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3-torsion in the Homology of Complexes of Graphs of Bounded Degree

Jakob Jonsson

Abstract. For $\delta \geq 1$ and $n \geq 1$, consider the simplicial complex of graphs on n vertices in which each vertex has degree at most δ; we identify a given graph with its edge set and admit one loop at each vertex. This complex is of some importance in the theory of semigroup algebras. When $\delta = 1$, we obtain the matching complex, for which it is known that there is 3-torsion in degree d of the homology whenever $(n-4)/3 \leq d \leq (n-6)/2$. This paper establishes similar bounds for $\delta \geq 2$. Specifically, there is 3-torsion in degree d whenever

$$\frac{(3\delta-1)n-8}{6} \leq d \leq \frac{\delta(n-1)-4}{2}.$$

The procedure for detecting torsion is to construct an explicit cycle z that is easily seen to have the property that $3z$ is a boundary. Defining a homomorphism that sends z to a non-boundary element in the chain complex of a certain matching complex, we obtain that z itself is a non-boundary. In particular, the homology class of z has order 3.

1 Introduction

The aim of this paper is to examine the integral homology of certain simplicial complexes defined in terms of degree bounds of graphs. Specifically, each face in a given complex corresponds to a graph such that the degree of each vertex is bounded from above by a certain fixed value. The rational homology has been computed [7], but not very much is known about the integral homology. This paper makes some progress on the latter problem, detecting 3-torsion in the homology for various choices of parameters.

Let us formulate the problem more precisely, starting with basic graph-theoretic definitions. We refer to the positive integers as *vertices*. An *edge* is an unordered pair $\{v, w\}$ of vertices, where we allow $v = w$. We will often write vw instead of $\{v, w\}$. An edge of the form vv is a *loop*. The vertices of an edge are the *endpoints* of the edge. We refer to an edge set E as being *on* a vertex set V if the endpoints of the edges in E all belong to V. A *graph* (more precisely, a simple graph admitting loops) is a pair (V, E) such that E is an edge set on the vertex set V. We will mainly speak of edge sets and only involve graphs when the underlying vertex set is not clear from context.

For an edge set σ, the degree $\deg_\sigma(v)$ of a vertex v is the number of occurrences of v in σ; we adopt the convention that v occurs twice in the loop vv. For example,

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for the edge set $\sigma = \{aa, ab, ac, bc, bd\}$, we have that $\deg_\sigma(a) = 4$, $\deg_\sigma(b) = 3$, $\deg_\sigma(c) = 2$, and $\deg_\sigma(d) = 1$.

Let $n \geq 1$ and let $\lambda = (\lambda_1, \dots, \lambda_n)$ be an arbitrary sequence of integers. Define BD_n^λ to be the family of edge sets σ on the vertex set $[n] = \{1, \dots, n\}$ such that $\deg_\sigma(i) \leq \lambda_i$ for each $i \in [n]$. For an edge set E on the vertex set $[n]$, let $\mathsf{BD}_n^\lambda(E)$ be the subfamily of BD_n^λ obtained by restricting to subsets of E. The two families BD_n^λ and $\mathsf{BD}_n^\lambda(E)$ are closed under deletion of edges, which means that they are abstract simplicial complexes.

Write $\mathsf{BD}_n^{(\delta,\dots,\delta)} = \mathsf{BD}_n^\delta$. For $\delta = 1$, we obtain the *matching complex* $\mathsf{M}_n = \mathsf{BD}_n^1$. By the work of Bouc [2] and Shareshian and Wachs [8], the bottom nonvanishing homology group of M_n is an elementary 3-group for almost all n. One may use this fact to prove that $\widetilde{H}_d(\mathsf{M}_n; \mathbb{Z})$ contains 3-torsion whenever

$$\frac{n-4}{3} \leq d \leq \frac{n-6}{2};$$

see Jonsson [4, §11.2.3]. The goal of the present paper is to obtain analogous results about BD_n^δ for $\delta \geq 2$.

Theorem 1.1 *For $\delta \geq 2$, the group $\widetilde{H}_d(\mathsf{BD}_n^\delta; \mathbb{Z})$ contains 3-torsion whenever*

$$\frac{(3\delta - 1)n - 8}{6} \leq d \leq \frac{\delta(n-1) - 4}{2}.$$

We prove Theorem 1.1 by constructing an explicit cycle z in $\widetilde{C}_d(\mathsf{BD}_n^\delta; \mathbb{Z})$ for each pair (d, n) satisfying the inequalities in the theorem. As it turns out, the order of the homology class of z is easily seen to divide three. To show that the homology class is nonvanishing, we consider the natural epimorphism from the chain complex $\widetilde{C}_d(\mathsf{BD}_n^\delta; \mathbb{Z})$ to the chain complex of a certain link in BD_n^δ; we show that the homology class of the image of z under this epimorphism is nonvanishing.

The following conjecture states that Theorem 1.1 remains true for $\delta = 1$.

Conjecture 1.2 *We have that $\widetilde{H}_d(\mathsf{M}_n; \mathbb{Z})$ contains 3-torsion whenever*

$$\frac{n-4}{3} \leq d \leq \frac{n-5}{2}.$$

To settle the conjecture, it suffices to prove that $\widetilde{H}_d(\mathsf{M}_n; \mathbb{Z})$ contains 3-torsion whenever $d = (n-5)/2$ and $n \geq 7$ for n odd. Since 3-torsion is known to exist for $n \in \{7, 9, 11, 13, 15\}$ [2, 6, 8], one need only consider odd $n \geq 17$.

For $\delta \geq 2$, we do not know whether there are parameters (n, d) not satisfying the bounds in Theorem 1.1 such that there is 3-torsion in $\widetilde{H}_d(\mathsf{BD}_n^\delta)$. Computational results [6] show that the homology of BD_n^2 contains no 3-torsion for $n \leq 8$. In this context, it might be worth mentioning that the homology of BD_n^2 does contain 5-torsion for $n = 7$ and $n = 8$; Andersen [1] established the case $n = 7$ in the early 1990s.

One may also consider the subcomplex of BD_n^λ obtained by removing all loops vv. The reason for focusing on the variant admitting loops is that this variant appears naturally in algebra. Specifically, one may express the minimal free resolution of certain semigroup algebras [3, 7, 9] in terms of the homology of BD_n^λ. All constructions in this paper rely on the existence of loops and hence only apply to the full complex BD_n^λ.

2 Simplicial Chain Complexes

2.1 Notation

Most material in this section is standard, but we present a fairly detailed overview of the subject to avoid ambiguity in later sections.

Let Δ be a simplicial complex and let $\mathbb{F}$ be the ring of integers or a field. For $d \geq -1$, let $\widetilde{C}_d(\Delta;\mathbb{F})$ be the free $\mathbb{F}$-module with one basis element, denoted as $s_1 \wedge \cdots \wedge s_{d+1}$, for each d-dimensional face $\{s_1, \dots, s_{d+1}\}$ of Δ. We refer to $s_1 \wedge \cdots \wedge s_{d+1}$ as an *oriented simplex*. Let $\mathfrak{S}_n$ be the symmetric group on the set $[n] = \{1, \dots, n\}$. For any permutation $\pi \in \mathfrak{S}_{d+1}$ and any face $\sigma = \{s_1, \dots, s_{d+1}\}$, we define

$$s_{\pi(1)} \wedge s_{\pi(2)} \wedge \cdots \wedge s_{\pi(d+1)} = \mathrm{sgn}(\pi) \cdot s_1 \wedge s_2 \wedge \cdots \wedge s_{d+1}. \tag{2.1}$$

For convenience, we write

$$[\sigma] = s_1 \wedge s_2 \wedge \cdots \wedge s_{d+1},$$

implicitly assuming that we have a fixed linear order on the 0-cells in Δ.

Extend the definition of $s_1 \wedge \cdots \wedge s_{d+1}$ to arbitrary sequences $(s_1, \dots, s_{d+1})$ by defining $s_1 \wedge \cdots \wedge s_{d+1} = 0$ if $s_i = s_j$ for some $i \neq j$. Note that (2.1) implies that $2 \cdot s_1 \wedge \cdots \wedge s_{d+1} = 0$ for such a sequence.

The *boundary map* $\partial_d : \widetilde{C}_d(\Delta;\mathbb{F}) \to \widetilde{C}_{d-1}(\Delta;\mathbb{F})$ is the homomorphism defined by

$$\partial_d(s_1 \wedge \cdots \wedge s_{d+1}) = \sum_{i=1}^{d+1} (-1)^{i-1} s_1 \wedge \cdots \wedge s_{i-1} \wedge s_{i+1} \wedge \cdots \wedge s_{d+1}.$$

Combining all ∂_d, we obtain an operator ∂ on the direct sum $\widetilde{C}(\Delta;\mathbb{F})$ of all $\widetilde{C}_d(\Delta;\mathbb{F})$. It is well known and easy to see that $\partial^2 = 0$.

