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INTEGRAL CLOSURE OF STRONGLY GOLOD IDEALS

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Abstract. We prove that the integral closure of a strongly Golod ideal in a polynomial ring over a field of characteristic zero is strongly Golod, positively answering a question of Huneke. More generally, the rational power I_α of an arbitrary homogeneous ideal is strongly Golod for $\alpha \geqslant 2$ and, if I is strongly Golod, then I_α is strongly Golod for $\alpha \geqslant 1$. We also show that all the coefficient ideals of a strongly Golod ideal are strongly Golod.

§1. Introduction

Let K be a field of characteristic zero and let $A = K[x_1, \ldots, x_d]$ be the graded polynomial ring with $\deg(x_i) = a_i > 0$ $(i = 1, \ldots, d)$ and homogeneous maximal ideal M. Introduced by Herzog and Huneke in [3], the *strongly Golod* ideals in A are the proper homogeneous ideals I that satisfy the condition $\partial(I)^2 \subseteq I$, where $\partial(I)$ is the ideal generated by the partial derivatives of all the elements of I. The motivation for the introduction of this notion comes from a result proved in the same paper [3, Theorem 1.1] which shows that if I is strongly Golod, then the ring A/I is Golod, that is, the Poincaré series of A/I

$$P_{A/I}(t) = \sum_{i \geqslant 0} \dim_K \operatorname{Tor}_i^{A/I}(K, K) t^i$$

coincides with the rational series

$$H_{A/I}(t) = \frac{(1+t)^n}{1 - t \sum_{i \geqslant 1} \dim_K \mathrm{H}_i(\mathbf{y}; A/I) t^i},$$

where $\mathrm{H}_i(\mathbf{y}; A/I)$ is the ith Koszul homology of A/I with respect to a minimal homogeneous set of generators $\mathbf{y} = y_1, \ldots, y_n$ of the maximal homogeneous ideal of A/I. As proved by Serre, $P_{A/I}(t)$ is coefficientwise bounded above by $H_{A/I}(t)$, and the ring A/I is called *Golod* if $P_{A/I}(t) = H_{A/I}(t)$.

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Herzog and Huneke further studied the properties of the strongly Golod ideals. For every homogeneous ideal I of A, among other things, they proved that the powers I^s, the symbolic powers $I^{(s)}$, and the saturations $(I^s : M^\infty)$ are strongly Golod for all $s \geqslant 2$ [3, Theorem 2.3]. Moreover, if I is strongly Golod, then the Ratliff–Rush ideal $\tilde{I} = \bigcup_{i\geqslant 1}(I^{i+1} : I^i)$ is strongly Golod [3, Proposition 2.1] and the integral closures $\overline{I^s}$ are strongly Golod for $s \geqslant d+1$ [3, Theorem 2.11].

For monomial ideals, Herzog and Huneke improved the previous result by showing that if I is a monomial strongly Golod ideal, then the integral closure $\overline{I}$ is also strongly Golod [3, Proposition 3.1]. In particular, for an arbitrary monomial ideal I, the integral closures $\overline{I^s}$ are strongly Golod for $s \geqslant 2$. More generally, De Stefani later proved that the rational powers I_α of a monomial ideal I are strongly Golod for every rational $\alpha \geqslant 2$ [1, Proposition 3.7], and if I is a monomial strongly Golod ideal, then I_α is strongly Golod for every rational $\alpha \geqslant 1$ [1, Theorem 3.5].

In this article we show that the above results, which had been proved for monomial ideals, are actually valid for all homogeneous ideals. More precisely, we prove that if I is a homogeneous strongly Golod ideal, then the integral closure $\overline{I}$ is strongly Golod (Corollary 3.5), positively answering a question of Huneke [6, Problem 6.19]. More generally, if I is a homogeneous strongly Golod ideal, then the rational power I_α is strongly Golod for every $\alpha \geqslant 1$ (Corollary 4.6). Moreover, if I is an arbitrary homogeneous ideal, then all the rational powers I_α are strongly Golod for $\alpha \geqslant 2$ (Corollary 4.4).

Our techniques are very different from those of Herzog and Huneke [3] and De Stefani [1], which can only be used in the monomial case. The main ingredient in our arguments is the Rees-like algebra $A[\partial(I)t^a, It^{2a}, t^{-1}]$ with $a \in \mathbb{N}^*$ and the main tool used is a result of Seidenberg [7, Section 3] showing that a derivation on a noetherian domain R containing a field of characteristic zero, when extended to its quotient field, will map every element of the integral closure $\overline{R}$ to an element of $\overline{R}$ (see (2.5)). Using this approach, we prove that if I is a homogeneous strongly Golod ideal, then $\partial(\overline{I^m})\partial(\overline{I^n}) \subseteq \overline{I^{m+n-1}}$ for all positive integers m, n (Theorem 3.4). More generally, for rational powers of ideals, if I is a homogeneous strongly Golod ideal, then $\partial(I_\alpha) \subseteq I_{\alpha-(1/2)}$ for every rational $\alpha \geqslant \frac{1}{2}$ (Theorem 4.5). For arbitrary ideals, we also prove that $\partial(I_\alpha) \subseteq I_{\alpha-1}$ for every rational $\alpha \geqslant 1$ (Theorem 4.3). The main results of the paper regarding the strongly Golod property of the integral closure and the rational powers of a homogeneous ideal are then obtained as immediate consequences.

