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INFINITARY COMMUTATIVITY AND ABELIANIZATION IN FUNDAMENTAL GROUPS

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Abstract

Infinite product operations are at the forefront of the study of homotopy groups of Peano continua and other locally path-connected spaces. In this paper, we define what it means for a space X to have infinitely commutative π_1-operations at a point $x \in X$. Using a characterization in terms of the Specker group, we identify several natural situations in which this property arises. Maintaining a topological viewpoint, we define the transfinite abelianization of a fundamental group at any set of points $A \subseteq X$ in a way that refines and extends previous work on the subject.

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1. Introduction

Infinitary operations akin to infinite sums and products in analysis arise naturally in the context of fundamental groups and are often highly noncommutative. Such operations arise from the ability to form an infinite loop concatenation $\prod_{n=1}^{\infty} \alpha_n$ (with order type ω) and a transfinite loop concatenation $\prod_{\tau} \alpha_n$ (with dense order type) from a shrinking sequence of loops $\{\alpha_n\}$ based at a point. Even if the space in question has an abelian fundamental group, infinite permutation of the factors α_n may change the homotopy class of these loop products. In this paper, we define and study a notion of infinite commutativity for fundamental groups. In particular, we define a space X to be transfinitely π_1-commutative at a point $x \in X$ if the infinite permutation of the factors of transfinite loop concatenations based at x is a homotopy-invariant action.

Computations of singular homology groups of spaces such as the Hawaiian earring [15] and other 'wild' spaces that admit nontrivial infinitary π_1-operations [11,12, 18, 20] typically depend on the decomposition theory of infinite abelian groups (cf. [16]) since $H_1(X)$ is only the 'finitary', that is, ordinary abelianization of the group $\pi_1(X, x)$. The notion of 'strong' or 'infinite' abelianization of groups with natural infinite product operations is not new but has previously been considered in either a purely algebraic context or in more specialized situations. For instance, Cannon and Conner defined the strong abelianization of subgroups of the Hawaiian earring group [5, Section 4]. We define a notion of *infinitary abelianization* (Definition 5.1), which may be considered a topological refinement of the group-theoretic notion of σ-*abelianization* introduced in [9]. Since it is possible for a fundamental group $\pi_1(X, x)$ to be transfinitely π_1-commutative only at points within a proper subset of X, we define our notion of infinitary abelianization relative to a subset $A \subseteq X$. Our approach also refines the notion of strong abelianization defined in [8], which is defined using the quotient topology on π_1.

Remarkably, the difference between finite and infinite commutativity seems to disappear in higher dimensions. The work of Eda–Kawamura [14] on the higher homotopy groups of the n-dimensional Hawaiian earrings ($n \geq 2$) suggests that higher homotopy groups are always 'infinitely commutative'. We provide a formalization of this idea by observing that loop spaces satisfy our definition of transfinite π_1-commutativity at constant loops. In forthcoming work, the authors will show that all topological monoids (and slightly more general objects called pre-Δ-monoids) are transfinitely π_1-commutative at their identity elements. In particular, the transfinitely π_1-commutative property will play a key role in computations of fundamental groups of James reduced product spaces and similar constructions.

The remainder of this paper is structured as follows. In Section 2, we set notation and review the relevant theory of the Hawaiian earring group. In Section 3, we define the property of a space X being *transfinitely* π_1-*commutative* at a point $x \in X$ (Definition 3.2). Our main result in this section is Theorem 3.5, which characterizes this property in terms of canonical factorizations through the Specker group $\mathbb{Z}^{\mathbb{N}}$. In Section 4, we identify several natural examples and situations where the transfinitely π_1-commutative property holds. In Section 5, we define and study the infinitary commutator subgroup $C_\tau(A)$ and infinitary abelianization $\mathfrak{H}_A(X) = \pi_1(X, x)/C_\tau(A)$ of $\pi_1(X, x)$ at a subset $A \subseteq X$. In the case $A = X$, we write $\mathfrak{H}(X)$ for $\mathfrak{H}_X(X)$ and compare this group to the alternative functorial constructions in the literature, namely Eda's σ-commutator and Corson's 'strong abelianization' using the quotient topology on π_1 [8]. To illustrate the accessibility of $\mathfrak{H}_A(X)$, we identify the isomorphism type of this group for several important examples including the Hawaiian earring (Corollary 5.13), the double Hawaiian earring (Example 5.18), the harmonic archipelago (Example 5.15), and the Griffiths twin cone (Example 5.22). To analyze these examples, we develop the basic theory of the functor $\mathfrak{H}_A(X)$, including a van Kampen-type result (Theorem 5.17). It is well known that the first singular homology group of a one-point union $X \vee Y$ need not be isomorphic to $H_1(X) \oplus H_1(Y)$. We show that such an isomorphism $\mathfrak{H}(X \vee Y) \cong \mathfrak{H}(X) \oplus \mathfrak{H}(Y)$ does hold for arbitrary path-connected spaces X and Y (Theorem 5.19).

