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A Sufficient Condition for a Graph to be the Core of a Class 2 Graph

J. K. DUGDALE and A. J. W. HILTON

Department of Mathematics, University of Reading, Whiteknights, Reading RG6 6AX, England

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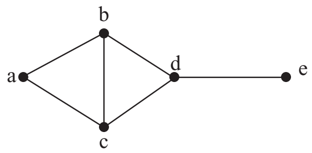
The *core* of a graph G is the subgraph G_Δ induced by the vertices of maximum degree. We define the deleted core $D(G)$ of G. We extend an earlier sufficient condition due to Hoffman [7] for a graph H to be the core of a Class 2 graph, and thereby provide a stronger sufficient condition. The new sufficient condition is in terms of $D(H)$. We show that in some circumstances our condition is necessary; but it is not necessary in general.

1. Introduction

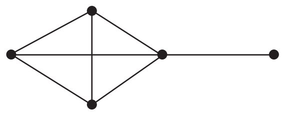
In this note all graphs will be simple, so they will have no loops or multiple edges. The *chromatic index* $\chi'(G)$ of a graph G is the least value of k such that $E(G)$ can be coloured with k colours so that no two adjacent edges receive the same colour. Vizing [9] showed that $\Delta(G) \leqslant \chi'(G) \leqslant \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G; the graphs G satisfying $\chi'(G) = \Delta(G)$ are called *Class 1*, and those satisfying $\chi'(G) = \Delta(G) + 1$ are called *Class 2*.

The *core* G_Δ of a graph G is the subgraph induced by the vertices of maximum degree. Every graph H is the core of a Class 1 graph (we can, using Vizing's theorem, say more: every graph H with $\Delta(H) \geqslant 1$ is the core of a Class 1 graph of degree $\Delta(H) + 1$ – we need only add $\Delta(H) + 1 - d_H(v)$ pendent edges to each vertex v of H to create a Class 1 graph of maximum degree $\Delta(H) + 1$ with core H). But not every graph is the core of a Class 2 graph. For example, Fournier [6] showed that no forest is the core of a Class 2 graph, and Hoffman [7] showed that the graph A of Figure 1 cannot be the core of a Class 2 graph. By contrast, the graph B of Figure 1 can be the core of a Class 2 graph. In [4], Chetwynd, Hilton and Hoffman showed that every graph G with minimum degree $\delta(G)$ satisfying $\delta(G) \geqslant 2$ can be the core of a Class 2 graph.

We call a graph G satisfying $|E(G)| > \Delta(G)\lfloor \frac{1}{2}|V(G)|\rfloor$ *overfull*. It is easy to see that an overfull graph must be Class 2. The result of Chetwynd, Hilton and Hoffman [4] was proved by showing that every graph G with $\delta(G) \geqslant 2$ is the core of an overfull graph. This result was carried a step further by Hoffman [7], who looked for a graph H having



The graph A



The graph B

Figure 1

G as its core such that a given subset S of $V(G)$ is of the form $S = V(G) \cap V(K)$, where K is an overfull subgraph with $\Delta(K) = \Delta(H)$. Hoffman's result was as follows.

Theorem 1.1. *Let H be a graph and let $S \subset V(H)$. Then there is a graph J having H as its core and containing an overfull subgraph K with $\Delta(K) = \Delta(J)$ and $V(K) \cap V(J) = S$ if and only if*

$$2 + \gamma_H(S) \leqslant \delta_H(S) < |S|,$$

where $\gamma_H(S)$ is the number of edges of H with exactly one end in S and $\delta_H(S)$ is the minimum degree (in H) of the vertices of S.

A fundamental lemma proved in [2] that follows from Vizing's Adjacency Lemma [9] (see also [5]) and lies behind Fournier's result, and also much of this paper, is the following. Here, for $v \in V(G)$, we let $d_G^\Delta(v)$ be the number of vertices of G of degree $\Delta = \Delta(G)$ to which v is adjacent. If there is no risk of ambiguity, we sometimes simplify this to $d^\Delta(v)$.

