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ON WEAKLY PERFECT ANNIHILATING-IDEAL GRAPHS

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Abstract

We prove that the annihilating-ideal graph of a commutative semigroup with unity is, in general, not weakly perfect. This settles the conjecture of DeMeyer and Schneider [‘The annihilating-ideal graph of commutative semigroups’, *J. Algebra* **469** (2017), 402–420]. Further, we prove that the zero-divisor graphs of semigroups with respect to semiprime ideals are weakly perfect. This enables us to produce a large class of examples of weakly perfect zero-divisor graphs from a fixed semigroup by choosing different semiprime ideals.

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1. Introduction

The notion of a zero-divisor graph was introduced by Beck [5] in 1988 to study colourings of commutative rings. He proved that the zero-divisor graphs of reduced rings are weakly perfect, that is, the chromatic number χ is equal to the clique number ω. He conjectured that the zero-divisor graphs of commutative rings are weakly perfect. However, Anderson and Naseer [1] provided a counterexample settling the conjecture negatively. Anderson and Livingston introduced and studied a modified version of the zero-divisor graph, which has only nonzero zero-divisors of the ring (see [4]). The concept of a zero-divisor graph has since then been extended to noncommutative rings (see [22]), semigroups (see [8, 9]) and partially ordered sets (see [12, 13, 18]).

In ring theory, the structure of a ring R is closely related to the behaviour of ideals. In the theory of zero-divisor graphs, a natural line of inquiry is to replace elements of the ring R by ideals. This gives rise to a zero-divisor graph on the set of ideals of R. This graph was introduced and studied by Behboodi and Rakeei [6, 7] and is called the annihilating-ideal graph of a commutative ring. Let R be a commutative ring (semigroup) with unity. The annihilating-ideal graph $\mathbb{AG}(R)$ of R is a simple undirected graph whose vertex set $V(\mathbb{AG}(R))$ is the set of nonzero ideals with nonzero annihilator. That is, a nonzero ideal I belongs to $V(\mathbb{AG}(R))$ if and only if there exists a

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nonzero ideal J of R such that $IJ = (0)$ and two distinct vertices I and J are adjacent if and only if $IJ = (0)$. In [7], Behboodi and Rakeei posed the following conjecture about annihilating-ideal graphs of commutative rings. (It is known [7, Corollary 2.11] that Conjecture 1.1 is true in the case when R is reduced.)

CONJECTURE 1.1. For every commutative ring R with unity, $\chi(\mathbb{AG}(R)) = \omega(\mathbb{AG}(R))$.

Recently, DeMeyer and Schneider [9, Theorem 22] proved that the annihilating-ideal graphs of reduced semigroups are weakly perfect and posed an analogue of Conjecture 1.1 for the annihilating-ideal graphs of commutative semigroups.

CONJECTURE 1.2. For every commutative semigroup S, $\chi(\mathbb{AG}(S)) = \omega(\mathbb{AG}(S))$.

In this paper, we prove that the zero-divisor graphs of semigroups with respect to semiprime ideals are weakly perfect. This result enables us to produce a large class of examples of weakly perfect zero-divisor graphs from a fixed semigroup with respect to different semiprime ideals. In particular, if we take (0) to be a semiprime ideal in a commutative ring (semigroup) S, we obtain the results of Beck [5, Theorem 3.8], Behboodi and Rakeei [7, Corollary 2.11] and DeMeyer and Schneider [9, Theorem 22]. Also, we settle Conjecture 1.2 negatively for the annihilating-ideal graph of commutative semigroups by giving an example of an annihilating-ideal graph of a nonreduced semigroup. Finally, we consider a partial order on a reduced semigroup and give the relation between the minimal prime ideals of a reduced semigroup S and the minimal prime ideals of S treated as a poset.

2. Zero-divisor graphs associated to semigroups

Throughout, we assume that S is a commutative semigroup with 0 *and* 1.

We begin with the necessary definitions and terminology. The vertex set of the zero-divisor graph $G(R)$ of a commutative ring R with unity is the set of all nonzero zero-divisors and two vertices x and y are adjacent if $xy = 0$. Based on this definition, the result of Beck [5] for reduced commutative rings takes the following form.