Finally, as already mentioned above, Herzog and Huneke proved that the Ratliff–Rush ideal of a strongly Golod ideal is strongly Golod as well. In the last section of the paper, we extend this result by showing that all the coefficient ideals $I_{\{k\}}$ $(1 \leqslant k \leqslant d)$ of a strongly Golod ideal I of finite colength are strongly Golod (Theorem 5.4), answering a question raised by De Stefani [1, Question 5.9]. (The coefficient ideal $I_{\{d\}}$ of an ideal of finite colength is the Ratliff–Rush ideal $\tilde{I}$.)

§2. Preliminaries

Let K be a field of characteristic zero and $A = K[x_1, \ldots, x_d]$ a graded polynomial ring with $\deg(x_i) = a_i > 0$.

2.1. For an ideal I of A and $i \in \{1, \ldots, d\}$, we denote by $\partial_i(I)$ the ideal generated by the partial derivatives $\partial_i f = \partial f/\partial x_i$ with $f \in I$, and by $\partial(I)$ the ideal generated by $\{\partial f/\partial x_i \mid f \in I, 1 \leqslant i \leqslant d\}$, that is, $\partial(I) = \partial_1(I) + \cdots + \partial_d(I)$.

Note that for an arbitrary ideal I we always have $I \subseteq \partial(I)$. Indeed, if $f \in I$, then $x_1 f \in I$, so $f = (\partial/\partial x_1)(x_1 f) - x_1(\partial/\partial x_1)(f) \in \partial(I)$.

2.2. If $f \in A$ is a homogeneous polynomial, then

$$\deg(f) f = \sum_{i=1}^{d} a_i x_i \frac{\partial f}{\partial x_i} \quad \text{(Euler's formula)},$$

and since the characteristic of K is zero, it follows that $f \in (\partial f/\partial x_1, \ldots, \partial f/\partial x_d)$.

Assume that $I = (f_1, \ldots, f_s)$ with f_i homogeneous polynomials in A. If $f = g_1 f_1 + \cdots + g_s f_s$ with $g_i \in A$, from the previous observation it follows that for each $\ell = 1, \ldots, d$ we have

$$\frac{\partial f}{\partial x_\ell} = \sum_{i=1}^{s} g_i \frac{\partial f_i}{\partial x_\ell} + \sum_{i=1}^{s} f_i \frac{\partial g_i}{\partial x_\ell} \in \left(\frac{\partial f_i}{\partial x_j} \mid 1 \leqslant i \leqslant s, 1 \leqslant j \leqslant d \right),$$

and therefore $\partial(I) = (\partial f_i/\partial x_j \mid 1 \leqslant i \leqslant s, 1 \leqslant j \leqslant d)$.

DEFINITION 2.3. [3] A proper homogeneous ideal I of A is said to be strongly Golod if $\partial(I)^2 \subseteq I$.

REMARK 2.4. As noted in [3, Remark 2.1], if A is standard graded and I is an ideal of A, the condition $\partial(I)^2 \subseteq I$ does not depend on the chosen coordinates in A. More precisely, if $y_1, \ldots, y_d$ are linear forms in A such that

$K[x_1, \ldots, x_d] = K[y_1, \ldots, y_d]$ and $\partial'(I)$ is the ideal of A generated by all the partial derivatives $\partial f/\partial y_j$ with $f \in I$ and $j = 1, \ldots, d$, then $\partial(I)^2 \subseteq I$ if and only if $\partial'(I)^2 \subseteq I$.

2.5. (Derivations and integral closure) Let R be a noetherian integral domain containing a field of characteristic zero, and let D be a derivation of its quotient field $Q(R)$. Seidenberg [7, Section 3] proved that if $D(R) \subseteq R$, then $D(\overline{R}) \subseteq \overline{R}$, where $\overline{R}$ is the integral closure of R in $Q(R)$. In fact, as Seidenberg shows, the result holds even without the noetherian assumption if the integral closure $\overline{R}$ is replaced by the quasi-integral closure of R, that is, the ring consisting of all the elements $\alpha \in Q(R)$ for which there exists $c \in R \setminus \{0\}$ such that $c\alpha^n \in R$ for all $n \geqslant 0$. In the literature, this is also referred to as the complete integral closure of R.

§3. Integral closure of strongly Golod ideals

Throughout this section K is a field of characteristic zero and $A = K[x_1, \ldots, x_d]$ is a polynomial ring with $\deg(x_i) = a_i > 0$. We show that the integral closure of a strongly Golod ideal is also strongly Golod. The results of this section are further extended to rational powers of ideals in the next section, and technically can be obtained as particular cases of them. However, for the sake of clarity, we prefer to present the main technique applied to the case of the integral closure of an ideal in this separate section.

