x) \geqslant 0\}). \tag{1.2}$$

Here and in the remaining part of this text, we denote by $x \cdot y := \sum_{j=1}^{k} x_j y_j$ the Euclidean scalar product of any two vectors $x := (x_1, \dots, x_k)$ and $y := (y_1, \dots, y_k)$ of $\mathbb{R}^k$, for $k \in \mathbb{N}^*$, and we put $|x| := (x \cdot x)^{1/2}$.

We fix G a portion of Γ taking the form $G := G' \times \mathbb{R}$, where G' is an arbitrary open set of $\partial\omega$ containing $\partial\omega_{\theta_0}^-$ and consider $K = K' \times \mathbb{R}$ with K' an arbitrary open set of $\partial\omega$ containing $\partial\omega_{\theta_0}^+$. The main goal of this paper consists of proving unique determination of q from the knowledge of the partial Dirichlet-to-Neumann (DN) map

$$\Lambda_q : f \mapsto \partial_\nu v_{|G}, \tag{1.3}$$

where ∂_ν is the normal derivative and $\mathrm{supp}(f) \subset K$.

1.1. Physical motivations

Let us recall that the problem under consideration in this paper is related to the so-called electrical impedance tomography (EIT) and its several applications in medical imaging and others. Note that the specific geometry of an infinite cylinder or closed waveguide can be considered for problems of transmission to long distance or transmission through particular structures, where the length-to-diameter ratio is really high, such as nanostructures. In this context, the problem addressed in this paper can correspond to the unique recovery of an impurity perturbing the guided propagation (see [11, 26]). Let us also observe that in Corollary 1.4, we show how one can apply our result to the problem stated in a slab, which is frequently used for modeling propagation in shallow-ocean acoustics (e.g. [1]), for coefficients supported in an infinite cylinder.

1.2. Known results

Since the pioneering work of [7], interest has grown in the Calderón or the EIT problem. In [47], Sylvester and Uhlmann provided one of the first and most important results related to this problem. They proved, in dimension $n \geqslant 3$, the unique recovery of a smooth conductivity from the full DN map. Since then, many authors have extended this result in several ways. The determination of an unknown coefficient from partial knowledge of the DN map was first addressed by Bukhgeim and Uhlmann in [6] and extended by Kenig, Sjöstrand and Uhlmann in [29] to the recovery of a potential from restriction of data to the back and the front face illuminated by a point lying outside the convex hull of the domain. Note that the result of [29] requires an overlap between the portion of the boundary where the measurements are made and the support of the test functions. In [28], Kenig and Salo removed this condition. We mention also the work of Isakov [25], who has considered this problem with inputs and measurements on the same portion of the boundary. In dimension two, similar results with full and partial data have been stated in [5, 22, 23]. Moreover, without being exhaustive, we refer to the work of [8, 9, 15, 36, 43, 44] dealing with the stability issue associated with this problem and some results inspired by this approach for other partial differential equations (PDEs) stated in [13, 20, 31–33].

Let us remark that all the above-mentioned results have been proved in a bounded domain. It appears that only a small number of mathematical papers deal with inverse BVPs in an unbounded domain. Combining results of unique continuation with complex geometric optics (CGO) solutions and a Carleman estimate borrowed from [6], Li and Uhlmann proved in [40] the unique recovery of compactly supported electric potentials of the stationary Schrödinger operator in a slab from partial boundary measurements. In [37], the authors extended this result to magnetic Schrödinger operators and [10] treated the stability issue for this inverse problem. We mention also [38, 39] dealing with more general Schrödinger equations, the work of [48] for bi-harmonic operators and the recovery of an embedded object in a slab treated by [21, 45]. More recently, [16, 17] proved the stable recovery of coefficients periodic along the axis of an infinite cylindrical domain. Finally, we mention [3, 4, 18, 27, 30, 34, 35] dealing with determination of non-compactly supported coefficients appearing in different PDEs from boundary measurements.

1.3. Statement of the main result and applications

In order to state the main result of this article, we start by recalling some results borrowed from [6, 16, 17] related to the well-posedness of the BVP (1.1) in the space $H_\Delta(\Omega) := \{u \in L^2(\Omega);\ \Delta u \in L^2(\Omega)\}$ with the norm

$$\|u\|^2_{H_\Delta(\Omega)} := \|u\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)}.$$

Since Ω is unbounded, for $X = \omega$ or $X = \partial\omega$ and any $s > 0$, we define the space $H^s(X \times \mathbb{R})$ by

$$H^s(X \times \mathbb{R}) := L^2(X; H^s(\mathbb{R})) \cap L^2(\mathbb{R}; H^s(X)).$$

We define also $H^{-s}(\Gamma)$ to be the dual space of $H^s(\Gamma)$. Combining [6, Lemma 1.1] with [16, Lemma 2.2], we deduce that the map

$$\mathcal{T}_0 u := u_{|\Gamma} \text{ (respectively, } \mathcal{T}_1 u := \partial_\nu u_{|\Gamma}), \quad u \in \mathcal{C}_0^\infty(\mathbb{R}^3)$$

extends into a bounded operator $\mathcal{T}_0 : H_\Delta(\Omega) \to H^{-\frac{1}{2}}(\Gamma)$ (respectively, $\mathcal{T}_1 : H_\Delta(\Omega) \to H^{-\frac{3}{2}}(\Gamma)$). We set the space

$$\mathcal{H}(\Gamma) := \mathcal{T}_0 H_\Delta(\Omega) = \{\mathcal{T}_0 u;\ u \in H_\Delta(\Omega)\},$$

and note from [16, Lemma 2.2] that $\mathcal{T}_0$ is bijective from $B := \{u \in L^2(\Omega);\ \Delta u = 0\}$ onto $\mathscr{H}(\Gamma)$. Thus, with reference to [6, 42], we consider

$$\|f\|_{\mathcal{H}(\Gamma)} := \|\mathcal{T}_0^{-1} f\|_{H_\Delta(\Omega)} = \|\mathcal{T}_0^{-1} f\|_{L^2(\Omega)}. \tag{1.4}$$

Note that [16, Lemma 2.2] is a consequence of [16, Lemma 2.1] and in [16, Lemma 2.1] we use the formula

$$\langle \Delta G, F\rangle_{L^2(\Omega)} = \langle G, \Delta F\rangle_{L^2(\Omega)}, \quad F \in H_0^2(\Omega),\ G \in H_\Delta(\Omega), \tag{1.5}$$

where $H_0^2(\Omega)$ denotes the closure of $\mathcal{C}_0^\infty(\Omega)$ in $H^2(\Omega)$. Since Ω is unbounded, the functions lying in $H^2(\Omega)$ or in $H_\Delta(\Omega)$ may have complicated behavior, and formula (1.5) needs some clarifications. Let us show (1.5). For this purpose, fix $(F_n)_{n\geqslant 1}$ a sequence of functions lying in $\mathcal{C}_0^\infty(\Omega)$ that converges to F with respect to $H^2(\Omega)$ and note that

$$\langle \Delta G, F_n\rangle_{L^2(\Omega)} = \langle \Delta G, F_n\rangle_{D'(\Omega),\mathcal{C}_0^\infty(\Omega)} = \langle G, \Delta F_n\rangle_{D'(\Omega),\mathcal{C}_0^\infty(\Omega)} = \langle G, \Delta F_n\rangle_{L^2(\Omega)}, \quad n \geqslant 1. \tag{1.6}$$

https://doi.org/10.1017/S1474748018000488 Published online by Cambridge University Press

Moreover, applying the Cauchy–Schwarz inequality, we get

$$\langle \Delta G, F_n\rangle_{L^2(\Omega)} - \langle \Delta G, F\rangle_{L^2(\Omega)} \leqslant \|\Delta G\|_{L^2(\Omega)}\|F_n - F\|_{L^2(\Omega)} \leqslant \|\Delta G\|_{L^2(\Omega)}\|F_n - F\|_{H^2(\Omega)},$$
$$\langle G, \Delta F_n\rangle_{L^2(\Omega)} - \langle G, \Delta F\rangle_{L^2(\Omega)} \leqslant \|G\|_{L^2(\Omega)}\|\Delta(F_n - F)\|_{L^2(\Omega)} \leqslant \|\Delta G\|_{L^2(\Omega)}\|F_n - F\|_{H^2(\Omega)}.$$

Combining this with the fact that

$$\lim_{n\to\infty} \|F_n - F\|_{H^2(\Omega)} = 0,$$

we deduce (1.5) from (1.6) by sending $n \to \infty$. This proves (1.5).

We define also $\mathcal{H}_K(\Gamma) := \{f \in \mathcal{H}(\Gamma) :\ \mathrm{supp}(f) \subset K\}$. Then, in view of [16, Proposition 1.1], assuming that 0 is not in the spectrum of $-\Delta + q$ with Dirichlet boundary condition on Ω, for any $f \in \mathscr{H}(\Gamma)$ we deduce that the BVP (1.1) admits a unique solution $v \in L^2(\Omega)$. Moreover, the DN map $\Lambda_q : f \mapsto \mathcal{T}_1 v_{|G}$ is a bounded operator from $\mathscr{H}_K(\Gamma)$ into $H^{-\frac{3}{2}}(G)$.

The main result of this paper can be stated as follows.

Theorem 1.1. *Let $q_1, q_2 \in L^\infty(\Omega)$ be such that $q_1 - q_2 \in L^1(\Omega)$ and 0 is not in the spectrum of $-\Delta + q_j$, $j = 1, 2$, with Dirichlet boundary condition on Ω. Then the condition*

$$\Lambda_{q_1} = \Lambda_{q_2} \tag{1.7}$$

implies $q_1 = q_2$.

From the main result of this paper, stated in Theorem 1.1, we deduce three other results related to other problems stated in an unbounded domain. The first application that we consider corresponds to the Calderón problem stated in the unbounded domain Ω. In order to state this problem, for $a_* \in (0, +\infty)$ and $a_0 \in W^{2,\infty}(\Omega)$ satisfying $a_0 \geqslant a_*$, we introduce the set of functions

$$\mathcal{A} := \{a \in \mathcal{C}^1(\overline{\Omega}) \cap W^{1,\infty}(\Omega) \cap H^2_{loc}(\Omega) :\ a \geqslant a_*,\ \Delta\left(a^{\frac{1}{2}}\right) - \Delta\left(a_0^{\frac{1}{2}}\right) \in L^1(\Omega) \cap L^\infty(\Omega)\}$$

and, for $a \in \mathcal{A}$, the BVP

$$\begin{cases} -\mathrm{div}(a\nabla u) = 0, & \text{in } \Omega,\\ u = f, & \text{on } \Gamma. \end{cases} \tag{1.8}$$

Recall that for any $a \in \mathcal{A}$ and any $f \in H^{\frac{1}{2}}(\Gamma)$, the BVP (1.8) admits a unique solution $u \in H^1(\Omega)$. Moreover, the full DN map associated with (1.8), defined by $f \mapsto a\mathcal{T}_1 u$ is a bounded operator from $H^{\frac{1}{2}}(\Gamma)$ to $H^{-\frac{1}{2}}(\Gamma)$. We define the partial DN map associated with (1.8) by

$$\Sigma_a : H^{\frac{1}{2}}(\Gamma) \cap a^{-\frac{1}{2}}(\mathcal{H}_K(\Gamma)) \ni f \mapsto a\mathcal{T}_1 u_{|G}, \tag{1.9}$$

where $a^{-\frac{1}{2}}(\mathcal{H}_K(\Gamma)) := \{a^{-\frac{1}{2}} f;\ f \in \mathcal{H}_K(\Gamma)\}$. The first application of Theorem 1.1 claims unique recovery of a conductivity $a \in \mathcal{A}$, from the knowledge of Σ_a. It is stated as follows.

Corollary 1.2. *Let ω be connected and pick $a_j \in \mathcal{A}$, for $j = 1, 2$, obeying*

$$a_1(x) = a_2(x), \quad x \in \Gamma \tag{1.10}$$

and

$$\partial_\nu a_1(x) = \partial_\nu a_2(x), \quad x \in K \cap G. \tag{1.11}$$

Then the condition $\Sigma_{a_1} = \Sigma_{a_2}$ *implies* $a_1 = a_2$.

For our second application we consider the recovery of potentials that are known in the neighborhood of the boundary outside a compact set. In the spirit of [2], we can improve Theorem 1.1 in a quite important way in that case. More precisely, we fix $R > 0$ and we consider γ_1 an arbitrary open subset of $K' \times (-\infty, -R)$, γ_2 an open subset of $\partial\omega \times (-\infty, -R)$, γ_1' an open subset of $K' \times (R, +\infty)$ and γ_2' an open subset of $\partial\omega \times (R, +\infty)$. Then, we consider the partial DN map given by

$$\Lambda^*_{q,R} : \{h \in \mathcal{H}(\Gamma) : \ \mathrm{supp}(h) \subset (K' \times [-R, R]) \cup \gamma_1 \cup \gamma_1'\} \ni f \mapsto \mathcal{T}_1 v_{|(\partial\omega\times[-R,R])\cup\gamma_2\cup\gamma_2'},$$

with v the solution of (1.1). Our second application can be stated as follows.