THEOREM 2.1. *Let R be a reduced commutative ring with unity and let $G(R)$ be its zero-divisor graph. Then $\omega(G(R)) = \chi(G(R)) = \#\{$minimal prime ideals of $R\}$.*

One of our aims is to identify the essential features of this result and adapt it to semigroups. A noteworthy feature is that the primary decomposition of ideals can still be achieved for semigroups so far as questions about the colouring of associated graphs are concerned. Recently, Anderson and Badawi [2] introduced the notion of a multiplicative prime subset of R. We next recall the notions of ideal, prime ideal and semiprime ideal in a semigroup.

DEFINITION 2.2. Let S be a semigroup. If a is an element of S, then the smallest ideal containing a is called the *principal ideal generated by a*. As in rings, this ideal is $aS = \{as \mid s \in S\}$, the set of multiples of a. The zero ideal will be denoted (0).

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A nonempty subset I of S is called an *ideal* if $sx \in I$ for every $x \in I$ and $s \in S$. A proper ideal I of S is said to be a *prime ideal* of S if, for $ab \in I$, either $a \in I$ or $b \in I$. For a nonempty subset $A \subseteq S$, the *annihilator* of A, denoted $\mathrm{ann}(A)$, is given by $\mathrm{ann}(A) = \{x \in S \mid xa = 0 \text{ for all } a \in A\}$.

Note that the product, union and intersection of ideals of S will again be an ideal of S, and that each nonzero ideal must necessarily be composed of a union of principal ideals. The concept of a semiprime ideal of a commutative ring with unity can be found in Rav [21].

Definition 2.3. Let S be a semigroup. We say that a proper ideal I of S is *semiprime* if $a^2 \in I$ implies that $a \in I$.

There are many examples of semiprime ideals. Let R be a commutative ring with unity and let $S = R$ be a semigroup with a multiplication induced by the multiplication of R.

(1) Every prime ideal is a semiprime ideal.
(2) The union of prime ideals is a semiprime ideal.
(3) Any radical ideal is a semiprime ideal.
(4) The nilradical of R is a semiprime ideal.
(5) The set of zero-divisors $Z(R)$ of R is a semiprime ideal. This can be seen as follows. It is easy to see that $Z(R)$ is an ideal of the semigroup $(R, \cdot)$. Now, if $a^2 \in Z(R)$, then $\exists\ x \neq 0$ such that $a^2x = 0$. If $ax = 0$, then $a \in Z(R)$. If $ax \neq 0$, then, as $a(ax) = 0$, we must have $a \in Z(R)$.
(6) If $U(R)$ is the set of units of R, then $R \setminus U(R)$ is a semiprime ideal.

In [23], Redmond introduced the zero-divisor graph of a commutative ring R with unity with respect to an ideal I. We consider this definition when I is an ideal of a semigroup S.

Definition 2.4. Let S be a semigroup and let I be an ideal of S. We associate to the semigroup S a simple undirected graph $G_I(S)$ with the vertex set

$$V(G_I(S)) = \{x \in S \mid x \notin I \text{ and } xy \in I \text{ for some } y \notin I\}$$

and two vertices a and b in $V(G_I(S))$ are adjacent if and only if $ab \in I$.

Remark 2.5. We note that if $I = (0)$, then $G_I(S)$ is the usual zero-divisor graph $G(S)$ associated to the semigroup S. Let R be a commutative ring with unity and let $S = R$ be the semigroup with respect to the multiplication of R. In this case, if we let $I = (0)$, then $G_I(R)$ is the usual zero-divisor graph of R, denoted $G(R)$.

It is clear that the set of all ideals of S forms a semigroup under the multiplication of ideals. Hence, if $\mathbb{A}(S)$ is the semigroup of all annihilating-ideals of S under the ideal multiplication, then the *annihilating-ideal graph* of S is the zero-divisor graph of $\mathbb{A}(S)$. We denote this graph by $\mathbb{AG}(S)$.

