

A PALEY–WIENER THEOREM FOR DISTRIBUTIONS ON REDUCTIVE SYMMETRIC SPACES

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Abstract Let $X = G/H$ be a reductive symmetric space and K a maximal compact subgroup of G . We study Fourier transforms of compactly supported K -finite distributions on X and characterize the image of the space of such distributions.

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1. Introduction

The well-known Paley–Wiener–Schwartz theorem for the Fourier transform on Euclidean space \mathbb{R}^n characterizes the Fourier image of the space $C_c^\infty(\mathbb{R}^n)$ of compactly supported smooth functions. The image is the space of entire functions $\varphi \in \mathcal{O}(\mathbb{C}^n)$ with decay of exponential type. The theorem has a counterpart, also well known and also called the Paley–Wiener–Schwartz theorem, where smooth functions are replaced by distributions, and where the exponential decay condition is replaced by a similar exponential condition of slow growth (see [16, Theorem 7.3.1]). The theorem for smooth functions was generalized to the Fourier transform of a reductive symmetric space G/H in [9]. It is the purpose of the present paper to establish an analogue of the theorem for distributions in the same spirit and generality.

In the more restricted case of a Riemannian symmetric space G/K , where $H = K$ is compact, a Paley–Wiener theorem for K -invariant smooth functions was obtained by work of Helgason and Gangolli [13, 14], and for general smooth functions by Helgason [15]. A counterpart for distributions was given by Eguchi, Hashizume and Okamoto in [12]. A different proof of the latter result is given in [11].

Another important special case is that of a reductive Lie group, considered as a symmetric space. In this case the Paley–Wiener theorem of [9] specializes to a theorem of Arthur [1], which describes the Fourier image of the space of compactly supported K -finite smooth functions on G (K being a maximal compact subgroup). The theorem

for distributions, which is obtained in the present paper, is new in this ‘group case’. The specialization to the group case is described in [8], to which we refer for further details.

The Paley–Wiener theorem of [9] describes the Fourier image of the space of K -finite compactly supported smooth functions by an exponential type condition, combined with a set of so-called Arthur–Campoli conditions. In the present theorem the exponential type condition is replaced by a condition of slow growth which is similar to its Euclidean analogue, whereas the additional Arthur–Campoli conditions remain the same as in [9]. The precise statement of our main result is given in Theorem 4.6, and its proof is given in §§ 5–13. The main tool in the proof is a Fourier inversion formula, through which a function is determined from its Fourier transform by means of certain ‘residual’ operators. Given a function φ in the conjectured image space, we construct the distribution f , which is the candidate for the inverse Fourier image, by means of this formula. The proof that f has compact support and transforms to φ is carried out by regularization with a Dirac sequence. The mentioned inversion formula is generalized to distributions in Corollary 13.2, after the proof of Theorem 4.6. Finally, in §§ 14–15 we discuss the topology on the image space by which the Fourier transform becomes a topological isomorphism.

For general background about harmonic analysis and Paley–Wiener theorems on reductive symmetric spaces we refer to the survey articles [2, 19].

2. Notation

As in [9] we use the notation and basic assumptions from [3, §§ 2, 3, 5, 6] and [4, § 2]. Only the most essential notions will be recalled.

Let G be a real reductive Lie group of Harish-Chandra’s class, and let H be an open subgroup of the group of fixed points for an involution σ . Then $X = G/H$ is a reductive symmetric space. Let K be a maximal compact subgroup of G , invariant under σ , and let θ denote the corresponding Cartan involution. Let \mathfrak{g} denote the Lie algebra of G , which decomposes in ± 1 eigenspaces for σ and θ as $\mathfrak{g} = \mathfrak{h} + \mathfrak{q} = \mathfrak{k} + \mathfrak{p}$. Then \mathfrak{h} and \mathfrak{k} are the Lie algebras of H and K . Let $\mathfrak{a}_{\mathfrak{q}}$ be a maximal abelian subspace of $\mathfrak{q} \cap \mathfrak{p}$, and choose a positive system Σ^+ for the root system Σ of $\mathfrak{a}_{\mathfrak{q}}$ in \mathfrak{g} . This positive system determines a parabolic subgroup P of G , which will be fixed throughout the paper. We also fix a finite-dimensional unitary representation (τ, V_{τ}) of K . The normalized Eisenstein integrals associated with these choices are denoted by $E^{\circ}(\psi : \lambda) : X \rightarrow V_{\tau}$, where $\psi \in {}^{\circ}\mathcal{C}$ and $\lambda \in \mathfrak{a}_{\mathfrak{q}\mathbb{C}}^*$, as in [3, p. 283]. Here ${}^{\circ}\mathcal{C} = {}^{\circ}\mathcal{C}(\tau)$ is the finite-dimensional Hilbert space defined in [3, equation (5.1)]. The Eisenstein integrals depend linearly on the parameter ψ in this space, and as functions on X they belong to the space $C^{\infty}(X : \tau)$ of smooth V_{τ} -valued functions on X which are τ -spherical, that is, which satisfy the transformation rule

$$f(kx) = \tau(k)f(x), \quad k \in K, x \in X. \tag{2.1}$$

The adjoint of the linear map $\psi \mapsto E^{\circ}(\psi : -\bar{\lambda} : x)$ is denoted by $E^*(\lambda : x)$ (see [4, equation (2.3)]), and the *Fourier transform* for the K -type τ on G/H is then defined by

$$\mathcal{F}f(\lambda) = \int_X E^*(\lambda : x)f(x) dx \in {}^{\circ}\mathcal{C} \tag{2.2}$$

for $\lambda \in \mathfrak{a}_{\mathbb{Q}\mathbb{C}}^*$ and for f in the space $C_c^\infty(X : \tau)$ of compactly supported functions in $C^\infty(X : \tau)$ (see [9, equation (2.1)]). Here dx is an invariant measure on G/H , normalized as in [3, § 3].

The normalized Eisenstein integrals $E^\circ(\psi : \lambda : x)$ depend meromorphically on the parameter $\lambda \in \mathfrak{a}_{\mathbb{Q}\mathbb{C}}^*$, in a uniform way with respect to the parameters ψ and x . The nature of this meromorphic dependence is crucial. It can be described as follows. By a real Σ -configuration in $\mathfrak{a}_{\mathbb{Q}\mathbb{C}}^*$ we mean a locally finite collection \mathcal{H} of affine hyperplanes Y in $\mathfrak{a}_{\mathbb{Q}\mathbb{C}}^*$ of the form $Y = \{\lambda \mid \langle \lambda, \alpha_Y \rangle = s_Y\}$, where $\alpha_Y \in \Sigma$ and $s_Y \in \mathbb{R}$. Let $d : \mathcal{H} \rightarrow \mathbb{N}$ be an arbitrary map. For $\omega \subset \mathfrak{a}_{\mathbb{Q}\mathbb{C}}^*$ we write

$$\mathcal{H}(\omega) = \{Y \in \mathcal{H} \mid Y \cap \omega \neq \emptyset\}$$

and, if the set $\mathcal{H}(\omega)$ is finite,

$$\pi_{\omega,d}(\lambda) = \prod_{Y \in \mathcal{H}(\omega)} (\langle \lambda, \alpha_Y \rangle - s_Y)^{d(Y)}.$$

Let V be an arbitrary complete locally convex vector space. The linear space of meromorphic functions $\varphi : \mathfrak{a}_{\mathbb{Q}\mathbb{C}}^* \rightarrow V$, such that $\pi_{\omega,d}\varphi$ is holomorphic on ω for all bounded open sets $\omega \subset \mathfrak{a}_{\mathbb{Q}\mathbb{C}}^*$, is denoted $\mathcal{M}(\mathfrak{a}_{\mathbb{Q}\mathbb{C}}^*, \mathcal{H}, d, V)$. It follows from [9, Lemma 2.1] and [4, Proposition 3.1] that there exist a real Σ -configuration \mathcal{H} and a map $d : \mathcal{H} \rightarrow \mathbb{N}$ such that the normalized Eisenstein integrals $\lambda \mapsto E^\circ(\psi : \lambda)$ belong to $\mathcal{M}(\mathfrak{a}_{\mathbb{Q}\mathbb{C}}^*, \mathcal{H}, d, C^\infty(X) \otimes V_\tau)$ for all $\psi \in {}^\circ\mathcal{C}$.

Clearly, the dualized Eisenstein integrals $E^*(\lambda : x)$ have the same type of meromorphic dependence on λ . We define $\mathcal{H} = \mathcal{H}(X, \tau)$ and $d = d_{X,\tau} : \mathcal{H} \rightarrow \mathbb{N}$ as in [9, § 2]. Then \mathcal{H} is a real Σ -configuration, and the map $E^* : \lambda \mapsto E^*(\lambda)$ satisfies

$$E^* \in \mathcal{M}(\mathfrak{a}_{\mathbb{Q}\mathbb{C}}^*, \mathcal{H}, d, C^\infty(X) \otimes V_\tau^* \otimes {}^\circ\mathcal{C}). \tag{2.3}$$

Moreover, $\mathcal{H}(X, \tau)$ and $d_{X,\tau}$ are minimal with respect to this property.

3. The Fourier transform of a distribution

The concept of ‘distributions’ used in this paper is that of *generalized functions*. By this we mean the following. A generalized function on a smooth manifold X is a continuous linear form on the space of compactly supported smooth densities on X . We denote by $C^{-\infty}(X)$ the space of generalized functions on X , and by $C_c^{-\infty}(X)$ the subspace of generalized functions with compact support. If a nowhere vanishing smooth density dx is given on X , then the multiplication with dx induces linear isomorphisms of the spaces $C^{-\infty}(X)$ and $C_c^{-\infty}(X)$ onto the topological linear duals of $C_c^\infty(X)$ and $C^\infty(X)$, respectively. If $f \in C^{-\infty}(X)$ and $\phi \in C_c^\infty(X)$, or if $f \in C_c^{-\infty}(X)$ and $\phi \in C^\infty(X)$, then we write, accordingly,

$$\int_X \phi(x) f(x) dx = f dx(\phi). \tag{3.1}$$

Let $X = G/H$, as in § 2, be equipped with the invariant measure dx . Then dx is a nowhere vanishing smooth density. For $f \in C^{-\infty}(X)$ the continuity of $f dx$, as a linear

form on $C_c^\infty(X)$, can be expressed as follows. Let X_1, \dots, X_n be a linear basis for \mathfrak{g} , and for $\alpha = (\alpha_1, \dots, \alpha_n)$ a multi-index let $X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n} \in \mathcal{U}(\mathfrak{g})$. Then for each compact $\Omega \subset X$ there exist constants C, k such that

$$\left| \int_X \phi(x)f(x) dx \right| \leq C \sup_{|\alpha| \leq k, x \in \Omega} |L_{X^\alpha} \phi(x)| \tag{3.2}$$

for all $\phi \in C^\infty(X)$ with support in Ω .

For each positive number M we denote by $C_M^{-\infty}(X)$ the space of generalized functions with support in the compact set $K \exp B_M H$. Here B_M is the closed ball in \mathfrak{a}_q centred at 0 and of radius M . In view of the generalized Cartan decomposition $G = KA_qH$ we have $C_c^{-\infty}(X) = \bigcup_M C_M^{-\infty}(X)$.