For the chain complex $(\widetilde{C}(\Delta;\mathbb{F}), \partial)$ on the simplicial complex Δ, we refer to elements in $\partial^{-1}(\{0\})$ as *cycles* and elements in $\partial(\widetilde{C}(\Delta;\mathbb{F}))$ as *boundaries*. Define the i-th *reduced homology group* of Δ with coefficients in $\mathbb{F}$ as the quotient $\mathbb{F}$-module

$$\widetilde{H}_d(\Delta;\mathbb{F}) = \frac{\partial_d^{-1}(\{0\})}{\partial_{d+1}(\widetilde{C}_{d+1}(\Delta;\mathbb{F}))} = \frac{\ker \partial_d}{\operatorname{im} \partial_{d+1}}.$$

2.2 Some Useful Constructions

Whenever $\sigma = \{s_1, \dots, s_a\}$ and $\tau = \{t_1, \dots, t_b\}$ are faces such that $\sigma \cup \tau \in \Delta$, we define the product of the oriented simplices $[\sigma] = s_1 \wedge \cdots \wedge s_a$ and $[\tau] = t_1 \wedge \cdots \wedge t_b$ to be the element

$$[\sigma] \wedge [\tau] = s_1 \wedge \cdots \wedge s_a \wedge t_1 \wedge \cdots \wedge t_b. \tag{2.2}$$

Note that $[\sigma] \wedge [\tau]$ is zero whenever $\sigma \cap \tau$ is nonempty, because this means that $s_i = t_j$ for some i and j.

Let Δ_1 and Δ_2 be subcomplexes of Δ such that $\sigma_1 \cup \sigma_2 \in \Delta$ whenever $\sigma_1 \in \Delta_1$ and $\sigma_2 \in \Delta_2$. Given elements $c_i \in \widetilde{C}_{d_i-1}(\Delta_i;\mathbb{F})$ for $i = 1, 2$, we define the product $c_1 \wedge c_2 \in \widetilde{C}_{d_1+d_2-1}(\Delta;\mathbb{F})$ by extending the product (2.2) bilinearly. We have that

$$\partial(c_1 \wedge c_2) = \partial(c_1) \wedge c_2 + (-1)^{d_1} c_1 \wedge \partial(c_2). \tag{2.3}$$

In particular, if c_1 and c_2 are cycles, then so is $c_1 \wedge c_2$.

For a face σ, let the link $\mathrm{lk}_\Delta(\sigma)$ be the complex $\{\tau : \tau \cup \sigma \in \Delta, \tau \cap \sigma = \varnothing\}$, and let the face deletion $\mathrm{fdel}_\Delta(\sigma)$ be the complex $\{\tau : \tau \in \Delta, \sigma \not\subseteq \tau\}$. Let $\sigma = \{s_1, \dots, s_r\} \in \Delta$ and let $c \in \widetilde{C}_{d-1}(\Delta;\mathbb{F})$. There is a unique decomposition of c as

$$c = s_1 \wedge \cdots \wedge s_r \wedge c' + x,$$

where $c' \in \widetilde{C}_{d-r-1}(\mathrm{lk}_\Delta(\sigma);\mathbb{F})$ and $x \in \widetilde{C}_{d-1}(\mathrm{fdel}_\Delta(\sigma);\mathbb{F})$. We write $\mathrm{lk}_c([\sigma]) = c'$ and $\mathrm{fdel}_c([\sigma]) = x$; thus

$$c = [\sigma] \wedge \mathrm{lk}_c([\sigma]) + \mathrm{fdel}_c([\sigma]).$$

Since

$$\partial(c) = \partial([\sigma]) \wedge \mathrm{lk}_c([\sigma]) + (-1)^r \cdot [\sigma] \wedge \partial\big(\mathrm{lk}_c([\sigma])\big) + \partial\big(\mathrm{fdel}_c([\sigma])\big),$$

we have that

$$\begin{aligned}\mathrm{lk}_{\partial(c)}([\sigma]) &= (-1)^r \cdot \partial(\mathrm{lk}_c([\sigma])),\\ \mathrm{fdel}_{\partial(c)}([\sigma]) &= \partial([\sigma]) \wedge \mathrm{lk}_c([\sigma]) + \partial(\mathrm{fdel}_c([\sigma])).\end{aligned}$$

Most importantly, up to the irrelevant sign $(-1)^r$, the map $c \mapsto \mathrm{lk}_c([\sigma])$ defines a homomorphism from the chain complex of Δ to the chain complex of $\mathrm{lk}_\Delta(\sigma)$. In particular, this map induces a homomorphism in homology.

Let $\Delta_1, \dots, \Delta_k$ be subcomplexes of Δ such that $\bigcup_{i=1}^k \sigma_i \in \Delta$ whenever $\sigma_i \in \Delta_i$ for each i. Suppose that we are given an element $c = c_1 \wedge \cdots \wedge c_k$, where c_i is an element in $\widetilde{C}_{d_i-1}(\Delta_i;\mathbb{F})$ for each i.

Lemma 2.1 *Let σ be a face of Δ. We have that*

$$[\sigma] \wedge \mathrm{lk}_c([\sigma]) = \sum_{(\tau_1,\dots,\tau_k)} [\tau_1] \wedge \mathrm{lk}_{c_1}([\tau_1]) \wedge \cdots \wedge [\tau_k] \wedge \mathrm{lk}_{c_k}([\tau_k]),$$

where the sum is over all ordered partitions $(\tau_1, \dots, \tau_k)$ of σ such that $\tau_i \in \Delta_i$.

Proof By linearity, we need only prove the lemma in the case that each c_i coincides with an oriented simplex $[\rho_i]$. For any $\tau_i \subseteq \rho_i$, we have that $[\tau_i] \wedge \mathrm{lk}_{[\rho_i]}([\tau_i]) = [\rho_i]$. Moreover, if $\tau_i \not\subseteq \rho_i$, then $[\tau_i] \wedge \mathrm{lk}_{[\rho_i]}([\tau_i]) = 0$. In particular, each summand in the right-hand side is either c or 0. As a consequence, if some element appears in both ρ_i

and ρ_j for some $i \neq j$, meaning that $c = 0$, then the right-hand side is zero. Clearly, the left-hand side is also zero in this case.

Assume that $\rho_1, \dots, \rho_k$ are pairwise disjoint and write $\rho = \rho_1 \cup \cdots \cup \rho_k$. If ρ does not contain σ, then both sides in the lemma are zero. Assume that ρ does contain σ. Then $[\sigma] \wedge \mathrm{lk}_c([\sigma]) = c$. Moreover,

$$\begin{aligned} c &= [\rho_1] \wedge \cdots \wedge [\rho_k] \\ &= [\sigma \cap \rho_1] \wedge \mathrm{lk}_{[\rho_1]}([\sigma \cap \rho_1]) \wedge \cdots \wedge [\sigma \cap \rho_k] \wedge \mathrm{lk}_{[\rho_k]}([\sigma \cap \rho_k]). \end{aligned}$$

The latter expression coincides with the right-hand side in the lemma, because $(\sigma \cap \rho_1, \dots, \sigma \cap \rho_k)$ is the only partition $(\tau_1, \dots, \tau_k)$ of σ such that $\mathrm{lk}_{[\rho_i]}([\tau_i])$ is nonzero for each i. ■

3 Basic Properties of Cycle Products in BD_n^λ

Let X be a finite multiset consisting of r distinct elements $x_1, \dots, x_r$ with associated multiplicities $m_1, \dots, m_r$, respectively. Define

$$\mu(X) = m_1! m_2! \cdots m_r!.$$

Let $A = \{a_1, \dots, a_{q-1}\}$ be a multiset of elements from $[n]$, and let $B = \{b_1, \dots, b_q\}$ be a subset of $[n]$, not necessarily disjoint from A. Define

$$\phi_{A,B} = \frac{1}{\mu(A)} \cdot \sum_{\pi \in \mathfrak{S}_q} \mathrm{sgn}(\pi) \cdot a_1 b_{\pi(1)} \wedge \cdots \wedge a_{q-1} b_{\pi(q-1)}.$$

For example,

$$\begin{aligned} \phi_{\{a_1,a_2\},\{b_1,b_2,b_3\}} = k \cdot (&a_1 b_1 \wedge a_2 b_2 - a_1 b_2 \wedge a_2 b_1 + a_1 b_2 \wedge a_2 b_3 \\ &- a_1 b_3 \wedge a_2 b_2 + a_1 b_3 \wedge a_2 b_1 - a_1 b_1 \wedge a_2 b_3), \end{aligned}$$

where $k = 1$ if $a_1 \neq a_2$ and $k = 1/2$ if $a_1 = a_2$. The reason for not admitting repetitions in B is that $\phi_{A,B} = 0$ whenever $b_i = b_j$ for some $i \neq j$; this is easy to see in the given example.