With this preparation, we are ready to prove the first main result of the paper.

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THEOREM 2.6. *Let I be a semiprime ideal of S. If $\omega(G_I(S)) < \infty$, then $G_I(S)$ is a weakly perfect graph. In general, however, $G_I(S)$ is not weakly perfect. Thus, Conjecture 1.2 is not true.*

PROOF. Suppose that $\omega(G_I(S)) = n$. Then there exists a clique $C = \{x_1, x_2, \ldots, x_n\}$ of $G_I(S)$. Note here that $x_i \notin I$, as $x_i \in V(G_I(S))$. We recall the familiar notation $(I : x) = \{y \in S \mid xy \in I\}$. Clearly, it is an ideal in S.

Claim 1. $(I : x_i) \neq (I : x_j)$ *for* $i \neq j$.

Suppose on the contrary that $(I : x_i) = (I : x_j)$ with $i \neq j$. Since the x_i's form a clique, we have $x_ix_j \in I$ for $i \neq j$. Hence, $x_j \in (I : x_i) = (I : x_j)$. So, $x_j^2 \in I$. But, since I is semiprime, this gives $x_j \in I$, which is a contradiction.

Claim 2. $(I : x_i)$ *is a prime ideal of* S *for all* i.

Clearly, $(I : x_i) \neq S$ for any x_i. It is easy to see that $(I : x_i)$ is an ideal. We now show that $(I : x_i)$ is a prime ideal of S. On the contrary, suppose that $ab \in (I : x_i)$ and $a, b \notin (I : x_i)$. Then $ax_i \notin I$ and $bx_i \notin I$. Hence, $ax_i, bx_i \in V(G_I(S))$. Now, consider the set $C' = \{x_1, \ldots, x_{i-1}, ax_i, bx_i, x_{i+1}, \ldots, x_n\}$. We show that C' is a clique in $G_I(S)$ of size $n + 1$. To do this, we first show that all the elements of C' are distinct.

Suppose that $ax_i = bx_i$. Since $ab \in (I : x_i)$, we have $abx_i \in I$ and so $abx_i = a^2x_i \in I$. Since I is an ideal, $a^2x_i^2 \in I$, that is, $(ax_i)^2 \in I$. Since I is semiprime, this gives $ax_i \in I$, contradicting our assumption that $a \notin (I : x_i)$. Thus, $ax_i \neq bx_i$.

Next, suppose that $ax_i = x_j$ with $i \neq j$. In this case $x_j^2 = ax_ix_j \in I$. As I is semiprime, this gives $x_j \in I$, again a contradiction.

Finally, suppose that $ax_i = x_i$. In this case, since $abx_i \in I$, we have $bx_i \in I$, that is, $b \in (I : x_i)$, contradicting our initial assumption that $b \notin (I : x_i)$. Hence, $ax_i \neq x_i$.

Thus, all the vertices of C' are distinct, showing that $|C'| = n + 1$. Clearly, C' forms a clique of size $n + 1$, which is a contradiction since $\omega(G_I(S)) = n$. This shows that $(I : x_i)$ is a prime ideal of S for all i.

Claim 3. $(I : x_i)$ *is minimal among all prime ideals of* S *containing* I.

Suppose on the contrary that Q is any prime ideal of S with $I \subseteq Q \subsetneq (I : x_i)$. Then there exists $y \in (I : x_i)$ such that $y \notin Q$. But then $yx_i \in I \subseteq Q$, so $x_i \in Q \subsetneq (I : x_i)$, since Q is prime. So, $x_i \in (I : x_i)$ and hence $x_i^2 \in I$. Again, since I is semiprime, $x_i \in I$, which is a contradiction. Thus, $(I : x_i)$ is minimal, as claimed.