A generalized function on X with values in V_τ is called τ -spherical if it satisfies (2.1). We denote by $C^{-\infty}(X : \tau)$ the space of τ -spherical generalized functions on X , and by $C_c^{-\infty}(X : \tau)$ and $C_M^{-\infty}(X : \tau)$ the subspaces of τ -spherical distributions with compact support, respectively with support in $K \exp B_M H$.

If $f \in C^{-\infty}(X : \tau)$ and $\phi \in C_c^\infty(X) \otimes V_\tau^*$, or if $f \in C_c^{-\infty}(X : \tau)$ and $\phi \in C^\infty(X) \otimes V_\tau^*$, then equation (3.1) still has a natural interpretation. Via this pairing (3.1) we have thus established linear isomorphisms of $C^{-\infty}(X : \tau)$ and $C_c^{-\infty}(X : \tau)$ with the topological linear duals of $C_c^\infty(X : \tau^*)$ and $C^\infty(X : \tau^*)$, respectively.

Having established (3.1) in this generality we can define the Fourier transform $\mathcal{F}f(\lambda)$ for $f \in C_c^{-\infty}(X : \tau)$ and $\lambda \in \mathfrak{a}_{q\mathbb{C}}^*$ by the very same formula (2.2) by which it was defined for $f \in C_c^\infty(X : \tau)$. The Fourier transform $\mathcal{F}f(\lambda)$ is a \mathcal{C} -valued meromorphic function of λ , and it follows from (2.3) that

$$\mathcal{F}f \in \mathcal{M}(\mathfrak{a}_q^*, \mathcal{H}(X, \tau), d_{X, \tau}, \mathcal{C}).$$

4. The distributional Paley–Wiener space

Recall the following definitions from [9].

Definition 4.1. Let \mathcal{H} be a real Σ -configuration in $\mathfrak{a}_{q\mathbb{C}}^*$, and let $d \in \mathbb{N}^{\mathcal{H}}$. By $\mathcal{P}(\mathfrak{a}_q^*, \mathcal{H}, d)$ we denote the linear space of functions $\varphi \in \mathcal{M}(\mathfrak{a}_q^*, \mathcal{H}, d)$ with polynomial decay in the imaginary directions, that is

$$\nu_{\omega, n}(\varphi) := \sup_{\lambda \in \omega + i\mathfrak{a}_q^*} (1 + |\lambda|)^n \|\pi_{\omega, d}(\lambda)\varphi(\lambda)\| < \infty \tag{4.1}$$

for all compact $\omega \subset \mathfrak{a}_q^*$ and all $n \in \mathbb{N}$. The union of these spaces over all $d : \mathcal{H} \rightarrow \mathbb{N}$ is denoted $\mathcal{P}(\mathfrak{a}_q^*, \mathcal{H})$.

The space $\mathcal{P}(\mathfrak{a}_q^*, \mathcal{H}, d)$ is a Fréchet space with the topology defined by means of the seminorms $\nu_{\omega, n}$ in (4.1).

We recall the following result from [9, Lemma 3.7].

Lemma 4.2. *The Fourier transform is continuous:*

$$\mathcal{F} : C_c^\infty(X : \tau) \rightarrow \mathcal{P}(\mathfrak{a}_q^*, \mathcal{H}(X, \tau), d_{X, \tau}) \otimes \mathcal{C}.$$

For $R \in \mathbb{R}$ we define $\bar{\mathfrak{a}}_q^*(P, R) = \{\lambda \in \mathfrak{a}_{q\mathbb{C}}^* \mid \forall \alpha \in \Sigma^+ : \operatorname{Re}\langle \lambda, \alpha \rangle < R\}$.

Definition 4.3. Let $\mathcal{H} = \mathcal{H}(X, \tau)$ and $d = d_{X, \tau}$. Let $\pi = \pi_{\bar{\mathfrak{a}}_q^*(P, 0), d}$. For each $M > 0$ we define $\text{PW}_M(X : \tau)$ as the space of functions $\varphi \in \mathcal{P}(\mathfrak{a}_q^*, \mathcal{H}, d) \otimes {}^\circ\mathcal{C}$ for which

- (i) $\mathcal{L}\varphi = 0$ for all $\mathcal{L} \in \text{AC}_{\mathbb{R}}(X : \tau)$,
- (ii) $\sup_{\lambda \in \bar{\mathfrak{a}}_q^*(P, 0)} (1 + |\lambda|)^n e^{-M|\operatorname{Re} \lambda|} \|\pi(\lambda)\varphi(\lambda)\| < \infty$ for each $n \in \mathbb{N}$

(see [9, Definition 3.1] for the definition of $\text{AC}_{\mathbb{R}}(X : \tau)$). Furthermore, the *Paley–Wiener space* $\text{PW}(X : \tau)$ is defined as

$$\text{PW}(X : \tau) = \bigcup_{M > 0} \text{PW}_M(X : \tau).$$

The main result of [9], Theorem 3.6, asserts that the Fourier transform is a linear isomorphism of $C_M^\infty(X : \tau)$ onto $\text{PW}_M(X : \tau)$ for each $M > 0$, and hence also of $C_c^\infty(X : \tau)$ onto $\text{PW}(X : \tau)$.

We now introduce the following definitions. If Ω is a topological space, we denote by $\mathcal{C}(\Omega)$ the collection of compact subsets $\omega \subset \Omega$, and by $\mathcal{N}(\Omega)$ the set of maps $n : \mathcal{C}(\Omega) \rightarrow \mathbb{N}$.

Definition 4.4. Let \mathcal{H} be a real Σ -configuration in $\mathfrak{a}_{q\mathbb{C}}^*$, and let $d \in \mathbb{N}^{\mathcal{H}}$, $n \in \mathcal{N}(\mathfrak{a}_q^*)$. By $\mathcal{P}^*(\mathfrak{a}_q^*, \mathcal{H}, d, n)$ we denote the linear space of functions $\varphi \in \mathcal{M}(\mathfrak{a}_q^*, \mathcal{H}, d)$ with at most polynomial growth of order n in the imaginary directions, that is

$$\nu_{\omega, n}^*(\varphi) := \sup_{\lambda \in \omega + i\mathfrak{a}_q^*} (1 + |\lambda|)^{-n(\omega)} \|\pi_{\omega, d}(\lambda)\varphi(\lambda)\| < \infty \tag{4.2}$$

for all $\omega \in \mathcal{C}(\mathfrak{a}_q^*)$. The union $\bigcup_n \mathcal{P}^*(\mathfrak{a}_q^*, \mathcal{H}, d, n)$ is denoted by $\mathcal{P}^*(\mathfrak{a}_q^*, \mathcal{H}, d)$, and the union $\bigcup_d \mathcal{P}^*(\mathfrak{a}_q^*, \mathcal{H}, d)$ is denoted by $\mathcal{P}^*(\mathfrak{a}_q^*, \mathcal{H})$.

The space $\mathcal{P}^*(\mathfrak{a}_q^*, \mathcal{H}, d, n)$ is a locally convex topological vector space with the topology defined by means of the seminorms $\nu_{\omega, n}^*$ in (4.2). This topological vector space is discussed further in § 15, where it is shown to be Fréchet under a natural condition on the map n . However, this property is not needed at present.

Definition 4.5. Let $\mathcal{H} = \mathcal{H}(X, \tau)$ and $d = d_{X, \tau}$. For each $M > 0$ we define $\text{PW}_M^*(X : \tau)$ as the space of functions $\varphi \in \mathcal{P}^*(\mathfrak{a}_q^*, \mathcal{H}, d) \otimes {}^\circ\mathcal{C}$ for which

- (i) $\mathcal{L}\varphi = 0$ for all $\mathcal{L} \in \text{AC}_{\mathbb{R}}(X : \tau)$,
- (ii) $\sup_{\lambda \in \bar{\mathfrak{a}}_q^*(P, 0)} (1 + |\lambda|)^{-n} e^{-M|\operatorname{Re} \lambda|} \|\pi(\lambda)\varphi(\lambda)\| < \infty$ for some $n \in \mathbb{N}$.

The *Paley–Wiener space* $\text{PW}^*(X : \tau)$ is then defined by

$$\text{PW}^*(X : \tau) = \bigcup_{M > 0} \text{PW}_M^*(X : \tau).$$

It is clear that $\text{PW}_M(X : \tau) \subset \text{PW}_M^*(X : \tau)$ for all $M > 0$ and that $\text{PW}(X : \tau) \subset \text{PW}^*(X : \tau)$.

We can now state our main theorem.

Theorem 4.6. *The Fourier transform \mathcal{F} is a linear isomorphism of $C_M^{-\infty}(X : \tau)$ onto the Paley–Wiener space $\text{PW}_M^*(X : \tau)$ for each $M > 0$, and hence also of $C_c^{-\infty}(X : \tau)$ onto $\text{PW}^*(X : \tau)$.*

The proof will be given in the course of the following §§ 5–13.

5. An estimate

Let a real Σ -configuration \mathcal{H} in $\mathfrak{a}_{\mathbb{Q}}^*$ and a map $d : \mathcal{H} \rightarrow \mathbb{N}$ be given. Let $\Pi_{\Sigma}(\mathfrak{a}_{\mathbb{Q}}^*)$ denote the set of polynomials on $\mathfrak{a}_{\mathbb{Q}}^*$ which are products of functions of the form $\lambda \mapsto \langle \alpha, \lambda \rangle - s$ with $\alpha \in \Sigma$ and $s \in \mathbb{C}$. Let V be a finite-dimensional normed vector space.

Lemma 5.1. *Let $\omega_0 \subset \omega_1 \subset \mathfrak{a}_{\mathbb{Q}}^*$, and assume that $\mathcal{H}(\omega_1)$ is finite. Assume also that for some $\delta > 0$ the open set*

$$\omega = \{ \lambda + \mu \mid \lambda \in \omega_0, \ |\mu| < \delta \},$$

is contained in ω_1 and satisfies $\mathcal{H}(\omega) = \mathcal{H}(\omega_0)$ (for example, this condition is fulfilled if ω_0 is compact and contained in the interior of ω_1).

Let $p \in \Pi_{\Sigma}(\mathfrak{a}_{\mathbb{Q}}^)$, $n \in \mathbb{Z}$ and $M \geq 0$ be given. There exists a constant $C > 0$ such that*

$$\begin{aligned} \sup_{\omega_0 + i\mathfrak{a}_{\mathbb{Q}}^*} (1 + |\lambda|)^n e^{-M|\text{Re}(\lambda)|} \|\pi_{\omega_0, d}(\lambda)\varphi(\lambda)\| \\ \leq C \sup_{\omega_1 + i\mathfrak{a}_{\mathbb{Q}}^*} (1 + |\lambda|)^n e^{-M|\text{Re}(\lambda)|} \|\pi_{\omega_1, d}(\lambda)p(\lambda)\varphi(\lambda)\| \end{aligned} \quad (5.1)$$

for all $\varphi \in \mathcal{M}(\mathfrak{a}_{\mathbb{Q}}^*, \mathcal{H}, d, V)$.