Lemma 3.1 *The element $\phi_{A,B}$ is a cycle in $\widetilde{C}_{q-2}(\mathrm{BD}_n^\lambda; \mathbb{Z})$, where λ_i is the total number of occurrences of the vertex i in A and B (counting multiplicities in A).*

Proof Let H be the subgroup of $\mathfrak{S}_q$ consisting of those $\mu(A)$ permutations in $\mathfrak{S}_q$ that satisfy $a_{\kappa(i)} = a_i$ for $1 \leq i \leq q-1$ and $\kappa(q) = q$. Let R be a right transversal of

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H in $\mathfrak{S}_q$. To see that $\phi_{A,B}$ has integer coefficients, note that

$$\begin{aligned}\mu(A)\cdot\phi_{A,B} &= \sum_{\kappa\in H}\sum_{\pi\in R}\operatorname{sgn}(\kappa\pi)\cdot a_1b_{\kappa\pi(1)}\wedge\cdots\wedge a_{q-1}b_{\kappa\pi(q-1)}\\ &= \sum_{\kappa\in H}\sum_{\pi\in R}\operatorname{sgn}(\pi)\cdot a_{\kappa^{-1}(1)}b_{\pi(1)}\wedge\cdots\wedge a_{\kappa^{-1}(q-1)}b_{\pi(q-1)}\\ &= \sum_{\kappa\in H}\sum_{\pi\in R}\operatorname{sgn}(\pi)\cdot a_1b_{\pi(1)}\wedge\cdots\wedge a_{q-1}b_{\pi(q-1)}\\ &= \mu(A)\cdot\sum_{\pi\in R}\operatorname{sgn}(\pi)\cdot a_1b_{\pi(1)}\wedge\cdots\wedge a_{q-1}b_{\pi(q-1)}.\end{aligned}$$

To see that $\phi_{A,B}$ is a cycle, let $t_{\pi,i}$ be the oriented simplex obtained by removing $a_ib_{\pi(i)}$ from $a_1b_{\pi(1)}\wedge\cdots\wedge a_{q-1}b_{\pi(q-1)}$. We get that

$$\partial(\phi_{A,B}) = \sum_{i=1}^{q-1}(-1)^{i-1}\sum_{\pi}\operatorname{sgn}(\pi)t_{\pi,i}.$$

Letting $g_i\colon \mathfrak{S}_q \to \mathfrak{S}_q$ be the involution given by $g_i(\pi) = \pi\circ(i,q)$, we see that $t_{\pi,i} = t_{g_i(\pi),i}$ and $\operatorname{sgn}(\pi) = -\operatorname{sgn}(g_i(\pi))$; hence another standard argument yields that the sum is zero. ∎

We refer to $\phi_{A,B}$ as a *chessboard cycle*. To explain this terminology, if A and B are disjoint ordinary sets, then $\phi_{A,B}$ is the fundamental cycle of the chessboard complex with rows indexed by A and columns indexed by B; see Shareshian and Wachs [8]. We say that the chessboard cycle $\phi_{A,B}$ is an $(|A|,|B|)$*-cycle*. Note that $\phi_{\{a\},\{b,c\}} = ab - ac$ and that $\phi_{\varnothing,\{b\}} = [\varnothing]$ for any b. The latter cycle is the generator of $\widetilde{C}_{-1}(\mathsf{M}_{\{b\}};\mathbb{Z}) \cong \mathbb{Z}$, where M_X denotes the matching complex on the vertex set X.

We will use chessboard cycles as building blocks when constructing homology elements of order three. A *chessboard product* is a cycle of the form

$$w = \phi_{A_1,B_1}\wedge\phi_{A_2,B_2}\wedge\cdots\wedge\phi_{A_t,B_t},$$

By some abuse of notation, we refer to the value t as the *codegree* of w. If $M = \sum_{i=1}^{t}(|A_i|+|B_i|)$, then $M = 2|A_i|+t$, and w is a cycle of degree $(M-t)/2-1$. Note that the codegree always has the same parity as the sum M.

The following result is due to Bouc [2] and Shareshian and Wachs [8].

Proposition 3.2 *Let $\eta\in\{0,1,2\}$ and $\alpha\geq 0$, and let X be a set of size $n = 3\alpha+2\eta+1$. Let $X = \bigcup_{i=0}^{\alpha}(A_i\cup B_i)$ be a partition of X into sets such that $|A_i| = 1$ and $|B_i| = 2$ for $1\leq i\leq\alpha$ and such that $|A_0| = \eta$ and $|B_0| = \eta+1$. Then the homology class of the chessboard product*

$$z = \bigwedge_{i=0}^{\alpha}\phi_{A_i,B_i}$$

is a nonzero element of the group

$$\widetilde{H}_{\alpha+\eta-1}(\mathsf{M}_X;\mathbb{Z}) \cong \widetilde{H}_{\alpha+\eta-1}(\mathsf{M}_n;\mathbb{Z}).$$

This group is an elementary 3*-group for* $n \geq 15$ *and for* $n \in \{7, 10, 12, 13\}$*, a finite group of exponent divisible by three for* $n = 14$*, and an infinite group for* $n \in \{1, 3, 4, 5, 6, 8, 9, 11\}$.

The group in the proposition is the bottom nonvanishing homology group of M_n [2, 8]. For $n = 14$, the exponent of the group is in fact divisible by 15 [5].

Let $k \geq 1$. For $1 \leq i \leq k$, let

$$\lambda^i = (\lambda^i_1, \dots, \lambda^i_n)$$

be a sequence of nonnegative integers, and let E be a set of edges on the vertex set $[n]$. Let $d_i \geq 0$ and $\gamma_i \in \widetilde{C}_{d_i-1}(\mathsf{BD}^{\lambda^i}_n(E);\mathbb{Z})$. Write

$$\lambda = \sum_{i=1}^{k} \lambda^i, \quad d = \sum_{i=1}^{k} d_i, \quad \text{and} \quad \gamma = \gamma_1 \wedge \cdots \wedge \gamma_k.$$

Lemma 3.3 *We have that* γ *is an element in* $\widetilde{C}_{d-1}(\mathsf{BD}^{\lambda}_n(E);\mathbb{Z})$*. If each* γ_i *is a cycle, then so is* γ*. Moreover, the order of the homology class of* γ *in the group* $\widetilde{H}_{d-1}(\mathsf{BD}^{\lambda}_n(E);\mathbb{Z})$ *divides the order of the homology class of* γ_i *in the group* $\widetilde{H}_{d_i-1}(\mathsf{BD}^{\lambda^i}_n(E);\mathbb{Z})$ *for* $1 \leq i \leq k$.

Proof By construction, if $e_1 \wedge \cdots \wedge e_d$ appears in the expansion of $\gamma_1 \wedge \cdots \wedge \gamma_k$, then the sequence $(\deg_\sigma(1), \dots, \deg_\sigma(n))$ is bounded by $\sum_i \lambda^i = \lambda$, where $\sigma = \{e_1, \dots, e_d\}$. As a consequence, γ is indeed an element in $\widetilde{C}_{d-1}(\mathsf{BD}^{\lambda}_n(E);\mathbb{Z})$. The identity (2.3) and a straightforward induction argument yield that γ is a cycle whenever each γ_i is a cycle. Finally, if the homology class of, say, γ_1 has finite order a, then there is an element $c \in \widetilde{C}_{d_1}(\mathsf{BD}^{\lambda^1}_n(E);\mathbb{Z})$ such that $\partial(c) = a \cdot \gamma_1$. Since $c \wedge \gamma_2 \wedge \cdots \wedge \gamma_k$ belongs to $\widetilde{C}_d(\mathsf{BD}^{\lambda}_n(E);\mathbb{Z})$ and

$$\partial(c \wedge \gamma_2 \wedge \cdots \wedge \gamma_k) = (a\gamma_1) \wedge \gamma_2 \wedge \cdots \wedge \gamma_k = a \cdot \gamma_1 \wedge \cdots \wedge \gamma_k,$$

it follows that the order of the homology class of $\gamma_1 \wedge \cdots \wedge \gamma_k$ divides a. By symmetry, the same is true for γ_i instead of γ_1 for each $i \in \{2, \dots, k\}$. ∎

From now on, assume that each γ_i is a cycle. We will make repeated use of the following result.