Lemma 1.2. *Let G be a graph and let $v \in V(G)$ satisfy $d_G^\Delta(v) \leqslant 1$. Then if $\Delta(G-v) = \Delta(G)$ then $\chi'(G-v) = \chi'(G)$.*

This result is most useful, for if one wishes to know $\chi'(G)$, then one need only find a suitable vertex v to be deleted, and then the problem is reduced to finding $\chi'(G-v)$. Quite often this procedure can be iterated, and thus an unmanageable graph G is reduced to a manageable graph, say $G\backslash\{v_1, \dots, v_r\}$. This technique was employed by Chetwynd and Hilton in [2, 3] and was formalized by Hoffman and Rodger [8]; it has been employed many times since.

An extension of this lemma to the case when an edge is removed instead of a vertex is also true [2].

Lemma 1.3. *Let G be a graph and let u and v be adjacent vertices, joined by an edge e.*

(i) *If $\Delta(G) = \Delta(G-w)$ and $d_G^\Delta(w) \leqslant 1$ then $\chi'(G-w) = \chi'(G)$.*
(ii) *If $\Delta(G) = \Delta(G-e)$ and $d_G^\Delta(w) + d(u) \leqslant \Delta$ then $\chi'(G-e) = \chi'(G)$.*

Suppose that a graph H is the core G_Δ of a graph G. In this paper we use Lemma 1.3 to define the deleted core $D(H)$ of H. Roughly speaking, this is the graph that one obtains from H by repeatedly removing vertices and edges as permitted by Lemma 1.3. We then

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show that if the deleted core $D(H)$ of H satisfies certain specified conditions, then it is possible for the graph G to be Class 2.

Although the converse is not true in general, we show that it is true if $|V(D(H))| \leqslant 4$. An intriguing question is whether a minor variation of our condition can be found which would be both necessary and sufficient for a graph H to be the core of a Class 2 graph. Although this might seem to be unlikely, we recall that we do have a necessary and sufficient condition for a graph H to be the core of a critical graph: call a graph G *critical* if G is connected, Class 2, and $\chi'(G-e) < \chi'(G)$ for each edge $e \in E(G)$. Chetwynd, Hilton and Hoffman [7] proved the following.

Theorem 1.4. *Let H be a graph. Then H is the core of a critical graph if and only if $\delta(H) \geqslant 2$.*

2. Some useful results

We first note that, with the aid of both parts of Lemma 1.3, we can prove the following further analogue of Lemma 1.2.

Theorem 2.1. *Let G be a graph and let $w \in V(G)$. Suppose that $\Delta(G-w) = \Delta(G)$. If $d_G(w) \leqslant \Delta(G)-1$ and the neighbours of w in G_Δ are $u_1, u_2, \ldots, u_s$, with $d_G^\Delta(u_i) \leqslant i$ $(1 \leqslant i \leqslant s)$, then $\chi'(G-w) = \chi'(G)$.*

Proof. Let e_i join u_i to w $(1 \leqslant i \leqslant s)$. Since $d_G^\Delta(u_1) + d_G(w) \leqslant \Delta$, by Lemma 1.3(ii), $\chi'(G-e_1) = \chi'(G)$. Then $d_{G-e_1}(w) \leqslant \Delta - 2$, so $d_{G-e_1}^\Delta(u_2) + d_{G-e_1}(w) \leqslant \Delta$. Proceeding inductively in this way we find that, for $1 \leqslant i \leqslant s-1$, $d_{G\backslash\{e_1,e_2,\ldots,e_i\}}(w) \leqslant \Delta - i - 1$ and $\chi'(G\backslash\{e_1,\ldots,e_i\}) = \chi'(G)$. Then $d_G^\Delta\backslash\{e_1,e_2,\ldots,e_i\}(u_{i+1}) + d_{G\backslash\{e_1,\ldots,e_i\}}(w) \leqslant \Delta$, so by Lemma 1.3(ii), $\chi'(G\backslash\{e_1,\ldots,e_{i+1}\}) = \chi'(G)$ $(1 \leqslant i \leqslant s-1)$. Eventually we find that $d^\Delta_{G\{e_1,e_2,\ldots,e_s\}}(w) = 0$, so by Lemma 1.3(i), $\chi'((G\backslash\{e_1,\ldots,e_s\}) - w) = \chi'(G\backslash\{e_1,\ldots,e_s\}) = \chi'(G)$. But $(G\backslash\{e_1,\ldots,e_s\}) - w = G - w$, so $\chi'(G-w) = \chi'(G)$. □

We also note the following results (see [2] for a proof of Lemmas 2.2 and 2.3, and [1, 3] for a proof of Lemma 2.4).