Proof. We first prove the result under the assumption that $d = 0$ on $\mathcal{H}(\omega_1)$. Then $\pi_{\omega_0, d} = \pi_{\omega_1, d} = 1$ and every function from $\mathcal{M}(\mathfrak{a}_{\mathbb{Q}}^*, \mathcal{H}, d, V)$ is holomorphic on the open set $\omega + i\mathfrak{a}_{\mathbb{Q}}^*$.

It suffices to prove the estimate for $p(\lambda) = \langle \alpha, \lambda \rangle - s$, with $\alpha \in \Sigma$ and $s \in \mathbb{C}$. We choose a number $\delta < 1$ with the properties mentioned above, and fix $\mu \in \mathfrak{a}_{\mathbb{Q}}^*$ such that $|\mu| < \delta$ and $c := \langle \alpha, \mu \rangle > 0$. Then for every $\lambda \in \omega_0 + i\mathfrak{a}_{\mathbb{Q}}^*$ and $0 < r \leq 1$, we have, by Cauchy’s integral formula,

$$\varphi(\lambda) = \frac{1}{2\pi i} \int_{|z|=r} \frac{\varphi(\lambda + z\mu)}{z} dz = \frac{1}{2\pi i} \int_{|z|=r} \frac{p(\lambda + z\mu)\varphi(\lambda + z\mu)}{(p(\lambda) + cz)z} dz.$$

If $|p(\lambda)| > 2c/3$, we fix $r = 1/3$, and if $|p(\lambda)| \leq 2c/3$, we fix $r = 1$. In all cases we have $|p(\lambda) + cz| \geq c/3$ for $|z| = r$. Using the above integral formula we thus obtain the estimate

$$(1 + |\lambda|)^n \|\varphi(\lambda)\| \leq \frac{3}{c} (1 + |\lambda|)^n \sup_{|z|=r} \|[p\varphi](\lambda + z\mu)\|.$$

We now observe that, for all $\lambda \in \mathfrak{a}_{\mathbb{Q}}^*$ and $z \in \mathbb{C}$, $|z| \leq 1$,

$$(1 - \delta)(1 + |\lambda|) \leq 1 + |\lambda| - \delta \leq 1 + |\lambda + z\mu|$$

and hence

$$(1 + |\lambda|)^n \|\varphi(\lambda)\| \leq \frac{3}{c(1 - \delta)^n} \sup_{|z|=r} (1 + |\lambda + z\mu|)^n \|[p\varphi](\lambda + z\mu)\|.$$

Since $|\operatorname{Re}(\lambda + z\mu)| \leq |\operatorname{Re} \lambda| + \delta$ for all z with $|z| = r$ we further obtain

$$(1 + |\lambda|)^n e^{-M|\operatorname{Re}(\lambda)|} \|\varphi(\lambda)\| \leq \frac{3e^{M\delta}}{c(1 - \delta)^n} \sup_{|z|=r} (1 + |\lambda + z\mu|)^n e^{-M|\operatorname{Re}(\lambda + z\mu)|} \|[p\varphi](\lambda + z\mu)\|.$$

Now (5.1) follows, and we can proceed to the general case.

From $\omega_0 \subset \omega_1$ it follows that $\pi_{\omega_1, d} = q\pi_{\omega_0, d}$ with $q \in \Pi_{\Sigma}(\mathfrak{a}_q^*)$. Define $d' : \mathcal{H} \rightarrow \mathbb{N}$ by $d' = d$ on $\mathcal{H} \setminus \mathcal{H}(\omega_0)$ and $d' = 0$ on $\mathcal{H}(\omega_0)$. By application of the first part of the proof, with pq , ω and d' in place of p , ω_1 and d , there exists a constant $C > 0$ such that for every function $\psi \in \mathcal{M}(\mathfrak{a}_q^*, \mathcal{H}, d', V)$, we have

$$\sup_{\omega_0 + i\mathfrak{a}_q^*} (1 + |\lambda|)^n e^{-M|\operatorname{Re}(\lambda)|} \|\psi(\lambda)\| \leq C \sup_{\omega + i\mathfrak{a}_q^*} (1 + |\lambda|)^n e^{-M|\operatorname{Re}(\lambda)|} \|q(\lambda)p(\lambda)\psi(\lambda)\|. \tag{5.2}$$

Let now $\varphi \in \mathcal{M}(\mathfrak{a}_q^*, \mathcal{H}, d, V)$. Then $\psi = \pi_{\omega_0, d}\varphi$ belongs to $\mathcal{M}(\mathfrak{a}_q^*, \mathcal{H}, d', V)$, so that (5.2) holds. This estimate remains valid if the supremum in the right-hand side is taken over the bigger set $\omega_1 + i\mathfrak{a}_q^*$. Since $\pi_{\omega_1, d} = q\pi_{\omega_0, d}$, the required estimate (5.1) follows. \square

6. The Fourier transform maps into PW*(X : τ)

We have already seen that $\mathcal{F}f \in \mathcal{M}(\mathfrak{a}_q^*, \mathcal{H}(X, \tau), d_{X, \tau}) \otimes \mathcal{C}$ for $f \in C_c^{-\infty}(X : \tau)$. In order to show that $\varphi = \mathcal{F}f$ belongs to the Paley–Wiener space we must verify both the estimate (4.2) for some $n \in \mathcal{N}(\mathfrak{a}_q^*)$, and the conditions (i) and (ii) of Definition 4.5.

Let $f \in C_M^{-\infty}(X : \tau)$. The following estimate for $\mathcal{F}f$, from which both (4.2) and (ii) follow easily by application of Lemma 5.1, will now be established. Let $R \in \mathbb{R}$, then there exists a polynomial $p \in \Pi_{\Sigma}(\mathfrak{a}_q^*)$ and a constant $n \in \mathbb{N}$ such that

$$\sup_{\lambda \in \bar{\mathfrak{a}}_q^*(P, R)} (1 + |\lambda|)^{-n} e^{-M|\operatorname{Re} \lambda|} \|p(\lambda)\mathcal{F}f(\lambda)\| < \infty. \tag{6.1}$$

The verification of (6.1) is based on the following estimate for the Eisenstein integral (see [4, Lemma 4.3]). There exists a polynomial $p \in \Pi_{\Sigma}(\mathfrak{a}_q^*)$ and for each $u \in U(\mathfrak{g})$ a constant $n \in \mathbb{N}$ such that

$$\sup_{x \in X_M, \lambda \in \bar{\mathfrak{a}}_q^*(P, R)} (1 + |\lambda|)^{-n} e^{-M|\operatorname{Re} \lambda|} \|p(\lambda)E^*(\lambda : u; x)\| < \infty \tag{6.2}$$

for all $M > 0$.

Let $\tau_X : X \rightarrow \mathbb{R}$ be the map defined by $\tau_X(kaH) = \|\log a\|$ for $k \in K$, $a \in A_q$. It is easily seen that $\tau_X(k \exp YH) = \|Y\|$ for $k \in K$, $Y \in \mathfrak{p} \cap \mathfrak{q}$, hence it follows from [18, Proposition 7.1.2] that τ_X is smooth on the open subset of X where $\tau_X > 0$. For $x \in X$ we have $x \in X_M$ if and only if $\tau_X(x) \leq M$.

Let $h \in C^\infty(\mathbb{R})$ be an arbitrary smooth function satisfying $h(s) = 1$ for $s \leq \frac{1}{2}$ and $h(s) = 0$ for $s \geq 1$. Then the function

$$\varphi_\lambda(x) = h(|\lambda|(\tau_X(x) - M))p(\lambda)E^*(\lambda : x)$$

is smooth and coincides with $p(\lambda)E^*(\lambda : x)$ in a neighbourhood of X_M . Hence

$$p(\lambda)\mathcal{F}f(\lambda) = \int_X \varphi_\lambda(x)f(x) dx, \tag{6.3}$$

and hence by (3.2)

$$\|p(\lambda)\mathcal{F}f(\lambda)\| \leq C \sup_{|\alpha| \leq k, x \in X} \|L_{X^\alpha} \varphi_\lambda(x)\|$$

with constants C and k independent of λ . It follows from the Leibniz rule that

$$\|L_{X^\alpha} \varphi_\lambda(x)\|$$

is bounded by a constant times the product of

$$\sup_{|\beta| \leq k} |h(|\lambda|(\tau_X(X^\beta; x) - M))|$$

and

$$\sup_{|\beta| \leq k} \|p(\lambda)E^*(\lambda : X^\beta; x)\|.$$

The former factor is bounded by a constant times $(1 + |\lambda|)^k$ and it vanishes outside $X_{M+|\lambda|^{-1}}$. By (6.2) the second factor is estimated on this set by a constant times

$$(1 + |\lambda|)^n e^{(M+|\lambda|^{-1})|\operatorname{Re} \lambda|} \leq (1 + |\lambda|)^n e^{1+M|\operatorname{Re} \lambda|}$$

so that the desired estimate (6.1) follows.

It remains to be established that $\mathcal{L}\mathcal{F}f = 0$ for $\mathcal{L} \in \operatorname{AC}_{\mathbb{R}}(X : \tau)$. Recall that $E^*(\lambda : \cdot)$ is meromorphic in λ with values in $C^\infty(X : \tau^*) \otimes \mathcal{C}$, and that by definition an element $\mathcal{L} \in \mathcal{M}(\mathfrak{a}_{\mathbb{Q}\mathbb{C}}^*, \Sigma)_{\text{laur}}^* \otimes \mathcal{C}^*$ with real support belongs to $\operatorname{AC}_{\mathbb{R}}(X : \tau)$ if and only if it annihilates $\lambda \mapsto E^*(\lambda : \cdot)$. The Fourier transform $\mathcal{F}f(\lambda) \in \mathcal{C}$ is obtained by applying the linear form $f dx \in C^\infty(X : \tau^*)'$ to $E^*(\lambda : \cdot) \in C^\infty(X : \tau^*) \otimes \mathcal{C}$. We claim that the applications of \mathcal{L} and $f dx$ commute, so that

$$\mathcal{L}\mathcal{F}f = \mathcal{L}(f dx(E^*(\cdot : \cdot))) = f dx(\mathcal{L}E^*(\cdot : \cdot)) = 0.$$

This claim is easily verified with the lemma below.

