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Proper minimal sets on compact connected 2-manifolds are nowhere dense

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Abstract. Let $\mathcal{M}^2$ be a compact connected two-dimensional manifold, with or without boundary, and let $f:\mathcal{M}^2\to\mathcal{M}^2$ be a continuous map. We prove that if $M\subseteq\mathcal{M}^2$ is a minimal set of the dynamical system $(\mathcal{M}^2, f)$ then either $M=\mathcal{M}^2$ or M is a nowhere dense subset of $\mathcal{M}^2$. Moreover, we add a shorter proof of the recent result of Blokh, Oversteegen and Tymchatyn, that in the former case $\mathcal{M}^2$ is a torus or a Klein bottle.

1. *Introduction*

A compact metric space X with a continuous map $f:X\to X$ can be viewed as a *dynamical system* in which the *orbit* of a point $x\in X$ is defined to be the set $\mathrm{Orb}_f(x)=\{x, f(x), f^2(x), \ldots\}$. The most fundamental dynamical systems are the minimal ones. These are systems which have no non-trivial subsystems. More precisely, a system (X, f) is called *minimal* if there is no proper subset $M\subseteq X$ which is non-empty, closed and f-invariant (i.e. $f(M)\subseteq M$). In such a case we also say that the map f itself is minimal. Clearly, a system (X, f) is minimal if and only if the orbit of every point $x\in X$ is dense in X. The classification, i.e. the full topological characterization, of compact metric spaces admitting minimal maps is a well-known open problem in topological dynamics. As a trivial example of a space which does not admit any minimal map let us mention any non-degenerate space with fixed/periodic point property. Probably the best-known examples of minimal systems are periodic orbits, irrational rotations on the tori and odometers (adding machines) on the Cantor set. For topological properties of minimal maps see [**15**], and for a survey on spaces admitting/not admitting minimal maps see [**5**]. For the special case of minimal maps on two-dimensional manifolds see [**3**].

Usually a dynamical system is not minimal. However, the basic and well-known fact due to Birkhoff is that any compact dynamical system (X, f) has minimal subsystems $(M, f|_M)$. Such (closed) sets M are called *minimal sets* of f or, more precisely, of (X, f).

The problem of understanding the behaviour of all points of a given system under forward iteration and, in particular, finding all minimal sets of the system is central in topological dynamics. It seems that Dowker [**7**] was the first to study the *topological structure of minimal sets* (of homeomorphisms). Since then it has been a topic of constant interest. Much is known in this direction on spaces with dimension at most one. It is folklore that if X is a compact *zero-dimensional* space, $f : X \to X$ is continuous and $M \subseteq X$ is its minimal set, then M is either a finite set (a periodic orbit of f) or a Cantor set. This is in fact a characterization, because also conversely, whenever $M \subseteq X$ is a finite or a Cantor set then there is a continuous map $f : X \to X$ such that M is a minimal set of f. Among *one-dimensional* spaces, the characterization of minimal sets is known for *graphs* (i.e. for one-dimensional compact connected polyhedra)—minimal sets on graphs are characterized as finite sets, Cantor sets and unions of finitely many pairwise disjoint simple closed curves, see [**2**] or [**18**]. On *dendrites* (i.e. on locally connected continua which contain no simple closed curve) the problem is very difficult and the full characterization of minimal sets has been found just recently, see [**1**].

However, with the exception of maps of zero- and some one-dimensional spaces, the dynamics of arbitrary continuous maps has not been extensively studied. This is quite understandable because continuity puts little restriction on maps of spaces of dimension higher than one. Therefore any substantial study of continuous maps of *higher dimensional* spaces requires some restriction. In topological dynamics such a restriction is usually the assumption that the map is a *homeomorphism*. For instance the study of minimal sets is then much easier. In fact, if (X, h) is a dynamical system and h is a homeomorphism, then the boundary of a minimal set M is h-invariant (and closed), hence is equal to the set M or is empty. Thus, a minimal set of a homeomorphism either has empty interior (i.e. it is nowhere dense in X) or it is a clopen subset of X. Consequently, if X is connected, then the homeomorphism h has only nowhere dense minimal sets, with one possible exception when the whole space X is minimal for h. It is well known that among the compact connected two-dimensional manifolds the latter case may happen only on the torus and the Klein bottle. (Any homeomorphism on a compact manifold with non-zero Euler characteristic has a periodic point, hence only the torus and the Klein bottle need to be checked; the case of the torus is trivial; for an example of a minimal homeomorphism on the Klein bottle see [**8, 21**].) By a two-dimensional manifold or, in short, by a *2-manifold* we mean a separable metric space such that every one of its points has a neighbourhood homeomorphic either to the (two-dimensional) Euclidean plane or to the closed half-plane. Thus, by manifolds we mean both manifolds without boundary and manifolds with boundary. If we consider just one of these two kinds of manifolds we will always state it explicitly.

Until relatively recently it was not known whether there exist minimal maps on 2-manifolds which are *not* homeomorphisms. The first examples of non-invertible minimal maps on the torus were found in [**15**] (where ideas from [**22**] were developed), while such examples on the Klein bottle are not available in the literature as far as we know.

Then a natural question appeared whether some other 2-manifolds, different from the torus and the Klein bottle, also admit minimal (not necessarily invertible) maps. Blokh *et al* in [**3**] proved that in fact among compact connected 2-manifolds there are no other spaces admitting minimal maps. This can be reformulated by saying that on compact connected 2-manifolds the whole manifold is a minimal *set* for an appropriate continuous map if and only if it is a torus or a Klein bottle. No other non-trivial result on minimal sets on 2-manifolds has been discovered so far. Our paper brings a new one. We have proved the following theorem.

THEOREM A. *Let $\mathcal{M}^2$ be a compact connected two-dimensional manifold, with or without boundary, and let $f : \mathcal{M}^2 \to \mathcal{M}^2$ be a continuous map. If $M \subseteq \mathcal{M}^2$ is a minimal set of the dynamical system $(\mathcal{M}^2, f)$ then either $M = \mathcal{M}^2$ or M is nowhere dense in $\mathcal{M}^2$.*

While we had been trying to prove this theorem, the paper [**3**] appeared where Blokh, Oversteegen and Tymchatyn proved the result (already mentioned above Theorem A) that if $M = \mathcal{M}^2$ then $\mathcal{M}^2$ is a 2-torus or a Klein bottle. They in fact proved slightly more.