Corollary 1.3. *Let* ω *be connected,* $R > 0$, $\delta \in (0, R)$, $q_1, q_2 \in L^\infty(\Omega)$ *be such that* $q_1 - q_2 \in L^1(\Omega)$ *and* 0 *is not in the spectrum of* $-\Delta + q_j$, $j = 1, 2$, *with Dirichlet boundary condition on* Ω. *We fix* $\omega_{1,*}$, $\omega_{2,*}$, *two arbitrary* $\mathcal{C}^2$ *open and connected subsets of* ω *satisfying* $\partial\omega \subset (\partial\omega_{1,*} \cap \partial\omega_{2,*})$. *We consider also* $\Omega_{j,*}$, $j = 1, 2$, *two* $\mathcal{C}^2$ *open and connected subsets of* Ω *such that*

$$\omega_{1,*} \times (-\infty, -R) \subset \Omega_{1,*} \subset \omega_{1,*} \times (-\infty, \delta - R),$$
$$\omega_{2,*} \times (R, +\infty) \subset \Omega_{2,*} \subset \omega_{2,*} \times (R - \delta, +\infty)$$

and we assume that

$$q_1(x) = q_2(x), \quad x \in \Omega_{1,*} \cup \Omega_{2,*}. \tag{1.12}$$

Then the condition $\Lambda^*_{q_1,R} = \Lambda^*_{q_2,R}$ *implies* $q_1 = q_2$.

In our third application we consider the recovery of potentials, supported in an infinite cylinder, appearing in a stationary Schrödinger equation on a slab. More precisely, for $L > 0$, we consider the set $\mathcal{O} := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_1 \in (0, L)\}$; then assuming that $q \in L^\infty(\mathcal{O})$ and that 0 is not in the spectrum of $-\Delta + q$ with Dirichlet boundary condition on $\mathcal{O}$, we consider the problem

$$\begin{cases} (-\Delta + q)v = 0, & \text{in } \mathcal{O}, \\ v_{|x_1=0} = 0, & \\ v_{|x_1=L} = f. & \end{cases} \tag{1.13}$$

Fixing $r > 0$, $\partial\mathcal{O}_+ := \{(L, x_2, x_3) : \ x_2, x_3 \in \mathbb{R}\}$ and

$$\partial\mathcal{O}_{-,r} := \{(0, x_2, x_3) : \ x_2 \in (-r, r), \ x_3 \in \mathbb{R}\},$$

we associate with this problem the partial DN map

$$\mathcal{N}_{q,r} : H^{\frac{1}{2}}(\partial\mathcal{O}_+) \ni f \mapsto -\partial_{x_1} v_{|\partial\mathcal{O}_{-,r}}.$$

Then, we prove the following result.

Corollary 1.4. *Let $q_1, q_2 \in L^\infty(\mathcal{O})$ be such that $q_1 - q_2 \in L^1(\mathcal{O})$ and 0 is not in the spectrum of $-\Delta + q_j$, $j = 1, 2$, with Dirichlet boundary condition on $\mathcal{O}$. Moreover, assume that there exists $r \in (0, +\infty)$ such that*

$$q_1(x_1, x_2, x_3) = q_2(x_1, x_2, x_3) = 0, \quad (x_1, x_2, x_3) \in \{(y_1, y_2, y_3) \in \mathcal{O} : \ |y_2| \geqslant r\}. \tag{1.14}$$

Then, for any $R > r$, the condition

$$\mathcal{N}_{q_1,R} = \mathcal{N}_{q_2,R} \tag{1.15}$$

implies $q_1 = q_2$.

1.4. Comments about the main result and the applications

To our best knowledge this is the first paper proving recovery of coefficients that are neither compactly supported nor periodic for elliptic equations in unbounded domains from boundary measurements. Indeed, beside the present paper it seems that only these two cases have been addressed so far (see [16, 17, 37, 40]).

Like several other papers, the main tools in our analysis are suitable solutions of the equation also called complex geometric optics solutions combined with Carleman estimates. It has been proved by [16, 17, 37, 40] that for compactly supported or periodic coefficients one can apply unique continuation or Floquet decomposition in order to transform the problem on an unbounded domain into a problem on a bounded domain. Then, one can use the CGO solutions for the problem on the bounded domain in order to prove the recovery of the coefficients under consideration. For a more general class of coefficients, one cannot apply such arguments and the construction of CGO solutions for the problem on unbounded domains seems unavoidable. The main difficulty in the construction of such CGO solutions for unbounded domains comes from the fact that the CGO solutions should admit some kind of decay in order to be square integrable in the domain. It seems that this condition is not fulfilled by any CGO solution considered so far. In this paper, using a suitable localization in space, that propagates along the infinite direction of the unbounded cylindrical domain, we introduce, for what seems to be the first time, CGO solutions that can be directly applied to the inverse problem on the unbounded domain. This is an important difference from previous related works and it allows also to derive results like Corollary 1.3 where the recovery of non-compactly supported coefficients is proved by means of measurements on a bounded subset of the unbounded boundary. In addition to the specific property of the principal part of our CGO solutions, we prove the extension of several arguments, required for our construction, to the unbounded domain such as Carleman estimate and construction of the decaying remainder term (see §§ 2–4). For these extensions, we use several arguments such as separation of variables and suitable Fourier decomposition of operators.

Note that Theorem 1.1 is related to a Carleman estimate with linear weight that we prove by taking advantage of the cylindrical shape of our domain. This property has already been observed by [28] for bounded domains.

Let us mention that the arguments used for the construction of the CGO solutions work only if the unbounded domain has one infinite direction (or a cylindrical shape). This approach fails if the unbounded domain has more than one infinite direction like the slab.

However, following the approach of [37, 40], by means of unique continuation properties we prove in Corollary 1.4 the recovery of coefficients supported in an unbounded cylinder. Here the cylinder can be arbitrary and this result extends that of [37, 40] to non-compactly supported coefficients. Note also that, combining the density results stated in Lemma 6.1, used for the proof of Corollary 1.3, with Corollary 1.4, one can check that the data used by [37, 40] for the recovery of compactly supported coefficients allow to recover a more general class of coefficients supported in an infinite cylinder and known in the neighborhood of the boundary outside a compact set.

In the main result of this paper, stated in Theorem 1.1, we show that the partial DN map Λ_q allows to recover coefficients q which are equivalent modulo integrable functions to a fixed bounded function. This last condition is not fulfilled by the class of potentials, periodic along the axis of the cylindrical domain, considered by [16, 17]. However, combining Theorem 1.1 with [16, 17], one can conclude that the partial DN map Λ_q allows to recover the class of coefficients q considered in the present paper as well as potentials q which are periodic along the axis of Ω.

Let us remark that in a similar way to [37, 40], with a suitable choice of admissible coefficients q, it is possible to formulate (1.13) with q replaced by $q-k^2$ and k^2 taking some suitable value in the absolute continuous spectrum of the operator $-\Delta+q$ with Dirichlet boundary condition. In this context, (1.13) admits a unique solution satisfying the Sommerfeld radiation condition on the infinite directions of the domain. Assuming that q is chosen in such a way that these conditions are fulfilled for (1.1) and (1.13), one can adapt the argument of the present paper to this problem. In this paper we do not consider such extension of our main result which requires a study of the forward problem.

Let us also observe that like in [37, 40], Corollary 1.4 can be formulated with different kinds of measurements on the side $x_1=0$ and $x_1=L$ of $\partial\mathcal{O}$.

1.5. Outline

This paper is organized as follows. In § 2, we start by considering the CGO solutions, without boundary conditions, for the problem in an unbounded cylindrical domain. For the construction of these solutions we combine different arguments such as localization of the CGO solutions along the axis of the waveguide and some arguments of separation of variables. Then, in the spirit of [6], we introduce in § 3 a Carleman estimate with linear weight stated in an infinite cylindrical domain. Using this Carleman estimate, we build in § 4 CGO solutions vanishing on some parts of the boundary. In § 5, we combine all these results in order to prove Theorem 1.1. Finally, § 6 is devoted to the applications of the main result stated in Corollaries 1.2–1.4.

2. CGO solutions without conditions

In this section we introduce the first class of CGO solutions of our problem without boundary conditions. These CGO solutions correspond to some specific solutions $u\in H^2(\Omega)$ of $-\Delta u+qu=0$ in Ω for $q\in L^\infty(\Omega)$. More precisely, we start by fixing $\theta\in\mathbb{S}^1:=\{y\in\mathbb{R}^2:\ |y|=1\}$, $\xi'\in\theta^\perp\setminus\{0\}$, with $\theta^\perp:=\{y\in\mathbb{R}^2:\ y\cdot\theta=0\}$, $\xi:=(\xi',\xi_3)\in\mathbb{R}^3$, with

https://doi.org/10.1017/S1474748018000488 Published online by Cambridge University Press

$\xi_3 \neq 0$. Then, we consider $\eta \in \mathbb{S}^2 := \{y \in \mathbb{R}^3 : \ |y| = 1\}$ defined by

$$\eta = \frac{(\xi', -\frac{|\xi'|^2}{\xi_3})}{\sqrt{|\xi'|^2 + \frac{|\xi'|^4}{\xi_3^2}}}.$$

In particular, we have

$$\eta \cdot \xi = (\theta, 0) \cdot \xi = (\theta, 0) \cdot \eta = 0. \tag{2.16}$$

We fix also $\chi \in \mathcal{C}_0^\infty(\mathbb{R}; [0, 1])$ such that $\chi = 1$ on a neighborhood of 0 in $\mathbb{R}$ and, for $\rho > 1$, we consider solutions $u \in H^2(\Omega)$ of $-\Delta u + qu = 0$ in Ω taking the form

$$u(x', x_3) = e^{-\rho\theta \cdot x'} \left(e^{i\rho\eta \cdot x} \chi\left(\rho^{-\frac{1}{4}} x_3\right) e^{-i\xi \cdot x} + w_\rho(x)\right), \quad x = (x', x_3) \in \Omega. \tag{2.17}$$

Here the remainder term $w_\rho \in H^2(\Omega)$ satisfies the decay property

$$\rho^{-1} \|w_\rho\|_{H^2(\Omega)} + \rho \|w_\rho\|_{L^2(\Omega)} \leqslant C\rho^{\frac{7}{8}}, \tag{2.18}$$

with C independent of ρ. This construction can be summarized in the following way.

Theorem 2.1. *There exists $\rho_0 > 1$ such that, for all $\rho > \rho_0$, the equation $-\Delta u + qu = 0$ admits a solution $u \in H^2(\Omega)$ of the form (2.17) with w_ρ satisfying the decay property (2.18).*

Remark 2.2. *Comparing to CGO solutions on bounded domains, the main difficulty in the construction of CGO solutions in our context comes from the fact that Ω is not bounded and the CGO solutions should be square integrable. This means that the usual principal parts of the CGO solutions considered by [6, 29, 47], taking the form $e^{-\rho\theta \cdot x'} e^{i\rho\eta \cdot x} e^{-i\xi \cdot x}$ in our context, are incompatible with the square integrability property. This is the main reason why we introduce the new expression involving the cut-off χ that allows to localize such expressions. The main difficulty in our choice consists in using this expression to localize without losing the decay property stated in (2.18). This will be done by assuming that the principal part of the CGO solutions given by*

$$e^{-\rho\theta \cdot x'} e^{i\rho\eta \cdot x} \chi\left(\rho^{-\frac{1}{4}} x_3\right) e^{-i\xi \cdot x}$$

propagates in some suitable way along the axis of the waveguide with respect to the large parameter ρ.

Clearly, u solves $-\Delta u + qu = 0$ if and only if w_ρ solves

$$P_{-\rho} w_\rho = -q w_\rho - e^{\rho\theta \cdot x'}(-\Delta + q) e^{-\rho\theta \cdot x'} e^{i\rho\eta \cdot x} \chi\left(\rho^{-\frac{1}{4}} x_3\right) e^{-i\xi \cdot x}, \tag{2.19}$$

with P_s, $s \in \mathbb{R}$, the differential operator defined by

$$P_s := -\Delta - 2s\theta \cdot \nabla' - s^2, \tag{2.20}$$

where $\nabla' = (\partial_{x_1}, \partial_{x_2})^T$. We need to consider here a solution of this equation in the unbounded domain Ω that satisfies the decay property (2.18). For bounded domains, the construction of such solutions is well known and goes back to [47]. Moreover, for potentials q admitting a decay of the form $|x_3|^{-1}$ along the axis of the waveguide, one can construct solutions of (2.19) satisfying the decay property (2.18), by applying [47, Theorem 2.3]. In this section we introduce a different construction which can be applied to any $q \in L^\infty(\Omega)$. More precisely, we take advantage of the cylindrical shape of Ω to make a Fourier decomposition of (2.19). Then we apply results on bounded domains in order to construct w_ρ with the required decay property.

In order to define a suitable set of solutions of (2.19), we start by considering the following equation

$$P_{-\rho} y(x) = F(x), \quad x \in \Omega. \tag{2.21}$$

Again, here we deal with an equation on the unbounded domain Ω, but this equation can be decomposed, by means of Fourier transform, into a family of equations stated on the bounded domain ω. More precisely, taking the Fourier transform with respect to x_3, denoted by $\mathcal{F}_{x_3}$, on both sides of this identity we get

$$P_{k,-\rho} y_k = F_k, \quad k \in \mathbb{R}, \tag{2.22}$$

with $F_k(x') = \mathcal{F}_{x_3} F(x', k)$, $y_k(x') = \mathcal{F}_{x_3} y(x', k)$ and

$$P_{k,-\rho} = -\Delta' + 2\rho\theta \cdot \nabla' - \rho^2 + k^2.$$

Here $\Delta' = \partial_{x_1}^2 + \partial_{x_2}^2$ and $\mathcal{F}_{x_3}$ is defined by

$$\mathcal{F}_{x_3} h(x', k) := (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} h(x', x_3) e^{-ikx_3}\, dx_3, \quad h \in L^1(\Omega).$$

We fix also $p_{k,-\rho}(\zeta) = |\zeta|^2 + 2i\rho\theta \cdot \zeta + k^2 - \rho^2$, $\zeta \in \mathbb{R}^2$, $k \in \mathbb{R}$, such that, for $D_{x'} = -i\nabla'$, we have $p_{k,-\rho}(D_{x'}) = P_{k,-\rho}$. Combining this decomposition with properties of solutions of (2.22) on the bounded domain ω, we will complete the construction of the expression w_ρ. This will be done, by applying first some results of [12, 19, 24] about solutions of PDEs with constant coefficients, given by the following.