THEOREM 3.1. *Let I be an arbitrary ideal of A. Then for every positive integer n,*

$$\partial(\overline{I^n}) \subseteq \overline{I^{n-1}}.$$

Proof. Consider the (extended) Rees algebra $\mathcal{R} = A[It, t^{-1}]$ with quotient field $Q(\mathcal{R})$. Since A is normal, the integral closure of $\mathcal{R}$ in its quotient field is $\overline{\mathcal{R}} = \bigoplus_{n\in\mathbb{Z}} \overline{I^n}t^n$, where $\overline{I^n} = A$ for $n \leqslant 0$. For every $i \in \{1, \ldots, d\}$, since $\partial_i = \partial/\partial x_i$ is a derivation on $Q(\mathcal{R}) = Q(A)(t) = K(x_1, \ldots, x_d, t)$, it follows that

$$D_i := \frac{1}{t}\partial_i = \frac{1}{t}\frac{\partial}{\partial x_i} : Q(\mathcal{R}) \to Q(\mathcal{R})$$

is also an additive group homomorphism that satisfies the Leibniz's rule, and hence D_i is a derivation on $Q(\mathcal{R})$.

Since $\partial_i(I^n) \subseteq I^{n-1}$, we have $D_i(\mathcal{R}) \subseteq \mathcal{R}$ and by Seidenberg's theorem (2.5) it follows that $D_i(\overline{\mathcal{R}}) \subseteq \overline{\mathcal{R}}$, or equivalently, $\partial_i(\overline{I^n}) \subseteq \overline{I^{n-1}}$ for every positive n. Since this holds for all i, we obtain $\partial(\overline{I^n}) \subseteq \overline{I^{n-1}}$. □

As an immediate consequence, we show that $\overline{I^n}$ is strongly Golod for every homogeneous ideal I and every $n \geqslant 2$. This result was already known to be true for monomial ideals [3, Proposition 3.1]. In the case of homogeneous ideals, it was only known for $n \geqslant d+1$ [3, Theorem 2.11].

COROLLARY 3.2. *Let I be a homogeneous ideal of A. Then $\overline{I^n}$ is strongly Golod for every integer $n \geqslant 2$.*

Proof. Using the previous theorem we have $\partial(\overline{I^n})^2 \subseteq (\overline{I^{n-1}})^2 \subseteq \overline{I^{2n-2}} \subseteq \overline{I^n}$ for $n \geqslant 2$. ∎

In order to be able to prove that the integral closure $\overline{I}$ of a strongly Golod ideal I is strongly Golod, we need a refinement of the first inclusion in the proof of Corollary 3.2. We obtain this by considering the Rees-like algebra $A[\partial(I)t, It^2, t^{-1}]$ instead of the classical Rees algebra $A[It, t^{-1}]$. We begin by identifying the homogeneous components of this new algebra.

LEMMA 3.3. *Let I be a strongly Golod ideal of A and let*

$$\mathcal{S} = A[\partial(I)t, It^2] = \bigoplus_{n \geqslant 0} J_n t^n.$$

Then

$$J_{2n} = I^n \qquad \textit{and} \qquad J_{2n+1} = \partial(I) I^n \quad \textit{for } n \geqslant 0.$$

Proof. We start by observing that $J_2 = \partial(I)^2 + I = I$, since I is strongly Golod. From the t-graded structure on $\mathcal{S}$, it is clear that $I^n \subseteq J_{2n}$ and $\partial(I)I^n \subseteq J_{2n+1}$ for $n \geqslant 0$. On the other hand, since $\partial(I)^2 \subseteq I$, we also have $J_{2n} = \sum_{k=0}^{n} I^k \partial(I)^{2n-2k} \subseteq \sum_{k=0}^{n} I^k I^{n-k} = I^n$ and $J_{2n+1} = \sum_{k=0}^{n} I^k \partial(I)^{2n+1-2k} \subseteq \sum_{k=0}^{n} I^k I^{n-k} \partial(I) = I^n \partial(I)$, finishing the proof. ∎

THEOREM 3.4. *Let I be a homogeneous ideal of A. If I is strongly Golod, then for every positive integers m, n,*

$$\partial(\overline{I^m})\partial(\overline{I^n}) \subseteq \overline{I^{m+n-1}}.$$

Proof. We consider the Rees-like A-algebra from the previous lemma $\mathcal{S} = A[\partial(I)t, It^2] = \bigoplus_{n \geqslant 0} J_n t^n$. As noted in Section 2.1, we have $I \subseteq \partial(I)$, and therefore $J_{n+1} \subseteq J_n$ for all $n \geqslant 0$. If we set $\mathcal{U} := \mathcal{S}[t^{-1}]$, this implies that $\mathcal{U} = \bigoplus_{n \in \mathbb{Z}} J_n t^n$ with $J_n = A$ for $n < 0$. On the quotient field $Q(\mathcal{U}) = Q(A)(t)$, for each $i = 1, \ldots, d$, we consider again the derivation

$$D_i := \frac{1}{t}\partial_i = \frac{1}{t}\frac{\partial}{\partial x_i} : Q(A)(t) \to Q(A)(t).$$

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We claim that $D_i(\mathcal{U}) \subseteq \mathcal{U}$ for every $i = 1, \ldots, d$. Indeed, for $n \geqslant 1$,

$$D_i(J_{2n}t^{2n}) = \frac{1}{t}t^{2n}\partial_i(I^n) \subseteq t^{2n-1}I^{n-1}\partial(I) = J_{2n-1}t^{2n-1}.$$