THEOREM B. [**3**] *Suppose that $f : \mathcal{M}^2 \to \mathcal{M}^2$ is a minimal map of a 2-manifold (compact or not, with or without boundary). Then f is a monotone map with tree-like point inverses and $\mathcal{M}^2$ is either a finite union of tori or a finite union of Klein bottles which are cyclically permuted by f.*

We state this result as a theorem, because our methods developed in the present paper, together with some ideas from [**3**], enabled us to find a new, shorter proof of it (see §4). Though [**3**] was an inspiration for us, neither our proof of Theorem A nor that of Theorem B uses the results of [**3**].

We wish to emphasize that to find a full topological characterization of minimal sets on compact, connected 2-manifolds is a very difficult task. Of course, some examples of 'strange' minimal sets of continuous maps on 2-manifolds are scattered in the literature (e.g. the Sierpinski curve on the 2-torus, see [**5**]) and one can also think of embedding known one-dimensional minimal systems into a 2-manifold. But all this is far from giving a characterization. To illustrate the problems we encounter here, notice that though non-degenerate sets with fixed point property are trivially non-minimal (a fact which could be useful when trying to characterize minimal sets), we are in fact unable to describe all such sets on 2-manifolds—recall at least the long-standing open problem in the continuum theory whether each non-separating plane continuum has the fixed point property, see [**9**, pp. 299 and 404]. We believe that the following weaker version of this problem, suggested by Auslander, could be interesting for both topologists and dynamists.

CONJECTURE. *No non-degenerate non-separating plane continuum admits a minimal map.*

In the present paper we study minimal sets on compact, *connected* 2-manifolds. Notice that on compact *disconnected* 2-manifolds one can have a minimal set which, though being different from the whole manifold, has non-empty interior. Consider for instance a disjoint union of two 2-tori, with an irrational rotation on one of them (so, this torus is a minimal set) and the identity on the other one.

It would be interesting to study minimal maps on *non-compact connected* 2-manifolds. Note that such a whole manifold is never a minimal set. To see this, recall Gottschalk's classical result [**13**] saying that a metric space which is locally compact but not compact does not admit any minimal map (in particular, this applies to non-compact manifolds). However, let us mention that it may be highly non-trivial to check whether such a space admits a *homeomorphism* with all full orbits dense. (By a *full orbit* of a point x under a homeomorphism h we mean the set $\{h^n(x) : n \in \mathbb{Z}\}$, i.e. the union of the forward and the backward orbits of x.) The difficulties appear due to non-compactness—recall that if a homeomorphism in a *compact* metric space has all full orbits dense that it also has all forward orbits dense, see [**13**]. The most famous result in this direction is that from [**17**] saying that a two-dimensional sphere with finitely many points removed does not admit any homeomorphism with all full orbits dense. For examples of (necessarily non-compact) metric spaces admitting homeomorphisms with all full orbits dense but not admitting minimal maps, see [**5**].

Since on manifolds a general theorem by Fathi and Herman [**11**] ties the existence of minimal *diffeomorphisms* to the existence of locally free diffeomorphisms, we in particular know that for instance all the odd-dimensional spheres admit minimal diffeomorphisms. We find the following problem challenging.

Problem. Prove or disprove that, for $n \geq 3$, on compact connected n-dimensional manifolds proper minimal sets with non-empty interior exist.

The paper is organized as follows. In §2 we recall some key facts and notions we will use in the proofs. We also introduce some terminology. In §3 we prove auxiliary technical results on continuous endomorphisms of 2-manifolds without boundary. Finally, in §4 we prove Theorems A and B.

2. *Preliminaries from topology and topological dynamics*

The topological terminology used in the paper is mostly that from [**24**]; see also [**6, 9, 16, 20**]. We are going to recall here some less well-known definitions and to fix some standing assumptions and some *ad hoc* terminology and notation. We assume that all spaces under consideration are compact metric (though some of the definitions and results we are going to recall work in more general spaces, see [**24**]).

2.1. *Monotone-light decomposition.*

A *continuum* is a compact connected set. A continuous map $g : X \to Y$ is *light* if, for each $y \in Y$, $f^{-1}(y)$ is totally disconnected. A continuous map $m : X \to Y$ is *monotone* if, for each $y \in Y$, the set $m^{-1}(y)$ is a continuum (we may equivalently ask that $m^{-1}(y)$ be connected since X and Y are compact metric spaces). We will strongly use the fact that if $f : X \to Y$ is a continuous map of compact metric spaces then there exist a compact metric space Z, a monotone map $m : X \to Z$ and a light map $g : Z \to Y$ such that $f = g \circ m$ (so called *monotone-light factorization* or *decomposition*, see [**24**, 4.1, p. 141] or [**20**, Theorem 13.3]).

2.2. *Order of a point, and cyclic elements.*

For a subset Y of a space X, we denote the *boundary* of Y by Bd Y. We denote by ord(p, X) the *order of a point* p in a continuum X

(in the sense of Menger-Urysohn, see [**16**, §51, I, p. 274]). It is defined as follows. Let $\mathfrak{n}$ stand for a cardinal number. We write $\operatorname{ord}(p, X) \leq \mathfrak{n}$ provided that, for every open set U containing p, there exists an open set V such that $p \in V \subseteq U$ and $\operatorname{card} \operatorname{Bd} V \leq \mathfrak{n}$. We write $\operatorname{ord}(p, X) = \mathfrak{n}$ provided that $\operatorname{ord}(p, X) \leq \mathfrak{n}$ and for each cardinal number $\mathfrak{m} < \mathfrak{n}$ the condition $\operatorname{ord}(p, X) \leq \mathfrak{m}$ does not hold. Points of order 1 in a continuum X are called *end points* and points of order at least 3 are called *ramification points* of X (notice that when a ramification point p has a neighbourhood which is a dendrite then the space really locally 'ramifies' in p). We will use the notation $\mathcal{E}(X)$ and $\mathcal{R}(X)$ for the set of all end points and ramification points of X, respectively.