Lemma 2.2. *If G has at most two vertices of maximum degree, then G is Class I.*

Lemma 2.3. *Let G be a connected graph with three vertices of maximum degree. Then G is Class 2 if and only if G has three vertices of degree $|V(G)|-1$, and the remaining vertices have degree $|V(G)|-2$.*

Lemma 2.4. *Let G be a connected graph with four vertices of maximum degree. Then G is Class 2 if and only if either*

(i) *G has 4 vertices of degree $|V(G)|-1$, $|V(G)|-5$ vertices of degree $|V(G)|-2$, and one vertex of degree $|V(G)|-3$, or*

(ii) *G has a cut edge whose removal separates G into G_1 and G_2, where $\Delta(G_1) = \Delta(G)$, and G_1 either satisfies* (i) *or is the graph of Lemma 2.3, or*

(iii) *G has 4 vertices of degree $|V(G)|-2$, and $|V(G)|-4$ vertices of degree $|V(G)|-3$.*

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3. The deleted core of a graph

We first define a deleted core $D(H)$ of a graph H. The definition of H is iterative, and it is not clear immediately from the definition that it is uniquely defined. However, we show in Lemma 2.3 that it is uniquely defined.

Let $d \geqslant \Delta(H)$. First from H define H^{*d} to be the graph obtained from H as follows. To each vertex $v \in V(H)$, adjoin $d - d_H(v)$ pendent edges. Then H^{*d} has vertices of degrees 1 and d, and H is the core of H^{*d}, that is, $H = H^{*d}_\Delta$. Write $J = H^{*d}$. J is a particular graph with core H. Also, let G be an arbitrary graph with core H and with $\Delta(G) = \Delta(J) = d$.

Set $J_0 = J_\Delta$ and $G_0 = G$. If $J_0(= H)$ contains a vertex v such that $d^\Delta_j(v) = 0$ or 1, then let f_1 be such a vertex, and let $J_1 = J_0\backslash\{f_1\}$ and $G_1 = G_0\backslash\{f_1\}$. Otherwise, if J_0 contains an edge $e = uv$ with $d_J(u) < \Delta(J)$ and $d^\Delta_J(v) + d_J(u) \leqslant \Delta(J)$, then let f_1 be such an edge and let $J_1 = J_0\backslash\{f_1\}$ and $G_1 = G_0\backslash\{f_1\}$. Otherwise put $J_1 = J_0$ and $G_1 = G_0$ and define a deleted core $D(H)$ of H to be $J_{0\Delta} = J_\Delta = H$. Then, if $\Delta(J_1) = \Delta(J)$ then $\Delta(G_1) = \Delta(G)$ and, by Lemma 1.3 (part (i) or (ii)), G_1 is Class 1 if and only if G is Class 1. If $\Delta(J_1) < \Delta(J)$ then $\Delta(G_1) = \Delta(J_1) < \Delta(J) = \Delta(G)$, and, by Lemma 2.2, G is Class 1.