Lemma 2.3. *For every $\rho > 1$ and $k \in \mathbb{R}$ there exists a bounded operator*

$$E_{k,\rho} : \ L^2(\omega) \to L^2(\omega)$$

such that:

$$P_{k,-\rho} E_{k,\rho} F = F, \quad F \in L^2(\omega), \tag{2.23}$$

$$\|E_{k,\rho}\|_{\mathcal{B}(L^2(\omega))} \leqslant C\rho^{-1}, \tag{2.24}$$

$$E_{k,\rho} \in \mathcal{B}(L^2(\omega); H^2(\omega)) \tag{2.25}$$

and

$$\|E_{k,\rho}\|_{\mathcal{B}(L^2(\omega);H^2(\omega))} + \|k^2 E_{k,\rho}\|_{\mathcal{B}(L^2(\omega))} \leqslant C\rho, \tag{2.26}$$

with $C > 0$ depending only on ω.

Proof. In view of [12, Theorem 2.3] (see also [19, Theorem 10.3.7]), one can find a bounded operator $E_{k,\rho} \in \mathcal{B}(L^2(\omega))$, defined by means of fundamental solutions associated with $P_{k,-\rho}$ (see [19, § 10.3]), satisfying (2.23). Moreover, for

$$\tilde{p}_{k,-\rho}(\zeta) := \left(\sum_{\alpha\in\mathbb{N}^2} |\partial_\zeta^\alpha p_{k,-\rho}(\zeta)|^2\right)^{\frac{1}{2}}, \quad \zeta \in \mathbb{R}^2,$$

and for all differential operators $Q(D_{x'})$ with Q a polynomial such that $\frac{Q(\zeta)}{\tilde{p}_{k,-\rho}(\zeta)}$ is bounded, we get $Q(D_{x'})E_{k,\rho} \in \mathcal{B}(L^2(\omega))$ and there exists a constant C depending only on ω such that

$$\|Q(D_{x'})E_{k,\rho}\|_{\mathcal{B}(L^2(\omega))} \leqslant C \sup_{\zeta\in\mathbb{R}^2} \frac{|Q(\zeta)|}{\tilde{p}_{k,-\rho}(\zeta)}. \tag{2.27}$$

Since

$$\tilde{p}_{k,-\rho}(\zeta) \geqslant \sqrt{|\Im\partial_{\zeta_1} p_{k,-\rho}(\zeta)|^2 + |\Im\partial_{\zeta_2} p_{k,-\rho}(\zeta)|^2} = 2\rho, \quad \zeta \in \mathbb{R}^2,$$

(2.27) implies

$$\|E_{k,\rho}\|_{\mathcal{B}(L^2(\omega))} \leqslant C \sup_{\zeta\in\mathbb{R}^2} \frac{1}{\tilde{p}_{k,-\rho}(\zeta)} \leqslant C\rho^{-1}$$

and (2.24) is fulfilled. In addition, for all $\zeta \in \mathbb{R}^2$, assuming that $k^2 + |\zeta|^2 \geqslant 2\rho^2$, we have

$$\tilde{p}_{k,-\rho}(\zeta) \geqslant |\Re p_{k,-\rho}(\zeta)| = k^2 + |\zeta|^2 - \rho^2 \geqslant \frac{k^2 + |\zeta|^2}{2}.$$

Thus, we get

$$\sup_{\zeta\in\mathbb{R}^2} \frac{|\zeta|^2 + k^2}{\tilde{p}_{k,-\rho}(\zeta)} \leqslant \sup_{k^2+|\zeta|^2\geqslant 2\rho^2} \frac{|\zeta|^2 + k^2}{\tilde{p}_{k,-\rho}(\zeta)} + \sup_{k^2+|\zeta|^2\leqslant 2\rho^2} \frac{|\zeta|^2 + k^2}{\tilde{p}_{k,-\rho}(\zeta)} \leqslant 2 + 2\rho^2 \sup_{\zeta\in\mathbb{R}^2} \frac{1}{\tilde{p}_{k,-\rho}(\zeta)} \leqslant 3\rho.$$

Then, in view of [12, Theorem 2.3], we deduce (2.25) with

$$\|E_{k,\rho}\|_{\mathcal{B}(L^2(\omega);H^2(\omega))} + \|k^2 E_{k,\rho}\|_{\mathcal{B}(L^2(\omega))} \leqslant C \sup_{\zeta\in\mathbb{R}^2} \frac{1 + |\zeta|^2 + k^2}{\tilde{p}_{k,-\rho}(\zeta)} \leqslant C\rho$$

which implies (2.26). □

Applying this lemma, we will define suitable solutions of (2.21) which will be given by an operator isometric to the direct sum of the operators $E_{k,\rho}$, $k \in \mathbb{R}$.

Lemma 2.4. *For every $\rho > 1$ there exists a bounded operator*

$$E_\rho : \; L^2(\Omega) \to L^2(\Omega)$$

satisfying the conditions:

$$P_{-\rho} E_\rho F = F, \quad F \in L^2(\Omega), \tag{2.28}$$

$$\|E_\rho\|_{\mathcal{B}(L^2(\Omega))} \leqslant C\rho^{-1}, \tag{2.29}$$

https://doi.org/10.1017/S1474748018000488 Published online by Cambridge University Press

$$E_\rho \in \mathcal{B}(L^2(\Omega); H^2(\Omega)) \tag{2.30}$$

and

$$\|E_\rho\|_{\mathcal{B}(L^2(\Omega); H^2(\Omega))} \leqslant C\rho, \tag{2.31}$$

with $C > 0$ depending only on Ω.

Proof. In light of Lemma 2.3, we can introduce E_ρ on $L^2(\Omega)$ given by

$$E_\rho F := \Omega \ni (x', x_3) \mapsto \mathcal{F}_k^{-1}(E_{k,\rho}\mathcal{F}_{x_3}F(\cdot, k))(x', x_3).$$

Clearly (2.23) implies (2.28). In addition, we get

$$\|E_\rho F\|^2_{L^2(\Omega)} = \int_{\mathbb{R}} \|E_{k,\rho}\mathcal{F}_{x_3}F(\cdot, k)\|^2_{L^2(\omega)}\, dk$$

and from (2.24) we find

$$\|E_\rho F\|^2_{L^2(\Omega)} \leqslant C^2\rho^{-2} \int_{\mathbb{R}} \|\mathcal{F}_{x_3}F(\cdot, k)\|^2_{L^2(\omega)}\, dk = C^2\rho^{-2}\|F\|^2_{L^2(\Omega)}.$$

This proves (2.29). According to (2.25)–(2.26), we have $E_\rho \in \mathcal{B}(L^2(\Omega); H^2(\Omega))$ and, for all $F \in L^2(\Omega)$, we obtain

$$\begin{aligned}\|E_\rho F\|^2_{H^2(\Omega)} &\leqslant C' \int_{\mathbb{R}} \left[\|E_{k,\rho}\mathcal{F}_{x_3}F(\cdot, k)\|^2_{H^2(\omega)} + \|k^2 E_{k,\rho}\mathcal{F}_{x_3}F(\cdot, k)\|^2_{L^2(\omega)}\right] dk \\ &\leqslant C'C^2\rho^2 \int_{\mathbb{R}} \|\mathcal{F}_{x_3}F(\cdot, k)\|^2_{L^2(\omega)}\, dk = C'C^2\rho^2\|F\|^2_{L^2(\Omega)},\end{aligned}$$

with C' depending only on ω. This proves (2.30)–(2.31). □

Using this last result, we can build geometric optics solutions of the form (2.17).

Proof of Theorem 2.1. Note first that

$$\begin{aligned}&-e^{\rho\theta\cdot x'}(-\Delta + q)e^{-\rho\theta\cdot x'}e^{i\rho\eta\cdot x}\chi\left(\rho^{-\frac{1}{4}}x_3\right)e^{-i\xi\cdot x} \\ &= -\left((|\xi|^2 + q)\chi\left(\rho^{-\frac{1}{4}}x_3\right) - 2i\eta_3\rho^{\frac{3}{4}}\chi'\left(\rho^{-\frac{1}{4}}x_3\right) + 2i\xi_3\rho^{-\frac{1}{4}}\chi'\left(\rho^{-\frac{1}{4}}x_3\right)\right. \\ &\quad \left. -\rho^{-\frac{1}{2}}\chi''\left(\rho^{-\frac{1}{4}}x_3\right)\right)e^{i\rho\eta\cdot x}e^{-i\xi\cdot x}.\end{aligned} \tag{2.32}$$

On the other hand, we have

$$\int_{\mathbb{R}} |\chi\left(\rho^{-\frac{1}{4}}x_3\right)|^2\, dx_3 = \rho^{\frac{1}{4}} \int_{\mathbb{R}} |\chi(t)|^2 dt$$

and we deduce that

$$\|\chi\left(\rho^{-\frac{1}{4}}x_3\right)\|_{L^2(\Omega)} = \|\chi\|_{L^2(\mathbb{R})}|\omega|^{\frac{1}{2}}\rho^{\frac{1}{8}}.$$

In the same way, one can check that

$$\|\chi\left(\rho^{-\frac{1}{4}}x_3\right)\|_{L^2(\Omega)} + \|\chi'\left(\rho^{-\frac{1}{4}}x_3\right)\|_{L^2(\Omega)} + \|\chi''\left(\rho^{-\frac{1}{4}}x_3\right)\|_{L^2(\Omega)} \leqslant C\rho^{\frac{1}{8}},$$

https://doi.org/10.1017/S1474748018000488 Published online by Cambridge University Press

with C depending only on ω and χ. Combining this with (2.32), we find

$$\begin{aligned}&\| -e^{\rho\theta\cdot x'}(-\Delta+q)e^{-\rho\theta\cdot x'}e^{i\rho\eta\cdot x}\chi\left(\rho^{-\frac{1}{4}}x_3\right)e^{-i\xi\cdot x}\|_{L^2(\Omega)}\\ &\quad = C\left((|\xi|^2+\|q\|_{L^\infty(\Omega)})\rho^{\frac{1}{8}}+2|\eta_3|\rho^{\frac{7}{8}}+2|\xi_3|\rho^{-\frac{1}{8}}+\rho^{-\frac{3}{8}}\right)\leqslant C\rho^{\frac{7}{8}},\end{aligned} \tag{2.33}$$

with $C>0$ depending on ω, ξ and $\|q\|_{L^\infty(\Omega)}$. In view of Lemma 2.4, equation (2.19) can be written in the following way:

$$w_\rho=-E_\rho\left(qw_\rho+e^{\rho\theta\cdot x'}(-\Delta+q)e^{-\rho\theta\cdot x'}e^{i\rho\eta\cdot x}\chi\left(\rho^{-\frac{1}{4}}x_3\right)e^{-i\xi\cdot x}\right),$$

with $E_\rho\in\mathcal{B}(L^2(\Omega))$ defined in Lemma 2.4. To this end, we will apply a fixed point argument to the map

$$\begin{aligned}&\mathcal{L}:L^2(\Omega)\to L^2(\Omega),\\ &\qquad G\mapsto -E_\rho\left[qG+e^{\rho\theta\cdot x'}e^{-i\rho\eta\cdot x}(-\Delta+q)e^{-\rho\theta\cdot x'}e^{i\rho\eta\cdot x}\chi\left(\rho^{-\frac{1}{4}}x_3\right)e^{-i\xi\cdot x}\right].\end{aligned}$$

Indeed, in view of (2.29) and (2.33), we have

$$\|\mathcal{L}w\|_{L^2(\Omega)}\leqslant C\rho^{-\frac{1}{8}}+C\rho^{-1}\|w\|_{L^2(\Omega)},\quad w\in L^2(\Omega),$$

$$\|\mathcal{L}w_1-\mathcal{L}w_2\|_{L^2(\Omega)}\leqslant\|E_\rho[q(w_1-w_2)]\|_{L^2(\Omega)}\leqslant C\rho^{-1}\|w_1-w_2\|_{L^2(\Omega)},\quad w_1,w_2\in L^2(\Omega),$$

with C depending on ω, ξ and $\|q\|_{L^\infty(\Omega)}$. Therefore, there exists $\rho_0>1$ such that for $\rho\geqslant\rho_0$ the map $\mathcal{L}$ admits a unique fixed point w_ρ in $\{w\in L^2(\Omega):\|w\|_{L^2(\Omega)}\leqslant 1\}$. In addition, conditions (2.29)–(2.31) imply that $w_\rho\in H^2(\Omega)$ fulfills (2.18). This completes the proof of Theorem 2.1. □

3. Carleman estimate

In this section we establish a Carleman estimate for the Laplace operator in the unbounded cylindrical domain Ω. Before the statement of this result, we will show how one can extend some applications of the classical Green formula for H^2 functions into the infinite cylindrical domain Ω. Note that functions $G\in H^2(\Omega)$ may have complicated behaviors on the cross section $\omega\times\{x_3\}$ as $|x_3|\to+\infty$. However, using the fact that $H^2(\Omega)$ embedded continuously into the spaces

$$H^k(\mathbb{R}_{x_3};H^{2-k}(\omega)):=\{H\in L^2(\Omega):\ x_3\mapsto H(\cdot,x_3)\in H^k(\mathbb{R};H^{2-k}(\omega))\}^1,\quad k=0,1,2,$$

we can show the following extension of applications of the Green formula already considered in [16].