A point p of a set M *separates* M between some two points a and b of the component C of M containing p if $M \setminus \{p\} = M_a \cup M_b$ where M_a and M_b are mutually separated (i.e. are disjoint and both open) and contain a and b, respectively. If M is a connected set then $p \in M$ is called a *cut point* of M if $M \setminus \{p\}$ is not connected.

Let X be a locally connected continuum. Two points $a, b \in X$ are called *conjugate* provided no point separates a and b in X. If a point $p \in X$ is neither a cut point nor an end point of X, there exists a point of X other than p which is conjugate to p—in such a case the set consisting of p together with all points of X conjugate to p is called a *simple link* of X. An equivalent definition is that a simple link is a non-degenerate subcontinuum which is maximal with respect to the property of containing no cut point of itself. The cut points and end points of X are called *degenerate cyclic elements* and the simple links are called *true cyclic elements* of X. True cyclic elements of X are identical with *maximal cyclic subsets* of X, i.e. with non-degenerate maximal (with respect to the inclusion) cyclically connected subsets of X (a set C is called *cyclically connected* provided every two points of C lie together on some simple closed curve in C). Hence every simple closed curve is a subset of a true cyclic element. Every point of X belongs to at least one cyclic element—in fact every point is either a degenerate cyclic element or a point of a single true cyclic element of X. Any two different true cyclic elements can intersect in at most one point and in such a case this point is a cut point of X. There are only countably many true cyclic elements in X and if there are infinitely many of them then their diameters go to zero, i.e. the true cyclic elements of a locally connected continuum form a so-called *null family* of sets, see [**24**, 4.2, p. 71] or [**16**, Theorem 9, p. 315].

2.3. *Dendrites, cactoids and generalized cactoids.* Recall that *dendrites* are locally connected continua which contain no simple closed curve. We will use the facts that every dendrite has the fixed point property and every subcontinuum of a dendrite is a dendrite. Note that, by [**20**, Theorem 10.7], a non-degenerate continuum X is a dendrite if and only if each point of X is either a cut point of X or an end point of X. A continuum having the property that all of its points have a neighbourhood whose closure is a dendrite is called a *local dendrite*. Equivalently, a local dendrite is a locally connected continuum containing only finitely many simple closed curves (see [**16**, Theorem 4, p. 303]). For more properties of dendrites and local dendrites see [**16, 20, 24**].

If every true cyclic element of a locally connected continuum X is a 2-sphere then X is called a *cactoid*. We emphasize that any dendrite is a cactoid, since it has *no* true cyclic element. Any cactoid which is not a dendrite is homeomorphic with the

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boundary of a bounded connected open set in $\mathbb{R}^3$ and any dendrite is embeddable in $\mathbb{R}^2$, see [**24**, pp. 76–77]. So, cactoids are metrizable (in [**19**] a cactoid is already defined as a subspace of $\mathbb{R}^3$). By Moore's theorem [**19**], cactoids are characterized as those compact metric spaces which can be obtained as a quotient space $\mathbb{S}^2/\mathcal{D}$, where $\mathbb{S}^2$ is the 2-sphere and $\mathcal{D}$ is an upper semicontinuous (usc) decomposition of $\mathbb{S}^2$ into continua. Equivalently, cactoids may be characterized as those compact metric spaces which are monotone images of the 2-sphere, see [**24**, 2.2, p. 171 and 3.3, p. 175]. In fact, for compact metric spaces, usc decompositions into continua are equivalent to monotone maps, see [**24**, 4.1, p. 127]—in particular, for a monotone map from $\mathbb{S}^2$ onto a compact metric space, the family of the preimages of points forms a usc decomposition of $\mathbb{S}^2$ into continua; and conversely, for every usc decomposition $\mathcal{D}$ of $\mathbb{S}^2$ into continua, the map sending any point from $\mathbb{S}^2$ into the element of $\mathbb{S}^2/\mathcal{D}$ containing that point is monotone, the preimages of points being just the elements of $\mathcal{D}$.

If every true cyclic element of a locally connected continuum X is a (compact connected) 2-manifold without boundary and only a finite number of these are different from a 2-sphere then X is called a *generalized cactoid*. Any cactoid is a generalized cactoid. Every generalized cactoid is a monotone image of a compact connected two-dimensional manifold without boundary, see [**23**] (the converse is not true—such an image may be slightly more complicated, see below).

2.4. *Monotone images of 2-manifolds, and tree-like continua.* The following classical result (see [**23**, Corollary]) plays a key role in the present paper.

THEOREM 1. (Roberts and Steenrod [**23**]) *The family of compact metric spaces which are monotone images of compact connected two-dimensional manifolds without boundary is composed of just those spaces each of which can be obtained from a generalized cactoid by performing, consecutively, finitely many (possibly zero) times the operation of the identification of just two points.*

Definition 2. Let a compact metric space K be a monotone image of a compact connected 2-manifold $\mathcal{M}^2$ without boundary, i.e. $K = \gamma(G)$ where G is a generalized cactoid and γ just makes identifications within a finite subset of G (and is the identity elsewhere). The γ-images of true cyclic elements of G are said to be *true atoms* of K. A point $z \in K$ is said to be a *dendritic point*, if z has an open neighbourhood whose closure is a dendrite. The set of all dendritic points of K will be denoted by $K_{\ddagger}$.

Given K, we will always work with a fixed G. (In general it is not determined uniquely; we emphasize that the notion of a true atom of K does not depend on the choice of G—in fact, one can see that T is a true atom of K if and only if it is a subset of K which can be obtained from a compact connected 2-manifold without boundary by finitely many identifications as in Theorem 1.) Since $\mathcal{M}^2$ is a locally connected metric continuum, so is K. Notice also that the set $K_{\ddagger}$ is open in K. The true atoms of K obviously form a null family. Every true atom of K is a cyclically connected subset of a true cyclic element of K. A true cyclic element of K can contain infinitely many of the true atoms of K.