If f_1 was selected, then we iterate this procedure. In general, for $i \geqslant 0$, suppose that $J_0, J_1, \ldots, J_i, G_0, G_1, \ldots, G_i$ and $f_1, f_2, \ldots, f_i \in E(J_\Delta) \cup (J_\Delta)$ have been selected, and that $J_j = J_{j-1}\backslash\{f_j\}$ and $G_j = G_{j-1}\backslash\{f_j\}$, for $1 \leqslant j \leqslant i$. Put $J_i = J\backslash\{f_1, \ldots, f_i\}$ and $G_i = G\backslash\{f_1, \ldots, f_i\}$. If $\Delta(J_i) = \Delta(J)$ and $J_\Delta\backslash\{f_1, \ldots, f_i\}$ contains a vertex v such that $d^\Delta_{J_i}(v) = 0$ or 1, then we let f_{i+1} be such a vertex and let $J_{i+1} = J_i\backslash\{f_{i+1}\}$. We also let $G_{i+1} = G_i\backslash\{f_{i+1}\}$. Otherwise, if $\Delta(J_i) = \Delta(J)$ and $J_\Delta\backslash\{f_1, \ldots, f_i\}$ contains an edge $e = uv$ with $d_{J_i}(u) < \Delta(J)$ and $d^\Delta_{J_i}(v) + d_{J_i}(u) \leqslant \Delta(J)$ then let f_{i+1} be such an edge and let $J_{i+1} = J_i\backslash\{f_{i+1}\}$. Also let $G_{i+1} = G_i\backslash\{f_{i+1}\}$. If $\Delta(J\backslash\{f_1, \ldots, f_i\}) < \Delta(J)$ and $J_\Delta\backslash\{f_1, \ldots, f_i\}$ is not the empty graph, then choose any vertex of $J_\Delta\backslash\{f_1, \ldots, f_i\}$ as f_{i+1}, and let $J_{i+1} = J_i\backslash\{f_{i+1}\}$. Also put $G_{i+1} = G_i\backslash\{f_{i+1}\}$. If none of these is possible, put $J_{i+1} = J_i$ and $G_{i+1} = G_i$, and define a *deleted core* $D(H)$ of H to be $J_\Delta\backslash\{f_1, f_2, \ldots, f_i\}$. If $\Delta(J_i) = \Delta(J_{i+1}) = \Delta(J)$ then $\Delta(G_i) = \Delta(J_i) = \Delta(J_{i+1}) = \Delta(G_{i+1}) = \Delta(G)$, and, by Lemma 1.3, G_{i+1} is Class 2 if and only if G_i is Class 2. If $\Delta(J_{i+1}) < \Delta(J_i) = \Delta(J)$ then $\Delta(G_{i+1}) < \Delta(G_i) = \Delta(G)$ and then, by Lemma 2.2, G_i is Class 1, and, working back, G is Class 1. If $\Delta(J_{i+1}) = \Delta(J_i) < \Delta(J)$ then, similarly, G is Class 1. If the deleted core is the empty graph then again, similarly, G is Class 1.

This process is iterated until it has to stop. The graph obtained finally from J_Δ is called a *deleted core* of H, and is denoted by $D(H)$. It is clear from the definition that if $D(H)$ is obtained from J_Δ by the removal of $f_1, \ldots, f_r$ as described above, then either $\Delta(J_r) = \Delta(J)$ (so that $\Delta(G_r) = \Delta(G)$) or $\Delta(J_r) < \Delta(J)$ (so that $\Delta(G_r) < \Delta(G)$). In the former case, G is Class 2 if and only if G_r is Class 2. In the latter case G is Class 1.

The graph in Figure 2 is an example of a graph H in which the removal of vertices alone will not suffice to produce $D(H)$. The edges e_1, e_2 that must be removed at some point are indicated. $D(H)$ is induced by the vertex set $\{a, b, c\}$.

Now let us consider the question of the uniqueness of $D(H)$.

Lemma 3.1. *Let H be a graph. Then the deleted core $D(H)$ is uniquely determined.*

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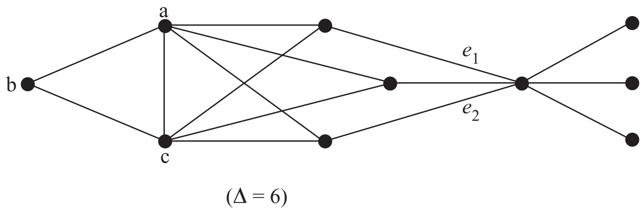

Figure 2

Proof. Let $D_a(H)$ be a deleted core obtained from H by removal in succession of elements $a_1, a_2, \dots, a_r$ of H, as described in the definition of $D(H)$ (so that $V(D_a(H)) = V(H)\backslash\{a_1, a_2, \dots, a_r\}$ and $E(D_a(H)) = E(H)\backslash\{a_1, a_2, \dots, a_r\}$); similarly, let $D_b(H)$ be a deleted core obtained from H by the removal in succession of elements $b_1, \dots, b_s$ of H.