[1]Here, for $H\in L^2(\Omega)$ and for a.e. $x_3\in\mathbb{R}$, $H(\cdot,x_3)$ denotes the function $\omega\ni x'\mapsto H(x',x_3)$.

https://doi.org/10.1017/S1474748018000488 Published online by Cambridge University Press

Lemma 3.1. *For an $F, G \in H^2(\Omega)$, the traces $F_{|\partial\Omega}, G_{|\partial\Omega}, \partial_\nu F_{|\partial\Omega}, \partial_\nu G_{|\partial\Omega}$ are well defined as elements of $L^2(\partial\Omega)$*[2] *and we have*

$$\int_\Omega \Delta F G\, dx = \int_{\partial\Omega} \partial_\nu F G d\sigma(x) - \int_{\partial\Omega} F \partial_\nu G d\sigma(x) + \int_\Omega \Delta G F\, dx, \tag{3.34}$$

$$\int_\Omega \partial_{x_3}^2 F \partial_{x'}^\alpha G\, dx = -\int_\Omega \partial_{x_3} F \partial_{x_3} \partial_{x'}^\alpha G\, dx, \quad \alpha \in \mathbb{N}^2,\ |\alpha| \leqslant 1. \tag{3.35}$$

Proof. Since $H^2(\Omega)$ embedded continuously into $L^2(\mathbb{R}_{x_3}; H^2(\omega))$ the traces $x_3 \mapsto F(\cdot, x_3)_{|\partial\omega}$ and $x_3 \mapsto \partial_\nu F(\cdot, x_3)_{|\partial\omega}$ are well defined as elements of $L^2(\mathbb{R}_{x_3}; L^2(\partial\omega)) = L^2(\partial\Omega)$. This proves the first claim of the lemma. Now let us prove (3.34)–(3.35). For this purpose, let us first remark that for F compactly supported one can apply the Green formula into a bounded neighborhood of $\mathrm{supp}(F) \cap \Omega$ in Ω in order to prove (3.34)–(3.35). This proves that (3.34)–(3.35) hold true for $F \in \{T_{|\Omega} :\ T \in \mathcal{C}_0^\infty(\mathbb{R}^3)\}$. Now let us consider the case of functions F which are not compactly supported. Following the proof of [35, Lemma 2.7], we can define the extension operator $P : H^2(\Omega) \longrightarrow H^2(\mathbb{R}^3)$ which is bounded and satisfies

$$P(H)_{|\Omega} = H, \quad H \in H^2(\Omega).$$

Then, applying the density of $\mathcal{C}_0^\infty(\mathbb{R}^3)$ in $H^2(\mathbb{R}^3)$, we consider the sequence $(\varphi_n)_{n\geqslant 1}$ lying in $\mathcal{C}_0^\infty(\mathbb{R}^3)$ such that

$$\lim_{n\to\infty} \|\varphi_n - PF\|_{H^2(\mathbb{R}^3)} = 0.$$

In particular we have

$$\|\varphi_n - F\|_{H^2(\Omega)} = \|\varphi_n - PF\|_{H^2(\Omega)} \leqslant \|\varphi_n - PF\|_{H^2(\mathbb{R}^3)}, \quad n \geqslant 1,$$

which implies that

$$\lim_{n\to+\infty} \|\varphi_n - F\|_{H^2(\Omega)} = 0. \tag{3.36}$$

This proves that

$$\lim_{n\to+\infty} \|\varphi_n - F\|_{L^2(\Omega)} = \lim_{n\to+\infty} \|\Delta\varphi_n - \Delta F\|_{L^2(\Omega)} = 0. \tag{3.37}$$

Moreover, using the continuity of the map

$$L^2(\mathbb{R}_{x_3}; H^2(\omega)) \ni T \mapsto T_{|\partial\Omega} \in L^2(\partial\Omega), \quad L^2(\mathbb{R}_{x_3}; H^2(\omega)) \ni T \mapsto \partial_\nu T_{|\partial\Omega} \in L^2(\partial\Omega),$$

and the fact that

$$\begin{aligned}\|T\|^2_{L^2(\mathbb{R}_{x_3}; H^2(\omega))} &= \int_{\mathbb{R}} \int_\omega \sum_{\substack{\alpha\in\mathbb{N}^2,\\ |\alpha|\leqslant 2}} |\partial_{x'}^\alpha T(x', x_3)|^2\, dx'\, dx_3 \\ &= \int_\Omega \sum_{\substack{\alpha\in\mathbb{N}^2,\\ |\alpha|\leqslant 2}} |\partial_{x'}^\alpha T(x', x_3)|^2\, dx \leqslant \|T\|^2_{H^2(\Omega)}, \quad T \in H^2(\Omega),\end{aligned}$$

[2] Actually using the fact that $\Omega = \omega \times \mathbb{R}$ and the fact that ω is a bounded subset of $\mathbb{R}^2$, in a similar way to [41, Theorem 8.3, Chapter 1], we can prove that $F_{|\partial\Omega} \in H^{\frac{3}{2}}(\partial\Omega)$ and $\partial_\nu F_{|\partial\Omega} \in H^{\frac{1}{2}}(\partial\Omega)$. This can be proved by following the proof of [41, Theorem 8.3, Chapter 1] with local coordinates considered only with respect to the part $x' \in \omega$ of variables $x = (x', x_3) \in \omega \times \mathbb{R} = \Omega$.

we obtain

$$\|\varphi_n - F\|_{L^2(\partial\Omega)} \leqslant C\|\varphi_n - F\|_{L^2(\mathbb{R}_{x_3};H^2(\omega))} \leqslant C\|\varphi_n - F\|_{H^2(\Omega)}, \quad n \geqslant 1,$$

$$\|\partial_\nu\varphi_n - \partial_\nu F\|_{L^2(\partial\Omega)} \leqslant C\|\varphi_n - F\|_{L^2(\mathbb{R}_{x_3};H^2(\omega))} \leqslant C\|\varphi_n - F\|_{H^2(\Omega)}, \quad n \geqslant 1,$$

with $C > 0$ independent of n. Combining this with (3.36), we obtain

$$\lim_{n\to+\infty} \|\varphi_n - F\|_{L^2(\partial\Omega)} = \lim_{n\to+\infty} \|\partial_\nu\varphi_n - \partial_\nu F\|_{L^2(\partial\Omega)} = 0. \tag{3.38}$$

Now using the fact that φ_n, $n \in \mathbb{N}$, is compactly supported, we can apply the Green formula to get

$$\int_\Omega \Delta\varphi_n G\,dx = \int_{\partial\Omega} \partial_\nu\varphi_n G d\sigma(x) - \int_{\partial\Omega} \varphi_n \partial_\nu G d\sigma(x) + \int_\Omega \Delta G\varphi_n\,dx, \quad n \geqslant 1. \tag{3.39}$$

Moreover, applying the Cauchy–Schwarz inequality, we find

$$\left|\int_\Omega \Delta\varphi_n G\,dx - \int_\Omega \Delta F G\,dx\right| \leqslant \|\Delta\varphi_n - \Delta F\|_{L^2(\Omega)}\|G\|_{L^2(\Omega)},$$

$$\left|\int_\Omega \varphi_n \Delta G\,dx - \int_\Omega F\Delta G\,dx\right| \leqslant \|\varphi_n - F\|_{L^2(\Omega)}\|\Delta G\|_{L^2(\Omega)},$$

$$\left|\int_{\partial\Omega} \varphi_n\partial_\nu G d\sigma(x) - \int_{\partial\Omega} F\partial_\nu G d d\sigma(x)\right| \leqslant \|\varphi_n - F\|_{L^2(\partial\Omega)}\|\partial_\nu G\|_{L^2(\partial\Omega)},$$

$$\left|\int_{\partial\Omega} \partial_\nu\varphi_n G d\sigma(x) - \int_{\partial\Omega} \partial_\nu F G d d\sigma(x)\right| \leqslant \|\partial_\nu\varphi_n - \partial_\nu F\|_{L^2(\partial\Omega)}\|G\|_{L^2(\partial\Omega)}.$$

Combining this with (3.37)–(3.38) and sending $n \to \infty$ in (3.39), we obtain (3.34). In a similar way, using the fact that φ_n, $n \in \mathbb{N}$, is compactly supported we obtain

$$\int_\Omega \partial_{x_3}^2\varphi_n\partial_{x'}^\alpha G\,dx = -\int_\Omega \partial_{x_3}\varphi_n\partial_{x_3}\partial_{x'}^\alpha G\,dx, \quad \alpha \in \mathbb{N}^2,\ |\alpha| \leqslant 1. \tag{3.40}$$

Moreover, we have

$$\begin{aligned}\left|\int_\Omega \partial_{x_3}^2\varphi_n\partial_{x'}^\alpha G\,dx - \int_\Omega \partial_{x_3}^2 F\partial_{x'}^\alpha G\,dx\right| &\leqslant \|\partial_{x_3}^2(\varphi_n - F)\|_{L^2(\Omega)}\|\partial_{x'}^\alpha G\| \\ &\leqslant \|\varphi_n - F\|_{H^2(\Omega)}\|\partial_{x'}^\alpha G\|_{L^2(\Omega)},\end{aligned}$$

$$\begin{aligned}\left|\int_\Omega \partial_{x_3}\varphi_n\partial_{x_3}\partial_{x'}^\alpha G\,dx - \int_\Omega \partial_{x_3}F\partial_{x_3}\partial_{x'}^\alpha G\,dx\right| &\leqslant \|\partial_{x_3}(\varphi_n - F)\|_{L^2(\Omega)}\|\partial_{x_3}\partial_{x'}^\alpha G\| \\ &\leqslant \|\varphi_n - F\|_{H^2(\Omega)}\|\partial_{x_3}\partial_{x'}^\alpha G\|_{L^2(\Omega)}.\end{aligned}$$

Combining this with (3.36) and sending $n \to \infty$ in (3.40) we obtain (3.35). This completes the proof of the lemma. □

Our Carleman inequality, which is similar to [6, Lemma 2.1] for unbounded cylindrical domains, takes the following form.

Proposition 3.2. *Let $\theta \in \mathbb{S}^1$. Then, there exists $d > 0$ depending only on ω such that the estimate*

$$\frac{8\rho^2}{d}\int_\Omega e^{-2\rho\theta\cdot x'}|u(x)|^2\,dx + 2\rho\int_{\partial\omega_\theta^+\times\mathbb{R}} e^{-2\rho\theta\cdot x'}\theta\cdot\nu|\partial_\nu u(x)|^2 d\sigma(x)$$
$$\leqslant \int_\Omega e^{-2\rho\theta\cdot x'}|\Delta u(x)|^2\,dx + 2\rho\int_{\partial\omega_\theta^-\times\mathbb{R}} e^{-2\rho\theta\cdot x'}|\theta\cdot\nu||\partial_\nu u(x)|^2 d\sigma(x) \quad (3.41)$$

holds for every $u \in H^2(\Omega)$ satisfying $u_{|\Gamma} = 0$.

Proof. We fix $u \in H^2(\Omega)$ satisfying $u_{|\Gamma} = 0$. We decompose the operator $e^{-\rho\theta\cdot x'}\Delta e^{\rho\theta\cdot x'}$ into three terms $P'_+ + P^3_+ + P_-$, with

$$P'_+ := \Delta' + \rho^2 \quad \text{and} \quad P_- := 2\rho\theta\cdot\nabla', \quad P^3_+ := \partial^2_{x_3},$$

where we recall that Δ' (respectively, ∇') denotes the Laplace (respectively, gradient) operator with respect to $x' \in \omega$. Thus, fixing $v(x) := e^{-\rho\theta\cdot x'}u(x)$, we get

$$\begin{aligned}
&\int_\Omega e^{-2\rho\theta\cdot x'}|\Delta u(x)|^2\,dx\\
&= \int_\Omega |e^{-\rho\theta\cdot x'}\Delta e^{\rho\theta\cdot x'}v(x)|^2\,dx\\
&= \int_\Omega |(P'_+ + P^3_+ + P_-)v(x)|^2\,dx\\
&= \int_\Omega |(P'_+ + P^3_+)v(x)|^2\,dx + \int_\Omega |P_-v(x)|^2\,dx + 2\Re\langle P^3_+v, P_-v\rangle_{L^2(\Omega)} + 2\Re\langle P'_+v, P_-v\rangle_{L^2(\Omega)},
\end{aligned}$$

and it follows

$$\int_\Omega |P_-v(x)|^2\,dx + 2\Re\langle P'_+v, P_-v\rangle_{L^2(\Omega)} \leqslant \int_\Omega e^{-2\rho\theta\cdot x'}|\Delta u(x)|^2\,dx - 2\Re\langle P^3_+v, P_-v\rangle_{L^2(\Omega)}. \quad (3.42)$$

In addition, applying (3.35), we obtain

$$\begin{aligned}
2\Re\langle P^3_+v, P_-v\rangle_{L^2(\Omega)} &= -\rho\int_\mathbb{R}\int_\omega \nabla'\cdot(|\partial_{x_3}v(x)|^2\theta)\,dx'\,dx_1\\
&= -\rho\int_\Gamma |\partial_{x_3}v(x)|^2\theta\cdot\nu(x)d\sigma(x) = 0. \quad (3.43)
\end{aligned}$$