Lemma 3.4 *Suppose that one cycle* γ_i *has the property that*

$$\gamma_i = \phi_{\{a_1\},\{b_1,c_1\}} \wedge \phi_{\{a_2\},\{b_2,c_2\}} \wedge \phi_{\varnothing,\{x\}},$$

where the seven elements in the vertex set $W = \{a_1, b_1, c_1, a_2, b_2, c_2, x\}$ *are all distinct. Furthermore, suppose that* E *contains all edges between vertices in* W*. Then the order of the homology class of* $\gamma = \gamma_1 \wedge \cdots \wedge \gamma_k$ *in* $\widetilde{H}_{d-1}(\mathsf{BD}^{\lambda}_n(E);\mathbb{Z})$ *divides three.*

Proof For simplicity, assume that $i = 1$. Write $\gamma = \gamma_1 \wedge \gamma'$, where $\gamma' = \gamma_2 \wedge \cdots \wedge \gamma_k$. We may view γ_1 as a cycle in the chain complex of M_W and γ' as a cycle in the chain complex of $\mathsf{BD}_n^{\lambda'}(E)$, where λ' is obtained from λ by subtracting one from λ_w for each $w \in W$. Proposition 3.2 yields that the order of the homology class of γ_1 in the chain complex of M_W is three. By Lemma 3.3, we are done. ∎

Suppose that we are given pairwise disjoint faces $\sigma_i \in \mathsf{BD}_n^{\lambda^i}(E)$, $1 \le i \le k$; thus each edge in E appears in at most one σ_i. Write $\sigma = \bigcup_{i=1}^k \sigma_i$. Note that $\gamma_i' = \mathrm{lk}_{\gamma_i}([\sigma_i])$ is a cycle in the chain complex of

$$\mathrm{lk}_{\mathsf{BD}_n^{\lambda^i}(E)}(\sigma_i) = \mathsf{BD}_n^{\lambda^i - \deg_{\sigma_i}}(E \setminus \sigma_i),$$

where $\deg_{\sigma_i} = (\deg_{\sigma_i}(1), \dots, \deg_{\sigma_i}(n))$.

Lemma 3.5 *With σ as above, suppose that the following condition is satisfied:*

- *If σ is the disjoint union of the sets $\tau_1, \dots, \tau_k$, and $\mathrm{lk}_{\gamma_i}([\tau_i])$ is nonzero for all i, then $\tau_i = \sigma_i$ for all i.*

Then

$$\mathrm{lk}_\gamma([\sigma]) = \pm \mathrm{lk}_{\gamma_1}([\sigma_1]) \wedge \cdots \wedge \mathrm{lk}_{\gamma_k}([\sigma_k]), \tag{3.1}$$

and the order of the homology class of $\mathrm{lk}_\gamma([\sigma])$ in $\widetilde{H}_{d-|\sigma|-1}(\mathsf{BD}_n^{\lambda-\deg_\sigma}(E \setminus \sigma); \mathbb{Z})$ divides the order of the homology class of γ in $\widetilde{H}_{d-1}(\mathsf{BD}_n^{\lambda}(E); \mathbb{Z})$.

Proof By Lemma 2.1 and the assumption in the present lemma,

$$[\sigma] \wedge \mathrm{lk}_\gamma([\sigma]) = [\sigma_1] \wedge \mathrm{lk}_{\gamma_1}([\sigma_1]) \wedge \cdots \wedge [\sigma_k] \wedge \mathrm{lk}_{\gamma_k}([\sigma_k]).$$

Thus (3.1) follows immediately. For the final statement, use the fact that the map $c \mapsto \mathrm{lk}_c([\sigma])$ induces a homomorphism between the given homology groups. ∎

Assume that $\mathrm{lk}_{\gamma_i}([\sigma_i])$ is nonzero for $1 \le i \le k$. Note that if the condition in Lemma 3.5 is satisfied, then $\mathrm{lk}_{\gamma_i}([\sigma_i])$ does not contain any edge from σ in its expansion for $1 \le i \le k$. Namely, suppose $e \in \sigma_j$ appears in $\mathrm{lk}_{\gamma_i}([\sigma_i])$ for some $j \ne i$. Then each of $\mathrm{lk}_{\gamma_i}([\sigma_i \cup \{e\}])$ and $\mathrm{lk}_{\gamma_j}([\sigma_j \setminus \{e\}])$ is nonzero, contradicting the uniqueness of the partition $(\sigma_1, \dots, \sigma_k)$.

Recall that our goal is to detect 3-torsion in the homology of BD_n^δ for various values of n and δ. To achieve this, we will build a chessboard product

$$z = \phi_{A_1,B_1} \wedge \cdots \wedge \phi_{A_k,B_k}$$

and apply Lemma 3.4 to conclude that the order of the homology class of z in the chain complex of BD_n^δ divides three. To prove that the order is indeed three and not one, we will construct a set σ such that Lemma 3.5 applies. Specifically, there is a unique partition $\sigma = \sigma_1 \cup \cdots \cup \sigma_k$ such that $\mathrm{lk}_{\phi_{A_i,B_i}}([\sigma_i])$ is nonzero for all i. In particular,

$$\mathrm{lk}_z([\sigma]) = \pm \mathrm{lk}_{\phi_{A_1,B_1}}([\sigma_1]) \wedge \cdots \wedge \mathrm{lk}_{\phi_{A_k,B_k}}([\sigma_k]).$$

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By Lemma 3.5, it suffices to show that the homology class of $\mathrm{lk}_z([\sigma])$ is nonzero in the chain complex of $\mathrm{BD}_n^{(\delta,\dots,\delta)-\deg_\sigma}(E_n \setminus \sigma)$, where E_n is the set of all edges on the vertex set $\{1,\dots,n\}$. In fact, it suffices to show that this is true in the chain complex of the larger complex $\mathrm{BD}_n^{(\delta,\dots,\delta)-\deg_\sigma}$.

Lemma 3.6 *Let A be a multiset and let B be a set such that $|B| = |A| + 1 = q$. Let $r \le q-1$, and let $\{x_1,\dots,x_r\} \subseteq A$ be a multiset and $\{y_1,\dots,y_r\} \subset B$ a set such that $x_i = y_i$ whenever $y_i \in A$ and $x_i \in B$. Writing $\sigma = \{x_1y_1,\dots,x_ry_r\}$, we have that*

$$\mathrm{lk}_{\phi_{A,B}}([\sigma]) = \pm\phi_{A\setminus\{x_1,\dots,x_r\},B\setminus\{y_1,\dots,y_r\}}.$$

Proof By a simple induction argument, it suffices to consider the case that $r=1$ and $\sigma = \{x_1y_1\}$. We may assume that $x_1 = a_1$ and $y_1 = b_1$. We obtain that

$$\begin{aligned}\mathrm{lk}_{\phi_{A,B}}([\sigma]) &= \mathrm{lk}_{\phi_{A,B}}(a_1b_1)\\ &= \frac{1}{\mu(A)}\cdot\sum_{j:a_j=a_1}\sum_{\pi\in\mathfrak{S}_q:\pi(j)=1}(-1)^{j-1}\mathrm{sgn}(\pi)\\ &\qquad\cdot\ a_1b_{\pi(1)}\wedge\cdots\wedge a_{j-1}b_{\pi(j-1)}\wedge a_{j+1}b_{\pi(j+1)}\wedge\cdots\wedge a_{q-1}b_{\pi(q-1)}.\end{aligned}$$

Here, we use the assumption that $a_1 = b_1$ if $b_1 \in A$ and $a_1 \in B$. Defining $\widehat{\pi} = \pi\circ(1,j)$ and moving the element $a_1b_{\pi(1)} = a_jb_{\widehat{\pi}(j)}$ to the position between $a_{j-1}b_{\pi(j-1)}$ and $a_{j+1}b_{\pi(j+1)}$, we obtain that this is equal to

$$\begin{aligned}&\frac{1}{\mu(A)}\cdot\sum_{j:a_j=a_1}\sum_{\widehat{\pi}:\widehat{\pi}(1)=1}\mathrm{sgn}(\widehat{\pi})\cdot a_2b_{\widehat{\pi}(2)}\wedge\cdots\wedge a_{q-1}b_{\widehat{\pi}(q-1)}\\ &= \frac{m_{a_1}(A)}{\mu(A)}\cdot\sum_{\pi:\pi(1)=1}\mathrm{sgn}(\pi)\cdot a_2b_{\pi(2)}\wedge\cdots\wedge a_{q-1}b_{\pi(q-1)}\\ &= \frac{1}{\mu(A\setminus\{a_1\})}\cdot\sum_{\pi:\pi(1)=1}\mathrm{sgn}(\pi)\cdot a_2b_{\pi(2)}\wedge\cdots\wedge a_{q-1}b_{\pi(q-1)}\\ &= \phi_{A\setminus\{a_1\},B\setminus\{b_1\}}.\end{aligned}$$

Here, $m_{a_1}(A)$ denotes the multiplicity of the element a_1 in A. ■

Without the assumption that $a_1 = b_1$ if $b_1 \in A$ and $a_1 \in B$, $\mathrm{lk}_{\phi_{A,B}}(a_1b_1)$ would be equal to the sum of $\pm\phi_{A\setminus\{a_1\},B\setminus\{b_1\}}$ and $\pm\phi_{A\setminus\{b_1\},B\setminus\{a_1\}}$. For example,

$$\mathrm{lk}_{\phi_{\{1,2\},\{1,2,3\}}}(12) = \phi_{\{2\},\{3,1\}} + \phi_{\{1\},\{2,3\}}.$$

4 Main Ideas and the Case $\delta = 2$

Before proceeding to the complicated proof of Theorem 1.1, we discuss the main ideas of the proof and consider the easiest case $\delta = 2$.

