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Estimate, existence and nonexistence of positive solutions of Hardy–Hénon equations

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We consider the boundary Hardy–Hénon equation

$$-\Delta u=(1-|x|)^{\alpha}u^{p},\quad x\in B_1(0),$$

where $B_1(0)\subset\mathbb{R}^N$ $(N\geqslant 3)$ is a ball of radial 1 centred at 0, $p>0$ and $\alpha\in\mathbb{R}$. We are concerned with the estimate, existence and nonexistence of positive solutions of the equation, in particular, the equation with Dirichlet boundary condition. For the case $0<p<(N+2)/(N-2)$, we establish the estimate of positive solutions. When $\alpha\leqslant -2$ and $p>1$, we give some conclusions with respect to nonexistence. When $\alpha>-2$ and $1<p<(N+2)/(N-2)$, we obtain the existence of positive solution for the corresponding Dirichlet problem. When $0<p\leqslant 1$ and $\alpha\leqslant -2$, we show the nonexistence of positive solutions. When $0<p<1$, $\alpha>-2$, we give some results with respect to existence and uniqueness of positive solutions.

Keywords: Singularity; Estimate; Existence; Nonexistence

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1. Introduction

This paper is devoted to the study of positive solutions of the following elliptic equation

$$-\Delta u=(1-|x|)^{\alpha}u^{p},\quad x\in B_1(0),\tag{1.1}$$

where $p>0$, $\alpha\in\mathbb{R}$, and $B_1(0)\subset\mathbb{R}^N$ $(N\geqslant 3)$ denotes a ball of radius 1 centred at 0. We will establish some estimate, existence and nonexistence of positive solutions of (1.1). In particular, we are interested in the existence and uniqueness of positive solutions of the following Dirichlet problem

$$\begin{cases} -\Delta u = (1-|x|)^{\alpha} u^p, & x \in B_1(0), \\ u = 0, & |x| = 1. \end{cases} \tag{1.2}$$

In the equation (1.1), it is clear that $d(x, \partial B_1(0)) = 1 - |x|$ for $x \in B_1(0)$. This weight function $(1-|x|)^{\alpha}$ is singular or vanishing on the boundary of $B_1(0)$ when $\alpha \neq 0$.

Over the last few decades, the following elliptic equation

$$-\Delta u = a(x) u^p \quad \text{in } \Omega, \tag{1.3}$$

where Ω is a domain in $\mathbb{R}^N$, has been extensively studied under various assumptions. When $a(x) \equiv 1$, the equation is the well-known Lane–Emden equation. For the case, there was a great deal of work such as the existence, nonexistence, symmetry and uniqueness of positive solutions. For example, some interesting results in [**5**, **10**] are related to the symmetry of positive solutions of (1.3) with $\Omega = \mathbb{R}^N$. If $a(x) = |x|^{\alpha}$ and $0 \in \Omega$, the equation (1.3) is called Hardy–Hénon equation. When $\alpha \leqslant -2$ and $p > 1$, the Hardy–Hénon equation (1.3) has no positive solutions in any domain Ω containing the origin (see [**6**]). For the case $\alpha > -2$ and $p < (N+2+2\alpha)/(N-2)$, the Hardy–Hénon equation (1.3) with $\Omega = \mathbb{R}^N$ has no positive radial solution (refer to [**14**]). In addition, Du and Guo in [**8**] investigated the Hardy–Hénon equation (1.3) for the case $p < 0$ and $\alpha > -2$. When $a(x) = -d(x, \partial\Omega)^{\alpha}$, there were much well-known study with respect to boundary blow-up solution (also called large solution) of (1.3), for instance, the existence and uniqueness of large solution, and blow-up rate of large solution of (1.3) (see [**7**]). When $a(x) = |x|^{\alpha}$ and $\Omega = B_1(0)$, Cao–Peng–Yan [**4**] analysed the profile of ground state solution and proved the existence of multi-peaked solutions with their asymptotic behaviour for equation (1.3) subject to Dirichlet boundary condition.

In this paper, we are more interested in

$$\text{the case: the weight function } a(x) = (1-|x|)^{\alpha} \text{ and } \Omega = B_1(0).$$

In fact, the weight function $a(x) = d(x, \partial\Omega)^{\alpha} (= d(x)^{\alpha})$. Clearly, for the case $\alpha > 0$, $(1-|x|)^{\alpha}$ converges to zero as $|x| \to 1$, and for the case $\alpha < 0$, $(1-|x|)^{\alpha}$ blows up as $|x| \to 1$. Corresponding to the boundary Hardy potential $1/d(x)^2$, the equation (1.1) is called boundary Hardy–Hénon equation in this paper in order to distinguish from the well-known Hardy–Hénon equations.

Some elliptic equations with coefficient function $d(x)$ were extensively considered. Especially, for the Hardy potential $1/d(x)^2$, there are many interesting problems and results. The well-known Hardy constant and Hardy inequality were established in [**3**, **12**]. For elliptic equations with such Hardy potential, in [**1**], Bandle, Moroz and Reichel considered

$$-\Delta u = \lambda \frac{u}{d(x)^2} - d(x)^{\alpha} u^p \quad \text{in } \Omega, \tag{1.4}$$

and gave some classification of positive solutions under conditions $p > 1$, $\alpha > -2$, and $\lambda \leqslant 1/4$. For the case $\lambda > 1/4$, $\alpha > -2$ and $p > 1$, the uniqueness and asymptotic behaviour of positive solutions of (1.4) were obtained in [**9**]. In [**2**], Bandle

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and Pozio investigated the equation (1.4) with the sublinear term. More recently, in [**13**] Mercuri and Santos analysed the quantitative symmetry breaking of ground states for the following weighted Emden–Fowler equations

$$\begin{cases} -\Delta u = V_\alpha(|x|)|u|^{p-1}u, & x \in B_1(0), \\ u = 0, & x \in \partial B_1(0), \end{cases} \tag{1.5}$$

where $B_1(0) \subset \mathbb{R}^N$ $(N \geqslant 1)$, $p \in (1, 2^* - 1)$ with $2^* = 2N/(N-2)$ if $N \geqslant 3$ and $2^* = +\infty$ if $N = 1, 2$, and V_α $(\alpha > 0)$ defined as:

(i) for $R \in (0,1)$, $V_\alpha(r) = (1 - (r/R))^\alpha$ if $r \in [0, R)$ and $V_\alpha(r) = (1 - ((1-r)/(1-R)))^\alpha$ if $r \in [R, 1]$;

(ii) for $R = 0$, $V_\alpha(r) = r^\alpha$ if $r \in [0,1]$; for $R = 1$, $V_\alpha(r) = (1-r)^\alpha$ if $r \in [0,1]$.

In [**13**], some interesting quantitative results in regard to (1.5) were presented, for example, [**13**, proposition 1.8], which indicate that for the positive ground state solution u_α of (1.5) with $\alpha > 0$, $R = 1$ and $N \geqslant 3$ (i.e. (1.2)), there exist positive constants C_1 and C_2 such that $C_1\alpha^{2/(p-1)} \leqslant \max_{x\in\overline{B}(0,1)} u_\alpha(x) \leqslant C_2\alpha^{2/(p-1)}$ for large α. In contrast to [**13**, proposition 1.8], our theorem 1.3 is to establish the existence of positive solutions for (1.2) with $\alpha > -2$ and $p \in (1, 2^* - 1)$.

In consideration of above interesting work, in this article, we consider the following equation

$$-\Delta u = d(x)^\alpha u^p \text{ in } \Omega.$$

For convenience and brevity, we mainly study the special domain $B_1 := B_1(0)$, that is equation (1.1). We will investigate the estimate, existence and nonexistence of positive solutions of (1.1) and (1.2) in view of the weight function $(1 - |x|)^\alpha$. Throughout this paper, unless otherwise stated, a solution u of (1.2) is referred to classical solution, that is $u \in C^2(B_1) \cap C(\bar{B}_1)$. For all conclusions in this paper, we need the condition $N \geqslant 3$, which plays an important role in some proofs, so we assume always $N \geqslant 3$ throughout this paper.

By using blow-up method and some analysis technique we can obtain the following estimate of positive solutions of (1.1).

Theorem 1.1.