Claim 4. $I = \bigcap_{i=1}^n (I : x_i)$ *and this decomposition is irredundant.*

Clearly, $I \subseteq (I : x_i)$ for all i and hence $I \subseteq \bigcap_{i=1}^n (I : x_i)$. To prove the opposite inclusion, suppose, if possible, that $I \subsetneq \bigcap_{i=1}^n (I : x_i)$. Then there exists $t \in \bigcap_{i=1}^n (I : x_i)$ such that $t \notin I$. Hence, $tx_i \in I$ for all i. Consider the set $C'' = \{t, x_1, x_2, \ldots, x_n\}$. We show that C'' is a clique in $G_I(S)$ of size $n + 1$. In order to show that C'' is of size $n + 1$, it is enough to show that $t \neq x_i$ for any i. So, suppose that $t = x_i$ for some i. Since $tx_i \in I$ for all i, we get $x_i^2 = tx_i \in I$. Now, I is semiprime, so $x_i \in I$, which is a contradiction. Hence, $t \neq x_i$ for all i, showing that all elements of C'' are distinct. Since $tx_i \in I$ for all i, it is clear that C'' is indeed a clique of size $n + 1$, contradicting $\omega(G_I(S)) = n$. Thus, $I = \bigcap_{i=1}^n (I : x_i)$. To show that this decomposition is irredundant,

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suppose if possible that $\bigcap_{j=1, j\neq i}^{n}(I : x_j) \subseteq (I : x_i)$ for some i. Since $(I : x_i)$ is prime, we get $(I : x_k) \subseteq (I : x_i)$ for some $k \neq i$ with $1 \leq k \leq n$. As $(I : x_i)$ is minimal, we get $(I : x_i) = (I : x_k)$, contradicting Claim 1. Hence, the decomposition is irredundant.

Claim 5. If I has two irredundant decompositions $I = \bigcap_{i=1}^{n}(I : x_i) = \bigcap_{j=1}^{m}(I : y_j)$, then $n = m$ and, for any i, $(I : x_i) = (I : y_j)$ for some j.

Without loss of generality, assume that $m < n$. For each $j \in \{1, \ldots, m\}$, we have $\bigcap_{i=1}^{n}(I : x_i) \subseteq (I : y_j)$. As $(I : y_j)$ is prime, $(I : x_{k_j}) \subseteq (I : y_j)$ for some $k_j \in \{1, \ldots, n\}$. From the minimality of $(I : y_j)$, it follows that $(I : y_j) = (I : x_{k_j})$. So, $I = \bigcap_{j=1}^{m}(I : y_j) = \bigcap_{j=1}^{m}(I : x_{k_j})$. Since $m < n$, this contradicts the fact that $I = \bigcap_{i=1}^{n}(I : x_i)$ is an irredundant decomposition of I. Hence, $m = n$ and, for any i, $(I : x_i) = (I : y_j)$ for some j.

Claim 6. $\chi(G_I(S)) = n$.

Let us denote $(I : x_i)$ by P_i. We, now, consider a vertex colouring (given by Beck [5]) of $G_I(S)$ as follows. For $x \in V(G_I(S))$, define $f(x) = \min\{i \mid x \notin P_i\}$. We show that f is indeed a proper colouring of $G_I(S)$, that is, for x adjacent to y, we show that $f(x) \neq f(y)$. Take x adjacent to y, that is, $x,\ y \notin I$ and $xy \in I$. Since $x \notin I$, we have $x \notin P_i$ for some i. Let $f(x) = k$, so that $x \notin P_k$ and $x \in P_j$ for $1 \leq j \leq k - 1$. Now, we claim that $f(y) \neq k$. Suppose on the contrary that $f(y) = k$. Then $y \notin P_k = (I : x_k)$, that is, $yx_k \notin I$. Also, $xx_k \notin I$ since $x \notin P_k$. Thus, $xx_k, yx_k \in V(G_I(S))$. Put $C''' = \{x_1, \ldots, x_{k-1}, xx_k, yx_k, x_{k+1}, \ldots, x_n\}$. It is easy to see that C''' is a clique of size $n + 1$. This contradicts $\omega(G_I(S)) = n$ and hence f is indeed a proper colouring. Thus, $\omega(G_I(S)) = \chi(G_I(S))$, showing that $G_I(S)$ is a weakly perfect graph.

