Pascal's Essais pour les Coniques.

By W. J. MACDONALD, M.A.

In 1640, when only 16 years of age, Pascal published a tract of a few pages with the above title. It contains only a few enunciations, and concludes with the statement that the author has several other theorems and problems, but that his inexperience, and the distrust he has of his own powers, do not allow him to publish them till they have been examined by competent judges. He afterwards wrote a complete work (*opus completum*) on the Conics, which was submitted to Leibnitz by M. Périer, Pascal's brother-in-law. Leibnitz recommended that it should be published; but this was not done, and we know its contents only from the analysis which Leibnitz sent back to M. Périer.

One feature which distinguishes Modern Geometry from the Ancient Geometry, is that a few propositions of great generality are proved, and from these a large number of others are deduced as corollaries. Now, Pascal's work presents this feature in a marked degree; for he takes up a single proposition—the well-known "Mystic Hexagram," as he called it—and from it deduces all his others, four hundred corollaries, we are told. It has been suggested that the proposition is really due to Desargues; but he himself speaks of a proposition as Pascal's which can be none other than this.

I propose now to give an account of the contents of the earlier work, modernizing the enunciations, and supplying demonstrations on the lines on which I imagine Pascal himself worked.

We have first a definition equivalent to that of *concurrent lines* (parallel lines are included), and then a *conic* is defined to include the *circle, parabola, ellipse, hyperbola,* and *a pair of intersecting lines.*

Then comes Lemma I, which is the Hexagram for the circle: If a hexagon* be inscribed in a circle the three points in which the pairs of opposite sides intersect are collinear. (Fig. 18)

* By a hexagon is to be understood the figure formed by joining consecutively any six points on the circumference of the circle. Sixty different figures are possible according to the order in which the points are joined.

In ΔPQR, with transversal* BC, $\frac{PC}{CQ} \cdot \frac{QY}{YR} \cdot \frac{RB}{BP} = -1$;

" " " DE, $\frac{PD}{DQ} \cdot \frac{QE}{ER} \cdot \frac{RX}{XP} = -1$;

" " " FA, $\frac{PZ}{ZQ} \cdot \frac{QF}{FR} \cdot \frac{RA}{AP} = -1$;

$$\therefore \quad \underbrace{\frac{PZ}{ZQ} \cdot \frac{QY}{YR} \cdot \frac{RX}{XP}}_{\alpha} \cdot \underbrace{\frac{PC \cdot PD \cdot QE \cdot QF \cdot RA \cdot RB}{PA \cdot PB \cdot QC \cdot QD \cdot RE \cdot RF}}_{\beta} = -1$$

But $\beta = +1$ (Eucl. III. 36); $\therefore \quad \alpha = -1$;

$\therefore$ X, Y, Z are collinear.

This proof holds for every disposition of the points A,B,C,D,E,F, provided they are joined in alphabetical order.

Lemma II. states that lines which are concurrent in any plane are concurrent when projected into any other. From this follows at once Lemma III., in which the circle of Lemma I. is replaced by any conic. Conversely also: If the intersections of the pairs of opposite sides of a hexagon are collinear, the vertices of the hexagon lie on a conic.

Since a conic is determined by five points, and this proposition gives us a relation between these and any sixth point on the conic, it is clear that the proposition is one of great generality, and admits of many corollaries, some of which I shall now indicate.

It enables us to construct a conic, five points on it being given.

Let 1, 2, 3, 4, and 5 be the given points.

Through 1 draw any line meeting 34 in X; join 12 and 45 meeting in Y; join 23 and XY meeting in Z; join Z5 meeting X1 in 6. Then 6 is a point in the conic.

By drawing different lines through 1 any number of points on the conic can thus be determined.

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* Both Pascal and Desargues appear to have made much use of the propositions known as Menelaus' and Ceva's Theorems (Fig. 17),

that if $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = -1$,

D, E, and F are collinear; and conversely;

and if $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$,

AD, BE, and CF are concurrent; and conversely;

and these I shall assume as known.