For the remainder of the paper, we assume that $\delta \geq 2$. Recall that our goal is to prove that $\widetilde{H}_d(\mathrm{BD}_n^\delta; \mathbb{Z})$ contains 3-torsion whenever

$$\frac{(3\delta - 1)n - 8}{6} \leq d \leq \frac{\delta(n-1) - 4}{2}. \tag{4.1}$$

The basic idea of the proof is to construct a cycle z of degree d in the chain complex of BD_n^δ such that the order of the homology class of z is three. The cycle z will be a chessboard product of the form

$$\phi_{A_1,B_1} \wedge \cdots \wedge \phi_{A_t,B_t}$$

such that each element in $[n]$ appears a total of δ times in the multisets $A_1, \ldots, A_t$ and the sets $B_1, \ldots, B_t$. Assuming that $|A_i| = |B_i| - 1$, we obtain that

$$\sum_{i=1}^{t} |A_i| = d + 1.$$

We deduce that

$$\delta n = \sum_{i=1}^{t} (2|A_i| + 1) = 2(d+1) + t,$$

which yields that

$$d = \frac{\delta n - t - 2}{2}.$$

Equivalently, $t = \delta n - 2d - 2$. Note that we may write the bounds in (4.1) in terms of the codegree t as

$$\delta + 2 \leq t \leq \frac{n+2}{3} \tag{4.2}$$

with the additional constraint that $t \equiv \delta n \pmod 2$.

Let us consider the special case $\delta = 2$. This case is significantly easier to handle than the general case, and the construction described in this section is not an immediate specialization of the general construction described in later sections. Yet the underlying ideas are the same. For integers $a \leq b$, we define

$$[a, b] = \{i : a \leq i \leq b\}.$$

Theorem 4.1 *For $4 \leq t \leq (n+2)/3$ and t even, there is a chessboard cycle z of codegree t in the chain complex of BD_n^2 such that the homology class of z has order three.*

Proof First, we construct a cycle as in the theorem whenever $n = 3t - 2$ and $t \geq 4$. Since t is even, n is also even. Let

$$A_1 = \{1\} \cup [1, \ldots, n/2],$$
$$B_1 = \{2, 3\} \cup [n/2 + 1, \ldots, n].$$

Let $X = [4, n]$, and let w be a chessboard product of codegree $t-1$ in the chain complex of $M[X]$ consisting of one $(0,1)$-cycle and $t-2$ $(1,2)$-cycles. More precisely, define

$$w = \phi_{\varnothing,\{4\}} \wedge \phi_{\{5\},\{6,7\}} \wedge \phi_{\{8\},\{9,10\}} \wedge \cdots \wedge \phi_{\{n-2\},\{n-1,n\}}.$$

Let $z = \phi_{A_1,B_1} \wedge w$; we have that z is chessboard product of codegree t in the chain complex of BD_n^2. By Lemma 3.4, the order of the homology class of z divides three.

It remains to prove that the order of the homology class is not one. For this, let $\sigma = \{i(i+n/2) : 1 \le i \le n/2\}$. The edges in σ only appear in the cycle ϕ_{A_1,B_1}, not in w. In particular,

$$\mathrm{lk}_z([\sigma]) = \mathrm{lk}_{\phi_{A_1,B_1}}([\sigma]) \wedge w = \pm\phi_{\{1\},\{2,3\}} \wedge w$$

by Lemma 3.6. This is a chessboard product of codegree t in the chain complex of M_n. By Proposition 3.2, the homology class of this cycle is nonzero. By Lemma 3.5, the same is then true for the cycle z, which concludes the proof in this particular case.

The remainder of the proof is specific for the case $\delta = 2$ and does not easily generalize to larger values of δ. For $n' \ge n = 3t-2 \ge 10$, define $A_1' = A_1 \cup [n+1, n']$, $B_1' = B_1 \cup [n+1, n']$, $\sigma' = \{ii : n+1 \le i \le n'\}$, and $z' = \phi_{A_1',B_1'} \wedge w$. We have that z' is a chessboard product of codegree t in the chain complex of $\mathsf{BD}_{n'}^2$. Moreover, it is clear that

$$\mathrm{lk}_{\sigma'}(z') = \mathrm{lk}_{\phi_{A_1',B_1'}}([\sigma']) \wedge w = \pm\phi_{A_1,B_1} \wedge w = \pm z,$$

which we know is a cycle in BD_n^2 such that the homology class is nonzero. Using exactly the same argument as before, we deduce that the order of the homology class of z' is three. ∎

5 Three Cases Yielding the Main Result

As we saw in the previous section, one single construction suffices to establish the result for $\delta = 2$. This does not appear to be the case for general δ. Instead, we need different constructions depending on the parity of n. Specifically, we divide into three cases, depending on the parity of n and δ:

A. n and δ are both odd or both even.
B. n is even and δ is odd.
C. n is odd and δ is even.

Let us describe the basics of the three constructions. In each case, we will define multisets $A_1, \dots, A_{\delta-1}$ and sets $B_1, \dots, B_{\delta-1}$ of elements from $[n]$ with the property that $|A_p| + 1 = |B_p|$ for $1 \le p \le \delta - 1$.

The total number of times each vertex $i \in [n]$ occurs in the multisets $A_1, \dots, A_{\delta-1}$ and the sets $B_1, \dots, B_{\delta-1}$ will be either $\delta - 1$ or δ; we will let X denote the set of vertices appearing only $\delta - 1$ times. We will form a chessboard product w of codegree $t - \delta + 1$ in the chain complex of M_X satisfying the conditions of Lemma 3.4.

Consider the element

$$z = \bigwedge_{p=1}^{\delta-1} \phi_{A_p,B_p} \wedge w,$$

Table 1: Definition of the multisets C_p and the sets D_p in the case that $\delta = \alpha = \beta = 5$; hence $n = 25$. There is one copy of i in the multiset C_p for each occurrence of C_p in column i, and analogously for D_p. C_p is a submultiset of A_p, and D_p is a subset of B_p. There is one star in a given column i for each additional occurrence of the vertex i in the sets A_q, B_q, and X.

1	2	3	4	5	6	7	8	9	10	11	12
C_1	C_1	C_1	$*$	$*$	$*$	$*$	$*$	D_1	D_1	D_1	D_1
C_1	C_1	C_1	C_2	C_2	C_2	$*$	$*$	$*$	$*$	$*$	D_2
C_1	C_1	C_1	C_2	C_2	C_2	C_3	C_3	C_3	$*$	$*$	$*$
C_1	C_1	C_1	C_2	C_2	C_2	C_3	C_3	C_3	C_4	C_4	C_4
$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$

13	14	15	16	17	18	19	20	21	22	23	24	25
D_1	D_1	D_1	D_1	D_1	D_1	D_1	D_1	$*$	$*$	$*$	$*$	$*$
D_2	D_2	D_2	D_2	D_2	D_2	D_2	D_2	$*$	$*$	$*$	$*$	$*$
$*$	$*$	D_3	D_3	D_3	D_3	D_3	D_3	$*$	$*$	$*$	$*$	$*$
$*$	$*$	$*$	$*$	$*$	D_4	D_4	D_4	$*$	$*$	$*$	$*$	$*$
$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$

which is a chessboard product of codegree t in the chain complex of BD_n^δ. To prove Theorem 1.1, we will first apply Lemma 3.4 to deduce that the homology class of z has order dividing three. Defining an edge set σ such that z satisfies Lemma 3.5, we obtain a new cycle $\mathrm{lk}_z([\sigma])$, which turns out to be a non-boundary in the chain complex of a certain matching complex. As a consequence, the homology class of z must be an element of order three.

6 First Step

The first step of the construction is identical for all three cases. Recall that $\delta \geq 2$, and let α and β be any positive integers. Define $n = 3\delta + \alpha + \beta$.