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LEMMA 3. *A point $z \in K$ is dendritic if and only if there exists an open neighbourhood of z disjoint with every true atom of K.*

Proof. One implication is trivial. To prove the non-trivial one, let an open neighbourhood U of z be disjoint with every true atom of K. We may assume that this neighbourhood U is small enough to have the property that even its closure $\overline{U}$ does not intersect any true atom of K. Further, owing to the local connectedness of K we may assume that U is connected, hence $\overline{U}$ is connected. Thus $\overline{U}$ is a continuum which does not contain any point belonging to the γ-image of a true cyclic element of the generalized cactoid G. Since all points of G which do not belong to the true cyclic elements of G are cut points or end points of G, $V := \gamma^{-1}(\overline{U})$ is a compact set consisting of such points only. Therefore each component C of V, being a continuum consisting of cut points and end points (of G and hence also of C), is a dendrite. Thus the continuum $\overline{U}$ is obtained from a family of pairwise disjoint dendrites by making finitely many identifications. Hence this family of dendrites is finite, which implies that $\overline{U}$ is a local dendrite. Then the point $z \in U \subseteq \overline{U}$ has a neighbourhood (in the topology of K) whose closure is a dendrite. Hence z is dendritic. □

We will need to know when the monotone image of a compact connected 2-manifold without boundary is homeomorphic to the original manifold. Roberts and Steenrod answered this question in [**23**, Theorems 1 and 4].

THEOREM 4. (Roberts and Steenrod [**23**]) *Let $\mathcal{M}^2$ be a compact connected 2-manifold without boundary and let $\mathcal{G}$ be a usc decomposition of $\mathcal{M}^2$ into continua. For $g \in \mathcal{G}$, let $R(g)$ denote the* mod 2 *one-dimensional Betti number of the set g. Then the following are equivalent:*

(a) *the quotient space $\mathcal{M}^2/\mathcal{G}$ is homeomorphic to $\mathcal{M}^2$; and*
(b) *$\mathcal{G}$ contains at least two elements and $R(g) = 0$ for each $g \in \mathcal{G}$.*

In the terminology from [**6**] and [**9**], a compact subset C of a space X is a *cell-like set* in X if for each neighbourhood U of C the inclusion $i_U : C \to U$ is homotopic to a constant. Cell-likeness is equivalent to having *trivial shape*, i.e. the shape of a point. Recall that a continuum has trivial shape if it is the intersection of a decreasing sequence of compact absolute retracts. A subset K of an n-manifold without boundary $\mathcal{M}^n$ is a *cellular set* provided there exists a sequence of n-cells $C_1, C_2, \ldots$ in $\mathcal{M}^n$ such that Int $C_i \supset C_{i+1}$ and $K = \bigcap_i C_i$. Since the n-cells, being homeomorphic images of closed n-discs, are compact absolute retracts, each cellular subset of $\mathcal{M}^n$ is cell-like. The converse holds for $n = 2$ but not for $n \geq 3$. For at most one-dimensional metric compacta, being cell-like is equivalent to being tree-like. A continuum is *tree-like* if it is the inverse limit of an inverse sequence of trees. An equivalent definition is that it is an at most one-dimensional continuum of trivial shape.

Using these facts, one can slightly reformulate a lemma due to Roberts and Steenrod as follows.

LEMMA 5. (Roberts and Steenrod [**23**]) *Let $\mathcal{M}^2$ be a compact 2-manifold without boundary and g be a continuum on $\mathcal{M}^2$ with zero* mod 2 *one-dimensional Betti number. Then g is cell-like (hence tree-like provided it is at most one-dimensional).*

2.5. *Almost one-to-one maps.* Recall that a map $f : X \to Y$ is called *quasi-interior* (or *feebly open* or *almost open*) if, for every non-empty open set U in X, the interior of $f(U)$ is non-empty. For a map $f : X \to Y$ denote by D_f the set of points $x \in X$ such that $f^{-1}(f(x)) = x$ and by R_f the set of points $y \in Y$ such that the set $f^{-1}(y)$ is a singleton. Clearly $f(D_f) = R_f$ and $f^{-1}(R_f) = D_f$. A map $f : X \to Y$ of compact metric spaces is called *almost one-to-one* if the set D_f is dense in X or, equivalently (see [**4**, Lemma 2.7]), if R_f is dense in $f(X)$ and f is quasi-interior as a map from X to $f(X)$.

One of the main ingredients of our proofs will be the following theorem.

THEOREM 6. (Blokh *et al* [**4**]) *Suppose that $f : M \to N$ is a light and almost one-to-one map from an n-manifold M into a connected n-manifold N. Then $f|_{M\setminus\partial M} : M \setminus \partial M \to N$ is an embedding. In particular, if M is a closed manifold, then f is a homeomorphism.*

Here ∂M denotes the *(manifold) boundary* of the manifold M and a *closed manifold* means a compact, connected manifold without boundary.

2.6. *Facts from topological dynamics.* We will repeatedly use the fact that, by [**15**, Theorems 2.4 and 2.7], every minimal map in a compact metric space is quasi-interior and almost one-to-one.

A system (X, f) is *totally minimal* if (X, f^n) is minimal for all $n = 1, 2, \dots$ (here f^n is the nth iterate of f). Minimality of f on a connected space implies its total minimality, see [**25**, Theorem 3.1].

Suppose that $f : X \to X$ is a continuous map of compact metric spaces. Then f is minimal if and only if no proper, closed non-empty subset A of X is such that $f(A) \supseteq A$ (see for instance [**3**, Lemma 3.10]).

3. *Continuous endomorphisms of compact connected 2-manifolds without boundary*

In this section we adopt the following hypotheses and notation.

Standing hypotheses and notation for §3. Throughout the section, $\mathcal{M}^2$ denotes a compact connected two-dimensional manifold without boundary and $f : \mathcal{M}^2 \to \mathcal{M}^2$ is a continuous selfmap of $\mathcal{M}^2$; $f = g \circ m$ denotes the monotone-light factorization of f. Thus the monotone image $K = m(\mathcal{M}^2)$ of $\mathcal{M}^2$ is a space which can be written in the form $K = \gamma(G)$ where G is a generalized cactoid and γ just makes identifications within a finite subset of G. Recall that the γ-images of true cyclic elements of G are said to be *true atoms* of K. Next, $M \subseteq \mathcal{M}^2$ will denote a minimal set of the dynamical system $(\mathcal{M}^2, f)$.