Let 1 and 6 coincide, then 16 becomes the tangent at that point. (Fig. 19.)

This gives the proposition : If a pentagon* be inscribed in a conic, the points of intersection of the first and fourth sides, and of the second and fifth sides, are collinear with the point in which the third side meets the tangent at the opposite vertex.

From this we obtain the solution of the problem : To draw a tangent to a conic from a point on it, by the ruler alone.

Take any other points 2, 3, 4, and 5 on the conic. Join 12 and 45 meeting in X ; join 23 and 51 meeting in Y ; join XY and 43 meeting in Z.
Join Z1, which is the required tangent.

The problem in construction which this case solves is : To construct a conic, having given three points on the conic and a tangent with its point of contact. The construction is similar to that given for five points.

The hexagon may be reduced to a quadrilateral in two ways :
1. By considering two adjacent vertices of the quadrilateral each to contain two consecutive vertices of the hexagon.
2. By so considering two opposite vertices.
These give the two propositions :
1. The tangents at two adjacent vertices of a quadrilateral inscribed in a conic, meet on the line joining the intersection of the diagonals with the intersection of the pair of opposite sides which pass through the vertices.
2. The tangents at two opposite vertices of a quadrilateral inscribed in a conic, meet on the line joining the intersections of pairs of opposite sides.

[These are really the same propositions, and differ only in the order in which the points are supposed to be joined.]

The problem corresponding to this case is : Given a pair of tangents with their points of contact, and any other point on the conic, to construct it.

This case implicitly contains the Theory of Pole and Polar ; and it is the opinion of both Chasles and Poncelet that Pascal had developed the equivalent of that Theory in Book III. of the *Opus Completum.* The title of that Book is *De quatuor tangentibus, et rectis*

* Pentagon here has the same extended meaning as hexagon.

puncta tactuum jungentibus, unde rectarum harmonice sectarum et diametrorum proprietates oriuntur; and Leibnitz expressly states that it was founded on the properties of the hexagram.

If now three pairs of points coincide we have this proposition. The points in which the sides of a triangle inscribed in a conic meet the tangents at the opposite vertices are collinear.

Lemma III. also implicitly contains Carnot's Theorem. For if X,Y,Z be collinear, (Fig. 18)

$$\frac{PZ}{ZQ} \cdot \frac{QY}{YR} \cdot \frac{RX}{XP} = -1;$$

$$\therefore \quad \frac{PC \cdot CD \cdot QE \cdot EF \cdot RA \cdot RB}{PA \cdot PB \cdot QC \cdot QD \cdot RE \cdot RF} = 1,$$

which is Carnot's Theorem.

This is another proposition of great generality concerning points on a conic. Pascal apparently knew it, for, as we shall see immediately, he extended it to eight points.

To return to the *Essais.*

In Fig. 20 he says

$$\frac{PM}{MA} \cdot \frac{AS}{SQ} = \frac{PL}{LA} \cdot \frac{AT}{TQ}.$$

Because PKNOVQ is a hexagon, QP, ON, and MS are concurrent in X_1 (say).

Because PKONVQ is a hexagon, QP, ON, and LT are concurrent in X_2 (say).

But two lines in each set are the same;

$\therefore$ X_1 and X_2 are the same point (say) X.

Hence in ΔAPQ,

with transversal SMX, $\dfrac{PM}{MA} \cdot \dfrac{AS}{SQ} \cdot \dfrac{QX}{XP} = -1$;

with transversal TLX, $\dfrac{PL}{LA} \cdot \dfrac{AT}{TQ} \cdot \dfrac{QX}{XP} = -1$;

$$\therefore \quad \frac{PM}{MA} \cdot \frac{AS}{SQ} = \frac{PL}{LA} \cdot \frac{AT}{TQ}.$$

The next proposition (Fig. 21) is equivalent to this: If from a point there be drawn three lines to cut the sides of an angle the anharmonic ratio of the segments made on one side is equal to that of the segments made on the other.