Lemma 6.1. *Let $\mathcal{L} \in \mathcal{M}(\mathfrak{a}_{\mathbb{Q}\mathbb{C}}^*, \Sigma)_{\text{laur}}^*$ be a Σ -Laurent functional on $\mathfrak{a}_{\mathbb{Q}\mathbb{C}}^*$, and let $\varphi \in \mathcal{M}(\mathfrak{a}_{\mathbb{Q}\mathbb{C}}^*, \Sigma, V)$, where V is a complete locally convex space. For each continuous linear form ξ on V , the function $\xi \circ \varphi$ belongs to $\mathcal{M}(\mathfrak{a}_{\mathbb{Q}\mathbb{C}}^*, \Sigma)$ and the following identity holds*

$$\mathcal{L}(\xi \circ \varphi) = \xi(\mathcal{L}\varphi).$$

Proof. We refer to [6, § 10] for notation. We may assume that \mathcal{L} is supported in a single point $a \in \mathfrak{a}_{\mathbb{Q}\mathbb{C}}^*$. If $\psi \in \mathcal{O}_a(\mathfrak{a}_{\mathbb{Q}\mathbb{C}}^*, V)$ then $\xi \circ \psi \in \mathcal{O}_a(\mathfrak{a}_{\mathbb{Q}\mathbb{C}}^*)$ and $u(\xi \circ \psi) = \xi(u\psi)$ for $u \in S(\mathfrak{a}_{\mathbb{Q}\mathbb{C}}^*)$. The proof is now straightforward from [6, Definition 10.1] (see also Remark 10.2 therein). □

7. Distributional wave packets

Recall that if $\varphi : \mathfrak{ia}_q^* \rightarrow \mathcal{C}$ is continuous and satisfies the estimate

$$\sup_{\lambda \in \mathfrak{ia}_q^*} (1 + |\lambda|)^n \|\varphi(\lambda)\| < \infty \tag{7.1}$$

for each $n \in \mathbb{N}$, then we define the wave packet $\mathcal{J}\varphi \in C^\infty(X : \tau)$ by

$$\mathcal{J}\varphi(x) = \int_{\mathfrak{ia}_q^*} E^\circ(\varphi(\lambda) : \lambda : x) \, d\lambda. \tag{7.2}$$

The wave packet is related to the Fourier transform by

$$\langle \mathcal{J}\varphi, g \rangle = \langle \varphi, \mathcal{F}g \rangle$$

for all $g \in C_c^\infty(X : \tau)$, that is,

$$\int_X \langle \mathcal{J}\varphi(x), g(x) \rangle \, dx = \int_{\mathfrak{ia}_q^*} \langle \varphi(\lambda), \mathcal{F}g(\lambda) \rangle \, d\lambda. \tag{7.3}$$

The brackets in the latter equation refer to the sesquilinear inner products on the finite-dimensional Hilbert spaces V_τ and \mathcal{C} , respectively.

The transform \mathcal{J} can be extended as follows to all continuous functions $\varphi : \mathfrak{ia}_q^* \rightarrow \mathcal{C}$ satisfying an estimate

$$\sup_{\lambda \in \mathfrak{ia}_q^*} (1 + |\lambda|)^{-n} \|\varphi(\lambda)\| < \infty, \tag{7.4}$$

for some $n \in \mathbb{N}$. For such a function φ we define the distributional wave packet $\mathcal{J}\varphi \in C^{-\infty}(X : \tau)$ by requiring (7.3) for all $g \in C_c^\infty(X : \tau)$. It follows from the estimate (7.4) together with Lemma 4.2, that the integral on the right-hand side of (7.3) is well defined and depends continuously on g , so that an element in $C^{-\infty}(X : \tau)$ is defined by this equation.

In particular, since for each $\varphi \in \text{PW}^*(X : \tau)$ the restriction $\varphi|_{\mathfrak{ia}_q^*}$ is well defined and satisfies (7.4) for some n , we thus have a well defined linear map

$$\mathcal{J} : \text{PW}^*(X : \tau) \rightarrow C^{-\infty}(X : \tau).$$

8. The Fourier transform is injective

The injectivity is established in the following theorem.

Theorem 8.1. *There exists an invariant differential operator $D \in \mathbb{D}(G/H)$ which is formally self-adjoint, injective as an operator $C_c^{-\infty}(X) \rightarrow C_c^{-\infty}(X)$ and which satisfies*

$$DJ\mathcal{F}f = \mathcal{F}Df = Df \tag{8.1}$$

for all $f \in C_c^{-\infty}(X : \tau)$.

In particular, the Fourier transform $\mathcal{F} : C_c^{-\infty}(X : \tau) \rightarrow \text{PW}^(X : \tau)$ is injective.*

The proof will be given after the following lemma.

Lemma 8.2. *Let $\varphi : \mathfrak{ia}_q^* \rightarrow \mathcal{C}$ be a continuous function satisfying (7.1) for all $n \in \mathbb{N}$. Then*

$$\int_X \langle f(x), \mathcal{J}\varphi(x) \rangle dx = \int_{\mathfrak{ia}_q^*} \langle \mathcal{F}f(\lambda), \varphi(\lambda) \rangle d\lambda \tag{8.2}$$

for all $f \in C_c^{-\infty}(X : \tau)$.

Proof. By taking adjoints in the estimate (6.2) a similar estimate is derived for the Eisenstein integral $E^\circ(\lambda : x)$ and its derivatives with respect to x . It follows that the definition (7.2) of $\mathcal{J}\varphi$ allows an interpretation as an integral over \mathfrak{ia}_q^* with values in the Fréchet space $C^\infty(X : \tau)$. In the left-hand side of (8.2) we apply $f dx$ to $\mathcal{J}\varphi$. By continuity we may then take $f dx$ inside the integral over \mathfrak{ia}_q^* and obtain

$$\int_X \langle f(x), \mathcal{J}\varphi(x) \rangle dx = \int_{\mathfrak{ia}_q^*} \int_X \langle f(x), E^\circ(\varphi(\lambda) : \lambda : x) \rangle dx d\lambda,$$

which, by definition of $E^*(\lambda : x)$, exactly equals the right-hand side of (8.2). □

Proof of Theorem 8.1. It follows from [3, Theorem 14.1, Proposition 15.2, Lemma 15.3] that there exists an invariant differential operator $D \in \mathbb{D}(G/H)$ which is formally self-adjoint, injective as an operator $C_c^\infty(X) \rightarrow C_c^\infty(X)$ and which satisfies (8.1) for all $f \in C_c^\infty(X : \tau)$. Since $D = D^*$, one obtains (8.1) for $f \in C_c^{-\infty}(X : \tau)$ by transposition using (7.3) and (8.2). Finally, it follows from Lemma 8.3 below that D is injective on the space of generalized functions as well. The injectivity of \mathcal{F} is an immediate consequence of (8.1) and the injectivity of D . □

Lemma 8.3. *Let $D \in \mathbb{D}(G/H)$. If D is injective $C_c^\infty(X) \rightarrow C_c^\infty(X)$ then D is injective $C_c^{-\infty}(X) \rightarrow C_c^{-\infty}(X)$.*

Proof. Recall that for $\phi \in C_c^\infty(G)$ and $f \in C^{-\infty}(X)$ we define $L(\phi)f \in C^\infty(X)$ by

$$L(\phi)f(x) = \int_G \phi(g)f(g^{-1}x) dg.$$

The integral can be interpreted as a $C^{-\infty}(X)$ -valued integral in the variable x , or it can be defined as the transpose of the operator $L(\phi^\vee) : C_c^\infty(X) \rightarrow C_c^\infty(X)$. In any case, $L(\phi)f$ is a smooth function on G and it is compactly supported when f has compact support. Furthermore, $L(\phi)$ commutes with every invariant differential operator D .

Let $\phi_j \in C_c^\infty(G)$, $j \in \mathbb{N}$, be an approximative unit, then it is well known that $L(\phi_j)f$ converges weakly (in fact, also strongly) to f , for each $f \in C^{-\infty}(X)$.

After these preparations the proof of the lemma is simple. If $f \in C_c^{-\infty}(X)$ and $Df = 0$ then $D(L(\phi_j)f) = L(\phi_j)Df = 0$ and hence $L(\phi_j)f = 0$ for all j . Hence $f = 0$. □

9. Generalized Eisenstein integrals and Fourier transforms

Let $F \subset \Delta$, where Δ is the set of simple roots for Σ^+ . We will use the notation of [9, § 5] and [7, § 9]. In particular, \mathcal{A}_F denotes the finite-dimensional Hilbert space and $E_F^\circ(\nu : x) \in \text{Hom}(\mathcal{A}_F, V_\tau)$ denotes the generalized Eisenstein integral defined in equations (5.5), (5.6) of [9], for $\nu \in \mathfrak{a}_{F\mathbb{Q}\mathbb{C}}^*$ and $x \in X$. The generalized Eisenstein integral is a meromorphic $\text{Hom}(\mathcal{A}_F, V_\tau)$ -valued function of ν , with singularities along a real $\Sigma_r(F)$ -configuration of hyperplanes in $\mathfrak{a}_{F\mathbb{Q}\mathbb{C}}^*$ (see [7, Lemma 9.8]). Here $\Sigma_r(F)$ is the set of all non-zero restrictions to $\mathfrak{a}_{F\mathbb{Q}\mathbb{C}}$ of elements in Σ . For $F = \emptyset$ the generalized Eisenstein integral $E_F^\circ(\nu : x)$ is identical with the normalized Eisenstein integral $E^\circ(\lambda : x)$.

The corresponding *generalized Fourier transform* is defined by

$$\mathcal{F}_F f(\nu) = \int_X E_F^*(\nu : x) f(x) dx \in \mathcal{A}_F$$

for $\nu \in \mathfrak{a}_{F\mathbb{Q}\mathbb{C}}^*$, $f \in C_c^\infty(X : \tau)$, where $E_F^*(\nu : x) = E_F^\circ(-\bar{\nu} : x)^* \in \text{Hom}(V_\tau, \mathcal{A}_F)$. The generalized Fourier transform is a meromorphic \mathcal{A}_F -valued function of ν ,

It is a remarkable property of the generalized Eisenstein integral $E_F^\circ(\nu : x)$ that it can be obtained from the ordinary Eisenstein integral $E^\circ(\lambda : x)$ by applying a suitable operator in the variable λ . More precisely, we have the following result.

Lemma 9.1. *There exists a Laurent functional $\mathcal{L} \in \mathcal{M}(\mathfrak{a}_{F\mathbb{Q}\mathbb{C}}^{\perp}, \Sigma_F)_{\text{laur}}^* \otimes \text{Hom}(\mathcal{A}_F, {}^\circ\mathcal{C})$ with real support, such that*

$$E_F^\circ(\nu : x) = \mathcal{L}[E^\circ(\nu + \cdot : x)]$$

for $\nu \in \mathfrak{a}_{F\mathbb{Q}\mathbb{C}}^*$.

Proof. This follows immediately from [7, Lemma 9.7] with ψ in a basis for the finite-dimensional space \mathcal{A}_F . □

By taking adjoints it follows from Lemma 9.1 that there exists a Laurent functional $\mathcal{L}_F \in \mathcal{M}(\mathfrak{a}_{F\mathbb{Q}\mathbb{C}}^{\perp}, \Sigma_F)_{\text{laur}}^* \otimes \text{Hom}({}^\circ\mathcal{C}, \mathcal{A}_F)$ with real support such that

$$E_F^*(\nu : x) = \mathcal{L}_F[E^*(\nu + \cdot : x)].$$

Let such a Laurent functional, denoted by \mathcal{L}_F , be fixed in the sequel. It follows immediately that

$$\mathcal{F}_F f(\nu) = \mathcal{L}_F[\mathcal{F}f(\nu + \cdot)]. \tag{9.1}$$

In order to study the consequences of (9.1) for the generalized Fourier transform, we need the following result.