For $1 \leq p \leq \delta - 1$, let C_p be the multiset consisting of $\delta - p$ copies of each of $3p-2$, $3p-1$, and $3p$. Moreover, let $D_p = \{i : 3p + \beta + 1 \leq i \leq 3\delta + \beta\}$. The multiset C_p and the set D_p both have size $3(\delta - p)$. The case $\delta = \alpha = \beta = 5$ and $n = 3\delta + \alpha + \beta = 25$ is illustrated in Table 1.

In all three cases, C_p will be a submultiset of A_p and D_p a subset of B_p. We will also construct an edge set σ and a cycle w of codegree t with properties as in Section 5. In each case, the following will hold.

(a) If i belongs to D_q (equivalently, $3q + \beta + 1 \leq i \leq 3\delta + \beta$), then i does not belong to A_q.
(b) If i belongs to C_p for some $p < q$ (equivalently, $1 \leq i \leq 3q - 3$), then i does not belong to A_q or B_q.
(c) No edge in the set σ is contained in the cycle w.

For $1 \leq p \leq \delta - 1$, define

$$\sigma_p^1 = \{i(i + 3k + \beta) : 3p - 2 \leq i \leq 3p, 1 \leq k \leq \delta - p\}.$$

Note that σ_p^1 constitutes a perfect matching between the multiset C_p and the set D_p

for each p. The set $\sigma^1 = \sigma^1_1 \cup \cdots \cup \sigma^1_{\delta-1}$ is a subset of the set σ to be constructed. Write $A'_p = A_p \setminus C_p$ and $B'_p = B_p \setminus D_p$.

Lemma 6.1 *Assuming* (a)–(c) *are true,* $\sigma^1 = \bigcup_{p=1}^{\delta-1} \sigma^1_p$ *is the unique partition* $\sigma^1 = \bigcup_{p=1}^{\delta-1} \tau_p$ *such that the link* $\mathrm{lk}_{\phi_{A_p,B_p}}([\tau_p])$ *is nonzero for all* p. *In particular,*

$$\mathrm{lk}_z([\sigma^1]) = \pm \bigwedge_{p=1}^{\delta-1} \phi_{A'_p,B'_p} \wedge w.$$

Proof Assume the opposite, and let $p \le \delta - 1$ be minimal such that some edge ij belongs to σ^1_p but not to τ_p; assume that $i < j$.

First, suppose that $ij \in \tau_q$ for some $q > p$. By properties of σ^1_p, we have that $i \le 3p$. Since $q > p$, this implies by (b) that $i \notin A_q \cup B_q$, which is a contradiction.

Next, suppose that $ij \in \tau_q$ for some $q < p$. By properties of σ^1_p, we have that

$$3p + \beta + 1 \le j \le 3\delta + \beta.$$

Since $q < p$, this implies by (a) that $j \notin A_q$, which yields that the total multiplicity of j in A_q and B_q is one. However, by minimality of p, τ_q contains σ^1_q, which implies that the vertex j already appears in an edge in τ_q. As a consequence, ij cannot belong to τ_q, as this would render $\mathrm{lk}_{\phi_{A_q,B_q}}([\tau_q])$ zero. This is another contradiction.

The last statement now follows from Lemma 3.5 and assumption (c) that no edge in σ is used in w. ∎

7 Second Step

Throughout this section, for $1 \le p \le \delta - 1$, we define

$$I_p = \{3p - 2\}, \quad J_p = \{3p - 1, 3p\}.$$

In all three cases, I_p is a subset of A'_p, whereas J_p is a subset of B'_p. Moreover, A'_p is an ordinary set in which no vertex has multiplicity exceeding one. In particular, there is no need to bother with multisets anymore.

Write

$$y = \bigwedge_{p=1}^{\delta-1} \phi_{A'_p,B'_p} \wedge w = \mathrm{lk}_z([\sigma^1]);$$

the second equality is by Lemma 6.1. In all three cases, we want to define a set σ^2 such that

$$\mathrm{lk}_y([\sigma^2]) = \pm \bigwedge_{p=1}^{\delta-1} \phi_{I_p,J_p} \wedge w.$$

Similarly to the situation for σ^1, the edges in σ^2 do not appear in w in any of the three cases. We will define the set σ as the union of σ^1 and σ^2.

Table 2: Definition of the sets E_p, F_p, G_p, H_p, I_p, J_p, and X in Case A for $\delta = \alpha = 5$ and $t = 7$. We have that $n = 25$ and $\ell = 3$. Each star denotes membership in C_p or D_p for some p; compare to Table 1.

1	2	3	4	5	6	7	8	9	10	11	12
*	*	*	E_1	E_1	E_1	E_1	E_1	*	*	*	*
*	*	*	*	*	*	E_2	E_2	E_2	E_2	E_2	*
*	*	*	*	*	*	*	*	*	E_3	E_3	E_3
*	*	*	*	*	*	*	*	*	*	*	*
I_1	J_1	J_1	I_2	J_2	J_2	I_3	J_3	J_3	I_4	J_4	J_4

13	14	15	16	17	18	19	20	21	22	23	24	25
*	*	*	*	*	*	*	*	F_1	F_1	F_1	F_1	F_1
*	*	*	*	*	*	*	*	F_2	F_2	F_2	F_2	F_2
E_3	E_3	*	*	*	*	*	*	F_3	F_3	F_3	F_3	F_3
G_4	G_4	G_4	E_4	E_4	*	*	*	F_4	F_4	G_4	G_4	G_4
H_4	H_4	H_4	X	X	X	X	X	X	X	H_4	H_4	H_4

7.1 Case A

In this case, $n \equiv \delta \pmod 2$. We need to a find a chessboard product of codegree t for each t satisfying $\delta + 2 \le t \le \frac{n+2}{3}$ and $t \equiv n \pmod 2$. We let $\alpha = \beta$; thus $n = 3\delta + 2\alpha$. The inequalities for t imply that $\alpha \ge 2$.

For integers a, b, c such that $b \le c$, we use the notation

$$a + [b, c] = [a + b, a + c] = \{a + b, a + b + 1, \dots, a + c\}.$$

We define this to be empty if $b > c$. The last α elements in the vertex set $[n] = [3\delta + 2\alpha]$ will play a special role in the construction; we define

$$F = 3\delta + \alpha + [1, \alpha].$$

For $1 \le p \le \delta - 2$, define

$$E_p = 3p + [1, \alpha], \quad F_p = F, \quad G_p = \varnothing, \quad H_p = \varnothing.$$

Moreover, define

$$\ell = \frac{n - 3t + 2}{2},$$

and let

$$L_1 = 3(\delta - 1) + [1, \ell], \quad L_2 = 3\delta + \alpha + [\alpha - \ell + 1, \alpha].$$

Define

$$E_{\delta-1} = 3(\delta - 1) + [\ell + 1, \alpha], \qquad F_{\delta-1} = 3\delta + \alpha + [1, \alpha - \ell],$$
$$G_{\delta-1} = L_1 \cup L_2, \qquad H_{\delta-1} = L_1 \cup L_2.$$

We let

$$A'_p = E_p \cup G_p \cup I_p, \quad B'_p = F_p \cup H_p \cup J_p.$$

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It is a straightforward exercise to show that the multiset $A_p = A'_p \cup C_p$ and the set $B_p = B'_p \cup D_p$ have the property that each $i \in [n]$ appears a total of either $\delta - 1$ or δ times in $A_1, \dots, A_{\delta-1}$ and $B_1, \dots, B_{\delta-1}$. Indeed, the elements in the set

$$X = 3(\delta - 1) + [\ell + 1, 3 + 2\alpha - \ell]$$

are exactly those elements that only appear $\delta - 1$ times. See Table 2 for an illustration.

Note that

$$|X| = 2\alpha + 3 - 2\ell = 3(t - \delta) + 1.$$

Writing $X = [a, b]$, we define

$$w = \phi_{\varnothing,\{a\}} \wedge \phi_{\{a+1\},\{a+2,a+3\}} \wedge \phi_{\{a+4\},\{a+5,a+6\}} \wedge \cdots \wedge \phi_{\{b-2\},\{b-1,b\}}.$$

We have that w is a chessboard product of codegree $t - \delta + 1$ in the chain complex of M_X.

For $1 \le p \le \delta - 2$, define

$$\sigma_p^{EF} = \{(3p + r)(n + 1 - r) : 1 \le r \le \alpha\}, \quad \sigma_p^{GH} = \varnothing.$$

Moreover, define

$$\sigma_{\delta-1}^{EF} = \{(3(\delta - 1) + r)(n + 1 - r) : \ell + 1 \le r \le \alpha\},$$
$$\sigma_{\delta-1}^{GH} = \{ii : i \in L_1 \cup L_2\}.$$

Each σ_p^{EF} is a perfect matching between E_p and F_p, and each σ_p^{GH} is a perfect matching between G_p and H_p. Write $\sigma_p^2 = \sigma_p^{EF} \cup \sigma_p^{GH}$ and $\sigma^2 = \bigcup_{p=1}^{\delta-1} \sigma_p^2$.