We, now, present a counterexample to Conjecture 1.2. This example is motivated by Example 2.5 in Joshi and Sarode [14].

Consider the commutative semigroup $S = \{0, a, b, c, d, e, f, 1\}$ with multiplication given by:

(1) $fx = 0$ for all $x \in S \setminus \{1\}$;
(2) $x^2 = f$ for all $x \in S \setminus \{0, 1, f\}$;
(3) $ab = bc = cd = de = ae = 0$ and $ac = ad = bd = be = ce = f$.

The annihilating-ideals of S are shown in Table 1.

The zero-divisor graph $G(S)$ and the annihilating-ideal graph $\mathbb{AG}(S)$ of the semigroup S are shown in Figure 1. It is easy to verify that

$$4 = \chi(\mathbb{AG}(S)) = \chi(G(S)) \neq \omega(\mathbb{AG}(S)) = \omega(G(S)) = 3. \qquad \square$$

TABLE 1. Annihilating-ideals of S.

(a)	(b)	(c)	(d)	(e)	(f)
$(b)\bigcup(e)$	$(a)\bigcup(c)$	$(b)\bigcup(d)$	$(c)\bigcup(e)$	$(a)\bigcup(d)$	

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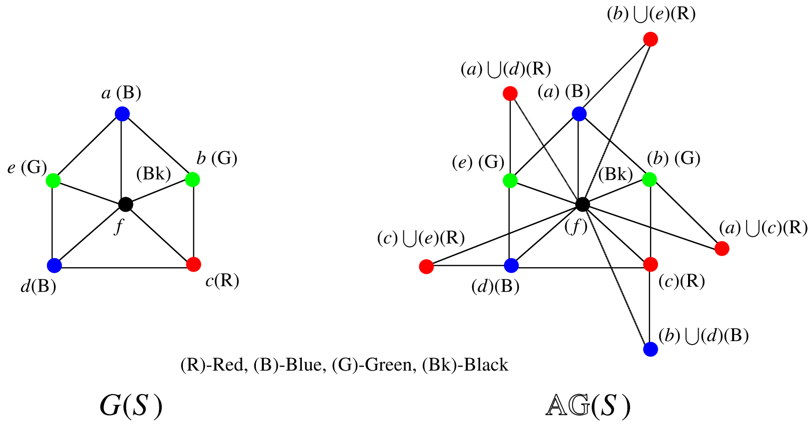

FIGURE 1. Neither of the graphs $G(S)$ and $\mathbb{AG}(S)$ is weakly perfect.

Clearly, in a reduced semigroup (or ring), the ideal (0) is semiprime. Hence, we have the following corollaries.

COROLLARY 2.7. *If S is a reduced semigroup (or ring), then $G(S)$ is a weakly perfect graph. Moreover, $\chi(G(S)) = \omega(G(S)) = |Min(S)|$, where $Min(S)$ denotes the set of minimal prime ideals over* (0).

COROLLARY 2.8. *If $I = \bigcup_{i=1}^{n} P_i$, where the P_i are prime ideals, then $G_I(R)$ is weakly perfect.*

3. Correspondence for minimal prime ideals

Let S be a reduced semigroup. In this section, we show that there is a one to one correspondence between minimal prime ideals of S and minimal primes of S treated as a meet-semilattice. We first state some definitions.

DEFINITION 3.1. Let L be a meet-semilattice. A nonempty subset $I \subseteq L$ is said to be a *semi-ideal* if $x \leq y \in I$ implies that $x \in I$. A proper semi-ideal P of L is said to be *prime* if $a \wedge b \in P$ implies that $a \in P$ or $b \in P$. A prime semi-ideal $P \subseteq L$ is said to be a *minimal prime semi-ideal* if there does not exist any prime semi-ideal Q such that $Q \subsetneq P$.

It has been a fruitful theme of research to study analogues of the concept of zero-divisor graphs of other algebraic and ordered structures (see Nimbhorkar *et al.* [19], Halaš and Jukl [12], Lu and Wu [18] and Joshi [13]).