Similarly, for $n \geqslant 1$,

$$\begin{aligned} D_i(J_{2n+1}t^{2n+1}) &= \frac{1}{t}t^{2n+1}\partial_i(\partial(I)I^n) \subseteq t^{2n}[\partial_i(\partial(I))I^n + \partial_i(I^n)\partial(I)] \\ &\subseteq t^{2n}(I^n + I^{n-1}\partial(I)^2) \subseteq I^n t^{2n} = J_{2n}t^{2n}. \end{aligned}$$

Since we also clearly have $D_i(J_n t^n) \subseteq J_{n-1}t^{n-1}$ for $n \leqslant 1$, it follows that $D_i(\mathcal{U}) \subseteq \mathcal{U}$. By Seidenberg's theorem applied to the noetherian A-algebra $\mathcal{U}$, we then obtain $D_i(\overline{\mathcal{U}}) \subseteq \overline{\mathcal{U}}$ for all i. On the other hand, since A is a normal domain, we have $\overline{\mathcal{U}} = \bigoplus_{n\in\mathbb{Z}} L_n t^n \subseteq A[t, t^{-1}]$ with L_n ideals of A and $L_n = A$ for $n \leqslant 0$. Therefore

$$D_i(L_n t^n) \subseteq L_{n-1}t^{n-1} \quad \text{for all } n \in \mathbb{Z}. \tag{3.4.1}$$

We now claim that $L_{2n} = \overline{I^n}$ for $n \geqslant 1$. First, if $f \in \overline{I^n}$, then $ft^{2n} \in \overline{I^n}t^{2n}$, so ft^{2n} belongs to the integral closure of $A[It^2]$ in $A[t^2]$. Then ft^{2n} is also integral over $\mathcal{U} = A[\partial(I)t, It^2, t^{-1}]$, hence $f \in L_{2n}$. For the opposite inclusion, let $f \in L_{2n}$. As ft^{2n} belongs to $\overline{\mathcal{U}}$, it satisfies an equation of integral dependence $(ft^{2n})^k + a_1(ft^{2n})^{k-1} + \cdots + a_{k-1}(ft^{2n}) + a_k = 0$ with $a_1, \ldots, a_k \in \mathcal{U}$. By considering the homogeneous part of t-degree $2nk$ in this equation, we may assume that $a_j \in I^{nj}t^{2nj}$ for $j = 1, \ldots, k$. If we write $a_j = b_j t^{2nj}$ with $b_j \in I^{nj}$ we then obtain

$$f^k + b_1 f^{k-1} + \cdots + b_{k-1}f + b_k = 0, \tag{3.4.2}$$

showing that $f \in \overline{I^n}$.

From (3.4.1) we now obtain $\partial_i(\overline{I^n})t^{2n-1} = D_i(L_{2n}t^{2n}) \subseteq L_{2n-1}t^{2n-1}$, that is, $\partial_i(\overline{I^n}) \subseteq L_{2n-1}$ for $n \geqslant 1$. Since this holds for every $i = 1, \ldots, d$, it follows that $\partial(\overline{I^n}) \subseteq L_{2n-1}$. Therefore, for $m, n \geqslant 1$, we have

$$\partial(\overline{I^m})\partial(\overline{I^n}) \subseteq L_{2m-1}L_{2n-1} \subseteq L_{2m+2n-2} = \overline{I^{m+n-1}},$$

where the second inclusion follows from the t-graded structure of $\overline{\mathcal{U}}$. ∎

COROLLARY 3.5. *Let I be a homogeneous ideal of A. If I is strongly Golod, then the integral closure $\overline{I}$ is strongly Golod, too.*

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Proof. From the previous theorem, $\partial(\overline{I})\partial(\overline{I}) \subseteq \overline{I}$, so $\overline{I}$ is strongly Golod. □

Note that the above result also recovers Corollary 3.2, for the power I^n is strongly Golod for every $n \geqslant 2$ [3, Theorem 2.3].

REMARK 3.6. If one removes the homogeneous requirement for a strongly Golod ideal in Definition 2.3, then the conclusions of Corollary 3.2, Theorem 3.4 and Corollary 3.5 are valid without requiring that I be homogeneous.

§4. Rational powers of ideals

As before, K is a field of characteristic zero and $A = K[x_1, \ldots, x_d]$ is a polynomial ring with $\deg(x_i) = a_i > 0$.

The strongly Golod property for rational powers of ideals in A was studied by De Stefani in [1]. If I is a monomial ideal and $\alpha \in \mathbb{Q}$, De Stefani proved that the rational power I_α is strongly Golod for $\alpha \geqslant 2$. Moreover, if I is a monomial strongly Golod ideal, then I_α is strongly Golod for $\alpha \geqslant 1$. Using techniques similar to those employed in the previous section, we extend these results to homogeneous ideals.

We begin with a brief overview of the rational powers of ideals.

DEFINITION 4.1. Let I be an ideal in a noetherian ring R and $\alpha = p/q \in \mathbb{Q}$ with p, q positive integers. The αth rational power of the ideal I is defined by $I_\alpha := \{x \in R \mid x^q \in \overline{I^p}\}$.