Lemma 7.1 *We have that $\sigma^2 = \bigcup_{p=1}^{\delta-1} \sigma_p^2$ is the unique partition $\sigma^2 = \bigcup_{p=1}^{\delta-1} \tau_p$ such that the link $\mathrm{lk}_{\phi_{A'_p,B'_p}}([\tau_p])$ is nonzero for all p. In particular,*

$$\mathrm{lk}_z([\sigma^1 \cup \sigma^2]) = \mathrm{lk}_y([\sigma^2]) = \pm \bigwedge_{p=1}^{\delta-1} \phi_{I_p,J_p} \wedge w.$$

Proof Assume the opposite; there is a partition $\sigma^2 = \bigcup_{q=1}^{\delta-1} \tau_q$ such that the link $\mathrm{lk}_{\phi_{A'_q,B'_q}}([\tau_q])$ is nonzero for all q and such that $\tau_p \ne \sigma_p^2$ for some p.

First, for $1 \le q \le \delta - 1$ and $1 \le k \le n$, we claim that there is at most one edge in τ_q containing the element k. Since $A'_q \cap B'_q$ is empty when $q \le \delta - 2$, the claim is true in this case. For the same reason, the loops in $\sigma_{\delta-1}^{GH}$ must be contained in $\tau_{\delta-1}$; hence the claim is true for $q = \delta - 1$ and $k \in L_1 \cup L_2$. For the remaining values of k, just observe that $A'_{\delta-1} \cap B'_{\delta-1} = L_1 \cup L_2$.

Most importantly, for $1 \le q \le \delta - 1$ and $k \in F$, there is exactly one edge in τ_q containing the vertex k; this is because σ^2 contains a total of $\delta - 1$ such edges.

Now, let $j \in F$ be minimal such that some edge ij containing j belongs to $\sigma_p^2 \setminus \tau_p$ for some p; choose p maximal with this property. We concluded above that the loops in $\sigma_{\delta-1}^{GH}$ all belong to $\tau_{\delta-1}$; hence we must have that $i \in E_p$ and $j \in F_p$.

Let q be such that $ij \in \tau_q$. For $q' > p$, we have that $\tau_{q'}$ contains the unique edge in $\sigma^2_{q'}$ that contains j; this is by maximality of p. In particular, $ij \notin \tau_{q'}$, which means that $q < p$.

Note that $3p + 1 \le i \le 3p + \alpha$. Writing $i = 3p + r$, we observe that $j = n + 1 - r$. If $i \le 3q + \alpha$, then σ^2_q contains the edge with endpoints

$$\begin{aligned} i &= 3p + r = 3q + (3p - 3q + r), \\ j' &= n + 1 - (3p - 3q + r) = j - 3(p - q). \end{aligned}$$

By minimality of j, we must have that ij' belongs to τ_q, which makes it impossible for ij to belong to τ_q. If $i > 3q + \alpha$, then i is not contained in $A'_q \cup B'_q$, which again makes it impossible for ij to belong to τ_q. In both cases, we obtain a contradiction; hence $\sigma^2_p = \tau_p$.

Since all edges ab in σ^2 have the property that $a = b$ or $|b - a| \ge 4$, no edges in σ^2 appear in the cycle w. As a consequence, we obtain the final statement of the lemma. ■

Lemma 7.2 *Let $n \equiv \delta \pmod 2$, and assume that*

$$\delta + 2 \le t \le \frac{n+2}{3} \quad \textit{and} \quad t \equiv n \pmod 2.$$

Then there is a cycle z of codegree t in the chain complex of BD^δ_n such that the homology class of z has order three.

Proof Let notation and assumptions be as above. Consider the cycle $z' = \mathrm{lk}_z([\sigma])$ in Lemma 7.1, where $\sigma = \sigma^1 \cup \sigma^2$; this is a chessboard product of codegree t. Note that each vertex appears in exactly $\delta - 1$ edges in σ. For vertices belonging to $L_1 \cup L_2$, one of these edges is a loop, which means that those vertices appear δ times in σ. In particular, we may view z' as a cycle in the chain complex of $\mathsf{M}_{[n]\setminus(L_1 \cup L_2)} \cong \mathsf{M}_{3t-2}$. By Proposition 3.2, the order of the homology class of z' is not one. By Lemma 3.5, this order divides the order of the homology class of z in the homology of BD^δ_n.

It remains to prove that the latter order divides three. For this, note that w is a chessboard product of codegree $t - \delta + 1$. Since $t - \delta + 1 \ge 3$, we may apply Lemma 3.4 to deduce that the homology class of z indeed divides three. ■

7.2 Case B

In this case, n is even and δ is odd. We need to a find a chessboard product of codegree t for each even t satisfying $\delta + 3 \le t \le \frac{n+2}{3}$. We let $\alpha = \beta - 1$; thus $n = 3\delta + 2\alpha + 1$. The inequalities for t imply that $\alpha \ge 3$.

We make small modifications to the construction in Case A, shifting all sets one step up. For example, $F = F_p$ was previously defined as $3\delta + \alpha + [1, \alpha]$ for $1 \le p \le \delta - 2$ and $3\delta + \alpha + [1, \alpha - \ell]$ for $p = \delta - 1$. This time, we define

$$F = F_p = 3\delta + \alpha + 1 + [1, \alpha]$$

Table 3: Definition of E_p, F_p, G_p, H_p, I_p, and J_p in Case B for $\delta = \alpha = 5$, and $t = 8$. We have that $n = 26$ and $\ell = 2$. Each star denotes membership in C_p or D_p for some p; compare to Table 1. Boxes denote positions yet to be filled.

1	2	3	4	5	6	7	8	9	10	11	12
$*$	$*$	$*$	$\square$	E_1	E_1	E_1	E_1	E_1	$*$	$*$	$*$
$*$	$*$	$*$	$*$	$*$	$*$	$\square$	E_2	E_2	E_2	E_2	E_2
$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$\square$	E_3	E_3
$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
I_1	J_1	J_1	I_2	J_2	J_2	I_3	J_3	J_3	I_4	J_4	J_4

13	14	15	16	17	18	19	20	21	22	23	24	25	26
$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	F_1	F_1	F_1	F_1	F_1
$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	F_2	F_2	F_2	F_2	F_2
E_3	E_3	E_3	$*$	$*$	$*$	$*$	$*$	$*$	F_3	F_3	F_3	F_3	F_3
$\square$	G_4	G_4	E_4	E_4	E_4	$*$	$*$	$*$	F_4	F_4	F_4	G_4	G_4
$\square$	H_4	H_4	$\square$	$\square$	$\square$	$\square$	$\square$	$\square$	$\square$	$\square$	$\square$	H_4	H_4

Table 4: A completed version of Table 3, including definitions of G_p', H_p', and X. As before, each star denotes membership in C_p or D_p for some p.

1	2	3	4	5	6	7	8	9	10	11	12
$*$	$*$	$*$	H_2'	E_1	E_1	E_1	E_1	E_1	$*$	$*$	$*$
$*$	$*$	$*$	$*$	$*$	$*$	G_2'	E_2	E_2	E_2	E_2	E_2
$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	H_4'	E_3	E_3
$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
I_1	J_1	J_1	I_2	J_2	J_2	I_3	J_3	J_3	I_4	J_4	J_4

13	14	15	16	17	18	19	20	21	22	23	24	25	26
$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	F_1	F_1	F_1	F_1	F_1
$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	F_2	F_2	F_2	F_2	F_2
E_3	E_3	E_3	$*$	$*$	$*$	$*$	$*$	$*$	F_3	F_3	F_3	F_3	F_3
G_4'	G_4	G_4	E_4	E_4	E_4	$*$	$*$	$*$	F_4	F_4	F_4	G_4	G_4
X	H_4	H_4	X	X	X	X	X	X	X	X	X	H_4	H_4

for $1 \le p \le \delta - 2$ and $F_{\delta-1} = 3\delta + \alpha + 1 + [1, \alpha - \ell]$. In the same manner, we shift the other sets E_p, G_p, H_p, L_1, and L_2 one step up; as before, $\ell = (n - 3t + 2)/2$. This shift leaves us with some gaps, marked with boxes in Table 3. Specifically, the vertices in the set $\{3p + 1 : 1 \le p \le \delta - 1\}$ appear fewer than δ times, as do the vertices in the set $3(\delta - 1) + 1 + [\ell + 1, 3 + 2\alpha - \ell]$. One vertex, $3(\delta - 1) + 1$, appears only $\delta - 2$ times.