Since $f|_M : M \to M$ is an almost one-to-one map (see §§2.5 and 2.6), both $m|_M : M \to m(M)$ and $g|_{m(M)} : m(M) \to M$ are also continuous almost one-to-one maps. (Observe that while $g|_{m(M)}$ is light, $m|_M$ need not be monotone and therefore $f|_M = g|_{m(M)} \circ m|_M$ is *not* the monotone-light factorization of $f|_M$ in general.)

LEMMA 7. *Let M be a connected minimal set of the dynamical system $(\mathcal{M}^2, f)$. Let $\mathcal{O} \subseteq M$ be an open set in $\mathcal{M}^2$ (hence open in M) such that $m(\mathcal{O})$ is open and connected in $K = m(\mathcal{M}^2)$. Then $m(\overline{\mathcal{O}})$ is not a dendrite.*

Proof. Suppose on the contrary that $m(\overline{\mathcal{O}})$ is a dendrite. The set $m(\mathcal{O})$ is connected and so the usual notion of an end point applies to it. Since this set is open, from the definition of an end point it follows that each end point of $m(\mathcal{O})$ is also an end point of K. The same is true for every ramification point of $m(\mathcal{O})$. We distinguish two cases:

Case 1. The set $m(\mathcal{O})$ has no end point. Since $m(\mathcal{O})$ is a subset of the dendrite $m(\overline{\mathcal{O}})$, we can find an ordinary point z in $m(\mathcal{O})$ (that is $\operatorname{ord}(z, K) = 2$). Owing to the local connectedness of $m(\mathcal{O})$, such z cannot be an accumulation point for $\mathcal{R}(K)$ (otherwise the open set $m(\mathcal{O})$ would contain an end point, a contradiction). Hence, we can find an open connected subset $\Lambda \subseteq m(\mathcal{O})$ which is disjoint with $\mathcal{R}(K) \cup \mathcal{E}(K)$. Since every point of Λ is ordinary, $\Lambda = (a, b)$ is an open arc in K (see [**16**, Theorem 5, p. 293]).

However, we are going to show that the existence of such Λ contradicts the fact that $m|_M$ is an almost one-to-one map. Indeed, fix a closed disc $B \subseteq \mathcal{O}$ with $m(B) \subseteq (a, b)$. The set $m(B)$ is closed, connected and, by minimality of $f|_M$, not a singleton. Thus there are points $\alpha \neq \beta$ in (a, b) with $m(B) = [\alpha, \beta]$. Choose preimages $x_\alpha, x_\beta \in B$ of α, β and fix any $x \in B$, $x \neq x_\alpha, x_\beta$. If $m(x) = \alpha$ or β then $x \notin D_{m|_M}$ (in the notation from §2.5). Otherwise choose a simple closed curve $\Gamma \subseteq B$ passing through x_α, x_β and x. Since $m(\Gamma)$ covers (at least) twice the interval $[\alpha, \beta]$, again $x \notin D_{m|_M}$. Hence $D_{m|_M}$ is not dense in M which contradicts the fact that $m|_M$ is almost one-to-one.

Case 2. The set $m(\mathcal{O})$ has at least one end point. Choose some $z \in \mathcal{E}(m(\mathcal{O})) = \mathcal{E}(K) \cap m(\mathcal{O})$. Then, owing to the local connectedness of $m(\mathcal{O})$ and by the definition of the end points, we can find a sufficiently small open connected neighbourhood $U \subseteq m(\mathcal{O})$ of z such that $\operatorname{card} \operatorname{Bd}_K U = 1$, that is $\operatorname{Bd}_K U = \{w\}$ for some $w \in K$. Let $X := \overline{U} = U \cup \{w\}$. Then X is a connected subset of the dendrite $m(\overline{\mathcal{O}})$ (notice that, by the boundary bumping theorem [**20**, Theorem 5.4], the closure $\overline{U}$ would be connected even if U were not assumed to be connected) and therefore X itself is a dendrite. Next, the minimality of $f|_M = g \circ m|_M : M \to M$ implies the minimality of $h := m \circ g|_{m(M)} : m(M) \to m(M)$ and so $h^j(w) \in U$ for some positive integer j. Now, since M is connected, the map $f^j : M \to M$ is minimal. Then $h^j : m(M) \to m(M)$ is also minimal, owing to the semiconjugacy $m \circ f^j = h^j \circ m$ (or to the connectedness of $m(M)$). Define a selfmap $p : X \to X$ by $p(x) = h^j(x)$ if $h^j(x) \in X$ and $p(x) = w$ otherwise. We are going to show that p is continuous. To this end notice that $p = \varphi \circ h^j$ where $\varphi : m(M) \to X$ is defined by $\varphi(x) = x$ if $x \in X$ and $\varphi(x) = w$ otherwise. So it is sufficient to show that φ is continuous. Consider the maps $\varphi_1 : X \to X$ and $\varphi_2 : m(M) \setminus U \to X$ defined by $\varphi_1(x) = x$ (the identity) and $\varphi_2(x) = w$ (a constant map). Notice that these two maps are continuous, coincide on the intersection of their domains (which is in fact just the singleton $\{w\}$) and their domains are closed in the domain of φ. Since $\varphi(x) = \varphi_1(x)$ if $x \in X$ and $\varphi(x) = \varphi_2(x)$ if $x \in m(M) \setminus U$, the continuity of φ follows.