The definition is independent of the representation of α as a quotient of integers. Moreover, I_α is an integrally closed ideal of R for every positive rational α. (If m is a positive integer, the mth rational power I_m of I is the integral closure $\overline{I^m}$.) For every positive rationals α, β, we have $I_\alpha I_\beta \subseteq I_{\alpha+\beta}$, and if $\alpha \geqslant \beta$, then $I_\alpha \subseteq I_\beta$. For these facts and other properties of the rational powers of ideals we refer the reader to [4, 10.5].

REMARK 4.2. Like the integral closures $\overline{I^m}$ $(m \geqslant 1)$, the rational powers can be obtained as homogeneous components of the integral closure of a Rees-like algebra. Let I be an ideal in a noetherian ring R and $a \geqslant 1$ an integer. Let $T = R[It^a, t^{-1}]$ and let $\overline{T} = \bigoplus_{n\in\mathbb{Z}} J_n t^n$ be the integral closure of T in $R[t, t^{-1}]$. Then $J_n = I_{n/a}$ for $n \geqslant 1$.

To see this we first note that $J_{na} = \overline{I^n}$ for all integers $n \geqslant 1$. Indeed, if $f \in \overline{I^n}$, then $ft^{na} \in \overline{I^n}t^{na}$, so ft^{na} belongs to the integral closure of $R[It^a]$ in $R[t^a]$. Then ft^{na} is integral over $T = R[It^a, t^{-1}]$, hence $f \in J_{na}$.

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For the other inclusion, let $f \in J_{na}$. Since $ft^{na} \in \overline{T}$, we have an equation of integral dependence $(ft^{na})^k + a_1(ft^{na})^{k-1} + \cdots + a_{k-1}(ft^{na}) + a_k = 0$ with $a_1, \ldots, a_k \in T$. Moreover, by taking the homogeneous part of t-degree nak in this equation, we may assume that $a_j \in I^{nj}t^{naj}$ for $j = 1, \ldots, k$, and if we write $a_j = b_j t^{naj}$ with $b_j \in I^{nj}$, we obtain an equation of integral dependence of f over I^n.

For the general case, if $f \in J_n$, then $f^a \in J_{na} = \overline{I^n}$, so $f \in I_{n/a}$. Conversely, if $f \in I_{n/a}$, then $f^a \in \overline{I^n}$, so $(ft^n)^a \in \overline{I^n}t^{na} = J_{na}t^{na} \subseteq \overline{T}$. This implies that ft^n is also integral over T, so $f \in J_n$.

Theorem 4.3. *Let I be an arbitrary ideal of A and $\alpha \in \mathbb{Q}$ with $\alpha \geqslant 1$. Then*

$$\partial(I_\alpha) \subseteq I_{\alpha-1}.$$

Proof. Write $\alpha = N/a$ with a, N positive integers and consider the A-algebra $T = A[It^a, t^{-1}]$. Since A is a normal domain, the integral closure of T in its quotient field $Q(A)(t)$ coincides with the integral closure $\overline{T} = \bigoplus_{n\in\mathbb{Z}} J_n t^n$ of T in $A[t, t^{-1}]$. For every $i \in \{1, \ldots, d\}$, let

$$D_i := \frac{1}{t^a}\partial_i = \frac{1}{t^a}\frac{\partial}{\partial x_i} : Q(A)(t) \to Q(A)(t).$$

This map is a derivation, and since $\partial_i(I^n) \subseteq I^{n-1}$, we have $D_i(I^n t^{na}) \subseteq I^{n-1}t^{(n-1)a}$ for all n, that is, $D_i(T) \subseteq T$. By Seidenberg's theorem, we then obtain $D_i(\overline{T}) \subseteq \overline{T}$, or equivalently, $\partial_i(J_n) \subseteq J_{n-a}$ for every integer n. Therefore $\partial(J_n) \subseteq J_{n-a}$ for all integers n. However, as explained in Remark 4.2, $J_N = I_{N/a} = I_\alpha$ and $J_{N-a} = I_{(N-a)/a} = I_{\alpha-1}$, so $\partial(I_\alpha) \subseteq I_{\alpha-1}$. ∎

Corollary 4.4. *Let I be a homogeneous ideal of A. Then I_α is strongly Golod for every $\alpha \in \mathbb{Q}$ with $\alpha \geqslant 2$.*

Proof. From the previous theorem, $\partial(I_\alpha)^2 \subseteq I_{\alpha-1}I_{\alpha-1} \subseteq I_{2\alpha-2} \subseteq I_\alpha$ for $\alpha \geqslant 2$. ∎

If the ideal I is strongly Golod, the inclusion proved in Theorem 4.3 can be improved.

Theorem 4.5. *Let I be a homogeneous ideal of A. If I is strongly Golod, then for every $\alpha \in \mathbb{Q}$ with $\alpha \geqslant \frac{1}{2}$,*

$$\partial(I_\alpha) \subseteq I_{\alpha-(1/2)}$$

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Proof. Write $\alpha = N/2a$ with N, a positive integers. (It will become clear later that it is important to represent α with an even denominator.) Let $\mathcal{C} = A[\partial(I)t^a, It^{2a}] = \bigoplus_{n\geqslant 0} C_{na}t^{na}$. From Lemma 3.3, it follows that $C_{2na} = I^n$ and $C_{(2n+1)a} = \partial(I)I^n$ for $n \geqslant 0$. Moreover, as noted in Section 2.1, $I \subseteq \partial(I)$, so $C_{(n+1)a} \subseteq C_{na}$ for every $n \geqslant 0$.