Let L be a meet-semilattice with 0. Associate with L, a simple undirected graph $\Gamma(L)$, the zero-divisor graph of L, whose vertex set is the set

$$\{x \in L \setminus \{0\} \mid x \wedge y = 0 \text{ for some nonzero } y \in L\}$$

and two vertices x and y are adjacent if $x \wedge y = 0$.

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On the lines of Beck's theorem for reduced rings, Lu and Wu [18], Halaš and Jukl [12] and Joshi [13] essentially proved the following theorem for posets. Recently, Devhare *et al.* [10] proved that the complement of the zero-divisor graph $\Gamma^c(P)$ of a poset P is weakly perfect when $\omega(\Gamma^c(P)) < \infty$. This gives us a large family of examples of weakly perfect graphs.

THEOREM 3.2. *Let P be a poset with 0 and assume that $\omega(\Gamma(P)) < \infty$. Then the number n of all minimal prime semi-ideals of P is finite and $\chi(\Gamma(P)) = \omega(\Gamma(P)) = n$.*

Any meet-semilattice is an idempotent semigroup and, hence, a reduced semigroup with respect to meet as the binary operation. Hence, choosing $I = (0)$ in Theorem 2.6, the following result is immediate.

THEOREM 3.3. *Let L be a meet-semilattice with 0 and assume that $\omega(\Gamma(L)) < \infty$. Then $\omega(\Gamma(L)) = \chi(\Gamma(L))$.*

Going the other way, any reduced semigroup S can be given the structure of a meet semilattice. To this end, let S be a reduced commutative semigroup with $0 \neq 1$. Define a relation $\leq$ such that $r \leq s$ in S if and only if either $\text{ann}(s) \subsetneq \text{ann}(r)$ or $r \leq s$ in some predetermined well-order on the set $[r] = \{x \in S \mid \text{ann}(r) = \text{ann}(x)\}$. LaGrange and Roy [17, Remark 3.4] proved that $\leq$ is a partial order on S. In fact, it follows from Remark 4.8 of [16] that S is a meet-semilattice. Both the statements of the following theorem follow from [3, Lemma 3.5(1), (2) and (5)] and [11, Theorem 3.4]. For the sake of completeness, we provide the proof.

THEOREM 3.4. *Let S be a reduced commutative semigroup. Then the following statements are true.*

(i) *$\langle S; \leq\rangle$ is a meet-semilattice.*
(ii) *For $a, b \in S$, we have $ab = 0$ if and only if $a \wedge b = 0$. Therefore, the zero-divisor graph $G(S)$ of a semigroup S and the zero-divisor graph $\Gamma(S)$ of S (treated as a meet-semilattice) are essentially the same, that is, $\Gamma(S) = G(S)$.*

PROOF. (i) From the above discussion, S is a poset. Now, we prove that S is a meet-semilattice. Let $a, b \in S$. If a and b are comparable, then $\inf\{a, b\}$ exists and we are done. Assume that a and b are incomparable. Consider the class $[ab]$ determined by ab and choose the largest element from $[ab]$, say y, with respect to the predetermined well-order on $[ab]$. Thus, $ab \leq y$ and $\text{ann}(ab) = \text{ann}(y)$. We first observe that $\text{ann}(a) \subsetneq \text{ann}(ab)$ and $\text{ann}(b) \subsetneq \text{ann}(ab)$. This can be seen as follows. There are two cases to rule out, namely if $\text{ann}(a) = \text{ann}(ab) = \text{ann}(b)$ and the case when $\text{ann}(a) \subsetneq \text{ann}(ab)$ and $\text{ann}(b) = \text{ann}(ab)$. In the first case, we obtain $\text{ann}(a) = \text{ann}(b)$, showing that a and b are comparable, a contradiction to the incomparability of a and b. In the second case, we find that $\text{ann}(a) \subsetneq \text{ann}(b)$, contradicting again the incomparability of a and b. Thus, we have $\text{ann}(a) \subsetneq \text{ann}(ab)$ and $\text{ann}(b) \subsetneq \text{ann}(ab)$. Hence, ab and y are lower bounds of $\{a, b\}$. Now, let t be any lower bound of $\{a, b\}$, that is, $t \leq a, b$. We then have $\text{ann}(a) \subseteq \text{ann}(t)$ and $\text{ann}(b) \subseteq \text{ann}(t)$. Again there are two cases to