(i) *If $1 < p < (N+2)/(N-2)$, $\alpha > -2$, then there exists $C = C(N, p, \alpha)$ such that any positive solution u of (1.1) satisfies*

$$u(x) \leqslant C(1 - |x|)^{-((2+\alpha)/(p-1))}, \quad x \in B_1. \tag{1.6}$$

(ii) *If $0 < p < 1$ and $\alpha > -2$, then there exists $C = C(N, p, \alpha)$ such that any positive solution u of (1.1) satisfies*

$$u(x) \geqslant C(1 - |x|)^{-((2+\alpha)/(p-1))}, \quad x \in B_1. \tag{1.7}$$

For the case $\alpha \leqslant -2$, some results on nonexistence of positive solutions of (1.1) are established, which are contained in the following theorem.

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Theorem 1.2.

(i) *Let $p > 1$ and $\alpha \leqslant -2$. Then* (1.1) *has no positive solutions with a positive lower bound.*

(ii) *Let $p > 1$ and $\alpha + p + 2 \leqslant 0$. Then* (1.1) *has no positive solutions.*

(iii) *Let $1 < p < (N+2)/(N-2)$ and $p + 1 + \alpha < 0$. Then* (1.1) *has no positive solutions.*

For the Dirichelt problem (1.2), under $\alpha > -2$ and subcritical nonlinear term, we can obtain the existence of positive classical solutions.

Theorem 1.3. *Let $-2 < \alpha$ and $1 < p < (N+2)/(N-2)$. Then the problem* (1.2) *has a positive solution.*

For the sublinear case $0 < p < 1$, together with some estimate of positive solutions and some analysis, we can obtain the following two results of nonexistence.

Theorem 1.4. *Let $0 < p < 1$ and $1 + p + \alpha < 0$. Then* (1.2) *has no positive solutions in $C^1(\bar{B}_1)$.*

Theorem 1.5. *Let $0 < p \leqslant 1$ and $\alpha \leqslant -2$. Then* (1.1) *has no positive solutions.*

Finally, by using the subsolution and supersolution method, the existence of positive solution is established for the case $0 < p < 1$ and $\alpha > -2$, and by the maximum principle and Hopf's Lemma, we can establish the uniqueness of positive solutions. Concretely, we have the following theorem.

Theorem 1.6.

(i) *Suppose that $\alpha > -2$ and $p < 1$. Then* (1.2) *has a positive classical solution. Moreover, if $p \leqslant 0$, the positive solution of* (1.2) *is unique.*

(ii) *Suppose that $\psi \in C^1(\bar{B}_1)$ is a nonnegative function, $\alpha \geqslant 0$ and $0 < p < 1$. Then the following problem*

$$\begin{cases} -\Delta u = (1-|x|)^\alpha u^p, & x \in B_1, \\ u = \psi, & |x| = 1, \end{cases} \tag{1.8}$$

has a unique positive solution in $C^2(B_1) \cap C^1(\bar{B}_1)$.

At the end of introduction, we point out that it is challenging to deal with the existence and nonexistence of positive solutions for the more general problem

$$-\Delta u = a(x) u^p \text{ in } \Omega,$$

where Ω is a bounded and smooth domain, and $a(x) \in C(\Omega)$ satisfies

$$c_1 d(x)^\alpha \leqslant a(x) \leqslant c_2 d(x)^\alpha \text{ in } \Omega,$$

with $d(x) = d(x, \partial\Omega)$ and constants $c_i > 0$ $(i = 1, 2)$. Due to the limited length of the paper, for the more general case we will develop some other technique and methods to establish the similar results to this paper in the near future.

The rest of this paper is organized as follows. In § 2 we mainly consider the equation (1.1) with the superlinear nonlinear term. Firstly, we give estimate of positive solutions, i.e., lemma 2.2, and then give the proof of theorem 1.2. We also establish the estimate of positive radial solutions under small perturbation, i.e., lemma 2.3, and then complete the proof of theorem 1.3. In § 3, we study the existence and nonexistence of positive solutions to (1.1) with sublinear nonlinear term. We establish the estimate of positive solutions, i.e., lemma 3.2. Finally, we prove theorems 1.4–1.6.

2. The case $p > 1$

The following lemma can be found in [**14**], which will be used for the estimate of positive solutions.

LEMMA 2.1. *Let $N \geqslant 3$, $1 < p < (N+2)/(N-2)$, and $\mu \in (0,1]$. Let $a \in C^{\mu}(\bar{B}_1)$ satisfy*

$$\|a\|_{C^{\mu}(\bar{B}_1)} \leqslant C_1 \text{ and } a(x) \geqslant C_2,\ x \in \bar{B}_1,$$

for some constants $C_1, C_2 > 0$. There exists $C > 0$, depending only on μ, C_1, C_2, p, N, such that, for any nonnegative classical solution u of

$$-\Delta u = a(x)u^p,\ x \in B_1,$$

u satisfies

$$|u(x)|^{\frac{p-1}{2}} + |\nabla u(x)|^{((p-1)/(p+1))} \leqslant C\left(1 + \frac{1}{1-|x|}\right),\quad x \in B_1. \tag{2.1}$$

Be based on lemma 2.1, the following lemma 2.2 can be derived.

LEMMA 2.2. *Let $1 < p < (N+2)/(N-2)$. There exists $C = C(N,p,\alpha)$ such that any nonnegative solution u of (1.1) satisfies*

$$\begin{aligned} u(x) &\leqslant C(1-|x|)^{-((2+\alpha)/(p-1))} \quad \text{and} \\ |\nabla u(x)| &\leqslant C(1-|x|)^{-((p+1+\alpha)/(p-1))},\quad 1/2 \leqslant |x| < 1. \end{aligned} \tag{2.2}$$

Proof. Let x_0 be an arbitrary point in B_1. We define a function by

$$U(x) = d(x_0, \partial B_1)^{((2+\alpha)/(p-1))} u\left(x_0 + \frac{d(x_0,\partial B_1)}{2}x\right),\quad x \in B_1.$$

Then U satisfies

$$-\Delta U = a(x;x_0)U^p,\quad x \in B_1,$$

where

$$a(x;x_0) = \frac{d\left(x_0 + \frac{d(x_0,\partial B_1)}{2}x, \partial B_1\right)^{\alpha}}{4d(x_0,\partial B_1)^{\alpha}}.$$

Clearly, for any $x \in B_1$, we have that

$$a(x;x_0) \geqslant \frac{1}{2^{\alpha+2}} \quad \text{as } \alpha \geqslant 0$$

and

$$a(x;x_0) \geqslant \frac{3^\alpha}{2^{2+\alpha}} \quad \text{as } \alpha < 0.$$

We claim that for all x_0 satisfying $|x_0| \geqslant 1/2$,

$$\|a(\cdot;x_0)\|_{C^1(\bar{B}_1)} \leqslant C, \tag{2.3}$$

where C depends only on α. In fact, for any $x \in B_1$ and $x_0 \in B_1$, $a(x;x_0)$ can be written by

$$a(x;x_0) = \frac{1}{4} \cdot \left(\frac{1 - \left| x_0 + \frac{1-|x_0|}{2} x \right|}{1 - |x_0|} \right)^\alpha .$$

It follows that for $\alpha \geqslant 0$

$$4a(x;x_0) = \left(\frac{1 - \left| x_0 + \frac{1-|x_0|}{2} x \right|}{1 - |x_0|} \right)^\alpha \leqslant \left(\frac{3(1-|x_0|)/2}{1-|x_0|} \right)^\alpha = \left(\frac{3}{2} \right)^\alpha .$$

As $\alpha < 0$, we have

$$4a(x;x_0) = \left(\frac{1 - \left| x_0 + \frac{1-|x_0|}{2} x \right|}{1 - |x_0|} \right)^\alpha \leqslant \left(\frac{(1-|x_0|)/2}{1-|x_0|} \right)^\alpha = \left(\frac{1}{2} \right)^\alpha .$$

In addition, it is clear that

$$|D_i a(x;x_0)| = \left| \frac{\alpha}{8} \left(\frac{1 - \left| x_0 + \frac{1-|x_0|}{2} x \right|}{1 - |x_0|} \right)^{\alpha-1} \cdot \frac{x_0^i + \frac{1-|x_0|}{2} x^i}{|x_0 + \frac{1-|x_0|}{2} x|} \right| ,$$

where x^i and x_0^i denote the i-th component of x and x_0 respectively. So, if $\alpha \geqslant 1$ and $|x_0| \geqslant 1/2$, then we obtain that

$$|D_i a(x;x_0)| \leqslant \frac{1}{4} \cdot \left(\frac{3}{2} \right)^{\alpha-1} \cdot \frac{\alpha}{8} \cdot \left| \frac{x_0^i + \frac{1-|x_0|}{2} x^i}{x_0 + \frac{1-|x_0|}{2} x} \right| \leqslant \frac{\alpha}{32} \left(\frac{3}{2} \right)^{\alpha-1}, \quad \forall x \in B_1.$$