Here we apply the fact that $\partial_{x_3}v \in L^2(\mathbb{R}_{x_3}; H^1(\omega))$ and the fact that the condition $v_{|\Gamma} = 0$ implies $\partial_{x_3}v_{|\Gamma} = 0$. Since $v \in H^2(\Omega)$, for a.e. $x_3 \in \mathbb{R}$, the function $v(\cdot, x_3) := x' \mapsto v(x', x_3) \in H^2(\omega)$ satisfies $v(\cdot, x_3)_{|\partial\omega} = 0$ and we can apply [6, Lemma 2.1] and deduce that there exists $d > 0$ depending on ω such that, for a.e. $x_3 \in \mathbb{R}$, we have

$$\frac{8\rho^2}{d}\int_\omega |v(x', x_3)|^2\,dx' + 2\rho\int_{\partial\omega} e^{-2\rho\theta\cdot x'}\theta\cdot\nu(x')|\partial_\nu e^{\rho\theta\cdot x'}v(x', x_3)|^2 d\sigma(x')$$
$$\leqslant \int_\omega |P_-v(x', x_3)|^2\,dx' + 2\Re\int_\omega P'_+v(x', x_3)\overline{P_-v(x', x_3)}\,dx'.$$

https://doi.org/10.1017/S1474748018000488 Published online by Cambridge University Press

It follows

$$\frac{8\rho^2}{d}\int_\omega |e^{-\rho\theta\cdot x'}u(x',x_3)|^2\,dx' + 2\rho\int_{\partial\omega} e^{-2\rho\theta\cdot x'}\theta\cdot\nu(x)|\partial_\nu u(x',x_3)|^2 d\sigma(x')$$
$$\leqslant \int_\omega |P_- v(x',x_3)|^2\,dx' + 2\Re\int_\omega P'_+ v(x',x_3)\overline{P_- v(x',x_3)}\,dx'.$$

Thus, using the fact that $u, v \in H^2(\Omega) \subset L^2(\mathbb{R}_{x_3}; H^2(\omega))$, we can integrate both sides of this inequality with respect to $x_3 \in \mathbb{R}$ and get

$$\frac{8\rho^2}{d}\int_\Omega |e^{-\rho\theta\cdot x'}u(x',x_3)|^2\,dx'\,dx_3 + 2\rho\int_\Gamma e^{-2\rho\theta\cdot x'}\theta\cdot\nu(x)|\partial_\nu u(x',x_3)|^2 d\sigma(x')\,dx_3$$
$$\leqslant \int_\Omega |P_- v(x',x_3)|^2\,dx'\,dx_3 + 2\Re\int_\Omega P'_+ v(x',x_3)\overline{P_- v(x',x_3)}\,dx'\,dx_3. \tag{3.44}$$

Putting (3.42)–(3.44) together, we end up getting (3.41). □

Combining (3.41) with the fact that

$$|\Delta u|^2 \leqslant 2\left(|(-\Delta+q)u|^2 + \|q\|^2_{L^\infty(\Omega)}|u|^2\right),$$

we get

$$\left(\frac{4\rho^2}{d} - \|q\|^2_{L^\infty(\Omega)}\right)\int_\Omega e^{-2\rho\theta\cdot x'}|u(x)|^2\,dx + \rho\int_{\partial\omega_\theta^+\times\mathbb{R}} e^{-2\rho\theta\cdot x'}\theta\cdot\nu|\partial_\nu u(x)|^2 d\sigma(x)$$
$$\leqslant \int_\Omega e^{-2\rho\theta\cdot x'}|(\Delta+q)u(x)|^2\,dx + \rho\int_{\partial\omega_\theta^-\times\mathbb{R}} e^{-2\rho\theta\cdot x'}|\theta\cdot\nu||\partial_\nu u(x)|^2 d\sigma(x).$$

As a consequence we obtain the following estimate.

Corollary 3.3. *For $M > 0$, let $q \in L^\infty(\Omega)$ satisfy $\|q\|_{L^\infty(\Omega)} \leqslant M$. Then, assuming that the conditions of Proposition 3.2 are fulfilled, we find*

$$\frac{2\rho^2}{d}\int_\Omega e^{-2\rho\theta\cdot x'}|u(x)|^2\,dx + \rho\int_{\partial\omega_\theta^+\times\mathbb{R}} e^{-2\rho\theta\cdot x'}\theta\cdot\nu|\partial_\nu u(x)|^2 d\sigma(x)$$
$$\leqslant \int_\Omega e^{-2\rho\theta\cdot x'}|(\Delta+q)u(x)|^2\,dx + \rho\int_{\partial\omega_\theta^-\times\mathbb{R}} e^{-2\rho\theta\cdot x'}|\theta\cdot\nu||\partial_\nu u(x)|^2 d\sigma(x), \tag{3.45}$$

for $\rho \geqslant \rho_1 := M(d/2)^{\frac{1}{2}} + 1$.

4. CGO solutions vanishing on parts of the boundary

We fix $q \in L^\infty(\Omega)$ and for all $y \in \mathbb{S}^1$, $r > 0$, we set

$$\partial\omega_{+,r,y} = \{x \in \partial\omega : \nu(x)\cdot y > r\}, \quad \partial\omega_{-,r,y} = \{x \in \partial\omega : \nu(x)\cdot y \leqslant r\}.$$

We recall that ν denotes both exterior unit vectors normal to $\partial\omega$ and to $\partial\Omega$. In the spirit of [29], we will apply the Carleman estimate (3.41) in order to build solutions $u \in H_\Delta(\Omega)$ to

$$\begin{cases} -\Delta u + qu = 0 & \text{in } \Omega, \\ u = 0, & \text{on } \partial\omega_{+,\varepsilon/2,-\theta}\times\mathbb{R}, \end{cases} \tag{4.46}$$

https://doi.org/10.1017/S1474748018000488 Published online by Cambridge University Press

taking the form

$$u(x',x_3)=e^{\rho\theta\cdot x'}e^{-i\rho\eta\cdot x}\left(\chi\left(\rho^{-\frac{1}{4}}x_3\right)+z_\rho(x)\right),\quad x=(x',x_3)\in\Omega. \tag{4.47}$$

Here $\theta\in\{y\in\mathbb{S}^1:|y-\theta_0|\leqslant\varepsilon\}$ and the remainder term $z_\rho\in e^{-\rho\theta\cdot x'}e^{i\rho\eta\cdot x}H_\Delta(\Omega)$ satisfies the boundary condition

$$z_\rho(x',x_3)=-\chi\left(\rho^{-\frac{1}{4}}x_3\right),\quad (x',x_3)\in\partial\omega_{+,\varepsilon/2,-\theta}\times\mathbb{R}$$

and the decay property

$$\|z_\rho\|_{L^2(\Omega)}\leqslant C\rho^{-\frac{1}{8}}, \tag{4.48}$$

with C depending on K', Ω and any $M\geqslant\|q\|_{L^\infty(\Omega)}$. Recalling that

$$(\partial\omega\setminus K')\subset(\partial\omega\setminus\partial\omega_{-,\varepsilon,-\theta})=\partial\omega_{+,\varepsilon,-\theta},$$

one can check that (4.46) implies $\mathrm{supp}(\mathcal{T}_0u)\subset K$ (recall that for $v\in\mathcal{C}_0^\infty(\overline{\Omega})$, $\mathcal{T}_0v=v_{|\Gamma}$).

The main result of this section can be stated as follows.

Theorem 4.1. *Let $q\in L^\infty(\Omega)$, $\theta\in\{y\in\mathbb{S}^1:|y-\theta_0|\leqslant\varepsilon\}$. For all $\rho>\rho_1$, one can find a solution $u\in H_\Delta(\Omega)$ of (4.46) taking the form (4.47) with z_ρ satisfying (4.48). Here ρ_1 denotes the constant introduced at the end of Corollary 3.3.*

To establish the existence of such solutions of (4.46) we recall some preliminary tools and an intermediate result. We consider first some weighted spaces. We set $s\in\mathbb{R}$, $\theta\in\{y\in\mathbb{S}^1:|y-\theta_0|\leqslant\varepsilon\}$ and we denote by γ the function

$$\gamma(x)=|\theta\cdot\nu(x)|,\quad x\in\Gamma.$$

We define the spaces $L_s(\Omega)$ and, for all non-negative measurable functions g on Γ, the spaces $L_{s,g,\pm}$ by

$$L_s(\Omega)=e^{-s\theta\cdot x'}L^2(\Omega),\quad L_{s,g,\pm}=\{f:\ e^{s\theta\cdot x'}g^{\frac{1}{2}}(x)f\in L^2(\omega_{\pm,\theta}\times\mathbb{R})\}$$

with the norm

$$\|u\|_s=\left(\int_\Omega e^{2s\theta\cdot x'}|u|^2\,dx\right)^{\frac{1}{2}},\quad u\in L_s(\Omega),$$

$$\|u\|_{s,g,\pm}=\left(\int_{\partial\omega_{\pm,\theta}\times\mathbb{R}}e^{2s\theta\cdot x'}g(x)|u|^2d\sigma(x')\,dx_3\right)^{\frac{1}{2}},\quad u\in L_{s,g,\pm}.$$

We consider also the space

$$\mathcal{D}_0=\{v_{|\Omega}:\ v\in\mathcal{C}_0^2(\mathbb{R}^3),\ v_{|\Gamma}=0\}$$

and, in light of Corollary 3.3, an application of the Carleman estimate (3.45) to any $h\in\mathcal{D}_0$ yields

$$\rho\|h\|_\rho+\rho^{\frac{1}{2}}\|\partial_\nu h\|_{\rho,\gamma,-}\leqslant C(\|(-\Delta+q)h\|_\rho+\|\partial_\nu h\|_{\rho,\rho\gamma,+}),\quad \rho\geqslant\rho_1. \tag{4.49}$$

We define also the space

$$\mathcal{M} = \{((-\Delta + q)v_{|\Omega}, \partial_\nu v_{|\partial\omega_{+,\theta}\times\mathbb{R}}) : \ v \in \mathcal{D}_0\}$$

and consider it as a subspace of $L_\rho(\Omega) \times L_{\rho,\rho\gamma,+}$. Combining the Carleman estimate (4.49) with a classical application of the Hahn Banach theorem (see [29, Proposition 7.1] and [16, Lemma 3.2] for more details) to a suitable linear form defined on $\mathcal{M}$, we obtain the following intermediate result.

Lemma 4.2. *We fix $\partial\omega^*_{-,\theta} = \{x \in \partial\omega : \ \theta \cdot \nu(x) < 0\}$. Given $\rho \geqslant \rho_1$, with ρ_1 the constant of Corollary 3.3, and*

$$v \in L_{-\rho}(\Omega), \quad v_- \in L_{-\rho,\gamma^{-1},-},$$

there exists $y \in L_{-\rho}(\Omega)$ such that the following conditions:

(1) $-\Delta y + q y = v$ *in* Ω,

(2) $y_{|\partial\omega^*_{-,\theta}\times\mathbb{R}} = v_-$,

(3) $\|y\|_{-\rho} \leqslant C(\rho^{-1}\|v\|_{-\rho} + \rho^{-\frac{1}{2}}\|v_-\|_{-\rho,\gamma^{-1},-})$ *with* C *depending on* Ω, $M \geqslant \|q\|_{L^\infty(\Omega)}$,

are fulfilled.

Armed with this lemma we are now in a position to prove Theorem 4.1.

Proof of Theorem 4.1. We need to consider z_ρ satisfying

$$\begin{cases} z_\rho \in L^2(\Omega) \\ (-\Delta + q)(e^{\rho\theta\cdot x'}e^{-i\rho\eta\cdot x}z_\rho) = -(-\Delta + q)e^{\rho\theta\cdot x'}e^{-i\rho\eta\cdot x}\chi\left(\rho^{-\frac{1}{4}}x_3\right) \quad \text{in } \Omega \\ z_\rho = -\chi\left(\rho^{-\frac{1}{4}}x_3\right) \quad \text{on } \partial\omega_{+,\varepsilon/2,-\theta} \times \times\mathbb{R}. \end{cases} \tag{4.50}$$

We choose $\psi \in \mathcal{C}^\infty_0(\mathbb{R}^2)$ satisfying $\mathrm{supp}(\psi) \cap \partial\omega \subset \{x \in \partial\omega : \theta \cdot \nu(x) < -\varepsilon/3\}$ and $\psi = 1$ on $\{x \in \partial\omega : \theta \cdot \nu(x) < -\varepsilon/2\} = \partial\omega_{+,\varepsilon/2,-\theta}$. Then, we fix

$$v_-(x', x_3) = -e^{\rho\theta\cdot x'}e^{-i\rho\eta\cdot x}\chi(\rho^{-\frac{1}{4}}x_3)\psi(x'), \quad x \in \partial\omega_{-,\theta} \times \mathbb{R}.$$

Using the fact that $v_-(x) = 0$ for

$$x \in \{x \in \Gamma : \ \theta \cdot \nu(x) \geqslant -\varepsilon/3\} \times \mathbb{R}$$

we deduce that $v_- \in L_{-\rho,\gamma^{-1},-}$. Set also

$$v(x) = -(-\Delta + q)e^{\rho\theta\cdot x'}e^{-i\rho\eta\cdot x}\chi\left(\rho^{-\frac{1}{4}}x_3\right), \quad x \in \Omega.$$

In view of Lemma 4.2, we can find $h \in H_\Delta(\Omega)$ such that

$$\begin{cases} (-\Delta + q)h = v & \text{in } \Omega, \\ h(x) = v_-(x), & x \in \partial\omega_{-,\theta} \times \mathbb{R}. \end{cases}$$