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consider, namely, when $\text{ann}(a) = \text{ann}(t) = \text{ann}(b)$ and the case when $\text{ann}(a) = \text{ann}(t)$ and $\text{ann}(b) \subsetneq \text{ann}(t) = \text{ann}(a)$. In both the cases we find that a and b are comparable, violating the incomparability of a and b. Thus, we have $\text{ann}(a) \subsetneq \text{ann}(t)$ and $\text{ann}(b) \subsetneq \text{ann}(t)$. Further, we claim that $\text{ann}(ab) \subseteq \text{ann}(t)$. To see this, let $x \in \text{ann}(ab)$. Hence, $xab = 0$, implying that $xa \in \text{ann}(b) \subsetneq \text{ann}(t)$. So, $xat = 0$, that is, $xt \in \text{ann}(a) \subsetneq \text{ann}(t)$. Thus, $xt^2 = 0$ and hence $(xt)^2 = 0$. Now, since S is reduced, we get $xt = 0$, that is, $x \in \text{ann}(t)$. This shows that $\text{ann}(y) = \text{ann}(ab) \subseteq \text{ann}(t)$. We, now, claim that $a \wedge b = y$. There are two possibilities, namely, $\text{ann}(ab) = \text{ann}(y) = \text{ann}(t)$ and $\text{ann}(ab) = \text{ann}(y) \subsetneq \text{ann}(t)$. In the first case, we have $y, t \in [ab]$. Since y is the largest element of $[ab]$, we get $t \leq y$. In the second case, clearly, we have $t \leq y$. So, in both the cases, $t \leq y$. Hence, $\inf\{a, b\} = y$, that is, $a \wedge b$ exists and equals y. Note in particular that, when a and b are incomparable, $ab \leq a \wedge b$. Thus, S is a meet-semilattice.

(ii) Let $a, b \in S$ and $ab = 0$. We show that $a \wedge b = 0$. To do this, let t be any lower bound of a and b. Hence, $\text{ann}(a) \cup \text{ann}(b) \subseteq \text{ann}(t)$. Since $ab = 0$, we have $b \in \text{ann}(a)$ and so $b \in \text{ann}(t)$. Thus, $tb = 0$ and so $t \in \text{ann}(b) \subseteq \text{ann}(t)$. This implies that $t^2 = 0$, giving $t = 0$. This shows that $a \wedge b = 0$.

Conversely, we assume that $a \wedge b = 0$. If a and b are comparable, say $a \leq b$, then in this case $a \wedge b = a = 0$. This gives $ab = 0$. If a and b are incomparable, then, by (i) above, $ab \leq a \wedge b = 0$, so $ab = 0$. □

The following well-known result is due to Kist [15].

LEMMA 3.5. *Let S be a reduced semigroup and P be a prime ideal of S. Then P is a minimal prime ideal if and only if it satisfies the following condition* (§).

(§) *For any $x \in P$, there exists $y \notin P$ such that $xy = 0$.*

The following result is a modified version of [20, Theorem 4] by Pawar and Thakare, which is an analogue of the above result.

LEMMA 3.6. *Let L be a meet-semilattice with 0 and let P be a prime semi-ideal. Then P is a minimal prime semi-ideal if and only if it satisfies the following condition* ($\star$).

($\star$) *For any $x \in P$, there exists $y \notin P$ such that $x \wedge y = 0$.*

With this preparation, we are now ready to prove our main result, which relates minimal prime ideals of a reduced semigroup S and minimal prime semi-ideals of S treated as a meet-semilattice.