If $\alpha < 1$ and $|x_0| \geqslant 1/2$, then we have that

$$|D_i a(x;x_0)| \leqslant \frac{1}{4} \cdot \left(\frac{3}{2} \right)^{\alpha-1} \cdot \frac{|\alpha|}{8} \cdot \left| \frac{x_0^i + \frac{1-|x_0|}{2} x^i}{x_0 + \frac{1-|x_0|}{2} x} \right| \leqslant \frac{|\alpha|}{32} \left(\frac{1}{2} \right)^{\alpha-1}, \quad \forall x \in B_1.$$

Therefore, applying lemma 2.1, we have

$$U(0) + |\nabla U(0)| \leqslant C.$$

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Further, we obtain

$$u(x_0) \leqslant Cd(x_0, \partial B_1)^{-((2+\alpha)/(p-1))},$$
$$|\nabla u(x_0)| \leqslant Cd(x_0, \partial B_1)^{-((p+1+\alpha)/(p-1))}.$$

By the arbitrariness of x_0 and $d(x_0, \partial B_1) = 1 - |x_0|$, we can obtain the desired conclusion. □

Applying lemmas 2.1 and 2.2, we can present a proof of theorem 1.1 (i).

Proof of theorem 1.1 (i). On the one hand, by lemma 2.2 there is $C = C(N, p, \alpha)$ such that any positive solution u of (1.1) satisfies

$$u(x) \leqslant C(1 - |x|)^{-\frac{2+\alpha}{p-1}}, \quad x \in B_1 \backslash B_{1/2}.$$

On the other hand, noticing that $\frac{1}{2} \leqslant 1 - |x| \leqslant 1$, $\forall x \in \overline{B}_{1/2}$ and that by lemma 2.1 there exists $C = C(N, p, \alpha)$ such that $|u(x)| \leqslant C$, $\forall x \in \overline{B}_{1/2}$ for any positive solution u of (1.1), we know that there is $C = C(N, p, \alpha)$ such that

$$u(x) \leqslant C(1 - |x|)^{-\frac{2+\alpha}{p-1}}, \quad x \in \overline{B}_{1/2}.$$

The proof is completed. □

Next, we are going to prove theorem 1.2 by analysing the corresponding integral average of positive solutions and together with lemma 2.2.

Proof of theorem 1.2. We argue indirectly by assuming that $u \in C^2(B_1)$ is a positive solution of (1.1). Using spherical coordinates to write $u(x) = u(r, \theta)$ with $r = |x|$ and $\theta = \frac{x}{|x|}$, we have

$$u_{rr} + \frac{N-1}{r} u_r + \frac{1}{r^2} \Delta_{S^{N-1}} u = -(1 - r)^\alpha u^p, \quad r \in (0, 1). \tag{2.4}$$

Let

$$\tilde{u}(r) = \frac{1}{|S^{N-1}|} \int_{S^{N-1}} u(r, \theta)\, \mathrm{d}\theta.$$

From the above equation for $u(r, \theta)$ it follows that

$$\tilde{u}_{rr} + \frac{N-1}{r} \tilde{u}_r = -\frac{(1-r)^\alpha}{|S^{N-1}|} \int_{S^{N-1}} u(r, \theta)^p\, \mathrm{d}\theta. \tag{2.5}$$

So, we have

$$(r^{N-1} \tilde{u}'(r))' < 0 \text{ for } r \in (0, 1).$$

Therefore, it is clear that $r^{N-1}\tilde{u}'$ is decreasing, and hence has a limit $m \in [-\infty, +\infty)$ as $r \to 1^-$. In addition, by Jensen's inequality for (2.5) we obtain

$$-(r^{N-1} \tilde{u}')' \geqslant (1 - r)^\alpha r^{N-1} \tilde{u}^p \text{ for } r \in (0, 1). \tag{2.6}$$

We firstly prove the conclusion (i). We divide the proof into two cases for clarity. Suppose that u has a positive lower bound.

Case 1. Suppose that $m \geqslant 0$. Then

$$r^{N-1}\tilde{u}'(r) > m \text{ for } r \in (0,1).$$

Therefore, $\tilde{u}'(r) > 0$ holds for all $r \in (0,1)$. So, we can assume $\tilde{u}(r) \to m_1 > 0$ as $r \to 1^-$. Take $r_1 \in (0,1)$ such that $\tilde{u}(r) > m_1/2$ for all $r \in (r_1, 1)$. From (2.6) it follows that for $r \in (r_1, 1)$

$$r_1^{N-1}\tilde{u}'(r_1) - r^{N-1}\tilde{u}'(r) = -\int_{r_1}^{r} (\tau^{N-1}\tilde{u}'(\tau))'\,\mathrm{d}\tau$$

$$\geqslant \int_{r_1}^{r} (1-\tau)^{\alpha}\tau^{N-1}\tilde{u}(\tau)^p\,\mathrm{d}\tau \geqslant 2^{-p} r_1^{N-1} m_1^p \int_{r_1}^{r} (1-\tau)^{\alpha}\,\mathrm{d}\tau.$$

Hence, we have

$$r_1^{N-1}\tilde{u}'(r_1) \geqslant 2^{-p} r_1^{N-1} m_1^p \int_{r_1}^{r} (1-\tau)^{\alpha}\,\mathrm{d}\tau.$$

Letting $r \to 1^-$, in view of $\alpha \leqslant -2$, we obtain a contradiction.

Case 2. $m \in [-\infty, 0)$. For the case, there exist $r_* > 0$ and $m_2 > 0$ such that

$$r^{N-1}\tilde{u}'(r) < -m_2 \text{ for all } r \in (r_*, 1),$$

and hence there is $m_* \in (0, m_2]$ such that

$$\tilde{u}'(r) < -m_* \quad \text{for } r \in (r_*, 1).$$

Since u has a positive lower bound, we can assume

$$\tilde{u}(r) \to m_3 \in (0, \infty) \text{ as } r \to 1^-.$$

Clearly,

$$\tilde{u}(r) > m_3 \quad \text{for all } r \in (r_*, 1).$$

From (2.6) it follows that

$$r_*^{N-1}\tilde{u}'(r_*) - r^{N-1}\tilde{u}'(r) \geqslant m_3^p \int_{r_*}^{r} (1-\tau)^{\alpha}\tau^{N-1}\,\mathrm{d}\tau \text{ for } r \in (r_*, 1).$$

Further, we have

$$-r^{N-1}\tilde{u}'(r) \geqslant m_3^p \int_{r_*}^{r} (1-\tau)^{\alpha}\tau^{N-1}\,\mathrm{d}\tau \geqslant m_3^p r_*^{N-1} \int_{r_*}^{r} (1-\tau)^{\alpha}\,\mathrm{d}\tau \text{ for } r \in (r_*, 1).$$

So, we obtain

$$\tilde{u}'(r) \leqslant -m_3^p r_*^{N-1} r^{1-N} \int_{r_*}^{r} (1-\tau)^{\alpha}\,\mathrm{d}\tau \leqslant -m_3^p r_*^{N-1} \int_{r_*}^{r} (1-\tau)^{\alpha}\,\mathrm{d}\tau \text{ for } r \in (r_*, 1).$$

Integrating the above inequality from r_* to r, we see

$$\tilde{u}(r) - \tilde{u}(r_*) \leqslant -m_3^p r_*^{N-1} \int_{r_*}^{r} \left(\int_{r_*}^{t} (1-\tau)^{\alpha}\,\mathrm{d}\tau \right) \mathrm{d}t.$$

By the condition $\alpha \leqslant -2$, the right-hand side converges to $-\infty$ as $r \to 1^-$. Therefore, we obtain a contradiction, and hence complete the proof of the conclusion (i).