Thus, $z_\rho = e^{-\rho\theta\cdot x'}e^{i\rho\eta\cdot x}h$ will fulfill (4.50). Repeating some arguments similar to Theorem 2.1, we obtain

$$\left\| e^{-\rho\theta\cdot x'}(-\Delta + q)e^{\rho\theta\cdot x'}e^{-i\rho\eta\cdot x}\chi\left(\rho^{-\frac{1}{4}}x_3\right)\right\|_{L^2(\Omega)} \leqslant C\rho^{\frac{7}{8}},$$

with C depending only on ω and $M \geqslant \|q\|_{L^\infty(\Omega)}$. Combining this with condition (3) of Lemma 4.2 we get

$$\begin{aligned}\|z_\rho\|_{L^2(\Omega)} = \|h\|_{-\rho} &\leqslant C\left(\rho^{-1}\|v\|_{-\rho} + \rho^{-\frac{1}{2}}\|v_-\|_{-\rho,\gamma^{-1},-}\right)\\ &\leqslant C\left(\rho^{-\frac{1}{8}} + \rho^{-\frac{1}{2}}\|\psi\gamma^{-\frac{1}{2}}\|_{L^2(\partial\omega_{-,\theta})}\|\chi\left(\rho^{-\frac{1}{4}}\cdot\right)\|_{L^2(\mathbb{R})}\right)\\ &\leqslant C\left(\rho^{-\frac{1}{8}} + \rho^{-\frac{3}{8}}\|\psi\gamma^{-\frac{1}{2}}\|_{L^2(\partial\omega_{-,\theta})}\|\chi\|_{L^2(\mathbb{R})}\right)\\ &\leqslant C\rho^{-\frac{1}{8}},\end{aligned}$$

with C depending only on Ω and $\|q\|_{L^\infty(\Omega)}$. This proves estimate (4.48). Since $e^{-\rho\theta\cdot x'}e^{i\rho\eta\cdot x}z_\rho = h \in H_\Delta(\Omega)$, u defined by (4.47) is lying in $H_\Delta(\Omega)$ and is a solution of (4.46). This completes the proof of Theorem 4.1. □

5. Uniqueness result

In this section we will complete the proof of Theorem 1.1. We fix $q = q_2 - q_1$ on Ω and we extend it by $q = 0$ on $\mathbb{R}^3 \setminus \Omega$. We assume that the constant ε given in the beginning of § 4 is chosen in such a way that for any $\theta \in \{y \in \mathbb{S}^1 : |y - \theta_0| \leqslant \varepsilon\}$ we have $\partial\omega_{-,\varepsilon,\theta} \subset G'$. We consider $\rho > \max(\rho_0, \rho_1)$ and fix $\theta \in \{y \in \mathbb{S}^1 : |y - \theta_0| \leqslant \varepsilon\}$, $\xi := (\xi', \xi_3) \in \mathbb{R}^3$ satisfying $\xi_3 \neq 0$ and $\xi' \in \theta^\perp \setminus \{0\}$. In light of Theorem 2.1, we can pick $u_1 \in H^2(\Omega)$ solving $-\Delta u_1 + q_1 u_1 = 0$ on Ω of the form (2.17) with w_ρ satisfying (2.18). In addition, according to Theorem 4.1, we can fix $u_2 \in H_\Delta(\Omega)$ a solution of (4.46), with $q = q_2$, taking the form (4.47) with $e^{-\rho\theta\cdot x'}e^{i\rho\eta\cdot x}z_\rho \in H_\Delta(\Omega)$ satisfying (4.48). Fix $w_1 \in H_\Delta(\Omega)$ solving

$$\begin{cases} -\Delta w_1 + q_1 w_1 = 0 & \text{in } \Omega,\\ \mathcal{T}_0 w_1 = \mathcal{T}_0 u_2. \end{cases} \tag{5.51}$$

It is clear that $u = w_1 - u_2$ solves

$$\begin{cases} -\Delta u + q_1 u = (q_2 - q_1)u_2 & \text{in } \Omega,\\ u(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.52}$$

Using the fact that $(q_2 - q_1)u_2 \in L^2(\Omega)$, in light of [14, Lemma 2.2], we have $u \in H^2(\Omega)$. Then, applying (3.34), we find

$$\int_\Omega (q_2 - q_1)u_2 u_1\, dx = -\int_\Omega \Delta u u_1\, dx + \int_\Omega q_1 u u_1\, dx = -\int_\Gamma \partial_\nu u u_1\, d\sigma(x) + \int_\Gamma \partial_\nu u_1 u\, d\sigma(x).$$

On the other hand, we have $u_{|\Gamma} = 0$ and, combining (1.7) with the fact that $\mathrm{supp}\mathcal{T}_0 u_2 \subset K$, we deduce that $\partial_\nu u_{|G} = 0$. It follows that

$$\int_\Omega q u_2 u_1\, dx = -\int_{\Gamma\setminus G} \partial_\nu u u_1\, d\sigma(x). \tag{5.53}$$

In view of (2.18), by interpolation, we have

$$\|w_\rho\|_{L^2(\Gamma)} \leqslant C\|w_\rho\|_{H^{\frac{9}{16}}(\Omega)} \leqslant C\left(\|w_\rho\|_{L^2(\Omega)}\right)^{\frac{23}{32}}\left(\|w_\rho\|_{H^2(\Omega)}\right)^{\frac{9}{32}} \leqslant C\rho^{\frac{7}{16}}.$$

Thus, applying the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}\left|\int_{\Gamma\setminus G}\partial_\nu u u_1\,d\sigma(x)\right| &\leqslant \int_{\mathbb{R}}\int_{\partial\omega_{+,\varepsilon,\theta}}|\partial_\nu u e^{-\rho x'\cdot\theta}\left(e^{i\rho\eta\cdot x}\chi\left(\rho^{-\frac14}x_3\right)e^{-i\xi\cdot x}+w_\rho(x)\right)|d\sigma(x')\,dx_3\\ &\leqslant C\left(\int_{\partial\omega_{+,\varepsilon,\theta}\times\mathbb{R}}|e^{-\rho x'\cdot\theta}\partial_\nu u|^2 d\sigma(x)\right)^{\frac12}\left(\|\chi\left(\rho^{-\frac14}\cdot\right)\|_{L^2(\mathbb{R})}+\|w_\rho\|_{L^2(\Gamma)}\right)\\ &\leqslant C\rho^{\frac{7}{16}}\left(\int_{\partial\omega_{+,\varepsilon,\theta}\times\mathbb{R}}|e^{-\rho x'\cdot\theta}\partial_\nu u|^2 d\sigma(x)\right)^{\frac12}\end{aligned}$$

for some C independent of ρ. This estimate and the Carleman estimate (3.45) imply

$$\begin{aligned}&\left|\int_\Omega (q_2-q_1)u_2u_1\,dx\right|^2\\ &\leqslant C\rho^{\frac78}\int_{\partial\omega_{+,\varepsilon,\theta}\times\mathbb{R}}|e^{-\rho x'\cdot\theta}\partial_\nu u|^2 d\sigma(x)\\ &\leqslant \varepsilon^{-1}C\rho^{\frac78}\int_{\partial\omega_{+,\theta}\times\mathbb{R}}|e^{-\rho x'\cdot\theta}\partial_\nu u|^2|\nu\cdot\theta|d\sigma(x)\\ &\leqslant \varepsilon^{-1}C\rho^{-\frac18}\left(\int_\Omega |e^{-\rho x'\cdot\theta}(-\Delta+q_1)u|^2\,dx\right)\\ &\leqslant \varepsilon^{-1}C\rho^{-\frac18}\left(\int_\Omega |e^{-\rho x'\cdot\theta}qu_2|^2\,dx\right)\\ &\leqslant \varepsilon^{-1}C\rho^{-\frac18}\left(\|q\|_{L^\infty(\Omega)}\|\chi\|_{L^\infty(\mathbb{R})}\int_\Omega |q(x)|\,dx+\|q\|^2_{L^\infty(\Omega)}\|z_\rho\|^2_{L^2(\Omega)}\right),\end{aligned}\tag{5.54}$$

where $C>0$ is a constant independent of ρ. Applying the fact that $q\in L^1(\Omega)$, we deduce that

$$\lim_{\rho\to+\infty}\int_\Omega qu_2u_1\,dx=0.\tag{5.55}$$

Moreover, we have

$$\int_\Omega qu_1u_2\,dx=\int_{\mathbb{R}^3}\chi^2(\rho^{-\frac14}x_3)q(x)e^{-i\xi\cdot x}\,dx+\int_\Omega Y(x)\,dx+\int_\Omega Z(x)\,dx,$$

with $Y(x)=q(x)e^{-i\rho\eta\cdot x}z_\rho(x)w_\rho(x)$ and

$$Z(x)=q(x)\chi\left(\rho^{-\frac14}x_3\right)\left[z_\rho e^{-ix\cdot\xi}+w_\rho e^{-i\rho\eta\cdot x}\right].$$

Applying the decay estimate given by (2.18) and (4.48), we obtain

$$\int_\Omega |Y(x)|\,dx\leqslant \|w_\rho\|_{L^2(\Omega)}\|z_\rho\|_{L^2(\Omega)}\leqslant C\rho^{-\frac14},$$

$$\begin{aligned}\int_\Omega |Z(x)|\,dx&\leqslant \|q\|_{L^2(\Omega)}\|\chi\left(\rho^{-\frac14}\cdot\right)\|_{L^\infty(\mathbb{R})}(\|w_\rho\|_{L^2(\Omega)}+\|z_\rho\|_{L^2(\Omega)})\\ &\leqslant C\|q\|^{\frac12}_{L^\infty(\Omega)}\|q\|^{\frac12}_{L^1(\Omega)}\|\chi\|_{L^\infty(\mathbb{R})}\rho^{-\frac18},\end{aligned}$$

with C independent of ρ. Combining this with (5.55), we deduce that

$$\lim_{\rho\to+\infty}\int_{\mathbb{R}^3}\chi^2(\rho^{-\frac14}x_3)q(x)e^{-i\xi\cdot x}\,dx=0.$$

On the other hand, since $q\in L^1(\mathbb{R}^3)$ and $\chi(0)=1$, by the Lebesgue dominated convergence theorem, we find

$$\lim_{\rho\to+\infty}\int_{\mathbb{R}^3}\chi^2(\rho^{-\frac14}x_3)q(x)e^{-i\xi\cdot x}\,dx=\int_{\mathbb{R}^3}q(x)e^{-i\xi\cdot x}\,dx.$$

This proves that, for all $\theta\in\{y\in\mathbb{S}^1:|y-\theta_0|\leqslant\varepsilon\}$, all $\xi'\in\mathbb{R}^2\setminus\{0\}$ orthogonal to θ and all $\xi_3\in\mathbb{R}\setminus\{0\}$, we have

$$\mathcal{F}\left[\mathcal{F}_{x_3}q(\cdot,\xi_3)\right](\xi')=(2\pi)^{-1}\int_{\mathbb{R}^2}\mathcal{F}_{x_3}q(x',\xi_3)e^{-i\xi'\cdot x'}\,dx'=0. \tag{5.56}$$

Since $q\in L^1(\mathbb{R}^3)$, $\xi_3\mapsto\mathcal{F}_{x_3}q(\cdot,\xi_3)$ is continuous from $\mathbb{R}$ to $L^1(\mathbb{R}^2)$ and

$$|\mathcal{F}_{x_3}q(\cdot,\xi_3)|\leqslant(2\pi)^{-\frac12}\int_{\mathbb{R}}|q(\cdot,x_3)|\,dx_3,$$

by the Fubini and the Lebesgue dominated convergence theorem, we deduce that (5.56) holds for all $\xi'\in\mathbb{R}^2$ orthogonal to θ and all $\xi_3\in\mathbb{R}$. Now using the fact that for any $\xi_3\in\mathbb{R}$, $\mathcal{F}_{x_3}q(\cdot,\xi_3)$ is supported on $\overline{\omega}$ which is compact, we deduce, by analyticity of $\mathcal{F}\left[\mathcal{F}_{x_3}q(\cdot,\xi_3)\right]$, that $\mathcal{F}_{x_3}q(\cdot,\xi_3)=0$. This proves that $q=0$ which completes the proof of Theorem 1.1.

6. Applications

In this section we will prove the three applications of Theorem 1.1 stated in Corollaries 1.2–1.4.

