106.19 Some observations on inequalities related to Huygens' inequality

Our starting point is a pair of trigonometric inequalities valid for $0 < x < \frac{\pi}{2}$: Huygens' inequality

$$\frac{2\sin x + \tan x}{3} > x \tag{1}$$

and the Cusa-Huygens' inequality

$$x > \frac{3\sin x}{2 + \cos x}. \tag{2}$$

These feature in Huygens' 1654 treatise *De circuli magnitudine inventa* which built on earlier ideas of Snell and Nicholas of Cusa for accelerating the convergence of Archimedes' method for calculating π. We look at refinements of these inequalities, using as a common tool the following technique for establishing inequalities: if $f'(x) > g(x)$ for $0 < x < a$, then, on integrating, $f(x) - f(0) > \int_0^x g(t)\,dt$ for $0 < x < a$. In all our applications, $f(0) = 0$ with $f(x) > \int_0^x g(t)\,dt$ then giving the inequality sought.

Inequalities related to (1) and (2) have attracted quite a lot of interest recently, including a lovely article by Nelsen, [1], and the wide-ranging survey by Bhayo and Sándor, [2]. These both feature the chain of inequalities

$$\frac{2 + \cos x}{3} > \frac{\sin x}{x} > \sqrt[3]{\cos x} > \frac{3\cos x}{2\cos x + 1}. \tag{3}$$

Here, $\frac{2 + \cos x}{3} > \frac{\sin x}{x}$ is equivalent to (2) while $\frac{\sin x}{x} > \frac{3\cos x}{2\cos x + 1}$ is equivalent to (1). Also note that, because the component pieces are all even functions, (3) is actually valid for $0 < |x| < \frac{\pi}{2}$.

The proofs in Nelsen's article use bounds for the partial sums of the Maclaurin expansions of $\sin x$ and $\cos x$. An alternative method for (1), used by Christopher Bradley in [3], is to consider $f(x) = \frac{2\sin x + \tan x}{3}$ for which $f'(x) = \frac{\cos x + \cos x + \sec^2 x}{3} > \sqrt[3]{\cos x \cos x \sec^2 x} = 1$, by the AM-GM inequality. On integrating, $f(x) - f(0) > x$, which gives (1), since $f(0) = 0$. The same method also works for (2). For if $g(x) = \frac{3\sin x}{2 + \cos x}$, then $g'(x) = \frac{6\cos x + 3}{(2 + \cos x)^2}$ so that $1 - g'(x) = \left(\frac{1 - \cos x}{2 + \cos x}\right)^2 > 0$ which, on integrating, gives $x - g(x) + g(0) > 0$, as required for (2).

Regarding (3), we have the following surprising observation:

$$\frac{\sin x}{x} > \sqrt[3]{\cos x} \Leftrightarrow \frac{\sin x}{x} > \frac{3\cos x}{2\cos x + 1}.$$

So the apparently weaker equivalent of Huygens' inequality on the right-hand side is equivalent to the stronger left-hand inequality.

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For the $\Rightarrow$ direction, we follow Nelsen's proof in [1]: the GM-HM inequality applied to $1,\ 1,\ \cos x$ gives $\sqrt[3]{\cos x} > \dfrac{3}{1 + 1 + \frac{1}{\cos x}} = \dfrac{3\cos x}{2\cos x + 1}$.

For the $\Leftarrow$ direction, recall that $\dfrac{\sin x}{x} > \dfrac{3\cos x}{2\cos x + 1}$ is equivalent to

$$2\sin x + \tan x > 3x. \tag{4}$$

To show that $\dfrac{\sin x}{x} > \sqrt[3]{\cos x}$, it suffices to prove that $\sin^2 x \tan x > x^3$. Consider $h(x) = \sin^2 x \tan x$. Then

$$h'(x) = 2\sin^2 x + \tan^2 x > \tfrac{1}{3}(2\sin x + \tan x)^2 > 3x^2,$$

using the power means inequality together with (4). On integrating, $h(x) - h(0) > x^3$, as required.

Huygens' inequality has an interesting geometrical interpretation, shown in Figure 1(a). Here, area $A_1 = \frac{1}{2}(\tan x - x)$ and area $A_2 = \frac{1}{2}(x - \sin x)$, so (1) is equivalent to $A_1 > 2A_2$.