Lemma 9.2. *Let \mathcal{H} be a real Σ -configuration in $\mathfrak{a}_{\mathbb{Q}\mathbb{C}}^*$, and let $F \subset \Delta$ be given. Let $\mathcal{L} \in \mathcal{M}(\mathfrak{a}_{F\mathbb{Q}\mathbb{C}}^{\perp}, \Sigma_F)_{\text{laur}}^*$ have real support. There exists a real $\Sigma_r(F)$ -configuration \mathcal{H}_F in $\mathfrak{a}_{F\mathbb{Q}\mathbb{C}}^*$ and for every map $d : \mathcal{H} \rightarrow \mathbb{N}$ a map $d' : \mathcal{H}_F \rightarrow \mathbb{N}$ such that the following two conditions hold.*

- (a) The operator \mathcal{L}_* defined by $\mathcal{L}_*\psi(\nu) = \mathcal{L}[\psi(\nu + \cdot)]$ for $\nu \in \mathfrak{a}_{F\mathbb{Q}}^*$ maps $\mathcal{M}(\mathfrak{a}_q^*, \mathcal{H}, d)$ continuously into $\mathcal{M}(\mathfrak{a}_{Fq}^*, \mathcal{H}_F, d')$.
- (b) The operator \mathcal{L}_* restricts to a continuous linear map $\mathcal{P}(\mathfrak{a}_q^*, \mathcal{H}, d) \rightarrow \mathcal{P}(\mathfrak{a}_{Fq}^*, \mathcal{H}_F, d')$, where $\mathcal{P}(\mathfrak{a}_q^*, \mathcal{H}, d)$ and $\mathcal{P}(\mathfrak{a}_{Fq}^*, \mathcal{H}_F, d')$ are defined in Definition 4.1.

Proof. Let \mathcal{H}_F and d' be as in [6, Corollary 11.6 (b)]. Then \mathcal{L}_* maps $\mathcal{M}(\mathfrak{a}_q^*, \mathcal{H}, d)$ continuously into $\mathcal{M}(\mathfrak{a}_{Fq}^*, \mathcal{H}_F, d')$. The assertion (b) is given in [9, Lemma 6.1 (v)], with a proof following [5, Lemmas 1.10, 1.11]. □

We fix a $\Sigma_r(F)$ -hyperplane configuration $\mathcal{H}(X, \tau, F)$ as \mathcal{H}_F in Lemma 9.2, where we take $\mathcal{H} = \mathcal{H}(X, \tau)$ and $\mathcal{L} = \mathcal{L}_F$. In addition, we fix a map $d_{X,\tau,F} : \mathcal{H}(X, \tau, F) \rightarrow \mathbb{N}$ as d' , where we take $d = d_{X,\tau}$. It follows from (a) that

$$E_F^*(\nu : x) \in \mathcal{M}(\mathfrak{a}_{Fq}^*, \mathcal{H}(X, \tau, F), d_{X,\tau,F}) \otimes \text{Hom}(V_\tau, \mathcal{A}_F).$$

Furthermore, the following result is obtained from (b), (9.1) and Lemma 4.2.

Lemma 9.3. *The generalized Fourier transform is continuous:*

$$\mathcal{F}_F : C_c^\infty(X : \tau) \rightarrow \mathcal{P}(\mathfrak{a}_{Fq}^*, \mathcal{H}(X, \tau, F), d_{X,\tau,F}) \otimes \mathcal{A}_F.$$

10. Generalized wave packets and Fourier inversion

Let $\mathcal{H}_F = \mathcal{H}(X, \tau, F)$. For $\varphi \in \mathcal{P}(\mathfrak{a}_{Fq}^*, \mathcal{H}_F) \otimes \mathcal{A}_F$ we introduce the *generalized wave packet*

$$\mathcal{J}_F\varphi(x) = \int_{\epsilon_F + i\mathfrak{a}_{Fq}^*} E_F^\circ(\nu : x)\varphi(\nu) d\mu_{\mathfrak{a}_{Fq}^*}(\nu), \tag{10.1}$$

where the element $\epsilon_F \in \mathfrak{a}_{Fq}^{*+}$ and the measure $d\mu_{\mathfrak{a}_{Fq}^*}$ on $\epsilon_F + i\mathfrak{a}_{Fq}^*$ are as defined in [4, p. 42]. The definition is justified by the following estimate, for which we refer to [4, Lemma 10.8]. Let $\omega \subset \mathfrak{a}_{Fq}^*$ be compact. There exists a polynomial on \mathfrak{a}_{Fq}^* , $p \in \Pi(\mathfrak{a}_{Fq})$, for each $u \in U(\mathfrak{g})$ a number $n \in \mathbb{N}$, and for each x a constant C , locally uniform in x , such that

$$\|p(\nu)E_F^\circ(\nu : u; x)\| \leq C(1 + |\nu|)^n \tag{10.2}$$

for all $\nu \in \omega + i\mathfrak{a}_{Fq}^*$ (see also [7, Proposition 18.10], where a stronger result is given). It follows that for $\epsilon_F \in \mathfrak{a}_{Fq}^{*+}$ sufficiently close to 0, the integral (10.1) is independent of ϵ_F and converges locally uniformly in x . Moreover, the resulting function $\mathcal{J}_F\varphi$ belongs to $C^\infty(X : \tau)$.

The Fourier inversion formula of [4] now takes the form

$$f(x) = \sum_{F \subset \Delta} c_F \int_{\epsilon_F + i\mathfrak{a}_{Fq}^*} E_F^\circ(\nu : x)\mathcal{L}_F[\mathcal{F}f(\nu + \cdot)] d\mu_{\mathfrak{a}_{Fq}^*}(\nu) = \sum_{F \subset \Delta} c_F \mathcal{J}_F(\mathcal{L}_{F*}\mathcal{F}f)(x) \tag{10.3}$$

for $f \in C_c^\infty(X : \tau)$ (see [9, Theorem 8.3]), where the asterisk on \mathcal{L}_F as in Lemma 9.2 indicates that it acts by

$$\mathcal{L}_{F*}\varphi(\nu) = \mathcal{L}_F[\varphi(\nu + \cdot)]$$

for $\varphi \in \mathcal{M}(\mathfrak{a}_q^*, \mathcal{H})$. The c_F are explicitly given constants. The explicit expression $c_F = |W|t(\mathfrak{a}_{Fq}^+)$ is not relevant for the proof of Theorem 4.6.

11. Generalized distribution wave packets

We shall see later that the inversion formula (10.3) is valid also for distributions. For this purpose we need to extend the generalized wave packet map $\mathcal{J}_F : \mathcal{P}(\mathfrak{a}_{Fq}^*, \mathcal{H}_F) \otimes \mathcal{A}_F \rightarrow C^\infty(X : \tau)$ to a map $\mathcal{P}^*(\mathfrak{a}_{Fq}^*, \mathcal{H}_F) \otimes \mathcal{A}_F \rightarrow C^{-\infty}(X : \tau)$. Here $\mathcal{P}^*(\mathfrak{a}_{Fq}^*, \mathcal{H}_F)$ is defined as $\mathcal{P}^*(\mathfrak{a}_q^*, \mathcal{H})$ in Definition 4.4, but with \mathfrak{a}_{Fq}^* in place of \mathfrak{a}_q^* and $\mathcal{H}_F = \mathcal{H}(X, \tau, F)$ in place of \mathcal{H} .

In analogy with the definition of the distributional wave packet $\mathcal{J}\varphi$ given in §7, we use an adjoint relation with \mathcal{F}_F . For this purpose we introduce the sesquilinear pairing

$$\mathcal{P}(\mathfrak{a}_{Fq}^*, \mathcal{H}_F) \times \mathcal{P}^*(\mathfrak{a}_{Fq}^*, \mathcal{H}_F) \rightarrow \mathbb{C}$$

given by

$$\langle \varphi, \psi \rangle_\epsilon = \int_{\epsilon_F + i\mathfrak{a}_{Fq}^*} \langle \varphi(\lambda), \psi(-\bar{\lambda}) \rangle d\mu_{\mathfrak{a}_{Fq}^*}(\lambda) \tag{11.1}$$

for $\epsilon_F \in \mathfrak{a}_{Fq}^{*+}$ sufficiently close to zero. The pairing $\langle \cdot, \cdot \rangle$ inside the integral is the standard sesquilinear pairing $\mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$. The condition on ϵ_F guarantees that the domain of integration is disjoint from the singular locus of the integrand; moreover, by Cauchy’s theorem the integral is independent of the precise location of ϵ_F .

For $d : \mathcal{H}_F \rightarrow \mathbb{N}$ and $n \in \mathcal{N}(\mathfrak{a}_{Fq}^*)$ the spaces $\mathcal{P}(\mathfrak{a}_{Fq}, \mathcal{H}_F, d)$ and $\mathcal{P}^*(\mathfrak{a}_{Fq}, \mathcal{H}_F, d, n)$ are defined and topologized as in §4. The following lemma is obvious from these definitions.

Lemma 11.1. *For each pair $d_1, d_2 : \mathcal{H}_F \rightarrow \mathbb{N}$ and every $n \in \mathcal{N}(\mathfrak{a}_{Fq}^*)$, the pairing (11.1) restricts to a continuous sesquilinear pairing of $\mathcal{P}(\mathfrak{a}_{Fq}, \mathcal{H}_F, d_1)$ and $\mathcal{P}^*(\mathfrak{a}_{Fq}, \mathcal{H}_F, d_2, n)$.*

The pairing (11.1) is extended to \mathcal{A}_F -valued functions on $\mathfrak{a}_{Fq\mathbb{C}}^*$ in the obvious fashion. It is then easily seen by Fubini’s theorem and (10.2) that

$$\langle \mathcal{J}_F \varphi, f \rangle = \langle \varphi, \mathcal{F}_F f \rangle_\epsilon \tag{11.2}$$

for $\varphi \in \mathcal{P}(\mathfrak{a}_{Fq}^*, \mathcal{H}_F) \otimes \mathcal{A}_F$, $f \in C_c^\infty(X : \tau)$.

The *generalized distribution wave packet* $\mathcal{J}_F \varphi \in C^{-\infty}(X : \tau)$ is defined for functions $\varphi \in \mathcal{P}^*(\mathfrak{a}_{Fq}^*, \mathcal{H}_F) \otimes \mathcal{A}_F$ by (11.2) for all $f \in C_c^\infty(X : \tau)$. The definition is justified by Lemmas 9.3 and 11.1.

The following lemma is immediate from the definition.