6.1. Application to the Calderón problem

This subsection is devoted to the proof of Corollary 1.2. Applying the Liouville transform, we deduce that for u the solution to (1.8), $v:=a^{\frac12}u$ solves the following BVP:

$$\begin{cases}(-\Delta+q_a)v=0, & \text{in } \Omega,\\ v=a^{\frac12}f, & \text{on } \Gamma,\end{cases}$$

where $q_a:=a^{-\frac12}\Delta(a^{\frac12})$. In addition, one can check that

$$\Sigma_a f=a^{\frac12}\Lambda_{q_a}a^{\frac12}f-a^{\frac12}\left(\partial_\nu a^{\frac12}\right)f,\quad f\in H^{\frac12}(\Gamma)\cap a_1^{-\frac12}(\mathcal{H}_K(\Gamma)),$$

where Σ_a is defined by (1.9). Combining this with (1.10)–(1.11), we find that

$$\Sigma_{a_j}f=a_1^{\frac12}\Lambda_{q_j}a_1^{\frac12}f-a_1^{\frac12}\left(\partial_\nu a_1^{\frac12}\right)f_{|G},\quad j=1,2,\quad f\in H^{\frac12}(\Gamma)\cap a_1^{-\frac12}(\mathcal{H}_K(\Gamma)),$$

https://doi.org/10.1017/S1474748018000488 Published online by Cambridge University Press

where, we denote by q_j the potential q_{a_j}, $j = 1, 2$. Consequently, the condition $\Sigma_{a_1} = \Sigma_{a_2}$ implies

$$(\Lambda_{q_1} - \Lambda_{q_2})f = a_1^{-\frac{1}{2}}(\Sigma_{a_1} - \Sigma_{a_2})a_1^{-\frac{1}{2}}f = 0, \quad f \in a_1^{\frac{1}{2}}H^{\frac{1}{2}}(\Gamma) \cap (\mathcal{H}_K(\Gamma)).$$

In particular, this proves that $\Lambda_{q_1} = \Lambda_{q_2}$. Since $a_j \in \mathcal{A}$, $j = 1, 2$, it is clear that $q_j \in L^\infty(\Omega)$ and $q_1 - q_2 \in L^1(\Omega)$. Then, according to Theorem 1.1, we have $q_1 = q_2$. Fixing $y := a_1^{\frac{1}{2}} - a_2^{\frac{1}{2}} \in H^2_{loc}(\Omega)$ we deduce that y satisfies

$$\begin{cases} (-\Delta + q_1)y = -a_2^{\frac{1}{2}}(q_1 - q_2) = 0, & \text{in } \Omega, \\ y_{|K\cap G} = \partial_\nu y_{|K\cap G} = 0. \end{cases}$$

Combining this with results of unique continuation for elliptic equations (e.g. [46, Theorem 1]) we obtain $y = 0$ and we deduce that $a_1 = a_2$. This completes the proof of Corollary 1.2.

6.2. Recovery of coefficients that are known in the neighborhood of the boundary outside a compact set

This subsection is devoted to the proof of Corollary 1.3. For this purpose, we assume that the conditions of Corollary 1.3 are fulfilled. Let us also introduce the following sets of functions

$$S_q := \{u \in L^2(\Omega) : -\Delta u + qu = 0, \ \mathrm{supp}(\mathcal{T}_0 u) \subset K\}, \quad Q_q := \{u \in L^2(\Omega) : -\Delta u + qu = 0\},$$

$$S_{q,\gamma_1,\gamma_1'} := \{u \in L^2(\Omega) : \ -\Delta u + qu = 0, \ \mathrm{supp}(\mathcal{T}_0 u) \subset (K' \times [-R, R]) \cup \gamma_1 \cup \gamma_1'\},$$

$$Q_{q,\gamma_2,\gamma_2'} := \{u \in L^2(\Omega) : \ -\Delta u + qu = 0, \ \mathrm{supp}(\mathcal{T}_0 u) \subset (\partial\omega \times [-R, R]) \cup \gamma_2 \cup \gamma_2'\}.$$

We consider first the following result of density for these spaces.

Lemma 6.1. *The space $Q_{q_1,\gamma_2,\gamma_2'}$ (respectively, $S_{q_2,\gamma_1,\gamma_1'}$) is dense in Q_{q_1} (respectively, S_{q_2}) for the topology induced by $L^2(\Omega \setminus (\Omega_{1,*} \cup \Omega_{2,*}))$.*

Proof. Due to the similarity of these two results, we consider only the proof of the density of $Q_{q_1,\gamma_2,\gamma_2'}$ in Q_{q_1}. For this purpose, assume the contrary. Then, there exist $g \in L^2(\Omega \setminus (\Omega_{1,*} \cup \Omega_{2,*}))$ and $v_0 \in Q_{q_1}$ such that

$$\int_{\Omega\setminus(\Omega_{1,*}\cup\Omega_{2,*})} gv \, dx = 0, \quad v \in Q_{q_1,\gamma_2,\gamma_2'}, \tag{6.57}$$

$$\int_{\Omega\setminus(\Omega_{1,*}\cup\Omega_{2,*})} gv_0 \, dx = 1. \tag{6.58}$$

From now on, we extend g by 0 to Ω. Let $y \in H^2(\Omega)$ be the solution of

$$\begin{cases} (-\Delta + q_1)y = g, & \text{in } \Omega, \\ y = 0, & \text{on } \Gamma. \end{cases}$$

https://doi.org/10.1017/S1474748018000488 Published online by Cambridge University Press

Then, for any $v \in H^2(\Omega) \cap Q_{q_1,\gamma_2,\gamma_2'}$, (3.34) and (6.57) imply

$$0 = \int_\Omega g v \, dx = -\int_{\partial\omega\times[-R,R]} \partial_\nu y v \, d\sigma(x) - \int_{\gamma_2} \partial_\nu y v \, d\sigma(x) - \int_{\gamma_2'} \partial_\nu y v \, d\sigma(x).$$

Allowing $v \in H^2(\Omega) \cap Q_{q_1,\gamma_2,\gamma_2'}$ to be arbitrary, we deduce that

$$\partial_\nu y(x) = 0, \quad x \in (\partial\omega \times [-R, R]) \cup \gamma_2 \cup \gamma_2'. \tag{6.59}$$

Therefore, y satisfies

$$\begin{cases} (-\Delta + q_1) y = 0 & \text{in } \Omega_{1,*}, \\ y_{|\gamma_2} = \partial_\nu y_{|\gamma_2} = 0 & \end{cases}$$

and the unique continuation property for elliptic equations implies that $y_{|\Omega_{1,*}} = 0$. In the same way, we can prove that $y_{|\Omega_{2,*}} = 0$ and we deduce that

$$y_{|\partial\Omega_{j,*}} = \partial_\nu y_{|\partial\Omega_{j,*}} = 0, \quad j = 1, 2.$$

Combining this with (6.59), we obtain

$$y(x) = \partial_\nu y(x) = 0, \quad x \in \partial(\Omega \setminus (\overline{\Omega_{1,*}} \cup \overline{\Omega_{2,*}})) = \Gamma_*.$$

Then, for any $z \in \mathcal{C}_0^\infty(\mathbb{R}^3)$, applying the Green formula on a bounded neighborhood of $\mathrm{supp}(z)$ in $\Omega \setminus (\Omega_{1,*} \cup \Omega_{2,*})$, we find

$$\int_{\Omega\setminus(\Omega_{1,*}\cup\Omega_{2,*})} z \Delta y \, dx - \int_{\Omega\setminus(\Omega_{1,*}\cup\Omega_{2,*})} y \Delta z \, dx = 0. \tag{6.60}$$

Let us prove that (6.60) holds true for $z \in H_\Delta(\Omega)$. For this purpose, we fix $z \in H_\Delta(\Omega)$. According to [16, Lemma 2.1], there exists a sequence $(z_n)_{n\geqslant 1}$ lying in $\mathcal{C}_0^\infty(\mathbb{R}^3)$ such that

$$\lim_{n\to\infty} \|z - z_n\|_{L^2(\Omega)} = \lim_{n\to\infty} \|\Delta z - \Delta z_n\|_{L^2(\Omega)} = 0. \tag{6.61}$$

In light of (6.60), we have

$$\int_{\Omega\setminus(\Omega_{1,*}\cup\Omega_{2,*})} z_n \Delta y \, dx - \int_{\Omega\setminus(\Omega_{1,*}\cup\Omega_{2,*})} y \Delta z_n \, dx = 0, \quad n \geqslant 1. \tag{6.62}$$

Moreover, (6.61) implies

$$\begin{aligned} &\limsup_{n\to\infty} \left| \int_{\Omega\setminus(\Omega_{1,*}\cup\Omega_{2,*})} z_n \Delta y \, dx - \int_{\Omega\setminus(\Omega_{1,*}\cup\Omega_{2,*})} z \Delta y \, dx \right| \\ &\leqslant \left(\limsup_{n\to\infty} \|z - z_n\|_{L^2(\Omega)}\right) \|\Delta y\|_{L^2(\Omega)} = 0, \end{aligned}$$

$$\begin{aligned} &\limsup_{n\to\infty} \left| \int_{\Omega\setminus(\Omega_{1,*}\cup\Omega_{2,*})} y \Delta z_n \, dx - \int_{\Omega\setminus(\Omega_{1,*}\cup\Omega_{2,*})} y \Delta z \, dx \right| \\ &\leqslant \left(\limsup_{n\to\infty} \|\Delta z - \Delta z_n\|_{L^2(\Omega)}\right) \|y\|_{L^2(\Omega)} = 0. \end{aligned}$$

https://doi.org/10.1017/S1474748018000488 Published online by Cambridge University Press

Combining this with (6.62) we deduce that (6.60) holds true for $z \in H_\Delta(\Omega)$. Therefore, we get

$$\int_{\Omega\setminus(\Omega_{1,*}\cup\Omega_{2,*})} g v_0\, dx = \int_{\Omega\setminus(\Omega_{1,*}\cup\Omega_{2,*})} (-\Delta+q_1)y v_0\, dx = \int_{\Omega\setminus(\Omega_{1,*}\cup\Omega_{2,*})} y(-\Delta+q_1)v_0\, dx = 0.$$

This contradicts (6.58) and completes the proof of the lemma. □

Armed with this lemma we are now in a position to complete the proof of Corollary 1.3.

Proof of Corollary 1.3. Let $u_1 \in Q_{q_1,\gamma_2,\gamma_2'}$ and $u_2 \in S_{q_2,\gamma_1,\gamma_1'}$. Repeating the linearization process described in § 5 we deduce that $\Lambda^*_{q_1,R} = \Lambda^*_{q_2,R}$ implies

$$\int_\Omega (q_2-q_1)u_1u_2\, dx = 0.$$

Then, (1.12) implies

$$0 = \int_\Omega (q_2-q_1)u_1u_2\, dx = \int_{\Omega\setminus(\Omega_{1,*}\cup\Omega_{2,*})} (q_2-q_1)u_1u_2\, dx. \tag{6.63}$$

Combining this with the density result of Lemma 6.1 and applying again (1.12), we obtain

$$\int_\Omega (q_2-q_1)u_1u_2\, dx = \int_{\Omega\setminus(\Omega_{1,*}\cup\Omega_{2,*})} (q_2-q_1)u_1u_2\, dx = 0, \quad u_1 \in Q_{q_1},\ u_2 \in S_{q_2}.$$

Finally, choosing u_1, u_2 in a similar way to § 5, we can deduce that $q_1 = q_2$. This completes the proof of the corollary. □

6.3. Recovery of non-compactly supported coefficients in a slab

In this subsection we consider Corollary 1.4. Applying the construction of CGO solutions and the Carleman estimate of the previous sections, we will prove how one can extend the result of [40] to coefficients supported on an unbounded cylinder. For this purpose, we start by fixing $\delta \in (0, R-r)$ and ω an open smooth and connected subset of $(0, L)\times\mathbb{R}$ such that $(0, L)\times(-r-\delta, r+\delta) \subset \omega \subset (0, L)\times(-R, R)$. Then, we fix $\Omega := \omega\times\mathbb{R}$ and we consider the set of functions

$$\mathcal{V}_q(\Omega) := \{u \in H^1(\Omega) :\ -\Delta u + qu = 0 \ \text{ in } \Omega\},$$

$$\mathcal{W}_q(\mathcal{O}) := \{u_{|\Omega} :\ u \in H^1(\mathcal{O}),\ -\Delta u + qu = 0 \ \text{ in } \mathcal{O},\ u_{|x_1=0} = 0\},$$

$$\mathcal{W}_q(\Omega) := \{u \in H^1(\Omega) :\ -\Delta u + qu = 0 \ \text{ in } \Omega,\ u_{|x_1=0} = 0\}.$$

Following [40, Lemma 9], one can check the following result of density.

Lemma 6.2. *Let $q \in L^\infty(\mathcal{O})$ be such that 0 is not in the spectrum of $-\Delta+q$ with Dirichlet boundary condition on $\mathcal{O}$. Then the set $\mathcal{W}_q(\mathcal{O})$ is dense in $\mathcal{W}_q(\Omega)$ with respect to the topology of $L^2(\Omega)$.*

In the same way, combining Lemma 6.2 with the Carleman estimate (3.45) and [40, Lemma 10], we obtain the following important estimate.

Lemma 6.3. *Let $\theta := (\theta_1, \theta_2) \in \mathbb{S}^1$ be such that $\theta_1 > 0$ and assume that (1.14)–(1.15) are fulfilled. Then we have*

$$\left|\int_{\mathcal{O}} (q_1 - q_2) v_1 v_2 \, dx\right| = \left|\int_{\Omega} (q_1 - q_2) v_1 v_2 \, dx\right| \leqslant C\rho^{-\frac{1}{2}} (\theta_1)^{-\frac{1}{2}} \left(\int_{\Omega} |e^{-\rho x' \cdot \theta} (q_1 - q_2) v_2|^2 \, dx\right)^{\frac{1}{2}} \times \left(\int_{\Gamma \cap \{x_1 = L\}} |e^{\rho x' \cdot \theta} v_1|^2 d\sigma(x)\right)^{\frac{1}{2}} \tag{6.64}$$

for all $v_1 \in \mathcal{V}_{q_1}(\Omega)$ and for all $v_2 \in \mathcal{W}_{q_2}(\Omega)$.

Armed with these two results, we will complete the proof of Corollary 1.4 by choosing suitably the solutions v_j, $j = 1, 2$, of the equation $-\Delta v_j + q_j v_j = 0$ in Ω.