Now, we prove the conclusion (ii). As the arguments of the proof for conclusion (i), we can derive a contradiction for the case 1. For the case 2, we have that

$$\tilde{u}'(r) < -m_* \text{ for } r \in (r_*, 1), \text{ and } \tilde{u}(r) \to m_3 \in [0, +\infty) \text{ as } r \to 1^-.$$

If $m_3 \neq 0$ holds, we can obtain a contradiction as the arguments for case 2 in the proof of (i). Now, we assume $m_3 = 0$. By the differential mean value theorem, there holds

$$\tilde{u}(r) \geqslant m_*(1-r) \text{ for } r \in (r_*, 1).$$

From (2.6) it follows that for $r \in (r_*, 1)$

$$r_*^{N-1}\tilde{u}'(r_*) - r^{N-1}\tilde{u}'(r) \geqslant m_*^p \int_{r_*}^{r} (1-\tau)^{\alpha+p}\tau^{N-1}\,\mathrm{d}\tau.$$

Hence, we see

$$-\tilde{u}'(r) \geqslant m_*^p r^{1-N} \int_{r_*}^{r} (1-\tau)^{\alpha+p}\tau^{N-1}\,\mathrm{d}\tau \geqslant m_*^p \int_{r_*}^{r} (1-\tau)^{\alpha+p}\tau^{N-1}\,\mathrm{d}\tau.$$

Therefore, we obtain that

$$\tilde{u}(r_*) - \tilde{u}(r) \geqslant m_*^p r_*^{N-1} \int_{r_*}^{r} \int_{r_*}^{t} (1-\tau)^{\alpha+p}\,\mathrm{d}\tau\,\mathrm{d}t.$$

Since $\alpha + p + 2 \leqslant 0$, letting $r \to 1^-$, we can derive a contradiction.

For the proof of the conclusion (iii), it suffices to deduce a contradiction for the case $m_3 = 0$ as the proof of conclusion (ii). Since $1 < p < (N+2)/(N-2)$ holds, together with lemma 2.2, (2.5) and (2.6), we have

$$(1-r)^\alpha r^{N-1}\tilde{u}^p \leqslant -(r^{N-1}\tilde{u}')' \leqslant r^{N-1}(1-r)^\alpha C(1-r)^{-((2+\alpha)/(p-1))p} \text{ in } (0,1),$$

where $C > 0$ is a positive constant. So, we obtain

$$r^{N-1}(1-r)^\alpha C(1-r)^{-((2+\alpha)/(p-1))p} \geqslant m_*^p(1-r)^\alpha r^{N-1}(1-r)^p \text{ in } (r_*, 1).$$

Therefore, we have

$$(1-r)^{-((2+\alpha)/(p-1))p-p} \geqslant \frac{m_*^p}{C} \text{ in } (r_*, 1).$$

By the condition $p + 1 + \alpha < 0$, it is clear that $-((2+\alpha)/(p-1))p - p > 0$, and hence we can deduce a contradiction. □

In order to obtain the existence of positive solution of (1.2), we need to consider the corresponding perturbation problem, which has no singularity at the boundary, and establish the estimate of its solutions.

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LEMMA 2.3. *Suppose that* $1 < p < (N+2)/(N-2)$, $\alpha > -2$, $\epsilon_0 > 0$ *and* $\epsilon \in (0, \epsilon_0]$. *Then there exists* $C > 0$ *depending only on* α, p, ϵ_0, N *such that any positive radial solution* $u_\epsilon \in C^2(B_1) \cap C^1(\bar{B}_1)$ *of*

$$\begin{cases} -\Delta u = (1+\epsilon-|x|)^\alpha u^p, & x \in B_1(0), \\ u = 0, & |x| = 1 \end{cases} \tag{2.7}$$

satisfies

$$\|\nabla u_\epsilon\|_{L^\infty(B_1)} + \|u_\epsilon\|_{L^\infty(B_1)} \leqslant C. \tag{2.8}$$

Proof. We divide the proof into two steps.

Step 1. We prove that

$$\|u_\epsilon\|_{L^\infty(B_1)} \leqslant C,$$

where $C > 0$ depends only on α, p, ϵ_0, N. We use indirect method to prove the conclusion. Suppose that the assertion is false. Then there is a sequence of solutions u_k, ϵ_k and $P_k \in B_1$ such that

$$M_k = \max_{x \in \bar{B}_1} u_k(x) = u_k(P_k) \to +\infty \quad \text{as } k \to \infty.$$

Since u_k is a radially symmetric function, by the maximum principle we claim $P_k = 0$. In fact, if $P_k \neq 0$, then the symmetric property implies that there exists $Q_k \in B_1$ such that u_k takes minimum at Q_k and $|P_k| > |Q_k|$. So, we see

$$0 \geqslant -\Delta u_k(Q_k) = (1+\epsilon_k - |Q_k|)^\alpha u_k(Q_k)^p > 0.$$

This is a contradiction.

Without loss of generality, we assume $\epsilon_k \to \tilde{\epsilon} \in [0, \epsilon_0]$. We define

$$U_k(y) = \frac{1}{M_k} u_k(M_k^{-((p-1)/2)} y).$$

Then U_k satisfies

$$-\Delta U_k = \left(1 + \epsilon_k - |M_k^{-(p-1)/2)} y|\right)^\alpha U_k^p$$

with $0 \leqslant U_k \leqslant 1$ and $U_k(0) = 1$. By the standard arguments of elliptic equations, we can extract a subsequence of $\{U_k\}$ converging to a function U in $C^2_{\text{loc}}(\mathbb{R}^N)$, which satisfies

$$-\Delta U = (1+\tilde{\epsilon})^\alpha U^p \quad \text{in } \mathbb{R}^N \text{ and } U(0) = 1.$$

Since $1 < p < (N+2)/(N-2)$, this contradicts the corresponding Liouville-type results [**11**].

Step 2. We prove that

$$\|\nabla u_\epsilon\|_{L^\infty(B_1)} \leqslant C,$$

where $C > 0$ depends only on α, p, ϵ_0, N. By step 1, we assume that $\|u_\epsilon\|_{L^\infty(B_1)} \leqslant C$ for any $\epsilon \in (0, \epsilon_0]$. Since u_ϵ is radially symmetric, we also denote $u_\epsilon(r) = u_\epsilon(x)$ as

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$|x| = r$. For the case $\alpha \geqslant 0$, according to the regularity of elliptic equations, the conclusion can be obtained directly.

Now, we consider the case $-2 < \alpha < 0$. We still use indirect method to prove it. Suppose that the assertion is false. Then, there exist $\epsilon_k \in (0, \epsilon_0]$ and positive solution u_k of (2.7) with $\epsilon = \epsilon_k$ such that

$$\|\nabla u_k\|_{L^\infty(B_1)} \to \infty \text{ as } k \to \infty.$$

Since $u_k'(0) = 0$ and

$$-r^{N-1}u_k'(r) = \int_0^r (1+\epsilon_k-\tau)^\alpha \tau^{N-1} u_k(\tau)^p \, \mathrm{d}\tau,$$

we can deduce $u_k'(r) < 0$ for all $r \in (0,1]$. Let $r_k \in (0,1]$ be the minimum point of u_k'. From the interior estimate of elliptic equations it follows that $\{r_k\}$ has a subsequence, which converges to 1. Without loss of generality, we assume $r_k \to 1$ as $k \to \infty$. Hence, we have

$$-u_k'(r_k) \to +\infty \quad (k \to \infty).$$

From the equation (2.7), it follows that

$$-u_k'(r) = r^{1-N} \int_0^r \tau^{N-1}(1+\epsilon_k-\tau)^\alpha u_k(\tau)^p \, \mathrm{d}\tau.$$

By $\alpha > -2$, we can take a small constant $\eta > 0$ such that $\alpha + 1 - \eta > -1$. By the differential mean value theorem, it follows that

$$u_k(r) \leqslant (1-r)|u_k'(r_k)| \text{ for any } r \in (0,1).$$

Therefore, we have

$$\begin{aligned} |u_k'(r_k)| &\leqslant r_k^{1-N} \int_0^{r_k} \tau^{N-1}(1-\tau)^\alpha u_k(\tau)^p \, \mathrm{d}\tau \\ &\leqslant r_k^{1-N} \int_0^{r_k} (1-\tau)^\alpha u_k(\tau)^{1-\eta} u_k(\tau)^{p+\eta-1} \, \mathrm{d}\tau \\ &\leqslant (1+o(1))|u_k'(r_k)|^{1-\eta} C^{p+\eta-1} \int_0^{r_k} (1-\tau)^{\alpha+1-\eta} \, \mathrm{d}\tau. \end{aligned}$$

Since $\alpha + 1 - \eta > -1$ and $\eta > 0$, we can obtain

$$|u_k'(r_k)| \leqslant K|u_k'(r_k)|^{1-\eta} \text{ for all } k,$$

where K is a positive constant. This contradicts $|u_k'(r_k)| \to \infty$ $(k \to \infty)$. □

Next, we prove theorem 1.3 by using Nehari manifold method and together with lemma 2.3.