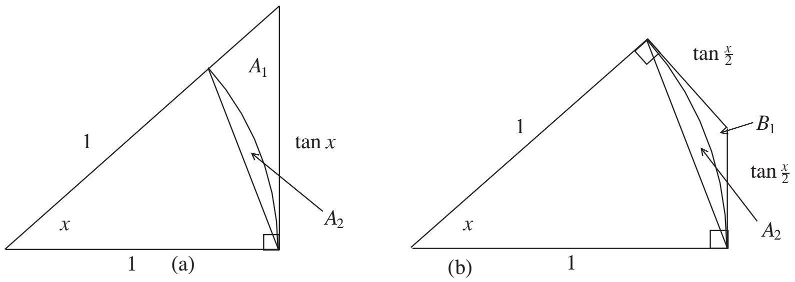

FIGURE 1

The related Figure 1(b) concerns area $B_1 = \frac{1}{2}\left(2\tan\frac{x}{2} - x\right)$ and area $A_2 = \frac{1}{2}(x - \sin x)$. In this case we have the inequality $2B_1 > A_2$ or

$$4\tan\tfrac{x}{2} + \sin x > 3x. \tag{5}$$

A quick proof of this is to note that, if $k(x) = 4\tan\frac{x}{2} + \sin x$, then

$$k'(x) = 2\sec^2\frac{x}{2} + 2\cos^2\frac{x}{2} - 1 > 4\sqrt{\sec^2\frac{x}{2}\cos^2\frac{x}{2}} - 1 = 3,$$

by the AM-GM inequality. This then integrates to give $k(x) - k(0) > 3x$, as required. Note that (5) is actually valid for $0 < x < \pi$.

Alternatively, we can co-opt the central inequality in (3),

$$\frac{4\tan\frac{x}{2} + \sin x}{3} = \frac{2\tan\frac{x}{2} + 2\tan\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}{3} > \sqrt[3]{\frac{8\sin^3\frac{x}{2}}{\cos\frac{x}{2}}} > x,$$

using the AM-GM inequality together with $\dfrac{\sin\frac{x}{2}}{\frac{x}{2}} > \sqrt[3]{\cos\dfrac{x}{2}}$ from (3).

How does (5) compare with (1)?

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We claim that $2\sin x + \tan x > 4\tan\frac{1}{2}x + \sin x > 3x$, for $0 < x < \frac{1}{2}\pi$. The left-hand inequality is equivalent to

$$\sin x + \tan x - 4\tan\frac{x}{2} > 0. \tag{6}$$

Setting $t = \tan\frac{x}{2}$ with $0 < t < 1$ we see that

$$\frac{2t}{1+t^2} + \frac{2t}{1-t^2} - 4t = \frac{4t^5}{1-t^4} > 0.$$

as required.

On the last two pages of [2], Bhayo and Sándor give a proof that Huygens' arc-length approximation formula, which occurs in Huygens' 1654 treatise mentioned above and is discussed in [4], features in the following strengthening of (2):

$$\frac{3\sin x}{2+\cos x} < \frac{8\sin\frac{x}{2} - \sin x}{3} < x. \tag{7}$$

For the left-hand inequality, they substitute $t = \cos\frac{x}{2}$ to show that

$$\frac{8\sin\frac{x}{2} - \sin x}{3} - \frac{3\sin x}{2+\cos x} = \frac{4\sin\frac{x}{2}}{3(2+\cos x)}(t-1)^2(2-t) > 0.$$

For the right-hand inequality, if $l(x) = \frac{1}{3}\left(8\sin\frac{x}{2} - \sin x\right)$, then

$$1 - l'(x) = \tfrac{2}{3}\left(1 - \cos\tfrac{x}{2}\right)^2 > 0$$

so, on integrating, $x > l(x) - l(0)$, as required.

Finally, the authors of [1, 2] discuss the hyperbolic analogues of (3):

$$\frac{2+\cosh x}{3} > \frac{\sinh x}{x} > \sqrt[3]{\cosh x} > \frac{3\cosh x}{2\cosh x + 1}, \text{ for } x > 0.$$

These may be established by applying exactly the same methods as those used above for their trigonometric counterparts. It is worth noting that (5), (6) and the right-hand side of (7) also remain valid in their hyperbolic versions; but (mimicking the proof above), the left-hand side of (7) is only valid for $0 < x < 2\cosh^{-1}2 = \ln\left(7 + 4\sqrt{3}\right)$.

References

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