In Δ ABE, with the transversals DH and CH,

$$\frac{AD}{DB}\cdot\frac{BH}{HE}\cdot\frac{EG}{GA}=-1=\frac{AC}{CB}\cdot\frac{BH}{HE}\cdot\frac{EF}{FA};$$

$$\therefore \qquad \frac{AD.BC}{AC.BD}=\frac{AG.EF}{AF.EG}.$$

Then follows the extension of Carnot's Theorem to the quadrilateral, to which we have already referred.

If (Fig. 22) the sides of the quadrilateral ACLH be cut by a conic,

AB.AE.CP.CR.HF.HK.LM.LO = AF.AK.CB.CE.HO.HM.LR.LP.

Apply Carnot's Theorem successively to the Δs ACG and LHG, and we have

1. AB.AE.CP.CR.GF.GK = AF.AK.CB.CE.GP.GR;
2. LM.LO.HF.HK.GP.GR = LR.LP.GF.GK.HM.HO.

From the multiplication of these the proposition follows.

A particular case of this theorem gives us an important property of conics, that if through any point a pair of secants to a conic be drawn parallel to two fixed directions, the rectangles under their segments are in a fixed ratio.

For if A and L are at infinity, $\dfrac{CP.CR}{CB.CE}=\dfrac{HO.HM}{HF.HK}$.

Next we have Desargues' proposition: That any transversal is cut by a conic, and the sides of an inscribed quadrilateral in six points which are in involution.*

In Fig. 23, apply Carnot's Theorem to ΔAA′F and the conic; then AB.AB′.A′L.A′M.FR.FS = AS.AR.FM.FL.A′B′.A′B.

Apply the same Theorem to ΔAA′F and the pair of lines LS, RM; then AS.AR.FM.FL.A′C′.A′C = AC.AC′.A′L.A′M.FR.FS.

Multiply and suppress common factors;

$$\therefore \qquad \frac{AB.AB'}{A'B'.A'B}=\frac{AC.AC'}{A'C'.A'C}.$$

He then suggests one or two problems, among which is: To draw a pair of tangents to a conic from an external point. His solution was, no doubt, that which depends on the polar properties of the complete quadrilateral, as we have seen that he probably knew these.

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* Six points A, A′, B, B′, C, C′ are in involution, if $\dfrac{AB.AB'}{A'B'.A'B}=\dfrac{AC.AC'}{A'C'.A'C}$.

In a communication dated 1654, and addressed to a society of savants, which in 1666 became the Academy of Sciences, Pascal stated that he had written a complete treatise on the conics, founded mainly on a single proposition. This work, as we have already stated, was, after its author's death, sent for examination to Leibnitz; and though it has been lost, we have the analysis of it which Leibnitz made for M. Périer, Pascal's brother-in-law. In spirit and method it anticipates the Modern Geometry of our century, and entitles Pascal to the credit of having been one of its founders.

Sixth Meeting, April 10*th*, 1884.

THOMAS MUIR, Esq., M.A., F.R.S.E., President, in the Chair.

On the Teaching of Elementary Geometry.

By A. J. G. BARCLAY, M.A.

[*Abstract.*]

This paper was prepared at the suggestion of the committee as the first of a series on the teaching of elementary mathematics, in the belief that an occasional paper of this nature, with discussions, would be useful.

In the introduction it was suggested that, as secondary education in this country was apparently on the eve of considerable changes, the present was an opportune time for discussing the whole subject of school mathematics; and also that the Society should be prepared to form a scheme of a mathematical course for both teaching and examination purposes.

The following points were specially referred to:

(1) That the most suitable time for a pupil to commence geometry is about the age of twelve. (2) That the introduction to the subject should be made with the usual definitions, along with numerous exercises in the making and naming of figures; this, rather than the course of geometrical drawing, unaccompanied by definitions, suggested by the Society for the Improvement of Geometrical Teach-

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