THEOREM 3.7. *Let S be a reduced semigroup with 0 and 1. Let P be a nonempty subset of S. Then P is a minimal prime ideal of S (treated as a semigroup) if and only if P is a minimal prime semi-ideal of S (treated as a meet-semilattice under the partial order given in Lemma 3.4).*

PROOF. Let P be a minimal prime ideal of S (treated as a semigroup). First, we prove that it is a semi-ideal of S treated as a meet-semilattice. Let $x \leq y \in P$. We claim that $x \in P$. Since $y \in P$, by Lemma 3.5, there exists $z \notin P$ such that $yz = 0$. By Theorem 3.4, we have $y \wedge z = 0$. Therefore $x \wedge z = 0$. Again by Theorem 3.4, $xz = 0$. This gives $x \in P$. Thus, P is a semi-ideal.

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Next, we prove that P is a prime semi-ideal. Let $x \wedge y \in P$. If x and y are comparable, say $x \leq y$, then $x \wedge y = x \in P$. If x and y are incomparable, then from the proof of Theorem 3.4(i), $xy \leq x \wedge y \in P$. Since P is a semi-ideal, we have $xy \in P$. Therefore, either $x \in P$ or $y \in P$, as P is a prime ideal (treated as a semigroup). This proves that P is a prime semi-ideal of S (treated as a meet-semilattice). Minimality of a prime semi-ideal P follows from the condition ($\star$) of Lemma 3.6, Lemma 3.5 and Theorem 3.4(ii).

Conversely, let P be a minimal prime semi-ideal of S (treated as a meet-semilattice). We first show that P is an ideal of a semigroup S. Let $a \in P$ and $r \in S$. By the condition ($\star$) of Lemma 3.6, there exists $c_1 \notin P$ such that $a \wedge c_1 = 0$. This gives $ac_1 = 0$ (by Theorem 3.4). Hence, $(ar)c_1 = 0$, which yields $(ar) \wedge c_1 = 0 \in P$. This gives $ar \in P$. Hence, P is an ideal of S (treated as a semigroup).

Next, we claim that P is a prime ideal of a semigroup S. Let $ab \in P$. Then, by the condition ($\star$) of Lemma 3.6, there exists $c \notin P$ such that $ab \wedge c = 0$. Hence, by Theorem 3.4(ii), $abc = 0$. We, now, claim that if $abc = 0$, then $a \wedge b \wedge c = 0$. For this, let t be a lower bound of $\{a, b, c\}$. Then $\mathrm{ann}(a) \subseteq \mathrm{ann}(t)$, $\mathrm{ann}(b) \subseteq \mathrm{ann}(t)$ and $\mathrm{ann}(c) \subseteq \mathrm{ann}(t)$. Clearly, $bc \in \mathrm{ann}(a) \subseteq \mathrm{ann}(t)$. Hence, $bct = 0$, which further implies that $ct \in \mathrm{ann}(b) \subseteq \mathrm{ann}(t)$. Since S is reduced, we have $ct = 0$. Therefore, $t \in \mathrm{ann}(c) \subseteq \mathrm{ann}(t)$ implies that $t^2 = 0$. Thus, we have $t = 0$. Thus, $a \wedge b \wedge c = 0 \in P$ implies that $a \wedge b \in P$ since $c \notin P$. As P is a prime semi-ideal, we have either $a \in P$ or $b \in P$. This proves that P is a prime ideal of a semigroup S. Minimality of the prime ideal P follows from the condition (§) of Lemma 3.5, Lemma 3.6 and Theorem 3.4(ii). □

Let $\mathrm{Min}(S)$ denote the set of all minimal prime ideals of S treated as a semigroup and $\mathrm{Min}_s^p(S)$ denote the set of all minimal prime semi-ideals of S treated as a meet-semilattice. By Theorem 3.4, $\Gamma(S) \cong G(S)$. Using Theorem 3.3, Theorem 3.4 and Theorem 3.7, we have the following result.

THEOREM 3.8. *Let S be a reduced commutative semigroup with 0 and 1. Assume that $\omega(\Gamma(S)) < \infty$. Then*

$$\omega(\Gamma(S)) = \chi(\Gamma(S)) = \omega(G(S)) = \chi(G(S)) = |Min(S)| = |Min_s^p(S)|.$$

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