If $D_a(H) = D_b(H)$ there is nothing to prove, so suppose that $D_a(H) \neq D_b(H)$. In that case we may assume that either (I) $V(D_a(H))\backslash V(D_b(H)) \neq \phi$, or (II) $V(D_a(H)) = V(D_b(H))$ but $E(D_a(H))\backslash E(D_b(H)) \neq \phi$. Let b_c be the element from $\{b_1, b_2, \dots, b_s\}$ of least index such that, if b_c is a vertex, then $b_c \in V(D_a((H))\backslash V(D_b(H))$, and if b_c is an edge then $b_c \in E(D_a(H))\backslash E(D_b(H))$ and $b_c = uv$, where either, in case (I), $u, v \in V(D_a(H))\backslash V(D_b(H))$ or, in case (II), $u, v \in V(D_b(H))$. Then, since $b_1, \dots, b_c, \dots, b_s$ are removed from H in succession, reducing H to $D_b(H)$, either

(i) $\Delta(J\backslash\{b_1, \dots, b_{c-1}\}) = \Delta(J), b_c \in V(H)$ and $d^{\Delta}_{J\backslash\{b_1,\dots,b_{c-1}\}}(b_c) = 0$ or 1, or

(ii) $\Delta(J\backslash\{b_1, \dots, b_{c-1}\}) = \Delta(J)$ and b_c is an edge $uv \in E(H)$ with $d^{\Delta}_{J\backslash\{b_1,\dots,b_{c-1}\}}(u) + d_{J\backslash\{b_1,\dots,b_{c-1}\}}(v) \leqslant \Delta(G)$, or

(iii) $\Delta(J\backslash\{b_1, \dots, b_{c-1}\}) < \Delta(J)$ and b_c is an arbitary vertex of $V(H)\backslash\{b_1, \dots, b_{c-1}\}$).

In each case $b_1, \dots, b_{c-1} \notin (V(D_a(H))\backslash V(D_b(H))) \cup (E(D_a(H))\backslash E(D_b(H)))$, so $V(H) \cap \{b_1, \dots, b_{c-1}\}) \subseteq V(H) \cap \{a_1, \dots, a_r\}$ and $E(H) \cap \{b_1, \dots, b_{c-1}\}) \subseteq E(H) \cap \{a_1, \dots, a_r\}$. If (i) is true and $\Delta(J) = \Delta(J\backslash\{a_1, \dots, a_r\})$, we have a contradiction since $d^{\Delta}_{J\backslash\{a_1,\dots,a_r\}}(b_c) \leqslant d^{\Delta}_{J\backslash\{b_1,\dots,b_{c-1}\}}(b_c) = 0$ or 1 so $d^{\Delta}_{J\backslash\{a_1,\dots,a_r\}}(b_c) = 0$ or 1, and so b_c is a vertex of $D_a(H)$ that is eligible to be chosen as the next vertex to be removed under case (i). If (i) is true and $\Delta(J) > \Delta(J\backslash\{a_1, \dots, a_r\})$, we again have a contradiction, since b_c is a vertex of $D_a(H)$ that is eligible to be removed under case (iii). If (iii) is true, then we have the same contradiction, since b_c is a vertex eligible to be removed under case (iii).

If (ii) is true and $b_c = uv$, where $u, v \in V(D_a(H))\backslash V(D_b(H))$, then either $\Delta(J) = \Delta(J\backslash\{a_1, \dots, a_r\})$, in which case we have a contradiction since $d^{\Delta}_{J\backslash\{a_1,\dots,a_r\}}(u) + d_{J\backslash\{a_1,\dots,a_r\}}(v) \leqslant d^{\Delta}_{J\backslash\{b_1,\dots,b_{c-1}\}}(u) + d_{J\backslash\{b_1,\dots,b_{c-1}\}}(v) \leqslant \Delta(J)$, so e is an edge of $D_a(H)$ that is eligible to be removed under case (ii), or $\Delta(J) > \Delta(J\backslash\{a_1, \dots, a_r\})$, in which case we have a contradiction since u and v are both eligible to be removed under case (iii). If (ii) is true and $b_c = uv$, where $u, v \in V(D_a(H))$, then we have a contradiction since $d^{\Delta}_{J\backslash\{a_1,\dots,a_r\}}(u) + d_{J\backslash\{a_1,\dots,a_r\}}(v) \leqslant d^{\Delta}_{J\backslash\{b_1,\dots,b_{c-1}\}}(u) + d_{J\backslash\{b_1,\dots,b_{c-1}\}}(v) \leqslant \Delta(J)$, so that b_c is an edge of $D_a(H)$ that is eligible to be removed under case (ii).