Now let

$$\mathcal{V} = \mathcal{C}[t^{-1}] = A[\partial(I)t^a, It^{2a}, t^{-1}] = \left(\bigoplus_{n\geqslant 0} C_{na}t^{na}\right)[t^{-1}].$$

From the description of the graded components C_{na} given above and the fact that $C_{(n+1)a} \subseteq C_{na}$ for every $n \geqslant 0$, it follows that

$$\begin{aligned}\mathcal{V} = A[t^{-1}] &\oplus \partial(I)t \oplus \partial(I)t^2 \oplus \cdots \oplus \partial(I)t^a \\ &\oplus It^{a+1} \oplus It^{a+2} \oplus \cdots \oplus It^{2a} \\ &\oplus I\partial(I)t^{2a+1} \oplus I\partial(I)t^{2a+2} \oplus \cdots \oplus I\partial(I)t^{3a} \\ &\oplus I^2t^{3a+1} \oplus I^2t^{3a+2} \oplus \cdots \oplus I^2t^{4a} \\ &\oplus I^2\partial(I)t^{4a+1} \oplus I^2\partial(I)t^{4a+2} \oplus \cdots \oplus I^2\partial(I)t^{5a} \\ &\oplus \cdots\end{aligned}$$

More precisely, if $\mathcal{V} = \bigoplus_{n\in\mathbb{Z}} V_nt^n$ (with $V_n = A$ for $n \leqslant 0$), for each $n \geqslant 1$ let $b = \lfloor (n-1)/a \rfloor$ so that we can write $n = ab + r$ with $b, r \in \mathbb{N}$ and $1 \leqslant r \leqslant a$. Then $V_n = I^{(b+1)/2}$ if b is odd and $V_n = I^{b/2}\partial(I)$ if b is even.

We now consider $\overline{\mathcal{V}}$ the integral closure of $\mathcal{V}$ in its quotient field $Q(A)(t)$. Since A is integrally closed, $\overline{\mathcal{V}} = \bigoplus_{n\in\mathbb{Z}} W_nt^n$ is a graded subalgebra of $A[t, t^{-1}]$ with $W_n = A$ for $n \leqslant 0$.

We first claim that $W_{2na} = \overline{I^n}$ for all $n \geqslant 1$. Indeed, for $f \in \overline{I^n}$ we have $ft^{2an} \in \overline{I^n}t^{2an}$, so ft^{2an} belongs to the integral closure of $A[It^{2a}]$ in $A[t^{2a}]$. Then ft^{2an} is also integral over $\mathcal{V}$, so $f \in W_{2an}$. Conversely, if $f \in W_{2an}$, then ft^{2an} satisfies an equation of integral dependence over $\mathcal{V}$, $(ft^{2na})^k + a_1(ft^{2na})^{k-1} + \cdots + a_{k-1}(ft^{2na}) + a_k = 0$, with $a_1, \ldots, a_k \in \mathcal{V}$. By considering the homogeneous part of t-degree $2nak$ in this equation, we may assume that $a_j \in I^{nj}t^{2naj}$ for $j = 1, \ldots, k$. This becomes an equation of integral dependence over $A[It^{2a}]$, so $f \in \overline{I^n}$.

We now claim that $W_n = I_{n/2a}$ for all $n \geqslant 1$. For $f \in I_{n/2a}$ we have $f^{2a}t^{2na} \in \overline{I^n}t^{2na} = W_{2na}t^{2na} \subseteq \overline{\mathcal{V}}$, so $(ft^n)^{2a}$ is integral over $\mathcal{V}$. Then ft^n is integral over $\mathcal{V}$ as well, so $f \in W_n$. Conversely, if $f \in W_n$, then $f^{2a} \in W_{2na} = \overline{I^n}$, so $f \in I_{n/2a}$.

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For $i = 1, \ldots, d$, consider again the derivation

$$D_i := \frac{1}{t^a}\partial_i = \frac{1}{t^a}\frac{\partial}{\partial x_i} : Q(A)(t) \to Q(A)(t).$$

We now show that $D_i(\mathcal{V}) \subseteq \mathcal{V}$, or equivalently, $\partial_i(V_n) \subseteq V_{n-a}$ for all integers n. This is clear for $n \leqslant a$, so we may assume $n \geqslant a+1$. If $n = (2k+1)a + r$ with integers $k \geqslant 0$ and $1 \leqslant r \leqslant a$, then $V_n = I^{k+1}$, $V_{n-a} = I^k\partial(I)$, and $\partial_i(I^{k+1}) \subseteq I^k\partial(I)$. In the other case, if $n = 2ka + r$ with integers $k \geqslant 1$ and $1 \leqslant r \leqslant a$, we have $V_n = I^k\partial(I)$, $V_{n-a} = I^k$, and $\partial_i(I^k\partial(I)) \subseteq I^k\partial(\partial(I)) + I^{k-1}\partial(I)\partial(I) \subseteq I^k$ since I is strongly Golod.