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Proof of theorem 1.3. **Case 1.** $-2 < \alpha \leqslant 0$. Consider the following problem

$$\begin{cases} -\Delta u = \left(1 + \dfrac{1}{n} - |x|\right)^{\alpha} |u|^{p-1}u, & x \in B_1(0), \\ u = 0, & |x| = 1. \end{cases} \tag{2.9}$$

Define a functional by

$$F_n(u) = \frac{1}{2}\int_{B_1} |\nabla u|^2 \,\mathrm{d}x - \frac{1}{p+1}\int_{B_1} \left(1 + \frac{1}{n} - |x|\right)^{\alpha} (u^+)^{p+1} \,\mathrm{d}x, \; u \in H_0^1(B_1),$$

where $u^+ = \max\{u, 0\}$.

We will show that F_n has a radially symmetric critical point in $H_0^1(B_1)$. We denote the norm in $H_0^1(B_1)$ by

$$\|u\| = \left(\int_{B_1} |\nabla u|^2 \,\mathrm{d}x\right)^{1/2}, \quad u \in H_0^1(B_1).$$

Let

$$X = \{u \in H_0^1(B_1) : u \text{ is a radially symmetric function }\}.$$

Clearly, for any given n, F_n satisfies the condition of mountain-pass lemma in X. By the theory of critical points on symmetric function space, F_n has a critical point, which is a radially symmetric function in $H_0^1(B_1)$. By the standard arguments, we assume that u_n is a nontrivial nonnegative solution of (2.9), and u_n is a radially symmetric function.

By the regularity and strong maximum principle, it follows that $u_n \in C^2(B_1) \cap C^1(\bar{B}_1)$ and $u_n > 0$. From lemma 2.3, there exists $C > 0$ such that for all n

$$\|u_n\|_{C^1(\bar{B}_1)} \leqslant C.$$

By the regularity of elliptic equations, $\{u_n\}$ is bounded in $C^{2+\mu}_{\mathrm{loc}}(B_1)$, where $\mu \in (0, 1)$. In view of Arzela–Ascoli theorem, without loss of generality, we assume $u_n \to u$ in $C^2_{\mathrm{loc}}(B_1)$. So, we can obtain that $u \in C^2(B_1) \cap C(\bar{B}_1)$ is a radially symmetric solution of (1.2).

We claim that u is a nontrivial solution. Without loss of generality, we assume that $u_n \to u$ in $C^1_{\mathrm{loc}}(B_1)$. Suppose that $u \equiv 0$ holds. Since $u_n \to u$ in $C(\bar{B}_1)$, we have

$$\|u_n\|_{L^\infty(B_1)} = o(1) \;\; (n \to \infty).$$

From the equations for u_n and u_{n+1}, it follows that

$$\begin{aligned} -\Delta(u_{n+1} - u_n) &= \left(1 + \frac{1}{n+1} - |x|\right)^{\alpha} u_{n+1}^p - \left(1 + \frac{1}{n} - |x|\right)^{\alpha} u_n^p \\ &> \left(1 + \frac{1}{n} - |x|\right)^{\alpha} (u_{n+1}^p - u_n^p) \\ &= \left(1 + \frac{1}{n} - |x|\right)^{\alpha} (u_{n+1} - u_n) w_n(x), \quad x \in B_1, \end{aligned}$$

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where $\|w_n\|_{L^\infty(B_1)} = o(1)$ $(n \to \infty)$. So, we have

$$\begin{cases} -\Delta(u_{n+1} - u_n) - \left(1 + \dfrac{1}{n} - |x|\right)^{\alpha} w_n(x)(u_{n+1} - u_n) > 0, & x \in B_1(0), \\ u_{n+1} - u_n = 0, & |x| = 1. \end{cases} \tag{2.10}$$

Denote by $\lambda_1[b(x), \omega]$ the first eigenvalue of

$$\begin{cases} -\Delta\phi + b(x)\phi = \lambda\phi, & x \in \omega, \\ \phi = 0, & x \in \partial\omega. \end{cases}$$

When $b(x) = -\frac{1}{4}(1 + \frac{1}{n} - |x|)^{-2}$, by lemma 2.3 in [9] it follows that

$$\lambda_1[b(x), B_1] > 0.$$

Since $\|w_n\|_{L^\infty(B_1)}$ is sufficiently small for sufficiently large n, by $\alpha > -2$ it follows that

$$\left(1 + \frac{1}{n} - |x|\right)^{\alpha} w_n(x) \leqslant \frac{1}{4}\left(1 + \frac{1}{n} - |x|\right)^{-2} \text{ in } B_1.$$

So, we have

$$\lambda_1\left[-\left(1 + \frac{1}{n} - |x|\right)^{\alpha} w_n(x), B_1\right] > 0 \text{ for large } n.$$

Together with (2.10) and the strong maximum principle, we obtain

$$u_{n+1}(x) > u_n(x) \text{ in } B_1 \text{ for large } n.$$

This contradicts $u_n \to 0$ in $C(\bar{B}_1)$ as $n \to \infty$.

Case 2. Suppose $\alpha > 0$. Define a functional F in $H_0^1(B_1(0))$ by

$$F(u) = \frac{1}{2}\int_{B_1} |\nabla u|^2 \, \mathrm{d}x - \frac{1}{p+1}\int_{B_1} (1 - |x|)^{\alpha}(u^+)^{p+1} \, \mathrm{d}x.$$

By the mountain pass lemma and the standard arguments, F has a positive critical point $v \in H_0^1(B_1(0))$. In view of $1 < p < (N+2)/(N-2)$ and $\alpha > 0$, together with the regularity of elliptic equations, $v \in C^2(B_1) \cap C^1(\bar{B}_1)$ is a positive solution of (1.2). □

3. The case $0 < p \leqslant 1$

Here, we firstly establish the following Liouville-type result and estimate of positive solutions, which should be useful for some estimates of positive solutions of some sublinear or negative exponent problems. We are not sure whether the following result is new, but we did not find it in some existing references, and here the method of study is basic.

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THEOREM 3.1. *Suppose that $p < 1$, $\gamma < 2$, $\epsilon > 0$ and $a(x) \geqslant \epsilon |x|^{-\gamma}$ in $\mathbb{R}^N$. For the differential inequality*

$$-\Delta v \geqslant a(x) v^p \quad \text{in } \mathbb{R}^N, \tag{3.1}$$

the following conclusions hold:

(i) *for $0 \leqslant \gamma < 2$, (3.1) has no positive classical solutions;*

(ii) *for $\gamma < 0$, any positive solution $v \in C^2(\mathbb{R}^N \backslash \{0\})$ of (3.1) satisfies*

$$v(x) \geqslant C|x|^{(2-\gamma)/(1-p)} \quad \text{for all} \quad x \neq 0,$$

where C depends only on ϵ, γ, p.

Proof. We firstly prove the conclusion (i). For $0 \leqslant \gamma < 2$, we suppose that (3.1) has a positive classical solution v. We are going to deduce a contradiction. For any positive integer n, we consider the following problem

$$\begin{cases} -\Delta w = c_n w^p, & x \in B_n(0), \\ w = m_n, & x \in \partial B_n(0), \end{cases} \tag{3.2}$$

where $m_n = \min_{|x| \leqslant n} v(x)$ and $c_n = \inf_{|x| \leqslant n} a(x)$. Clearly, m_n and v can act as a subsolution and supersolution of (3.2). By the supersolution and subsolution method, (3.2) has a minimal positive solution v_n in the interval $[m_n, v]$. This means that for any positive solution w of (3.2) with $m_n \leqslant w \leqslant v$ must satisfy $w(x) \geqslant v_n(x)$ in $B_n(0)$. In fact, v_n is the limit of the iteration sequence with the initial value m_n. Take $K > 0$ such that

$$|c_n s^p - c_n t^p| \leqslant K|s - t| \text{ for all } s, t \in [m_n, M_n],$$

where $M_n = \max_{|x| \leqslant n} v(x)$. Let w_1 be the unique positive solution of

$$\begin{cases} -\Delta w + Kw = Km_n + c_n m_n^p, & x \in B_n(0), \\ w = m_n, & x \in \partial B_n(0). \end{cases}$$

By the maximum principle, we obtain $w_1(x) > m_n$ in $B_n(0)$. By the uniqueness of solutions and invariant property of rotations for the operator Δ, it follows that w_1 is a radially symmetric function. Let w_2 be the unique positive solution of

$$\begin{cases} -\Delta w + Kw = Kw_1 + c_n w_1^p, & x \in B_n(0), \\ w = m_n, & x \in \partial B_n(0). \end{cases}$$

By the maximum principle, we obtain $w_2(x) \geqslant w_1(x)$ in B_n. By using the uniqueness and invariant property of rotations for the operator Δ again, it follows that w_2 is a radially symmetric function. Successively, by w_{k+1} denote the unique positive solution of

$$\begin{cases} -\Delta w + Kw = Kw_k + c_n w_k^p, & x \in B_n(0), \\ w = m_n, & x \in \partial B_n(0). \end{cases}$$

Similarly, any term of the iteration sequence $\{w_k\}$ is a radially symmetric function. By the standard arguments, $v_n(x) := \lim_{k \to \infty} w_k(x)$ is a minimal positive solution

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of (3.2) in the interval $[m_n, v]$. Clearly, v_n is radially symmetric. By the maximum principle, it is easy to show that v_n is the minimal positive solution in $[m_n, v]$.