We fill these gaps in the following manner. For odd p, define

$$G_p' = \varnothing, \quad H_p' = \varnothing.$$

For even p, define

$$G_p' = \{3p + 1\}, \quad H_p' = \{3p - 2\}.$$

For $1 \le p \le \delta - 1$, let

$$A_p' = E_p \cup G_p \cup G_p' \cup I_p, \quad B_p' = F_p \cup H_p \cup H_p' \cup J_p.$$

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Finally, define

$$X = \{3(\delta-1)+1\} \cup \big(3(\delta-1)+1+[\ell+1, 3+2\alpha-\ell]\big).$$

See Table 4 for an illustration.

Note that

$$|X| = 1+2\alpha-2\ell+3 = 3(t-\delta)+1.$$

Writing $X = \{3\delta-2\} \cup [a+1, b]$, we define

$$w = \phi_{\varnothing,\{3\delta-2\}} \wedge \phi_{\{a+1\},\{a+2,a+3\}} \wedge \phi_{\{a+4\},\{a+5,a+6\}} \wedge \cdots \wedge \phi_{\{b-2\},\{b-1,b\}}.$$

As before, w is a chessboard product of codegree $t-\delta+1$.

For $1 \le p \le \delta-2$, define

$$\sigma_p^{EF} = \{(3p+1+r)(n+1-r) : 1 \le r \le \alpha\},$$

$$\sigma_p^{GH} = \begin{cases} \{3p-2, 3p+1\} & \text{if } p \text{ is even,} \\ \varnothing & \text{if } p \text{ is odd.} \end{cases}$$

Moreover, define

$$\sigma_{\delta-1}^{EF} = \{(3(\delta-1)+1+r)(n+1-r) : \ell+1 \le r \le \alpha\},$$

$$\sigma_{\delta-1}^{GH} = \{ii : i \in L_1 \cup L_2\} \cup \{3(\delta-1)-2, 3(\delta-1)+1\}.$$

Each σ_p^{EF} is a perfect matching between E_p and F_p, and each σ_p^{GH} is a perfect matching between $G_p \cup G'_p$ and $H_p \cup H'_p$. Write

$$\sigma_p^2 = \sigma_p^{EF} \cup \sigma_p^{GH} \quad \text{and} \quad \sigma^2 = \bigcup_{p=1}^{\delta-1} \sigma_p^2.$$

Lemma 7.3 *We have that $\sigma^2 = \bigcup_{p=1}^{\delta-1} \sigma_p^2$ is the unique partition $\sigma^2 = \bigcup_{p=1}^{\delta-1} \tau_p$ such that the link $\mathrm{lk}_{\phi_{A'_p,B'_p}}([\tau_p])$ is nonzero for all p. In particular,*

$$\mathrm{lk}_z([\sigma^1 \cup \sigma^2]) = \mathrm{lk}_y([\sigma^2]) = \pm \bigwedge_{p=1}^{\delta-1} \phi_{I_p,J_p} \wedge w.$$

Proof We proceed as in the proof of Lemma 7.1, thus assuming the opposite. Look at the edges in σ_r^{GH} for even r. We have that $3r-2$ is contained in B'_q if and only if $q = r$, and $3r+1$ is not contained in any B'_q. Therefore, we must have that $(3r-2)(3r+1) \in \tau_r$. The remainder of the proof is identical to the proof of Lemma 7.1. Again, no edges in σ^2 appear in w, as every edge $ab \in \sigma^2$ satisfies $a = b$ or $|b-a| \ge 3$. ■

Lemma 7.4 *Let n be even and δ odd, and assume that*

$$\delta+3 \le t \le \frac{n+2}{3}$$

and t is even. Then there is a cycle z of codegree t in the chain complex of BD_n^δ such that the homology class of z has order three.

Proof The proof is exactly the same as that of Lemma 7.2, except that the first reference in the proof should be to Lemma 7.3 rather than to Lemma 7.1. ■

Table 5: Definition of the sets $E_p, F_p, G_p, G'_p, H_p, H'_p, I_p, J_p$, and X in Case C for $\delta = 6$, $\alpha = 5$, and $t = 8$. We have that $n = 29$ and $\ell = 3$.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$*$	$*$	$*$	H'_2	E_1	E_1	E_1	E_1	E_1	$*$	$*$	$*$	$*$	$*$	$*$
$*$	$*$	$*$	$*$	$*$	$*$	G'_2	E_2	E_2	E_2	E_2	E_2	$*$	$*$	$*$
$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	H'_4	E_3	E_3	E_3	E_3	E_3
$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	G'_4	E_4	E_4
$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
I_1	J_1	J_1	I_2	J_2	J_2	I_3	J_3	J_3	I_4	J_4	J_4	I_5	J_5	J_5

16	17	18	19	20	21	22	23	24	25	26	27	28	29
$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	F_1	F_1	F_1	F_1	F_1
$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	F_2	F_2	F_2	F_2	F_2
$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	F_3	F_3	F_3	F_3	F_3
E_4	E_4	E_4	$*$	$*$	$*$	$*$	$*$	$*$	F_4	F_4	F_4	F_4	F_4
G'_5	G_5	G_5	G_5	E_5	E_5	$*$	$*$	$*$	F_5	F_5	G_5	G_5	G_5
H'_5	H_5	H_5	H_5	X	X	X	X	X	X	X	H_5	H_5	H_5

7.3 Case C

In this final case, n is odd and δ is even. We need to a find a chessboard product of codegree t for each even t satisfying $\delta+2 \le t \le \frac{n+1}{3}$ (we cannot have $t = (n+2)/3$ if t is even and n is odd). Again, we let $\alpha = \beta - 1$; thus $n = 3\delta + 2\alpha + 1$. The inequalities for t imply that $\alpha \ge 2$.

This case is very similar to Case B. For $p < \delta - 1$, the sets E_p, F_p, G_p, G'_p, H_p, H'_p, σ_p^{EF}, and σ_p^{GH} are defined in exactly the same manner as in that case. The sets L_1, L_2, $E_{\delta-1}$, $F_{\delta-1}$, $G_{\delta-1}$, $H_{\delta-1}$, and $\sigma_{\delta-1}^{EF}$ are also defined as before, except that we now define

$$\ell = \frac{n - 3t + 1}{2}.$$

We do make one small modification, defining

$$G'_{\delta-1} = \{3(\delta-1)+1\}, \quad H'_{\delta-1} = \{3(\delta-1)+1\}.$$

We modify the set $\sigma_{\delta-1}^{GH}$ accordingly by setting

$$\sigma_{\delta-1}^{GH} = \{ii : i \in L_1 \cup L_2 \cup \{3(\delta-1)+1\}\}.$$

Let

$$A'_p = E_p \cup G_p \cup G'_p \cup I_p, \quad B'_p = F_p \cup H_p \cup H'_p \cup J_p.$$

Finally, define

$$X = 3(\delta-1) + 1 + [\ell+1, 3+2\alpha-\ell].$$

See Table 4 for an illustration.

Note that

$$|X| = 2\alpha - 2\ell + 3 = 3(t-\delta) + 1.$$

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Writing $X = [a, b]$, we define

$$w = \phi_{\varnothing,\{a\}} \wedge \phi_{\{a+1\},\{a+2,a+3\}} \wedge \phi_{\{a+4\},\{a+5,a+6\}} \wedge \cdots \wedge \phi_{\{b-2\},\{b-1,b\}}.$$

Again, w is a chessboard product of codegree $t - \delta + 1$.

Lemma 7.5 *We have that $\sigma^2 = \bigcup_{p=1}^{\delta-1} \sigma_p^2$ is the unique partition $\sigma^2 = \bigcup_{p=1}^{\delta-1} \tau_p$ such that the link $\mathrm{lk}_{\phi_{A'_p,B'_p}}([\tau_p])$ is nonzero for all p. In particular,*

$$\mathrm{lk}_z([\sigma^1 \cup \sigma^2]) = \mathrm{lk}_y([\sigma^2]) = \pm \bigwedge_{p=1}^{\delta-1} \phi_{I_p,J_p} \wedge w.$$

Proof Use exactly the same argument as in the proof of Lemma 7.3. ■

Lemma 7.6 *Let n be odd and δ even, and assume that*

$$\delta + 2 \le t \le \frac{n+1}{3}$$

and t is even. Then there is a cycle z of codegree t in the chain complex of BD_n^δ such that the homology class of z has order three.

Proof The proof is exactly the same as that of Lemma 7.2, except that the first reference in the proof should be to Lemma 7.5 rather than to Lemma 7.1. ■

7.4 Conclusion

Combining Lemmas 7.2, 7.4, and 7.6, and using the reformulation (4.2) in terms of codegree of the bounds (4.1), we obtain Theorem 1.1.

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Department of Mathematics, KTH Royal Institute of Technology, Stockholm, Sweden
e-mail: jakobj@math.kth.se