It must therefore follow that $D_a(H) = D_b(H)$, as required. □

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Lemma 3.2. *Let G be a graph with core H. Let $D(H)$ be obtained from H by the removal of the elements (vertices and edges) $f_1, \ldots, f_r$ of H, as described in the definition of $D(H)$. Then, if $\Delta(G) = \Delta(G \backslash \{f_1, \ldots, f_r\})$, it follows that $\chi'(G) = \chi'(G \backslash \{f_1, \ldots, f_r\})$.*

Proof. As in the definition of $D(H)$, let $G_o = G$ and let $G_{i+1} = G_i \backslash \{f_{i+1}\}$ $(0 \leqslant i \leqslant r-1)$. Suppose that $\Delta(G) = \Delta(G \backslash \{f_1, \ldots, f_r\})$. Then $\Delta(G_0) = \cdots = \Delta(G_r) = \Delta(G)$. Then, as pointed out in the iterative definition of $D(H)$, G_{i+1} is Class 1 if and only if G_i is Class 1 $(0 \leqslant i \leqslant r-1)$. Therefore $\chi'(G) = \chi'(G \backslash \{f_1, \ldots, f_r\})$, as asserted. □

4. A sufficient condition

We first use the idea of a deleted core to give a refinement of Hoffman's condition for a graph H to be the core of a Class 2 graph.

Theorem 4.1. *Let H be a graph with deleted core $D(H)$. Suppose that*

$$2 + \gamma_H(V(D(H))) \leqslant \delta_H(V(D(H))) < |V(D(H))|. \tag{4.1}$$

Then H is the core of a Class 2 graph.

Proof. We set $S = V(D(H))$ and apply Theorem 1.1. Then H is the core of a Class 2 graph J (and it has the extra property that there is an overfull subgraph K with $\Delta(K) = \Delta(J)$ and $V(K) \cap V(J) = V(D(H))$). □

Under certain conditions, Hoffman's condition applied to the deleted core is also necessary. The first example we give of this is when $|V(D(H))| = 3$.

Theorem 4.2. *Let H be a graph with deleted core $D(H)$, where $|V(D(H))| = 3$. Suppose that H is the core of a Class 2 graph. Then $2 + \gamma_H(V(D(H))) \leqslant \delta_H(V(D(H))) < |V(D(H))|$.*

Proof. Suppose that $D(H)$ is obtained from H by the removal of the elements (vertices or edges) $f_1, \ldots, f_r$ of H as described in the definition of $D(H)$. Suppose also that H is the core of a Class 2 graph G. Let $J = G \backslash \{f_1, \ldots, f_r\}$. Since $|V(D(H))| \geqslant 3 > 0$, it follows from the construction of $D(H)$ that $\Delta(G) = \Delta(J)$. Then, by Lemma 3.1, J is Class 2. We can now adjoin $\Delta(G) - d_J(v)$ pendent edges to each vertex v of $D(H)$, so that $D(H)$ is the core of the graph J^* obtained. Then J^* is a connected Class 2 graph with three vertices of maximum degree. By Lemma 2.3, J^* is K_{2n+1} less $(n-1)$ independent edges, where $2n = \Delta(G)$. Since J^* has in fact no pendent edges, $J^* = J$. But it is not possible to join any vertex of $V(D(H))$ to any further vertex without raising the maximum degree of J. Since $\Delta(J) = \Delta(G)$ this is not possible, so it follows that $H = D(H)$. Therefore $\gamma_H(V(D(H))) = 0$, $\delta_H(V(D(H))) = 2$ and $|V(D(H))| = 3$. Therefore $2 + \gamma_H(V(D(H))) \leqslant \delta_H(V(D(H))) < |V(D(H))|$, as asserted. □

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Theorem 4.3. *Let H be a graph with deleted core $D(H)$, where $|V(D(H))| = 4$. Suppose that H is the core of a Class 2 graph. Then* (4.1) *holds, that is,*

$$2 + \gamma_H(V(D(H))) \leqslant \delta_H(V(D(H))) < |V(D(H))|.$$

Proof. Suppose that $D(H)$ is obtained from H by the removal of the elements (vertices or edges) $f_1, \ldots, f_r$ of H, as described in the definition of $D(H)$. Suppose also that H is the core of a Class 2 graph G. Let $J = G\backslash\{f_1, \ldots, f_r\}$. Since $|V(D(H))| = 4 > 0$, it follows from the construction of $D(H)$ that $\Delta(G) = \Delta(J)$. Then, by Lemma 3.1, J is Class 2. We can now adjoin $\Delta(G) - d_J(v)$ pendent edges to each vertex v of $D(H)$, so that $D(H)$ is the core of the graph J^* obtained. Then J^* is a connected Class 2 graph with four vertices of maximum degree, and the possibilities for J^* are described in Lemma 2.4.