For convenience, by $\lambda(n)$ and ψ_n denote the first eigenvalue and the first eigenfunction of

$$\begin{cases} -\Delta\psi = \lambda\psi, & x \in \ B_n(0), \\ \psi(x) = 0, & |x| = n. \end{cases}$$

So, we have

$$\int_{B_n} \lambda(n)\psi_n v_n + \int_{|x|=n} m_n \frac{\partial\psi_n}{\partial\nu} = c_n \int_{B_n} v_n^p \psi_n.$$

By Hopf's lemma, we have $\partial\psi_n/\partial\nu < 0$ on ∂B_n, where ν is the outer unit normal vector. Therefore, we obtain

$$\int_{B_n} \psi_n v_n [\lambda(n) - c_n v_n^{p-1}] > 0.$$

By the maximum principle and the symmetric property, the origin is the maximum point of v_n. So, there holds

$$v_n(0) \geqslant \left(\frac{c_n}{\lambda(n)}\right)^{1/(1-p)}.$$

It is well-known that $\lambda(1) = n^2\lambda(n)$. By $a(x) \geqslant \epsilon|x|^{-\gamma}$, it follows that

$$c_n \geqslant \epsilon n^{-\gamma}.$$

Hence, we have

$$v_n(0) \geqslant \left(\frac{c_n}{\lambda(n)}\right)^{1/(1-p)} \geqslant \left(\frac{\epsilon n^{2-\gamma}}{\lambda(1)}\right)^{1/(1-p)}.$$

So, we obtain

$$v(0) \geqslant \left(\frac{\epsilon n^{2-\gamma}}{\lambda(1)}\right)^{1/(1-p)}.$$

Letting $n \to \infty$, we obtain a contradiction. Therefore, the conclusion (i) is proved.

Now, we prove the conclusion (ii). Suppose that $v \in C^2(\mathbb{R}^N \backslash \{0\})$ is any positive solution of (3.1) with $\gamma < 0$. Let x_R be an arbitrary point, where $R > 0$ and $|x_R| =$

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$2R$. Denote

$$\tilde{m}_R = \min_{|x-x_R|\leqslant \frac{3}{2}R} v(x), \quad \tilde{c}_R = \inf_{|x-x_R|<R} a(x).$$

We consider the Dirichlet problem

$$\begin{cases} -\Delta\vartheta = \tilde{c}_R\vartheta^p, & x \in B_R(x_R), \\ \vartheta = \tilde{m}_R, & x \in \partial B_R(x_R). \end{cases} \tag{3.3}$$

Similar to the proof of (i), it follows that (3.3) has a minimal solution $\eta_R(x)$ in $[\tilde{m}_R, v]$. Denote

$$\tilde{v}_R(x) = \eta_R(x + x_R) \text{ for } x \in B_R(0).$$

Then we have

$$\tilde{v}_R(x) \leqslant v(x + x_R) \text{ for } x \in B_R(0).$$

By the similar arguments of (i), it follows that

$$\tilde{v}_R(0) \geqslant \left(\frac{\tilde{c}_R}{\lambda(R)}\right)^{1/(1-p)},$$

where $\lambda(R)$ denotes the first eigenvalue of $-\Delta$ with Dirichlet boundary condition on B_R. By $a(x + x_R) \geqslant \epsilon|x + x_R|^{-\gamma}$, it follows that

$$\tilde{c}_R \geqslant \epsilon R^{-\gamma}.$$

So, we can obtain that

$$v(x_R) \geqslant \left(\frac{\epsilon R^{2-\gamma}}{\lambda(1)}\right)^{1/(1-p)}.$$

By the arbitrariness of x_R, it follows that

$$v(x) \geqslant \left(\frac{\epsilon}{2^{2-\gamma}\lambda(1)}\right)^{1/(1-p)} |x|^{(2-\gamma)/(1-p)} \text{ for all } x \neq 0. \qquad \square$$

In order to establish the lower estimate of positive solutions of (1.1). We need the following lemma.

LEMMA 3.2. *Let $N \geqslant 3$, $p < 1$, and $\mu \in (0, 1)$. Let $a \in C^\mu(\bar{B}_1)$ satisfy*

$$a(x) \geqslant C, \quad x \in \bar{B}_1,$$

for some constants $C > 0$. Then for any positive classical solution u of

$$-\Delta u = a(x)u^p, \quad x \in B_1,$$

u satisfies

$$|u(0)| \geqslant \left(\frac{C}{\lambda_1(B_1)}\right)^{1/(1-p)}, \qquad x \in B_1, \tag{3.4}$$

where $\lambda_1(B_1)$ is the first eigenvalue of $-\Delta$ with Dirichlet boundary condition on $B_1(0)$.

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Proof. Fix a small positive constant $\delta > 0$. For any large positive integer n, we consider

$$\begin{cases} -\Delta w = Cw^p, & x \in B_{1-\delta}(0), \\ w = \dfrac{1}{n}, & |x| = 1-\delta. \end{cases} \tag{3.5}$$

Let e satisfy

$$-\Delta e = 1 \text{ in } B_1, \quad e = 0 \text{ on } \partial B_1.$$

Then we can take a large constant $M > 0$ such that

$$-\Delta(Me) = M > CM^p e^p \text{ in } B_1$$

and

$$Me > \frac{1}{n} \text{ as } |x| \leqslant 1-\delta.$$

So, $(\frac{1}{n}, Me)$ is a pair of subsolution and supersolution of (3.5). As the arguments in the proof of theorem 3.1, (3.5) has a minimal positive solution in $[\frac{1}{n}, Me]$, and denote it by w_n. Clearly, w_n is a radial function. By the iteration method and maximum principle, it follows that

$$w_{n+1}(x) \leqslant w_n(x) \text{ for all } x \in B_{1-\delta},$$

where w_{n+1} is the minimal solution in $[(1/(n+1)), Me]$ of (3.5) with $w = 1/(n+1)$ on $|x| = 1-\delta$. By the maximum principle, the origin is the maximum value point of w_n. Let $\phi_\delta > 0$ and λ_δ be the first eigenfunction and the first eigenvalue of

$$\begin{cases} -\Delta \phi = \lambda_1 \phi, & x \in B_{1-\delta}, \\ \phi = 0, & |x| = 1-\delta. \end{cases}$$

So, for any n, we have

$$\int_{|x|<1-\delta} \lambda_\delta w_n \phi_\delta \, \mathrm{d}x + \int_{|x|=1-\delta} w_n \frac{\partial \phi_\delta}{\partial \nu} = C \int_{|x|<1-\delta} w_n^p \phi_\delta.$$

Hence we see

$$\int_{|x|<1-\delta} w_n \phi_\delta [\lambda_\delta - C w_n^{p-1}] > 0.$$

By the maximum principle, it follows that the origin is the maximum point of w_n. Therefore we obtain

$$w_n(0) > \left(\frac{C}{\lambda_\delta}\right)^{1/(1-p)}.$$

By the monotone property of $\{w_n\}$ in n, $w_\delta := \lim_{n\to\infty} w_n$ is well-defined in $B_{1-\delta}$ and $w_\delta(0) > 0$. By the regularity and the strong maximum principle, w_δ is a positive

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radially symmetric function in $B_{1-\delta}$. Clearly, w_δ satisfies

$$\begin{cases} -\Delta w_\delta = C w_\delta^p, & x \in B_{1-\delta}, \\ w_\delta = 0, & |x| = 1-\delta \end{cases}$$

and

$$w_\delta(0) \geqslant \left(\frac{C}{\lambda_\delta}\right)^{1/(1-p)}.$$

For any positive classical function u satisfying

$$-\Delta u = a(x) u^p, \quad x \in B_1,$$

when n is sufficiently large, it follows that

$$\begin{cases} -\Delta u = a(x) u^p \geqslant C u^p, & x \in B_{1-\delta}, \\ u(x) > \dfrac{1}{n}, & |x| = 1-\delta. \end{cases}$$

So, we obtain that

$$w_n(x) \leqslant u(x) \text{ in } B_{1-\delta}.$$

Therefore, we have

$$u(0) \geqslant w_\delta(0) \geqslant \left(\frac{C}{\lambda_\delta}\right)^{1/(1-p)}.$$

Letting $\delta \to 0^+$, we can obtain (3.4). □

The conclusion (ii) in theorem 1.1 can be expressed by the following theorem.