Finally, in view of the fixed point property of dendrites, there is $c \in \overline{U} = U \cup \{w\}$ such that $p(c) = c$. Observe that $c \neq w$ because $p(w) \in U \not\ni w$. Now, if it were $c \in U$ and $h^j(c) \notin \overline{U}$, then it would be $p(c) = w \neq c$. Therefore we have $c \in U$ and $h^j(c) \in \overline{U}$. This implies that $c = p(c) = h^j(c)$ and thus c is a fixed point of h^j, a contradiction with minimality of h^j. □

LEMMA 8. *Let M be a connected minimal set of the dynamical system $(\mathcal{M}^2, f)$. Let two open sets W_1, W_2 be such that $\overline{W_1} \subseteq W_2 \subseteq \operatorname{Int} M$, both $m(W_1)$ and $m(W_2)$ are open in K and $m(W_1)$ is connected. Then $m(W_1)$ is a connected 2-manifold without boundary (possibly non-compact).*

Proof. First, we claim that:
(i) $m(W_2) \cap K_\ddagger = \emptyset$.
Suppose on the contrary that $m(W_2)$ contains a dendritic point. Since $m(W_2)$ is assumed to be open and dendrites are locally connected, $m(W_2)$ contains an open connected set V whose closure is a dendrite. Put $\mathcal{O} := m^{-1}(V) \cap W_2$. Then the set $\mathcal{O} \subseteq M$ is open in $\mathcal{M}^2$. Moreover, $m(\mathcal{O}) = V$ (use the definition of $\mathcal{O}$ and the inclusion $m(W_2) \supseteq V$). Since V is open and connected in K and $m(\overline{\mathcal{O}}) = \overline{V}$ is a dendrite, we have a contradiction with Lemma 7.

Furthermore, we claim that:
(ii) every compact set $A \subseteq m(W_2)$ intersects only a finite number of true atoms of K.
Suppose on the contrary that some A does not satisfy this. Then, by a compactness argument, there exists a point $\alpha \in A$ such that any neighbourhood of α intersects infinitely many of the true atoms of K. Hence, since $m(W_2)$ is open and the true atoms of K form a null family, there exists an infinite sequence of true atoms of K inside $m(W_2)$. In this way, we have proved the existence of a topological 2-sphere $\mathbb{S}^2 \subseteq m(W_2) \subseteq m(M)$ (since K is obtained from the generalized cactoid G after identifying only a *finite* number of its points). In view of Theorem 6, the map $g|_{\mathbb{S}^2} : \mathbb{S}^2 \to \mathcal{M}^2$ is a homeomorphism (hence a surjection). This implies, on the one hand, that $M = f(M) = g(m(M)) \supseteq g(\mathcal{S}^2) = \mathcal{M}^2$ (hence $M = \mathcal{M}^2$) and, on the other hand, that $\mathcal{M}^2$ is a topological 2-sphere. This contradicts the fact that a 2-sphere does not admit any minimal map (every continuous selfmap of a 2-sphere has a fixed point or a periodic point of period 2).

Now, take an open set U with $m(\overline{W_1}) \subseteq U \subseteq \overline{U} \subseteq m(W_2)$. By (i) and (ii), $\overline{U}$ does not contain any dendritic point of K and intersects only a finite number of true atoms of K; denote them by $A_i = \gamma(G_i)$, $i = 1, \dots, k$ (here G_i are some of the true cyclic elements of G, i.e. compact connected 2-manifolds without boundary). Then, by Lemma 3, every point from U belongs to some of these finitely many true atoms, i.e. $U \subseteq \bigcup_{i=1}^k \gamma(G_i)$. In consequence, U can be viewed as an open subset of such a subspace of K which is of the form $\Gamma(\bigsqcup_{i=1}^k G_i^*)$ where $\bigsqcup_{i=1}^k G_i^*$ is a disjoint sum of compact connected 2-manifolds without boundary and Γ just makes some identifications within a finite set P of their points (i.e. for every $p \in P$ there exists $p' \in P$, $p' \neq p$, such that $\Gamma(p) = \Gamma(p')$).

Suppose that U contains a point from $\Gamma(P)$. Then U, being open, contains a finite wedge sum $\bigvee_{j=1}^r D_j$ of open discs ($r \geq 2$). For each of the discs D_j, the restriction $g|_{D_j} : D_j \to \mathcal{M}^2$ is an embedding by virtue of Theorem 6. This means that, except for one point, every point from the non-empty open subset $\bigcap_{j=1}^r g(D_j) \subseteq M$ has at least r preimages under g in U and hence also at least r preimages under f in M, contradicting the fact that the quasi-interior surjective map $f|_M : M \to M$ is almost one-to-one.

Thus we have proved that $U \cap \Gamma(P) = \emptyset$ whence we get that the *connected* set $m(\overline{W_1})$ is a subset of only one of the true atoms $A_1, \dots, A_k$. We may assume that this true atom is $A_1 = \Gamma(G_1^*)$. The set $m(\overline{W_1}) \subseteq A_1$ is closed and disjoint with the finite set $\Gamma(P) \cap A_1$.

By removing, if necessary, from A_1 a sufficiently small open neighbourhood of the set $\Gamma(P) \cap A_1$ we get a compact connected 2-manifold $\mathcal{V}^2$ such that $m(W_1) \subseteq \mathcal{V}^2 \subseteq A_1 \setminus \Gamma(P)$ ($\mathcal{V}^2$ is a manifold with boundary if $\Gamma(P) \cap A_1 \neq \emptyset$). Since $m(W_1)$ is open in the topology of K, it does not intersect the (possibly empty) boundary of $\mathcal{V}^2$. Therefore $m(W_1)$ is a 2-manifold without boundary (possibly non-compact). By the assumption, it is also connected. □

As an important consequence of Lemma 8, we get a considerably shorter proof of Theorem 3.16 and Corollary 3.17 from [**3**].

COROLLARY 9. [**3**] *Assume that $f : \mathcal{M}^2 \to \mathcal{M}^2$ is a minimal map. Then f is monotone with $R(f^{-1}(x)) = 0$ whenever $x \in \mathcal{M}^2$, hence f has tree-like point inverses. Moreover, $\mathcal{M}^2$ is either the 2-torus or the Klein bottle.*

Proof. Applying Lemma 8 to the situation $W_1 = W_2 = M = \mathcal{M}^2$ we get that $K = m(\mathcal{M}^2) = m(W_1)$ is a connected (and obviously compact) 2-manifold without boundary. Therefore, by Theorem 6, g is a homeomorphism. This implies, on the one hand, that $m(\mathcal{M}^2)$ is homeomorphic to $\mathcal{M}^2$ and, on the other hand, that the family of point inverses of f coincides with the family of point inverses of m, whence we get that f is monotone.