If J^* has 4 vertices of degree $|V(J^*)| - 1$, $|V(J^*)| - 5$ of degree $|V(J^*)| - 2$, and one vertex of degree $|V(J^*)| - 3$, then J^* has no pendent edges, so $J = J^*$. But it is not possible to join any vertex of $D(H)$ to any further vertex without raising the maximum degree of J. Since $\Delta(J) = \Delta(G)$, this is not possible, so it follows that $H = D(H)$. Then $\gamma_H(V(D(H))) = 0$, $\delta_H(V(D(H))) = 3$ and $|V(D(H))| = 4$. Therefore (4.1) is satisfied.

If J^* has 4 vertices of degree $|V(J^*)| - 2$, and $|V(J^*)| - 4$ vertices of degree $|V(J^*)| - 3$, the argument is similar, and one finds that $\gamma_H(V(D(H))) = 0$, $3 \geqslant \delta_H(V(D(H))) \geqslant 2$ and $|V(D(H))| = 4$, so again (4.1) is satisfied.

In the remaining possibility, J^* has a cut edge e that separates J^* into J_1 and J_2, where $\Delta(J_1) > \Delta(J_2)$ and J_1 either (a) has four vertices of degree $|V(J_1)| - 1$, $|V(J_1)| - 5$ vertices of degree $|V(J_1)| - 2$ and one vertex of degree $|V(J_1)| - 3$, or (b) has three vertices of degree $|V(J_1)| - 1$ and $|V(J_1)| - 3$ vertices of degree $|V(J_1)| - 2$. Case (a) is virtually a repetition of the first case described above, and it follows as there that $\gamma_H(V(D(H))) = 0$, $\delta_H(V(D(H))) = 3$ and $|V(D(H))| = 4$, so (4.1) is satisfied. In case (b) e could be a pendent edge, but it makes no difference. We have that $D(H)$ is a K_4, and then $\gamma_H(V(D(H))) = 1$, $\delta_H(D(H)) = 3$ and $|V(D(H))| = 4$, so (4.1) is satisfied. □

In [7] there was one graph of order 5 for which an *ad hoc* argument was needed to decide whether or not it could be the core of a Class 2 graph. This was the graph A of Figure 1. Theorem 4.3 can be used to decide this question. For $D(A)$ is the subgraph induced by $\{a, b, c, d\}$, and we have $\gamma_A(V(D(A))) = 1$ and $\delta_A(D(A)) = 2$, so (4.1) is not satisfied. Therefore A cannot be the core of a Class 2 graph. By contrast, $\gamma_B(V(D(B))) = 1$, $\delta_B(D(B)) = 3$, and $|V(D(B))| = 4$, so by Theorem 4.3, B can be the core of a Class 2 graph.

Finally, it is interesting to consider the Petersen graph P. Clearly $D(P) = P$, so $\gamma_P(V(D(P))) = 0$, $\delta_P(D(P)) = 3$ and $|V(D(P))| = 10$, so that (4.1) is satisfied. Now consider P^*, the graph obtained from P by deleting one vertex. It is well known that P^* is Class 2. Let a graph H be formed by the addition of three vertices, each joined by an edge to a distinct vertex of P^* of degree 2. Since P^* is Class 2, it is easy to see that H can be the core of a Class 2 graph. But $D(H) = P^*$, so $\gamma_P(V(D(H))) = 3$, $\delta_H(D(H)) = 3$, and so (4.1) is not satisfied. Thus (4.1) is not sufficient to ensure that H is the core of a Class 2 graph.

However, since P satisfies (4.1), but this slight modification does not, it may not be an impossible hope that (4.1) could be altered slightly so as to provide a necessary and sufficient condition for a graph H to be the core of a Class 2 graph.

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