We can now apply Seidenberg's theorem. From $D_i(\overline{\mathcal{V}}) \subseteq \overline{\mathcal{V}}$ we obtain $D_i(W_n t^n) \subseteq W_{n-a}t^{n-a}$ for all n, or equivalently, $\partial_i(W_n) \subseteq W_{n-a}$. On the other hand, we know that $W_N = I_{N/2a} = I_\alpha$, so $\partial_i(I_\alpha) = \partial_i(I_{N/2a}) \subseteq I_{(N-a)/2a} = I_{\alpha-(1/2)}$. As this holds for every i, we obtain $\partial(I_\alpha) \subseteq I_{\alpha-(1/2)}$. □

Corollary 4.6. *Let I be a homogeneous ideal of A. If I is strongly Golod, then I_α is strongly Golod for every $\alpha \in \mathbb{Q}$ with $\alpha \geqslant 1$.*

Proof. By the previous theorem, $\partial(I_\alpha)^2 \subseteq I_{\alpha-(1/2)}I_{\alpha-(1/2)} \subseteq I_{2\alpha-1} \subseteq I_\alpha$ for $\alpha \geqslant 1$. □

Remark 4.7. In analogy to what we noted in Remark 3.6, if one removes the homogeneous requirement in the definition of a strongly Golod ideal, the conclusions of Corollary 4.4, Theorem 4.5 and Corollary 4.6 hold without requiring that I be homogeneous.

§5. Coefficient ideals

Let M denote the maximal homogeneous ideal of the polynomial ring $A = K[x_1, \ldots, x_d]$ with $\deg x_i = a_i$ and K an infinite field. If I is an M-primary ideal, for $n \gg 0$ the length $\lambda(A/I^n)$ becomes a polynomial function $P_I(n)$ of degree d, the Hilbert–Samuel polynomial of I. We write this polynomial

$$P_I(n) = \mathrm{e}_0(I)\binom{n+d-1}{d} - \mathrm{e}_1(I)\binom{n+d-2}{d-1} + \cdots + (-1)^d\,\mathrm{e}_d(I)$$

with integer coefficients $\mathrm{e}_i(I)$, subsequently referred to as the Hilbert coefficients of I. The coefficient $\mathrm{e}_0(I)$ (the multiplicity of I) is positive and a well-known result of Rees shows that the integral closure $\overline{I}$ is the largest ideal containing I having the same coefficient e_0. Similarly, the Ratliff–Rush

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ideal $\tilde{I} = \bigcup_{i \geqslant 0}(I^{i+1} : I^i)$ is the unique largest ideal containing I having the same Hilbert coefficients $\mathrm{e}_0, \mathrm{e}_1, \ldots, \mathrm{e}_d$. In fact, Shah [8] proved that there exists a chain of ideals

$$I \subseteq \tilde{I} = I_{\{d\}} \subseteq \cdots \subseteq I_{\{k\}} \subseteq \cdots \subseteq I_{\{0\}} = \overline{I},$$

where $I_{\{k\}}$ is the unique ideal containing I maximal with respect to the property of having the same coefficients $\mathrm{e}_0, \ldots, \mathrm{e}_k$ as those of I. The ideal $I_{\{k\}}$ is called the kth coefficient ideal of I.

REMARK 5.1. Shah proved the existence of this chain for an ideal primary to the maximal ideal in a formally equidimensional local ring with infinite residue field. However, since $\lambda(A/I^n) = \lambda(A_M/I^n A_M)$, Shah's result is valid in the polynomial setting as well and $I_{\{k\}} = (IA_M)_{\{k\}} \cap A$.

REMARK 5.2. Coefficient ideals in polynomial rings have been studied by Heinzer and Lantz in [2]. They observed that if I is monomial, then its coefficient ideals are monomial, too [2, Observation 3.3]. In fact, with a modification of their of argument, one can prove that if I is a homogeneous ideal, then all the coefficient ideals of I are homogeneous as well. To see this, for each $u \in K \setminus \{0\}$, let $\phi_u : A \to A$ be the K-algebra automorphism of A defined by $\phi_u(x_i) = u^{a_i} x_i$ for $i = 1, \ldots, d$. If K is infinite, it is known that I is a homogeneous ideal of A if and only if $\phi_u(I) = I$ for every $u \in K \setminus \{0\}$; see [5, Exercise 13.1] for a more general statement. With this automorphism, the argument detailed in [2, Observation 3.3] shows that $\phi_u(I_{\{k\}}) = I_{\{k\}}$ for every $u \in K \setminus \{0\}$ and $k \geqslant 1$, and therefore the coefficient ideals $I_{\{k\}}$ are homogeneous for $k \geqslant 1$. For $k = 0$, it is also well known that the integral closure $\overline{I} = I_{\{0\}}$ is homogeneous.

We now consider a homogeneous strongly Golod ideal I in the polynomial ring $A = K[x_1, \ldots, x_d]$ over a field K of characteristic zero. Herzog and Huneke proved that the Ratliff–Rush ideal $\tilde{I}$ is strongly Golod [3, Proposition 2.12]. Moreover, we proved in Corollary 3.5 that the integral closure $\overline{I}$ is also strongly Golod. Assuming that I is M-primary, we prove that all the other coefficient ideals of I are strongly Golod as well, answering a question raised in [1, Question 5.9].