The point inverses of m form a usc decomposition of $\mathcal{M}^2$ into continua (we discussed this general fact in §2.3). Since f is minimal, their interiors are empty. Thus they are at most one-dimensional continua. Since $m(\mathcal{M}^2)$ is homeomorphic to $\mathcal{M}^2$, from Theorem 4 we get that $R(m^{-1}(z)) = 0$ whenever $z \in m(\mathcal{M}^2)$. Equivalently, $R(f^{-1}(x)) = 0$ whenever $x \in \mathcal{M}^2$. By Lemma 5, each point inverse of f is tree-like. Therefore, by [**6**, Theorem 25.1 and Corollary 1A], f can be approximated by homeomorphisms. Now use the fact that for a given manifold $\mathcal{M}^2$ with non-zero Euler characteristic there is r such that every homeomorphism $\mathcal{M}^2 \to \mathcal{M}^2$ has a periodic point with period at most r (see [**12**] or [**14**, Corollary 9]). So, if $\mathcal{M}^2$ is neither the 2-torus nor the Klein bottle, then f is a uniform limit of a sequence $(h_n)_{n=1}^{\infty}$ of homeomorphisms such that h_n has a periodic point with period at most r. Then using compactness and passing to a subsequence we get that also f has a periodic point with period at most r, which contradicts the minimality of f. □

4. *Proofs of Theorems A and B*

In this section we still use the standing hypotheses and notation from §3 except for the assumption that the manifold under consideration has no boundary (we will always explicitly say what kind of manifolds we have in mind).

We first prove Theorem A for manifolds without boundary.

THEOREM 10. *Let $\mathcal{M}^2$ be a compact connected 2-manifold without boundary and let $f : \mathcal{M}^2 \to \mathcal{M}^2$ be a continuous map. If $M \subseteq \mathcal{M}^2$ is a minimal set of the dynamical system $(\mathcal{M}^2, f)$ then either $M = \mathcal{M}^2$ or M is nowhere dense in $\mathcal{M}^2$.*

Proof. Let $M \neq \mathcal{M}^2$. We wish to prove that the (closed) set M is nowhere dense. Suppose on the contrary that Int $M \neq \emptyset$. Since we are on a manifold, the assumption Int $M \neq \emptyset$ gives that M has a connected component with non-empty interior. Then the minimality of $f|_M$ implies that M has finitely many components which are cyclically permuted

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by f and the restriction of f^k to each of them is totally minimal, k being the number of components. Hence without loss of generality we may assume that the minimal set M is connected. The set Bd M is a non-empty closed proper subset of M and so the minimality of M implies that $\Delta := \text{Bd}\, M \setminus f(\text{Bd}\, M) \neq \emptyset$ (see §2.6).

Fix $q \in \text{Bd}\, M \setminus f(\text{Bd}\, M)$ and choose two open discs $D_1 \subseteq \overline{D_1} \subseteq D_2$ centred at q and disjoint with $f(\text{Bd}\, M)$. Fix $j \in \{1, 2\}$. The set $f^{-1}(D_j \cap M)$ is disjoint with Bd M and so each of its components is contained either in Int M or in $\mathcal{M}^2 \setminus M$ (note that $\mathcal{M}^2 = \text{Int}(\mathcal{M}^2 \setminus M) \cup \text{Int}\, M \cup \text{Bd}\, M$). Since $f(M) = M$, q has a preimage $p \in f^{-1}(q)$ in Int M. The component C_j of $f^{-1}(D_j \cap M)$ containing p is a subset of Int M. Moreover, C_j is open in $\mathcal{M}^2$. In fact, if $x \in C_j \subseteq \text{Int}\, M$ then $f(x) \in D_j \cap M$ and there is an open disc B centred at x such that it holds both $B \subseteq \text{Int}\, M$ and $f(B) \subseteq D_j$. Hence $f(B) \subseteq D_j \cap M$. Since x belongs to connected sets C_j and B, also $C_j \cup B$ is connected. Since $f(C_j \cup B) \subseteq D_j \cap M$, by the definition of C_j we have $B \subseteq C_j$. Hence both C_1 and C_2 are open neighbourhoods of p.

Notice that $C_1 \subseteq C_2$ and $f(\overline{C_1}) \subseteq \overline{D_1} \subseteq D_2$. Hence $\overline{C_1} \subseteq f^{-1}(D_2 \cap M)$ and since $\overline{C_1}$ is connected and intersects the component C_2 of the set $f^{-1}(D_2 \cap M)$, it holds that $\overline{C_1} \subseteq C_2$.

Since for $j = 1, 2$ the set C_j is open (in $\mathcal{M}^2$) and is a component of the f-preimage of a set, it consists of the whole components of f-preimages of points. Therefore, since m just collapses to points the components of f-preimages of points from $\mathcal{M}^2$, also the set $m(C_j)$ is open (in K). By Lemma 8, $m(C_1)$ is a connected 2-manifold without boundary. In consequence, by Theorem 6, the light almost one-to-one map $g|_{m(C_1)} : m(C_1) \to \mathcal{M}^2$ is an embedding. Hence, due to the classical invariance of domain theorem, $g(m(C_1)) = f(C_1)$ is open in $\mathcal{M}^2$. This contradicts the facts that the set $f(C_1)$ is a subset of M and contains the point $q \in \text{Bd}\, M$. □

Now we present a trick which shows how, when proving our main results, the case of manifolds with boundary can be converted to the case of manifolds without boundary.

PROPOSITION 11. *Let $\mathcal{M}^2$ be a compact connected 2-manifold with boundary and $f : \mathcal{M}^2 \to \mathcal{M}^2$ be continuous. Then there exist a compact connected 2-manifold $\mathcal{N}^2$ without boundary and a continuous map $F : \mathcal{N}^2 \to \mathcal{N}^2$ such that $\mathcal{M}^2 \subseteq \mathcal{N}^2$ and $f = F|_{\mathcal{M}^2}$. Thus, $(\mathcal{M}^2, f)$ is a subsystem of $(\mathcal{N}^2, F)$.*