Lemma 11.2. *Let $\mathcal{H}_F = \mathcal{H}(X, \tau, F)$. Let $d : \mathcal{H}_F \rightarrow \mathbb{N}$ and $n \in \mathcal{N}(\mathfrak{a}_{Fq}^*)$, be arbitrary. The distributional generalized wave packet map*

$$\mathcal{J}_F : \mathcal{P}^*(\mathfrak{a}_{Fq}^*, \mathcal{H}_F, d, n) \otimes \mathcal{A}_F \rightarrow C^{-\infty}(X : \tau)$$

is continuous, when $\mathcal{P}^(\mathfrak{a}_{Fq}^*, \mathcal{H}_F, d, n)$ is equipped with the weak topology with respect to the pairing (11.1), and $C^{-\infty}(X : \tau)$ is equipped with the dual topology.*

It follows from Lemma 11.1 that the map is continuous also for the original topology on $\mathcal{P}^*(\mathfrak{a}_{Fq}^*, \mathcal{H}_F, d, n)$ and the weak dual topology on $C^{-\infty}(X : \tau)$. We shall see later (in Lemma 14.2) that it is continuous for the original topology on $\mathcal{P}^*(\mathfrak{a}_{Fq}^*, \mathcal{H}_F, d, n)$ and the strong dual topology on $C^{-\infty}(X : \tau)$. However, this is not needed for the proof of Theorem 4.6.

12. Multiplication operators on the Paley–Wiener space

The main result of this subsection, Proposition 12.1, was announced jointly with Flensted-Jensen in the survey paper [10] (see Proposition 18 therein). However, the present proof is independent of the preceding results of that paper.

Let \mathfrak{b} be a Cartan subspace of \mathfrak{g} containing $\mathfrak{a}_{\mathfrak{q}}$. Then $\mathfrak{b} = \mathfrak{b}_{\mathfrak{k}} \oplus \mathfrak{a}_{\mathfrak{q}}$ with $\mathfrak{b}_{\mathfrak{k}} = \mathfrak{b} \cap \mathfrak{k}$. Let $W(\mathfrak{b})$ be the Weyl group of the restricted root system of $\mathfrak{b}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$. Let \mathfrak{b}^d denote the real form $i\mathfrak{b}_{\mathfrak{k}} \oplus \mathfrak{a}_{\mathfrak{q}}$ of $\mathfrak{b}_{\mathbb{C}}$.

Let $\mathcal{O}(\mathfrak{b}_{\mathbb{C}}^*)^{W(\mathfrak{b})}$ denote the space of $W(\mathfrak{b})$ -invariant entire functions on $\mathfrak{b}_{\mathbb{C}}^*$, and for $r > 0$ let $\text{PW}_r(\mathfrak{b}^d)^{W(\mathfrak{b})}$ denote the subspace of functions which are also rapidly decreasing of exponential type r , that is,

$$\sup_{\lambda \in \mathfrak{b}_{\mathbb{C}}^*} (1 + |\lambda|)^n e^{-r|\text{Re } \lambda|} |\psi(\lambda)| < \infty$$

for all $n \in \mathbb{N}$. The real part $\text{Re } \lambda$ is taken with respect to the decomposition $\mathfrak{b}_{\mathbb{C}} = \mathfrak{b}^d + i\mathfrak{b}^d$. Let $\text{PW}(\mathfrak{b}^d)^{W(\mathfrak{b})}$ denote the union over r of all the spaces $\text{PW}_r(\mathfrak{b}^d)^{W(\mathfrak{b})}$.

Given $\psi \in \mathcal{O}(\mathfrak{b}_{\mathbb{C}}^*)^{W(\mathfrak{b})}$ we define a multiplication operator $M(\psi)$ on $\mathcal{M}(\mathfrak{a}_{\mathfrak{q}\mathbb{C}}^*, \circ\mathcal{C})$ as follows. Recall the orthogonal decomposition

$$\circ\mathcal{C} = \bigoplus_{A \in L} \circ\mathcal{C}[A] \tag{12.1}$$

(see [3, equation (5.14)]), where $L \subset i\mathfrak{b}_{\mathfrak{k}}^*$ is a finite set depending on τ . We define, for each $\lambda \in \mathfrak{a}_{\mathfrak{q}\mathbb{C}}^*$, an endomorphism $M(\psi, \lambda)$ of $\circ\mathcal{C}$ by $M(\psi, \lambda)\eta = \psi(\lambda + A)\eta$ for $\eta \in \circ\mathcal{C}[A]$, and we define for each $\varphi \in \mathcal{M}(\mathfrak{a}_{\mathfrak{q}\mathbb{C}}^*, \circ\mathcal{C})$ a function $M(\psi)\varphi \in \mathcal{M}(\mathfrak{a}_{\mathfrak{q}\mathbb{C}}^*, \circ\mathcal{C})$ by

$$M(\psi)\varphi(\lambda) = M(\psi, \lambda)\varphi(\lambda)$$

for all $\lambda \in \mathfrak{a}_{\mathfrak{q}\mathbb{C}}^*$.

The motivation behind this definition is as follows. If ψ belongs to $\text{PW}_r(\mathfrak{b}^d)^{W(\mathfrak{b})}$ then the operator $M(\psi)$ on $\mathcal{M}(\mathfrak{a}_{\mathfrak{q}\mathbb{C}}^*, \circ\mathcal{C})$ corresponds, via the Fourier transform \mathcal{F} , to a linear operator M_{ψ} on $C_c^{\infty}(X : \tau)$, a so-called *multiplier*, so that

$$\mathcal{F}(M_{\psi}f) = M(\psi)\mathcal{F}f$$

for all $f \in C_c^{\infty}(X : \tau)$. The existence of the multiplier M_{ψ} is given a relatively elementary proof in [10] without reference to the Paley–Wiener theorem for $C_c^{\infty}(X : \tau)$, which was only a conjecture when that paper was written. However, with the Paley–Wiener theorem for $C_c^{\infty}(X : \tau)$ available from [9], the existence of the multiplier is an immediate consequence of Proposition 12.1 below.

Let $D \in \mathbb{D}(G/H)$. It follows from [3, Lemma 6.2] that

$$\mathcal{F}(Df)(\lambda) = \mu(D, \lambda)\mathcal{F}f(\lambda).$$

By the definition of the decomposition (12.1), the endomorphism $\mu(D, \lambda)$ of $\circ\mathcal{C}$ acts on $\circ\mathcal{C}[A]$ as multiplication with $\gamma(D, \lambda + A)$ where

$$\gamma : \mathbb{D}(G/H) \rightarrow P(\mathfrak{b}_{\mathbb{C}}^*)^{W(\mathfrak{b})}$$

is the Harish-Chandra isomorphism. Hence

$$\mathcal{F}(Df) = M(\gamma(D))\mathcal{F}f \tag{12.2}$$

for $f \in C_c^\infty(X : \tau)$.

Proposition 12.1. *Let $\psi \in \text{PW}(\mathfrak{b}^d)^{W(\mathfrak{b})}$. The multiplication operator $M(\psi)$ maps the space $\text{PW}^*(X : \tau)$ into $\text{PW}(X : \tau)$. More precisely, if $r, R > 0$ and $\psi \in \text{PW}_r(\mathfrak{b}^d)^{W(\mathfrak{b})}$, then $M(\psi)$ maps $\text{PW}_R^*(X : \tau)$ into $\text{PW}_{R+r}(X : \tau)$.*

Proof. The idea of the proof is taken from [1, p. 87]. Let $\psi \in \text{PW}_r(\mathfrak{b}^d)^{W(\mathfrak{b})}$, then since L is finite there exists, for each $n \in \mathbb{N}$ a constant $C > 0$ such that

$$|\psi(\lambda + A)| \leq C(1 + |\lambda|)^{-n} e^{r|\text{Re } \lambda|}$$

for all $\lambda \in \mathfrak{a}_{\mathbb{Q}\mathbb{C}}^*$ and $A \in L$. It is now easily seen that the estimates (4.1) and (ii) in Definition 4.3 of $\text{PW}_{R+r}(X : \tau)$ are satisfied by $M(\psi)\varphi$ for $\varphi \in \text{PW}_R^*(X : \tau)$. Only the annihilation by $\text{AC}_{\mathbb{R}}(X : \tau)$ in Definition 4.3 (i) remains to be verified.

Let $\varphi \in \mathcal{M}(\mathfrak{a}_{\mathbb{Q}\mathbb{C}}^*, \Sigma, \circ\mathcal{C})$ be a function annihilated by all $\mathcal{L} \in \text{AC}_{\mathbb{R}}(X : \tau)$, and let $\psi \in \mathcal{O}(\mathfrak{b}_{\mathbb{C}}^*)^{W(\mathfrak{b})}$. We claim that then $M(\psi)\varphi$ is also annihilated by all $\mathcal{L} \in \text{AC}_{\mathbb{R}}(X : \tau)$.

Let $\mathcal{L} \in \text{AC}_{\mathbb{R}}(X : \tau)$ and assume first that ψ is a polynomial. Then there exists an invariant differential operator $D \in \mathbb{D}(G/H)$ such that $\psi = \gamma(D)$. It follows from (12.2) that $\mathcal{L}(M(\psi)\mathcal{F}f) = \mathcal{L}(\mathcal{F}(Df)) = 0$ for all $f \in C_c^\infty(X : \tau)$. By [6, p. 674], there exists a Laurent functional $\mathcal{L}' \in \mathcal{M}(\mathfrak{a}_{\mathbb{Q}\mathbb{C}}^*, \Sigma)_{\text{laur}}^* \otimes \circ\mathcal{C}^*$ such that $\mathcal{L}(M(\psi)\phi) = \mathcal{L}'\phi$ for all $\phi \in \mathcal{M}(\mathfrak{a}_{\mathbb{Q}\mathbb{C}}^*, \Sigma, \circ\mathcal{C})$. Moreover, $\text{supp } \mathcal{L}' \subset \text{supp } \mathcal{L}$. Hence $\mathcal{L}' \in \text{AC}_{\mathbb{R}}(X : \tau)$ by [9, Lemma 3.8], and we conclude that $\mathcal{L}(M(\psi)\varphi) = \mathcal{L}'\varphi = 0$.

Consider now the case of a general function $\psi \in \mathcal{O}(\mathfrak{b}_{\mathbb{C}}^*)^{W(\mathfrak{b})}$. We expand ψ in its Taylor series around 0, and denote by ψ_k the sum of the terms up to degree k . Then $\psi_k \rightarrow \psi$, uniformly on compact sets, from which it follows that $\mathcal{L}(M(\psi_k)\varphi) \rightarrow \mathcal{L}(M(\psi)\varphi)$. Each ψ_k is a $W(\mathfrak{b})$ -invariant polynomial, hence $\mathcal{L}(M(\psi_k)\varphi) = 0$. It follows that $\mathcal{L}(M(\psi)\varphi) = 0$. □

13. The Fourier transform is surjective

Let $\varphi \in \text{PW}_M^*(X : \tau)$ be given. Inspired by equation (10.3) we define

$$f = \sum_{F \subset \Delta} c_F \mathcal{J}_F(\mathcal{L}_{F*}\varphi) \in C^{-\infty}(X : \tau), \tag{13.1}$$

where \mathcal{L}_F is chosen as in §9 (see (9.1)). For (13.1) to make sense we need that $\mathcal{L}_{F*}\varphi$ belongs to the space $\mathcal{P}^*(\mathfrak{a}_{F\mathbb{Q}}^*, \mathcal{H}_F) \otimes \mathcal{A}_F$ on which \mathcal{J}_F was defined (see (11.2)). This is secured by the following lemma.