Proof of Corollary 1.4. From now on, we assume that the condition (1.15) is fulfilled. Let us first start by considering the set $\tilde{\omega} := \{x := (x_1, x_2, x_3) : (-x_1, x_2, x_3) \in \omega\} \cup \omega$ and let us extend q_2 by symmetry to $\tilde{\omega} \times \mathbb{R}$ by assuming that

$$q_2(-x_1, x_2, x_3) = q_2(x_1, x_2, x_3), \quad (x_1, x_2, x_3) \in \tilde{\omega} \times \mathbb{R}.$$

Applying the results of § 2, we can consider $u_2 \in H^2(\tilde{\omega} \times \mathbb{R})$ solving $-\Delta u_2 + q_2 u_2 = 0$ in $\tilde{\omega} \times \mathbb{R}$ and taking the form

$$u_2(x) := e^{\rho\theta \cdot x'} \left(e^{-i\rho\eta \cdot x} \chi\left(\rho^{-\frac{1}{4}} x_3\right) e^{-ix\cdot\xi} + w_{2,\rho}(x)\right), \quad x := (x', x_3) \in \tilde{\omega} \times \mathbb{R}, \tag{6.65}$$

with $\theta := (\theta_1, \theta_2) \in \mathbb{S}^1$ such that $\theta_1 > 0$, $\eta, \xi \in \mathbb{R}^3$ chosen in a similar way to the beginning of § 2, and $w_{2,\rho} \in H^2(\tilde{\omega} \times \mathbb{R})$ satisfying

$$\rho^{-1} \|w_{2,\rho}\|_{H^2(\tilde{\omega}\times\mathbb{R})} + \rho \|w_{2,\rho}\|_{L^2(\tilde{\omega}\times\mathbb{R})} \leqslant C\rho^{\frac{7}{8}}. \tag{6.66}$$

Then, we fix $v_2 \in H^2(\Omega)$ defined by

$$v_2(x_1, x_2, x_3) := u_2(x_1, x_2, x_3) - u_2(-x_1, x_2, x_3), \quad (x_1, x_2, x_3) \in \Omega. \tag{6.67}$$

It is clear that $v_2 \in \mathcal{W}_{q_2}(\Omega)$. In the same way, we fix $v_1 \in \mathcal{V}_{q_1}(\Omega)$:

$$v_1(x) := e^{-\rho\theta \cdot x'} \left(e^{i\rho\eta \cdot x} \chi\left(\rho^{-\frac{1}{4}} x_3\right) + w_{1,\rho}(x)\right), \quad x := (x', x_3) \in \Omega, \tag{6.68}$$

with $w_{1,\rho} \in H^2(\Omega)$ satisfying

$$\rho^{-1} \|w_{1,\rho}\|_{H^2(\Omega)} + \rho \|w_{1,\rho}\|_{L^2(\Omega)} \leqslant C\rho^{\frac{7}{8}}. \tag{6.69}$$

Applying (6.64)–(6.69) and the fact that $q_1 - q_2 \in L^\infty(\Omega) \cap L^1(\Omega) \subset L^2(\Omega)$, in a similar way to § 5 we deduce that

$$\lim_{\rho \to +\infty} \int_{\Omega} (q_1 - q_2) v_1 v_2 \, dx = 0.$$

On the other hand, we have

$$\int_\Omega (q_1 - q_2) v_1 v_2 \, dx = \int_\Omega (q_1 - q_2) \chi^2 \left(\rho^{-\frac{1}{4}} x_3\right) e^{-ix\cdot\xi} \, dx + \int_\Omega X_\rho \, dx, \tag{6.70}$$

where

$$\begin{aligned} X_\rho := & (q_1 - q_2) \left[e^{-\rho\theta\cdot x'} u_2 w_{1,\rho} + w_{2,\rho} e^{i\rho\eta\cdot x} \chi \left(\rho^{-\frac{1}{4}} x_3\right)\right] \\ & - (q_1 - q_2) e^{-2\rho\theta_1 x_1} \left(e^{i\rho\eta\cdot x} \chi \left(\rho^{-\frac{1}{4}} x_3\right) + w_{1,\rho}(x)\right) \\ & \times \left(e^{-i\rho\eta\cdot s(x)} \chi \left(\rho^{-\frac{1}{4}} x_3\right) e^{-i\xi\cdot s(x)} + w_{2,\rho}(s(x))\right), \end{aligned}$$

with $s(x_1, x_2, x_3) = (-x_1, x_2, x_3)$. Combining this with the decay estimates (6.66), (6.69) and using the fact that $\theta_1 x_1 > 0$, we deduce that

$$\lim_{\rho\to+\infty} \int_\Omega X_\rho \, dx = 0.$$

Then, (6.70) and the fact that $q_1 - q_2 \in L^1(\Omega)$ imply that

$$\int_\Omega (q_1 - q_2) e^{-ix\cdot\xi} \, dx = 0$$

and following the arguments used at the end of the proof of Theorem 1.1 we deduce that $q_1 = q_2$. □

Acknowledgments. The author would like to thank Gunther Uhlmann for his suggestions and remarks. The author is grateful to the anonymous referees for their careful reading and their suggestions that allowed improving the paper.

Conflict of Interest

The author declares that he has no conflict of interest.

References

1. D. Ahluwalia and J. Keller, *Exact and Asymptotic Representations of the Sound Field in a Stratified Ocean. Wave Propagation and Underwater Acoustics*, Lecture Notes in Physics, vol. 70, pp. 14–85 (Springer, Berlin, 1977).
2. H. Ammari and G. Uhlmann, Reconstuction from partial Cauchy data for the Schrödinger equation, *Indiana Univ. Math. J.* **53** (2004), 169–184.
3. M. Bellassoued, Y. Kian and E. Soccorsi, An inverse stability result for non compactly supported potentials by one arbitrary lateral Neumann observation, *J. Differential Equations* **260** (2016), 7535–7562.
4. M. Bellassoued, Y. Kian and E. Soccorsi, An inverse problem for the magnetic Schrödinger equation in infinite cylindrical domains, *Publ. Res. Inst. Math. Sci.* **54** (2018), 679–728.
5. A. Bukhgeim, Recovering the potential from Cauchy data in two dimensions, *J. Inverse Ill-Posed Probl.* **16** (2008), 19–34.
6. A. L. Bukhgeim and G. Uhlmann, Recovering a potential from partial Cauchy data, *Commun. Partial Differential Equations* **27**(3–4) (2002), 653–668.

7. A. P. Calderón, On an inverse boundary value problem, in *Seminar on Numerical Analysis and its Applications to Continuum Physics*, pp. 65–73 (Sociedade Brasileira de Matematica, Rio de Janeiro, 1980).
8. P. Caro, D. Dos Santos Ferreira and A. Ruiz, Stability estimates for the Radon transform with restricted data and applications, *Adv. Math.* **267** (2014), 523–564.
9. P. Caro, D. Dos Santos Ferreira and A. Ruiz, Stability estimates for the Calderón problem with partial data, *J. Differential Equations* **260** (2016), 2457–2489.
10. P. Caro and K. Marinov, Stability of inverse problems in an infinite slab with partial data, *Comm. Partial Differential Equations* **41** (2016), 683–704.
11. P.-Y. Chang and H.-H. Lin, Conductance through a single impurity in the metallic zigzag carbon nanotube, *Appl. Phys. Lett.* **95** (2009), 082104.
12. M. Choulli, *Une introduction aux problèmes inverses elliptiques et paraboliques*, Mathématiques et Applications, Volume 65 (Springer, Berlin, 2009).
13. M. Choulli and Y. Kian, Logarithmic stability in determining the time-dependent zero order coefficient in a parabolic equation from a partial Dirichlet-to-Neumann map. Application to the determination of a nonlinear term, *J. Math. Pures Appl. (9)* **114** (2018), 235–261.
14. M. Choulli, Y. Kian and E. Soccorsi, Stable determination of time-dependent scalar potential from boundary measurements in a periodic quantum waveguide, *SIAM J. Math. Anal.* **47**(6) (2015), 4536–4558.
15. M. Choulli, Y. Kian and E. Soccorsi, Double logarithmic stability estimate in the identification of a scalar potential by a partial elliptic Dirichlet-to-Neumann map, *Bulletin of the South Ural State University, Ser. Mathematical Modelling, Programming and Computer Software (SUSU MMCS)* **8**(3) (2015), 78–95.
16. M. Choulli, Y. Kian and E. Soccorsi, Stability result for elliptic inverse periodic coefficient problem by partial Dirichlet-to-Neumann map, *J. Spectr. Theory* **8**(2) (2018), 733–768.
17. M. Choulli, Y. Kian and E. Soccorsi, On the Calderón problem in periodic cylindrical domain with partial Dirichlet and Neumann data, *Math. Methods Appl. Sci.* **40** (2017), 5959–5974.
18. M. Choulli and E. Soccorsi, An inverse anisotropic conductivity problem induced by twisting a homogeneous cylindrical domain, *J. Spectr. Theory* **5** (2015), 295–329.
19. L. Hörmander, *The Analysis of Linear Partial Differential Operators*, vol. II (Springer, Berlin, Heidelberg, 1983).
20. G. Hu and Y. Kian, Determination of singular time-dependent coefficients for wave equations from full and partial data, *Inverse Probl. Imaging* **12** (2018), 745–772.
21. M. Ikehata, Inverse conductivity problem in the infinite slab, *Inverse Problems* **17** (2001), 437–454.
22. O. Imanuvilov, G. Uhlmann and M. Yamamoto, The Calderón problem with partial data in two dimensions, *J. Amer. Math. Soc.* **23** (2010), 655–691.
23. O. Imanuvilov, G. Uhlmann and M. Yamamoto, Partial Cauchy data for general second order elliptic operators in two dimensions, *Publ. Res. Inst. Math. Sci.* **48** (2012), 971–1055.
24. V. Isakov, Completness of products of solutions and some inverse problems for PDE, *J. Differential Equations* **92** (1991), 305–316.
25. V. Isakov, On uniqueness in the inverse conductivity problem with local data, *Inverse Probl. Imaging* **1** (2007), 95–105.
26. C. Kane, L. Balents and M. P. A. Fisher, Coulomb interactions and mesoscopic effects in carbon nanotubes, *Phys. Rev. Lett.* **79** (1997), 5086–5089.

27. O. Kavian, Y. Kian and E. Soccorsi, Uniqueness and stability results for an inverse spectral problem in a periodic waveguide, *J. Math. Pures Appl. (9)* **104**(6) (2015), 1160–1189.
28. C. E. Kenig and M. Salo, The Calderón problem with partial data on manifolds and applications, *Anal. PDE* **6**(8) (2013), 2003–2048.
29. C. E. Kenig, J. Sjöstrand and G. Uhlmann, The Calderon problem with partial data, *Ann. of Math. (2)* **165** (2007), 567–591.
30. Y. Kian, Stability of the determination of a coefficient for wave equations in an infinite waveguide, *Inverse Probl. Imaging* **8**(3) (2014), 713–732.
31. Y. Kian, Unique determination of a time-dependent potential for wave equations from partial data, *Ann. l'IHP (C) Nonlinear Anal.* **34** (2017), 973–990.
32. Y. Kian, Recovery of time-dependent damping coefficients and potentials appearing in wave equations from partial data, *SIAM J. Math. Anal.* **48**(6) (2016), 4021–4046.
33. Y. Kian and L. Oksanen, Recovery of time-dependent coefficient on Riemannian manifold for hyperbolic equations, *Int. Math. Res. Not. IMRN*, to appear, https://doi.org/10.1093/imrn/rnx263.
34. Y. Kian, Q. S. Phan and E. Soccorsi, Carleman estimate for infinite cylindrical quantum domains and application to inverse problems, *Inverse Problems* **30**(5) (2014), 055016.
35. Y. Kian, Q. S. Phan and E. Soccorsi, Hölder stable determination of a quantum scalar potential in unbounded cylindrical domains, *J. Math. Anal. Appl.* **426**(1) (2015), 194–210.
36. M. V. Klibanov, Convexification of restricted Dirichlet-to-Neumann map, *J. Inverse Ill-Posed Probl.* **25** (2017), 669–685.
37. K. Krupchyk, M. Lassas and G. Uhlmann, Inverse problems with partial data for a magnetic Schrödinger operator in an infinite slab or bounded domain, *Comm. Math. Phys.* **312** (2012), 87–126.
38. X. Li, Inverse boundary value problems with partial data in unbounded domains, *Inverse Problems* **28** (2012), 085003.
39. X. Li, Inverse problem for Schrödinger equations with Yang–Mills potentials in a slab, *J. Differential Equations* **253** (2012), 694–726.
40. X. Li and G. Uhlmann, Inverse problems on a slab, *Inverse Probl. Imaging* **4** (2010), 449–462.
41. J.-L. Lions and E. Magenes, *Non-homogeneous Boundary Value Problems and Applications*, vol. I (Dunod, Paris, 1968).
42. A. Nachman and B. Street, Reconstruction in the Calderón problem with partial data, *Commun. Partial Differential Equations* **35** (2010), 375–390.
43. L. Potenciano-Machado, Stability estimates for a Magnetic Schrodinger operator with partial data. Preprint, 2016, arXiv:1610.04015.
44. L. Potenciano-Machado, Optimal stability estimates for a Magnetic Schrödinger operator with local data, *Inverse Problems* **33** (2017), 095001.
45. M. Salo and J. N. Wang, Complex spherical waves and inverse problems in unbounded domains, *Inverse Problems* **22** (2006), 2299–2309.
46. J. C. Saut and B. Scheurer, Sur l'unicité du problème de Cauchy et le prolongement unique pour des équations elliptiques à coefficients non localement bornés, *J. Differential Equations* **43** (1982), 28–43.
47. J. Sylvester and G. Uhlmann, A global uniqueness theorem for an inverse boundary value problem, *Ann. of Math. (2)* **125** (1987), 153–169.
48. Y. Yang, Determining the first order perturbation of a bi-harmonic operator on bounded and unbounded domains from partial data, *J. Differential Equations* **257** (2014), 3607–3639.