Proof. The boundary of $\mathcal{M}^2$ consists of a finite number of simple closed curves $S_1, S_2, \ldots, S_k$ (denote their union by S). Put $\mathcal{M}_1^2 = \mathcal{M}^2 \times \{1\}$ with the topology coming from $\mathcal{M}^2$. Now glue the 2-manifolds $\mathcal{M}^2$ and $\mathcal{M}_1^2$ along the corresponding components of their boundary. Namely, define on $\mathcal{M}^2 \cup \mathcal{M}_1^2$ the decomposition $\mathcal{D}$ whose elements are the sets $\{s, (s, 1)\}$ ($s \in S$) and the remaining singletons. Then the quotient space $\mathcal{N}^2 := (\mathcal{M}^2 \cup \mathcal{M}_1^2)/\mathcal{D}$ is obviously a compact connected 2-manifold $\mathcal{N}^2$ without boundary containing a homeomorphic copy of $\mathcal{M}^2$. We may think of $\mathcal{M}^2$ as being a subset of $\mathcal{N}^2$ if we adopt the convention that to denote the points of $\mathcal{N}^2$ we keep the notation coming from $\mathcal{M}^2$ and $\mathcal{M}_1^2$. Thus, $\mathcal{M}^2 \subseteq \mathcal{N}^2$ and some points of $\mathcal{N}^2$ are of the form x where $x \in \mathcal{M}^2$, some are of the form $(x, 1)$ where $x \in \mathcal{M}^2$ and for each $s \in S$, the points s and $(s, 1)$ are identical. Now define $\varphi : \mathcal{N}^2 \to \mathcal{M}^2$ by $\varphi(x) = \varphi(x, 1) = x$ for every $x \in \mathcal{M}^2$. This is obviously a continuous map. Then the composition $F := f \circ \varphi$ is a continuous map $\mathcal{N}^2 \to \mathcal{N}^2$ which is an extension of f. □

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COROLLARY 12. *Let $\mathcal{M}^2$ be a compact connected 2-manifold with boundary and let $f: \mathcal{M}^2 \to \mathcal{M}^2$ be a continuous map. Then all the minimal sets of the system $(\mathcal{M}^2, f)$ are nowhere dense in $\mathcal{M}^2$.*

Proof. Let A be a minimal set of the system $(\mathcal{M}^2, f)$. By Proposition 11 there exists a compact connected 2-manifold $\mathcal{N}^2$ without boundary and a continuous map $F : \mathcal{N}^2 \to \mathcal{N}^2$ such that $(\mathcal{M}^2, f)$ is a subsystem of $(\mathcal{N}^2, F)$. Since $\mathcal{M}^2$ is a *proper* subset of $\mathcal{N}^2$, A is a proper minimal set in $(\mathcal{N}^2, F)$. By Theorem 10, $A \subseteq \mathcal{M}^2$ is nowhere dense in $\mathcal{N}^2$. Since $\mathcal{M}^2 \subseteq \mathcal{N}^2$ are 2-manifolds, A is nowhere dense also in $\mathcal{M}^2$. □

Now we easily get our main result. For convenience we repeat its statement here.

THEOREM A. *Let $\mathcal{M}^2$ be a compact connected two-dimensional manifold, with or without boundary, and let $f : \mathcal{M}^2 \to \mathcal{M}^2$ be a continuous map. If $M \subseteq \mathcal{M}^2$ is a minimal set of the dynamical system $(\mathcal{M}^2, f)$ then either $M = \mathcal{M}^2$ or M is nowhere dense in $\mathcal{M}^2$.*

Proof. Use Theorem 10 and Corollary 12. □

Using Corollaries 9 and 12 one can now prove Theorem B in a similar way as in [**3**]. For completeness we give a proof here.

THEOREM B. [**3**] *Suppose that $f : \mathcal{M}^2 \to \mathcal{M}^2$ is a minimal map of a 2-manifold (compact or not, with or without boundary). Then f is a monotone map with tree-like point inverses and $\mathcal{M}^2$ is either a finite union of tori or a finite union of Klein bottles which are cyclically permuted by f.*

Proof. Non-compact manifolds do not admit minimal maps (see Introduction). So $\mathcal{M}^2$ is compact. If it is also connected, Corollary 12 gives that it has no boundary and so one can apply Corollary 9. Therefore assume that the (compact) manifold $\mathcal{M}^2$ is not connected. Owing to the minimality of f, $\mathcal{M}^2$ has finitely many connected components $C_1, \ldots, C_n$ which are cyclically permuted by f (a component C_j is mapped *onto* the next one, $C_{j+1(\text{mod } n)}$). Denote $f_j = f|_{C_j}$, $j = 1, \ldots, n$. Fix one of the components C_j. Since $f^n|_{C_j} : C_j \to C_j$ is minimal, by Corollary 12 the compact connected 2-manifold C_j has no boundary. Then, by Corollary 9, C_j is either a 2-torus or a Klein bottle and $f^n|_{C_j}$ is monotone. Moreover, $f^{-n}(x)$ is tree-like for every $x \in C_j$. Therefore, by [**6**, Theorem 25.1 and Corollary 1A], $f^n|_{C_j}$ can be approximated by homeomorphisms. Since sufficiently close maps on C_j are homotopic (see, e.g., [**10**, 2.P.24, 3.2.5 and 5.S.2]), we find that $f^n|_{C_j}$ is a homotopy equivalence and induces an isomorphism on homology: $(f^n|_{C_j})_* : H_*(C_j) \to H_*(C_j)$. Now, from $(f^n|_{C_j})_* = (f_{j+n-1(\text{mod } n)})_* \circ \cdots \circ (f_j)_*$ we get that $(f_j)_*$ is an injection (monomorphism) and $(f_{j+n-1(\text{mod } n)})_*$ is a surjection (epimorphism). Since j is an arbitrary integer (mod n) here, we obtain that $(f_j)_* : H_*(C_j) \to H_*(C_{j+1(\text{mod } n)})$ is an isomorphism. In consequence, the components C_j and $C_{j+1(\text{mod } n)}$ are homeomorphic. Next, notice that the monotonicity of f^n proved above implies the monotonicity of f. In fact, every point inverse $f^{-1}(x) = f^{n-1}(f^{-n}(x))$, being a continuous image of a connected set, is connected. The monotonicity of f_j implies (see §2.3) that the family $\{f_j^{-1}(x) : x \in C_{j+1(\text{mod } n)}\}$ is a usc decomposition of C_j into continua. Hence, by Theorem 4, for each continuum g in this family we have $R(g) = 0$. Therefore, by Lemma 5, f_j has tree-like point inverses. □

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