Lemma 13.1. *Let $\mathcal{H}, \mathcal{L}, \mathcal{H}_F, d$ and d' be as in Lemma 9.2. For every $n \in \mathcal{N}(\mathfrak{a}_{\mathbb{Q}}^*)$ there exists $n' \in \mathcal{N}(\mathfrak{a}_{F\mathbb{Q}}^*)$ such that the operator \mathcal{L}_* restricts to a continuous linear map*

$$\mathcal{P}^*(\mathfrak{a}_{\mathbb{Q}}^*, \mathcal{H}, d, n) \rightarrow \mathcal{P}^*(\mathfrak{a}_{F\mathbb{Q}}^*, \mathcal{H}_F, d', n').$$

Proof. The proof is similar to the proof of Lemma 9.2 (b). □

It follows that (13.1) defines a distribution $f \in C^{-\infty}(X : \tau)$. We claim that $f \in C_M^{-\infty}(X : \tau)$ and $\mathcal{F}f = \varphi$.

Let $\psi_j \in \text{PW}(\mathfrak{b}^d)_{r_j}^W$ be a sequence of functions such that $r_j \rightarrow 0$ for $j \rightarrow \infty$, such that ψ_j is uniformly bounded on each set of the form $\omega + i\mathfrak{b}^{d*}$ with $\omega \subset \mathfrak{b}^{d*}$ compact, and such that $\psi_j \rightarrow 1$, locally uniformly on $\mathfrak{a}_{\mathbb{C}}^*$. Such a sequence can be constructed by application of the Euclidean Fourier transform to a smooth approximation of the Dirac measure on \mathfrak{b}^d . In particular, for each $\Lambda \in L$, the sequence of functions $\psi_j(\cdot + \Lambda)$ is uniformly bounded on each set $\omega + i\mathfrak{b}^{d*}$ as above, and converges to 1, locally uniformly on $\mathfrak{a}_{\mathbb{C}}^*$.

Consider the functions $\varphi_j := M(\psi_j)\varphi$. It follows from Proposition 12.1 that $\varphi_j \in \text{PW}_{M+r_j}(X : \tau)$. Hence, by the Paley–Wiener theorem of [9] there exists a unique function $f_j \in C_c^\infty(X : \tau)$, with support in X_{M+r_j} , such that $\mathcal{F}f_j = M(\psi_j)\varphi$.

Let d, n be such that $\varphi \in \mathcal{P}^*(\mathfrak{a}_{\mathbb{C}}^*, \mathcal{H}, d, n) \otimes \mathcal{C}$. Then it follows from the properties of ψ_j mentioned above that $\mathcal{F}f_j = \varphi_j \rightarrow \varphi$ for $j \rightarrow \infty$ as a sequence of functions in $\mathcal{M}(\mathfrak{a}_{\mathbb{C}}^*, \mathcal{H}, d)$. Moreover, the sequence is bounded as a sequence in $\mathcal{P}^*(\mathfrak{a}_{\mathbb{C}}^*, \mathcal{H}, d, n)$.

In view of Lemmas 9.2 and 13.1 it follows that the sequence $\mathcal{L}_{F^*}\mathcal{F}f_j$ is bounded in $\mathcal{P}^*(\mathfrak{a}_{F^*}^*, \mathcal{H}_F, d', n') \otimes \mathcal{A}_F$ and that $\mathcal{L}_{F^*}\mathcal{F}f_j \rightarrow \mathcal{L}_{F^*}\varphi$ as a sequence in $\mathcal{M}(\mathfrak{a}_{F^*}^*, \mathcal{H}_F, d')$. By dominated convergence it follows that $\mathcal{L}_{F^*}\mathcal{F}f_j \rightarrow \mathcal{L}_{F^*}\varphi$ weakly in $\mathcal{P}^*(\mathfrak{a}_{F^*}^*, \mathcal{H}_F, d', n')$ with respect to the pairing (11.1). In view of Lemma 11.2 this implies that

$$f_j = \sum_F c_F \mathcal{J}_F \mathcal{L}_{F^*} \mathcal{F}f_j \rightarrow \sum_F c_F \mathcal{J}_F \mathcal{L}_{F^*} \varphi = f$$

weakly in $C^{-\infty}(X : \tau)$. Since f_j belongs to $C_{M+r_j}^\infty(X : \tau)$ for each j , we conclude that $f \in C_M^{-\infty}(X : \tau)$. Moreover, it follows from the weak convergence $f_j \rightarrow f$ that $\mathcal{F}f_j(\lambda) \rightarrow \mathcal{F}f(\lambda)$ for all λ outside the hyperplanes in \mathcal{H} . Hence, $\mathcal{F}f = \varphi$ as claimed.

Theorem 4.6 has now been proved.

Corollary 13.2. *The Fourier inversion formula (10.3)*

$$f = \sum_{F \subset \Delta} c_F \mathcal{J}_F \mathcal{L}_{F^*} \mathcal{F}f \tag{13.2}$$

is valid for $f \in C_c^{-\infty}(X : \tau)$.

Proof. It was seen during the proof above that $\sum_F c_F \mathcal{J}_F \mathcal{L}_{F^*} \varphi \in C_c^{-\infty}(X : \tau)$ and

$$\varphi = \mathcal{F} \left(\sum_F c_F \mathcal{J}_F \mathcal{L}_{F^*} \varphi \right)$$

for all $\varphi \in \text{PW}^*(X : \tau)$. In particular, the latter identity applies to $\varphi = \mathcal{F}f$ for each $f \in C_c^{-\infty}(X : \tau)$. The formula (13.2) then follows from the injectivity of \mathcal{F} (Theorem 8.1). □

14. A topological Paley–Wiener theorem

We shall equip the spaces $C_M^{-\infty}(X : \tau)$ and $\text{PW}_M^*(X : \tau)$ with natural topologies for which the Fourier transform is an isomorphism.

On the space of generalized functions on X we use the *strong dual topology*, where we regard $C^{-\infty}(X)$ as the dual space of $C_c^\infty(X)$. Recall that by definition, the strong dual topology on $C_c^\infty(X)^*$ is the locally convex topology given by the seminorm system

$$p_B(f) = \sup_{\varphi \in B} |f(\varphi)|,$$

where B belongs to the family of all bounded subsets of $C_c^\infty(X)$. Notice that $C_c^\infty(X)$ is a Montel space, that is, it is reflexive and a subset is bounded if and only if it is relatively compact (see [17, p. 147]).

On the space $C_c^{-\infty}(X)$ of compactly supported generalized functions on X we use the strong dual topology, where we regard $C_c^{-\infty}(X)$ as the dual space of $C^\infty(X)$. As an immediate consequence of these dualities, the inclusion map $C_c^{-\infty}(X) \rightarrow C^{-\infty}(X)$ is continuous, and multiplication by a function $\psi \in C_c^\infty$ is continuous $C^{-\infty}(X) \rightarrow C_c^{-\infty}(X)$.

The topologies on $C^{-\infty}(X)$ and $C_c^{-\infty}(X)$ induce the same topology on the space of distributions supported in a fixed compact subset Ω of X . This follows from the last remark of the preceding paragraph, when we take as ψ any function which is identically 1 on a neighbourhood of Ω .

In particular, for each $M > 0$ the space $C_M^{-\infty}(X : \tau)$ of τ -spherical generalized functions with support in X_M is topologized in this fashion, as a topological subspace of $C^{-\infty}(X) \otimes V_\tau$, or equivalently, as a topological subspace of $C_c^{-\infty}(X) \otimes V_\tau$.

Recall that $\text{PW}_M^*(X : \tau) \subset \mathcal{P}^*(\mathfrak{a}_q^*, \mathcal{H}(X, \tau), d_{X, \tau}) \otimes {}^\circ\mathcal{C}$, and that

$$\mathcal{P}^*(\mathfrak{a}_q^*, \mathcal{H}, d) = \bigcup_{n \in \mathcal{N}(\mathfrak{a}_q^*)} \mathcal{P}^*(\mathfrak{a}_q^*, \mathcal{H}, d, n).$$

On each space $\mathcal{P}^*(\mathfrak{a}_q^*, \mathcal{H}, d, n)$, where $d : \mathcal{H} \rightarrow \mathbb{N}$ and $n \in \mathcal{N}(\mathfrak{a}_q^*)$, the topology was defined by means of the seminorms (4.2). On $\mathcal{N}(\mathfrak{a}_q^*)$ we define an order relation by $n_1 \leq n_2$ if and only if $n_1(\omega) \leq n_2(\omega)$ for all ω . It is easily seen that if $n_1 \leq n_2$ then $\mathcal{P}^*(\mathfrak{a}_q^*, \mathcal{H}, d, n_1) \subset \mathcal{P}^*(\mathfrak{a}_q^*, \mathcal{H}, d, n_2)$ with continuous inclusion. The family of spaces $\mathcal{P}^*(\mathfrak{a}_q^*, \mathcal{H}, d, n)$ indexed by $n \in \mathcal{N}$ is thus a directed family, and we can give $\mathcal{P}^*(\mathfrak{a}_q^*, \mathcal{H}, d)$ the inductive limit topology for the union over n . The Paley–Wiener space $\text{PW}_M^*(X : \tau)$ is given the relative topology of this space (where $d = d_{X, \tau}$), tensored by ${}^\circ\mathcal{C}$.

Theorem 14.1. *The Fourier transform is a topological isomorphism of $C_M^{-\infty}(X : \tau)$ onto $\text{PW}_M^*(X : \tau)$, for each $M > 0$.*

Proof. Only the topological statement remains to be proved. We will prove that \mathcal{F} is continuous with respect to the topology of $C_c^{-\infty}(X) \otimes V_\tau$, and that its inverse is continuous into $C^{-\infty}(X) \otimes V_\tau$. Since the topologies agree on $C_M^{-\infty}(X : \tau)$, as remarked above, this will prove the theorem.