THEOREM 3.3. *Let $N \geqslant 3$, $\alpha > -2$ and $p < 1$. Then there exists $C = C(N, p, \alpha)$ such that any nonnegative solution u of (1.1) satisfies*

$$u(x) \geqslant C(1-|x|)^{-((2+\alpha)/(p-1))} \text{ for all } x \in B_1. \tag{3.6}$$

Proof. Let x_0 be an arbitrary point in B_1. We define a function by

$$U(x) = (1-|x_0|)^{(2+\alpha)/(p-1)} u\left(x_0 + \frac{1-|x_0|}{2} x\right), \quad x \in B_1.$$

Then U satisfies

$$-\Delta U = a(x) U^p, \quad x \in B_1,$$

where

$$a(x) = \frac{1}{4} \cdot \left(\frac{1 - |x_0 + \frac{1-|x_0|}{2} x|}{1-|x_0|}\right)^\alpha.$$

Clearly, for any $x \in B_1$, we have

$$a(x) \geqslant \frac{1}{2^{\alpha+2}} \quad \text{as } \alpha \geqslant 0$$

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and

$$a(x) \geqslant \frac{3^{\alpha}}{2^{\alpha+2}} \quad \text{as} -2 < \alpha < 0.$$

Therefore, applying lemma 3.2, we have

$$U(0) \geqslant C.$$

Hence,

$$(1 - |x_0|)^{(2+\alpha)/(p-1)} u(x_0) \geqslant C.$$

By the arbitrariness of x_0, we can obtain (3.6). □

Proof of theorem 1.4. Suppose that $u \in C^1(\bar{B}_1)$ is a positive solution of (1.2). By Hopf's Lemma, there exist $c_1, c_2 > 0$ such that

$$c_1(1 - |x|) \leqslant u(x) \leqslant c_2(1 - |x|) \text{ in } B_1.$$

By the condition $1 + p + \alpha < 0$, it follows that $-((2+\alpha)/(p-1)) < 1$. By theorem 3.3, we see that

$$C(1 - |x|)^{-((2+\alpha)/(p-1))} \leqslant u(x) \leqslant c_2(1 - |x|) \text{ in } B_1.$$

This is impossible, and hence we obtain a contradiction. □

Remark 3.4. In fact, under the condition of theorem 1.4, problem (1.2) has no positive solution $u \in C^1(B_1) \cap C(\bar{B}_1)$, which has differential points on ∂B_1.

Proof of theorem 1.5. We prove this conclusion by using the indirect method. Suppose that (1.1) has a positive solution u.

Case 1. $p \in (0, 1)$. According to theorem 3.3 and $\alpha \leqslant -2$, there is a positive constant $C > 0$ such that

$$u(x) \geqslant C \quad \text{for all} \quad x \in B_1(0).$$

Denote

$$\tilde{u}(r) := \frac{1}{|S^{N-1}|} \int_{S^{N-1}} u(r, \theta)\, \mathrm{d}\theta,$$

and then $\tilde{u}$ satisfies

$$-(r^{N-1}\tilde{u}'(r))' \geqslant C^p (1-r)^{\alpha} r^{N-1} \text{ in } (0, 1),$$

As the arguments of the proof of the conclusion (i) in theorem 1.2, we can derive a contradiction.

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Case 2. $p = 1$. For such case, there holds

$$-\Delta u \geqslant \frac{u}{(1-|x|)^2} \quad \text{in } B_1(0).$$

This implies that the first eigenvalue $\lambda_1(n)$ of

$$\begin{cases} -\Delta\phi = \lambda \dfrac{\phi}{(1-|x|)^2}, & x \in B_{1-(1/n)}(0), \\ \phi = 0, & |x| = 1 - \dfrac{1}{n}. \end{cases}$$

is larger than 1. This is a contradiction to $\lim_{n\to\infty} \lambda_1(n) = \frac{1}{4}$, which can be found in [**9**]. □

REMARK 3.5. In fact, when the conditions $p < 0$ and $\alpha \leqslant -2$ hold, we can also prove that (1.1) has no positive solutions in $C^2(B_1) \cap C(\bar{B}_1)$. Suppose that $u \in C^2(B_1) \cap C(\bar{B}_1)$ is a positive solution of (1.1). For this case, by the equation and $f(t) = t^p$ is convex in t, we also can obtain

$$(r^N \tilde{u}'(r))' < 0 \text{ and} - (r^{N-1}\tilde{u}')' \geqslant (1-r)^\alpha r^{N-1} \tilde{u}^p \quad \text{in } (0,1). \tag{3.7}$$

So we obtain $r^{N-1}\tilde{u}'(r) \to m \in [-\infty, +\infty)$ as $(r \to 1)$. If $m < 0$ holds, by the similar argument in the proof of theorem 1.2 there exist $r_* > 0$, $m_* > 0$ and $m_1 \geqslant 0$ such that

$$\tilde{u}(r) \to m_1 \ (r \to 1) \text{ and } \tilde{u}'(r) < -m_* \text{ in } (r_*, 1).$$

Therefore, we can choose a small constant $\epsilon > 0$ such that $\tilde{u}(r)^p \geqslant \epsilon$ in $(r_*, 1)$. By integral for the second inequality in (3.7), it follows that

$$\tilde{u}(r) - \tilde{u}(r_*) \leqslant -\epsilon r_*^{N-1} \int_{r_*}^{r} \left(\int_{r_*}^{r} (1-\tau)^\alpha \right) \mathrm{d}t.$$

By $\alpha \leqslant -2$, letting $r \to 1$, we can derive a contradiction. If $m \geqslant 0$ holds, then there holds $\tilde{u}'(r) > 0$ in $(0,1)$. Since $u \in C(\bar{B}_1)$ is a positive function, $\tilde{u}(r)^p \geqslant m_0$ for all $r \in (0,1)$, where m_0 is a positive constant. For any fixed $r_1 \in (0,1)$, we obtain that for $r \in (r_1, 1)$

$$\tilde{u}'(r_1) \geqslant m_0 \int_{r_1}^{r} (1-\tau)^\alpha \, \mathrm{d}\tau.$$

In view of $\alpha \leqslant -2$, letting $r \to 1$, we can see a contradiction.

REMARK 3.6. When $p = 1$, there exists a unique $\alpha > -2$ such that (1.2) has a positive solution. In fact, let $\lambda_1(n, \alpha)$ be the first eigenvalue of

$$\begin{cases} -\Delta\phi = \lambda(1-|x|)^\alpha \phi, & x \in B_{1-(1/n)}(0), \\ \phi = 0, & |x| = 1 - \dfrac{1}{n}. \end{cases}$$

It is well-known that $\lim_{n\to\infty} \lambda_1(n, -2) = \frac{1}{4}$ and $\lambda_1(n, 0) > 5$ for sufficiently large n (refer to [**9**]). By the continuity property and monotone property of the first

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eigenvalue for weigh functions, we can choose $\gamma > -2$ satisfying $1 > \lambda_1(n,\gamma) > \frac{1}{2}$ for large n, and hence there exists a unique $\alpha_n \in (\gamma, 0)$ such that $\lambda_1(n, \alpha_n) = 1$. Clearly, $\{\alpha_n\}$ is an increasing sequence in n. Then it is clear that $\tilde{\alpha} := \lim_{n\to\infty} \alpha_n \in (\gamma, 0)$. By the regularity of elliptic equations, for the case $\alpha = \tilde{\alpha}$, (1.2) with $p = 1$ has positive solutions.

Finally, we prove the existence and uniqueness of positive solutions of (1.8).