5.3. We first recall the following structure theorem for coefficient ideals [8, Theorem 2]. If I is an ideal primary to the maximal ideal of a formally equidimensional local ring with infinite residue field and $1 \leqslant k \leqslant d$, then

$$I_{\{k\}} = \bigcup (I^{n+1} : (a_1, \ldots, a_k)), \tag{5.3.1}$$

where the union is taken over all $n \geqslant 1$ and all length k sequences $a_1, \ldots, a_k$ extendable to some minimal reduction of I^n. Moreover, this union can be replaced by a single quotient ideal [8, Theorem 3], that is, there exist $m \geqslant 1$ and $a_1, \ldots, a_k \in I^m$ extendable to some minimal reduction of I^m such that

$$I_{\{k\}} = (I^{m+1} : (a_1, \ldots, a_k)). \tag{5.3.2}$$

Note that this description is only valid for $k \geqslant 1$, hence excluding the integral closure of I.

In the polynomial setting with I primary to the maximal homogeneous ideal M of $A = K[x_1, \ldots, x_d]$, after localizing at M as in Remark 5.1, this means that there exist $m \geqslant 1$ and $a_1, \ldots, a_k \in I^m$ extendable to some minimal reduction of $I^m A_M$ such that (5.3.2) holds.

THEOREM 5.4. *Let I be an M-primary strongly Golod ideal of a polynomial ring $A = K[x_1, \ldots, x_d]$ over a field K of characteristic zero. Then the coefficient ideals $I_{\{k\}}$ are strongly Golod for all $k \in \{1, \ldots, d\}$.*

Proof. As in (5.3.2), choose $m \geqslant 1$ and $a_1, \ldots, a_k \in I^m$ extendable to some minimal reduction $(a_1, \ldots, a_k, a_{k+1}, \ldots, a_d)$ of $I^m A_M$ such that $I_{\{k\}} = (I^{m+1} : (a_1, \ldots, a_k))$. Let $f \in I_{\{k\}}$ and $s \in \{1, \ldots, k\}$. Since $f a_s \in I^{m+1}$, for every $i \in \{1, \ldots, d\}$ we have $\partial_i(f) a_s + f \partial_i(a_s) \in I^m \partial(I)$. However, since $a_s \in I^m$, we have $\partial_i(a_s) \in I^{m-1}\partial(I)$, and therefore $\partial_i(f) a_s \in f I^{m-1}\partial(I) + I^m \partial(I)$. Since $f a_s \in I^{m+1}$ and $a_s \in I^m$ we then obtain $\partial_i(f) a_s^2 \in I^{2m}\partial(I) + a_s I^m \partial(I) \subseteq I^{2m}\partial(I)$. This holds for every $f \in I_{\{k\}}$ and all $i = 1, \ldots, d$, and therefore $\partial(I_{\{k\}}) a_s^2 \subseteq I^{2m}\partial(I)$. Since I is strongly Golod, this implies that

$$\partial(I_{\{k\}})^2 a_s^4 \subseteq I^{4m}\partial(I)^2 \subseteq I^{4m+1}.$$

As this holds for all $s \in \{1, \ldots, k\}$, we get

$$\partial(I_{\{k\}})^2 \subseteq (I^{4m+1} : (a_1^4, \ldots, a_k^4)).$$

Finally, $a_1^4, \ldots, a_k^4$ is part of the minimal reduction $(a_1^4, \ldots, a_k^4, a_{k+1}^4, \ldots, a_d^4)$ of $I^{4m} A_M$, and from (5.3.1) it follows that $\partial(I_{\{k\}})^2 \subseteq (I A_M)_{\{k\}} \cap A = I_{\{k\}}$, and therefore $I_{\{k\}}$ is strongly Golod. ∎

References

[1] A. De Stefani, *Products of ideals may not be Golod*, J. Pure Appl. Algebra **220**(6) (2016), 2289–2306.

[2] W. Heinzer and D. Lantz, "*Coefficient and stable ideals in polynomial rings*", in *Factorization in Integral Domains (Iowa City, IA, 1996)*, Lecture Notes in Pure and Applied Mathematics **189**, Dekker, New York, 1997, 359–370.

[3] J. Herzog and C. Huneke, *Ordinary and symbolic powers are Golod*, Adv. Math. **246** (2013), 89–99.

[4] C. Huneke and I. Swanson, *Integral Closure of Ideals, Rings, and Modules*, London Mathematical Society Lecture Note Series **336**, Cambridge University Press, Cambridge, 2006.

[5] H. Matsumura, *Commutative Ring Theory*, 2nd ed., Cambridge Studies in Advanced Mathematics **8**, Cambridge University Press, Cambridge, 1989, Translated from the Japanese by M. Reid.

[6] J. McCullough and I. Peeva, "*Infinite graded free resolutions*", in *Commutative Algebra and Noncommutative Algebraic Geometry,* Vol. I, Mathematical Science Research Institute Publications **67**, Cambridge University Press, New York, 2015, 215–257.

[7] A. Seidenberg, *Derivations and integral closure*, Pacific J. Math. **16** (1966), 167–173.

[8] K. Shah, *Coefficient ideals*, Trans. Amer. Math. Soc. **327**(1) (1991), 373–384.

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