Let $\mathcal{H} = \mathcal{H}(X, \tau)$ and $d = d_{X, \tau}$. For the continuity of the Fourier transform

$$\mathcal{F} : C_c^{-\infty}(X : \tau) \rightarrow \mathcal{P}^*(\mathfrak{a}_q^*, \mathcal{H}, d) \otimes {}^\circ\mathcal{C} \tag{14.1}$$

we remark that by a theorem of Grothendieck, $C_c^{-\infty}$ is bornological, since it is the strong dual of the reflexive Fréchet space C^∞ (see [17, p. 154]). Therefore, it suffices to prove that the Fourier transform maps every bounded set $B \subset C_c^{-\infty}(X : \tau)$ to a bounded set in $\mathcal{P}^*(\mathfrak{a}_q^*, \mathcal{H}, d) \otimes {}^\circ\mathcal{C}$ (see [17, p. 62]). Since all such sets B are equicontinuous (see [17, p. 127]) we may assume that there exists a continuous seminorm ν on $C^\infty(X : \tau^*)$ such that

$$B \subset \left\{ f \mid \left| \int \varphi(x)f(x) \, dx \right| \leq \nu(\varphi), \forall \varphi \in C^\infty(X : \tau^*) \right\}. \tag{14.2}$$

As in (3.2) we may assume that the seminorm ν has the form

$$\nu(\varphi) = C \sup_{|\alpha| \leq k, x \in \Omega} \|L_{X^\alpha} \varphi(x)\|$$

for some $C > 0$, $k \in \mathbb{N}$ and $\Omega \subset X$ compact. Choose $M > 0$ such that $\Omega \subset X_M$, and let $R \in \mathbb{R}$. It follows from (6.2) that there exists a number $n \in \mathbb{N}$ and a polynomial $p \in \Pi_\Sigma(\mathfrak{a}_q^*)$ such that

$$\sup_{\lambda \in \bar{\mathfrak{a}}_q^*(P, R)} (1 + |\lambda|)^{-n} e^{-M|\operatorname{Re} \lambda|} \nu(p(\lambda)E^*(\lambda : \cdot)) < \infty. \tag{14.3}$$

It now follows from (14.2) with $\varphi = p(\lambda)E^*(\lambda : \cdot)$, combined with (14.3), that

$$\sup_{f \in B} \sup_{\lambda \in \bar{\mathfrak{a}}_q^*(P, R)} (1 + |\lambda|)^{-n} e^{-M|\operatorname{Re} \lambda|} \|p(\lambda)\mathcal{F}f(\lambda)\| < \infty. \tag{14.4}$$

For each compact set $\omega \subset \mathfrak{a}_q^*$ we choose $R \in \mathbb{R}$ such that $\omega \subset \bar{\mathfrak{a}}_q^*(P, R)$ and define $n(\omega)$ to be the number n in (14.4). By application of Lemma 5.1 it follows that the seminorm (4.2) is uniformly bounded on $\mathcal{F}(B)$. With $n \in \mathcal{N}(\mathfrak{a}_q^*)$ chosen in this fashion, we thus see that $\mathcal{F}(B)$ is contained and bounded in $\mathcal{P}^*(\mathfrak{a}_q^*, \mathcal{H}, d, n) \otimes {}^\circ\mathcal{C}$, hence also in the union $\mathcal{P}^*(\mathfrak{a}_q^*, \mathcal{H}, d) \otimes {}^\circ\mathcal{C}$ with the inductive limit topology. Thus (14.1) is continuous.

In order to establish the continuity of the inverse Fourier transform we use Corollary 13.2, according to which the inverse Fourier transform is given by the finite sum of c_F times $\mathcal{J}_F \mathcal{L}_{F^*}$. The operator \mathcal{L}_{F^*} is continuous from $\mathcal{P}^*(\mathfrak{a}_q^*, \mathcal{H}(X, \tau), d_{X, \tau}) \otimes {}^\circ\mathcal{C}$ to $\mathcal{P}^*(\mathfrak{a}_{Fq}^*, \mathcal{H}(X, \tau, F), d_{X, \tau, F}) \otimes \mathcal{A}_F$ by Lemma 13.1, and continuity of \mathcal{J}_F is established in the lemma below. □

Lemma 14.2. *The generalized wave packet operator \mathcal{J}_F is strongly continuous*

$$\mathcal{P}^*(\mathfrak{a}_{Fq}^*, \mathcal{H}(X, \tau, F), d_{X, \tau, F}) \otimes \mathcal{A}_F \rightarrow C^{-\infty}(X : \tau)$$

for each $F \subset \Delta$.

Proof. Let $B \subset C_c^\infty(X : \tau)$ be bounded, then $\mathcal{F}_F(B)$ is bounded by Lemma 9.3, and it follows from (11.2) that

$$p_B(\mathcal{J}_F \varphi) = \sup_{f \in B} |\langle f, \mathcal{J}_F \varphi \rangle| = \sup_{f \in B} |\langle \mathcal{F}_F f, \varphi \rangle|.$$

Hence $p_B \circ \mathcal{J}_F$ is continuous by Lemma 11.1. □

15. Further properties of the topology

In this final section we show that $\mathcal{P}^*(\mathfrak{a}_q^*, \mathcal{H}, d, n)$ is a Fréchet space, under a certain natural condition on $n \in \mathcal{N}(\mathfrak{a}_q^*)$, and that this condition is satisfied by sufficiently many elements n to give the same union $\mathcal{P}^*(\mathfrak{a}_q^*, \mathcal{H}, d)$.

Let \mathcal{H} be a real Σ -configuration in $\mathfrak{a}_{q\mathbb{C}}^*$, and let $d : \mathcal{H} \rightarrow \mathbb{N}$ be arbitrary. For $\omega \in \mathcal{C}(\mathfrak{a}_q^*)$ and $n \in \mathcal{N}(\mathfrak{a}_q^*)$, we denote by $\nu_{\omega, n}^*$ the seminorm on $\mathcal{P}^*(\mathfrak{a}_q^*, \mathcal{H}, d, n)$ defined in (4.2).

Definition 15.1. A function $n \in \mathcal{N}(\mathfrak{a}_q^*)$ is called *regular* if for every $\omega \in \mathcal{C}(\mathfrak{a}_q^*)$,

$$n(\omega) = \inf\{n(\omega') \mid \omega' \in \mathcal{C}(\mathfrak{a}_q^*), \omega \subset \text{int}(\omega')\}.$$

The set of such functions n is denoted $\mathcal{N}_0(\mathfrak{a}_q^*)$.

Notice that if n is regular, then $n(\omega_1) \leq n(\omega_2)$ for $\omega_1 \subset \omega_2$. It is also easy to see that if n_1, n_2 are regular, then so is $n = \max(n_1, n_2)$.

Lemma 15.2. *Let $n \in \mathcal{N}_0(\mathfrak{a}_q^*)$. Then $\mathcal{P}^*(\mathfrak{a}_q^*, \mathcal{H}, d, n)$ is a Fréchet space.*

Proof. It is easily seen that \mathcal{P}_n^* is complete (also without the condition of regularity). We have to show that it is metrizable. We will do this by pointing out a countable collection of compact sets ω , such that the corresponding family of seminorms $\nu_{\omega, n}^*$ generates the topology.

The topology of \mathfrak{a}_q^* is locally compact and second countable. Let \mathcal{B} be a countable basis of open sets with compact closures. For every finite collection $F \subset \mathcal{B}$, let

$$\omega_F = \bigcup_{B \in F} \bar{B}.$$

Then $\omega_F \in \mathcal{C}(\mathfrak{a}_q^*)$. We will show that the countable family $\nu_{\omega_F, n}^*$ generates the topology.

Indeed, let $\omega \in \mathcal{C}(\mathfrak{a}_q^*)$. Then by regularity there exists $\omega' \in \mathcal{C}(\mathfrak{a}_q^*)$, with $n(\omega') = n(\omega)$, and $\omega \subset \text{int}(\omega')$. If $\omega'' \in \mathcal{C}(\mathfrak{a}_q^*)$ satisfies $\omega \subset \omega'' \subset \omega'$, it follows that $n(\omega) = n(\omega'') = n(\omega')$. We now see that we may take ω' so that in addition $\pi_{\omega', d} = \pi_{\omega, d}$.

For every $\lambda \in \omega$ there exists $B \in \mathcal{B}$ with $\lambda \in B$ and $\bar{B} \subset \text{int}(\omega')$. By compactness, there exists a finite collection $F \subset \mathcal{B}$ such that $\omega \subset \omega_F \subset \text{int}(\omega')$. It follows that $n(\omega) = n(\omega_F)$ and $\pi_{\omega, d} = \pi_{\omega_F, d}$. Hence

$$\begin{aligned} \nu_{\omega, n}^*(\varphi) &= \sup_{\omega + i\mathfrak{a}_q^*} (1 + |\lambda|)^{-n(\omega_F)} \|\pi_{\omega_F, d}(\lambda)\varphi(\lambda)\| \\ &\leq \sup_{\omega_F + i\mathfrak{a}_q^*} (1 + |\lambda|)^{-n(\omega_F)} \|\pi_{\omega_F, d}(\lambda)\varphi(\lambda)\| \\ &= \nu_{\omega_F, n}^*(\varphi) \end{aligned}$$

for all φ . □

Proposition 15.3. *For each $n \in \mathcal{N}(\mathfrak{a}_q^*)$ there exists a function $\bar{n} \in \mathcal{N}_0(\mathfrak{a}_q^*)$ such that*

$$\mathcal{P}^*(\mathfrak{a}_q^*, \mathcal{H}, d, n) \subset \mathcal{P}^*(\mathfrak{a}_q^*, \mathcal{H}, d, \bar{n})$$

with continuous inclusion.

Proof. We define the map $\bar{n} : \mathcal{C}(\mathfrak{a}_q^*) \rightarrow \mathbb{N}$ by

$$\bar{n}(\omega) = \inf\{n(\omega') \mid \omega' \in \mathcal{C}(\mathfrak{a}_q^*), \omega \subset \text{int}(\omega')\}.$$

It is easily seen that \bar{n} is regular.

Let ω be a compact set. Then there exists a compact neighbourhood ω_1 of ω such that $\bar{n}(\omega) = n(\omega_1)$. It follows from Lemma 5.1 that there exists a constant $C > 0$ such that for all $\varphi \in \mathcal{M}(\mathfrak{a}_q^*, \mathcal{H}, d, \circ\mathcal{C})$,

$$\nu_{\omega, \bar{n}}^*(\varphi) = \nu_{\omega, -\bar{n}(\omega)}(\varphi) \leq C\nu_{\omega_1, -\bar{n}(\omega)}(\varphi) = C\nu_{\omega_1, -n(\omega_1)}(\varphi) = C\nu_{\omega_1, n}^*(\varphi).$$

The result follows. □

In the above we have proved that there exists a subset \mathcal{N}_0 of $\mathcal{N} = \mathcal{N}(\mathfrak{a}_q^*)$ with the following properties (where $\mathcal{P}_n^* = \mathcal{P}^*(\mathfrak{a}_q^*, \mathcal{H}, d, n)$ for simplicity):

- (a) for every $n \in \mathcal{N}_0$ the space \mathcal{P}_n^* is Fréchet;
- (b) for every $n_1, n_2 \in \mathcal{N}_0$ there exists $n \in \mathcal{N}_0$ such that $n_1 \leq n, n_2 \leq n$ (directed family);
- (c) for every $n \in \mathcal{N}$ there exists $m \in \mathcal{N}_0$ such that $\mathcal{P}_n^* \subset \mathcal{P}_m^*$ with continuous inclusion.

Because of these properties, the union $\bigcup_{n \in \mathcal{N}_0} \mathcal{P}_n^*$ is equal to $\bigcup_{n \in \mathcal{N}} \mathcal{P}_n^* = \mathcal{P}^*(\mathfrak{a}_q^*, \mathcal{H}, d)$ and the limit topologies defined by $\lim_{\mathcal{N}_0} \mathcal{P}_n^*$ and $\lim_{\mathcal{N}} \mathcal{P}_n^*$ are equal. In particular, we see that $\mathcal{P}^*(\mathfrak{a}_q^*, \mathcal{H}, d)$ is an inductive limit of Fréchet spaces (notice however, that it is not necessarily a *strict* inductive limit).

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