Proof of theorem 1.6. Step 1 We show the existence of positive solutions of (1.2) for the conclusion (i). Denote the first eigenfunction and eigenvalue by ϕ_1 and $\lambda_1(B_1)$ of

$$-\Delta\phi = \lambda\phi \text{ in } B_1, \quad \phi = 0 \text{ on } \partial B_1.$$

Define $\underline{u} = m\phi_1^{\beta}$, where $\beta = (2+\alpha)/(1-p)$. Clearly, we can choose $c_1, c_2 > 0$ such that

$$c_1\phi_1(x) \leqslant 1 - |x| \leqslant c_2\phi_1(x) \text{ in } B_1(0).$$

When m is a small positive constant, we obtain

$$\begin{aligned}
&-\Delta\underline{u} - (1-|x|)^{\alpha}\underline{u}^p \\
&= -m\beta\phi_1^{\beta-1}\Delta\phi_1 - m\beta(\beta-1)\phi_1^{\beta-2}|\nabla\phi_1|^2 - (1-|x|)^{\alpha}m^p\phi_1^{p\beta} \\
&\leqslant m\beta\lambda_1(B_1)\phi_1^{\beta} - m\beta(\beta-1)\phi_1^{\beta-2}|\nabla\phi_1|^2 - c^{\alpha}m^p\phi_1^{p\beta+\alpha} \\
&= m\beta\phi_1^{\beta-2}\left[\lambda_1(B_1)\phi_1^2 - (\beta-1)|\nabla\phi_1|^2 - \frac{c^{\alpha}m^{p-1}}{\beta}\right] \text{ in } B_1(0),
\end{aligned}$$

where c is a positive constant. According to $p-1<0$, $\beta>0$, it holds that

$$-\Delta\underline{u} - (1-|x|)^{\alpha}\underline{u}^p \leqslant 0 \text{ in } B_1(0).$$

In addition, let $\bar{u} = K\phi_1^{\rho}$, where $K > 0$ and $\rho > 0$ are constants. We can take large $M > 0$ and $0 < \rho \leqslant \min\{1, ((2+\alpha)/(1-p))\}$, then by a similar calculation we have

$$\begin{aligned}
&-\Delta\bar{u} - (1-|x|)^{\alpha}\bar{u}^p \\
&\geqslant K\rho\lambda_1(B_1)\phi_1^{\rho} - m\rho(\rho-1)\phi_1^{rho-2}|\nabla\phi_1|^2 - C^{\alpha}K^p\phi_1^{p\rho+\alpha} \\
&= K\rho\phi_1^{\rho-2}\left[\lambda_1(B_1)\phi_1^2 - (\rho-1)|\nabla\phi_1|^2 - \frac{C^{\alpha}K^{p-1}}{\rho}\right] \text{ in } B_1(0).
\end{aligned}$$

So, for sufficiently large K, it follows that

$$-\Delta\bar{u} - (1-|x|)^{\alpha}\bar{u}^p \geqslant 0 \text{ in } B_1(0).$$

By the supersolution and subsolution method, and together with the regularity of elliptic equations, (1.2) has a positive solution in $C^2(B_1) \cap C(\bar{B}_1)$.

Step 2 Prove the uniqueness of positive solution of (1.2) with $p < 0$ and $\alpha > -2$. By the estimate of positive solutions, there exists a constant $c_* > 0$ such that for

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any positive solution u of (1.2)

$$u(x) \geqslant c_* \phi_1^{(2+\alpha)/(1-p)} \text{ for all } x \in B_1(0).$$

When we take a small positive constant m satisfying $m < c_*$ in the step 1, then there is a minimal positive solution v_* in $[m\phi_1^\beta, K\phi_1^\rho]$. Suppose that v is any positive solution of (1.2). Then we have $v(x) \geqslant m\phi_1^\beta$. Therefore, $\min\{v, K\phi_1^\rho\}$ is a supersolution of (1.2) and $m\phi_1^\beta \leqslant \min\{v, K\phi_1^\rho\}$. So, it follows that $v_* \leqslant \min\{v, K\phi_1^\rho\}$, and hence we obtain $v_* \leqslant v$. This implies that v_* is a minimal positive solution. For the uniqueness, we need to show $v_* = v$. If it is not true, there exists $x_0 \in B_1(0)$ such that

$$v_*(x_0) - v(x_0) = \min\{v_*(x) - v(x) : x \in B_1(0)\} < 0.$$

In view of $p < 0$, we see

$$0 \geqslant -\Delta(v_* - v)(x_0) = (1 - |x_0|)^\alpha (v_*(x_0)^p - v(x_0)^p) > 0.$$

This is a contradiction.

Step 3 Prove the existence of positive solution of (1.8). Take a positive constant $\delta > 0$ and $1 < q < (N+2)/(N-2)$. Then the problem

$$-\Delta u = u^q \text{ in } B_{1+\delta}(0),\ u = 0 \text{ on } \partial B_{1+\delta}(0)$$

has a positive solution, and denote it by u_δ. We can choose a sufficiently large $M > 0$ such that $\bar{u} := Mu_\delta$ satisfies

$$-\Delta \bar{u} = Mu_\delta^q \geqslant (1 - |x|)^\alpha (Mu_\delta)^p = (1 - |x|)^\alpha \bar{u}^p \text{ in } B_1,$$

$$\bar{u}(x) \geqslant \underline{u}(x) \text{ in } B_1 \text{ and } \bar{u}(x) \geqslant \psi(x) \text{ on } \partial B_1.$$

Then by the supersolution and subsolution method and the standard arguments, there exists a minimal positive solution u_* and a maximal positive solution u^* of the interval $[m\phi_1^\beta, Mu_\delta]$.

Step 4 We show the uniqueness of the positive solutions for the case $\alpha \geqslant 0$ and $p \in (0, 1)$. Suppose that v is an arbitrary positive solution of (1.1). From theorem 3.3 it follows that there is a positive constant C such that

$$v(x) \geqslant C(1 - |x|)^\beta \text{ in } B_1.$$

Without loss of generality, we assume $m > 0$ satisfying

$$m\phi_1(x)^\beta \leqslant C(1 - |x|)^\beta \text{ in } B_1(0).$$

So, we obtain that $m\phi_1^\beta$ and $\min\{v, \bar{u}\}$ are a pair of subsolution and supersolution of (1.1). Therefore, we have

$$u_*(x) \leqslant v(x) \text{ in } B_1(0).$$

By the arbitrariness of v, u_* is the minimal positive solution of (1.8).

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Next we show that $v = u_*$. For the given solution v, we can take a suitable large $M > 0$ such that

$$Mu_\delta(x) \geqslant v(x) \text{ in } B_1(0).$$

So we have

$$u_*(x) \leqslant v(x) \leqslant u^*(x) \text{ in } B_1(0).$$

By $\alpha \geqslant 0$ and the regularity, u_* and u^* belong to $C^1(\bar{B}_1) \cap C^2(B_1)$. For our aim, it is sufficient to show $u_* = u^*$. Suppose that this conclusion is false. From the equation it follows that

$$\begin{cases} -\Delta(u_* - u^*) = (1-|x|)^\alpha [u_*^p - (u^*)^p], & x \in B_1(0), \\ u_* - u^* = 0, & |x| = 1. \end{cases}$$

Since $u^* \geqslant u_*$ and $u^* \not\equiv u_*$, in view of the strong maximum principle, we have

$$u^*(x) > u_*(x) \text{ in } B_1.$$

By Hopf's Lemma, we see

$$\frac{\partial(u^* - u_*)}{\partial \nu} < 0 \text{ on } \partial B_1(0),$$

where ν is the exterior unit normal on $\partial B_1(0)$. Multiplying the equation which u_* satisfies by u^* and multiplying the equation which u^* satisfies by u_*, respectively, and then integrating by parts the resulting identities over $B_1(0)$, we have that

$$\int_{\partial B_1} \psi \left[\frac{\partial u^*}{\partial \nu} - \frac{\partial u_*}{\partial \nu} \right] = \int_{B_1} (1-|x|)^\alpha u_* u^* \left[u_*^{p-1} - (u^*)^{p-1} \right].$$

From the sign of the two side, we can see a contradiction. Therefore, we deduce the uniqueness. □

REMARK 3.7.

(a) In this paper, we assume always $N \geqslant 3$. For the case $N = 2$, some estimate of positive solutions similar to lemma 2.1 should be established, which may be a challenge.

(b) With respect to theorem 1.2 part (i), we conjecture that (1.1) has no positive solutions when $p > 1$ and $\alpha \leqslant -2$, but we have not known how to prove it up to now.

(c) For the case $\alpha \in \mathbb{R}$, $p \geqslant (N+2)/(N-2)$ and $N \geqslant 3$, we have still made no progress.

(d) When domain Ω is a ball, this paper has revealed some interesting conclusions. For a general bounded smooth domain Ω, we guess that the corresponding conclusions should be also valid in which some new methods are perhaps developed.

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