2. Preliminaries and notation

For spaces X, Y, let Y^X denote the space of maps $f : X \to Y$ with the compact-open topology. If $A \subseteq X$, $B \subseteq Y$, then $(Y, B)^{(X,A)} \subseteq Y^X$ will denote the subspace of relative maps satisfying $f(A) \subseteq B$. If $A = \{x_0\}$ and $B = \{y_0\}$ contain only basepoints, we may simply write $(Y, y_0)^{(X,x_0)}$. The constant function $X \to Y$ at y_0 is denoted c_{y_0}. We write $\Omega(X, x_0)$ for the based loop space $(X, x_0)^{(I,\partial I)}$, where $I = [0, 1]$ is the closed unit interval. If $f : (X, x) \to (Y, y)$ is a based map, then $f_\# : \pi_1(X, x) \to \pi_1(Y, y)$ denotes the homomorphism induced on the fundamental group.

DEFINITION 2.1. A sequence $\{f_n\}_{n\in\mathbb{N}}$ in Y^X is *null at* $y \in Y$ if for every open neighborhood U of y, there is an $N \in \mathbb{N}$ such that $\mathrm{Im}\,(f_n) \subseteq U$ for all $n \geq N$, that is, if $\{f_n\} \to c_y$ in Y^X. We refer to $\{f_n\}$ as a *null sequence*.

If $\alpha_1, \alpha_2, \ldots, \alpha_n$ is a sequence of paths, that is, continuous functions $I \to X$, satisfying $\alpha_i(1) = \alpha_{i+1}(0)$, we write $\prod_{i=1}^{n} \alpha_i$ or $\alpha_1 \cdot \alpha_2 \cdot \cdots \cdot \alpha_n$ for the n-fold *concatenation* defined to be α_i on the interval $[(i-1)/n, i/n]$. We write $\alpha^-(t) = \alpha(1-t)$ for the *reverse* of a path α. If $[a, b], [c, d] \subseteq I$ and $\alpha : [a, b] \to X$, $\beta : [c, d] \to X$ are maps, we write $\alpha \equiv \beta$ if $\alpha = \beta \circ \lambda$ for some increasing homeomorphism $\lambda : [a, b] \to [c, d]$. Throughout this paper, any space in which paths are considered is assumed to be path connected.

Much of our work will require the use of the fundamental group of the Hawaiian earring, which we recall here. Let $C_n \subset \mathbb{R}^2$ denote the circle of radius $1/n$ centered at $(1/n, 0)$ and $\mathbb{H} = \bigcup_{n\in\mathbb{N}} C_n$ be the *Hawaiian earring* with basepoint $b_0 = (0, 0)$. We define some important loops in $\mathbb{H}$ as follows.

- For each $n \in \mathbb{N}$, let $\ell_n \in \Omega(C_n, b_0)$ be the canonical counterclockwise loop traversing the circle C_n.
- Let $\ell_\infty \in \Omega(\mathbb{H}, b_0)$ denote the loop defined as ℓ_n on the interval $[(n-1)/n, n/(n+1)]$ and $\ell_\infty(1) = b_0$.
- Let $C \subseteq I$ be the middle third Cantor set. Write $I \backslash C = \bigcup_{n\geq 1} \bigcup_{k=1}^{2^{n-1}} I_n^k$, where I_n^k is an open interval of length $1/3^n$ and, for fixed n, the sets I_n^k are indexed by their natural ordering in I. Let $\ell_\tau \in \Omega(\mathbb{H}, b_0)$ be the loop defined so that $\ell_\tau(C) = b_0$ and $\ell_\tau := \ell_{2^{n-1}+k-1}$ on $\overline{I_n^k}$ (see Figure 1).

The fundamental group $\pi_1(\mathbb{H}, b_0)$ is uncountable and is not isomorphic to a free group. However, $\pi_1(\mathbb{H}, b_0)$ is locally free and naturally isomorphic to a subgroup of an inverse limit of free groups. Let $\mathbb{H}_{\geq n} = \bigcup_{m\geq n} C_m$ be the smaller homeomorphic copies of $\mathbb{H}$ and let $\mathbb{H}_{\leq n} = \bigcup_{m=1}^{n} C_n$ be the wedge of the first n-circles so that $\pi_1(\mathbb{H}_{\leq n}, b_0) = F_n$

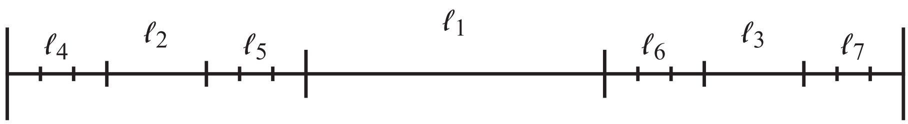

FIGURE 1. The transfinite concatenation loop ℓ_τ.

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is the group freely generated by the elements $[\ell_1], [\ell_2], \dots, [\ell_n]$. The retractions $r_{n+1,n} : \mathbb{H}_{\leq n+1} \to \mathbb{H}_{\leq n}$ collapsing C_{n+1} to b_0 induce an inverse sequence

$$\cdots \to F_{n+1} \to F_n \to \cdots \to F_2 \to F_1$$

on fundamental groups, in which $F_{n+1} \to F_n$ deletes the letter $[\ell_{n+1}]$ from a given word. The inverse limit $\check{\pi}_1(\mathbb{H}, b_0) = \varprojlim_n F_n$ is the first shape homotopy group [22]. The retractions $r_n : \mathbb{H} \to \mathbb{H}_{\leq n}$, which collapse $\mathbb{H}_{\geq n+1}$ to b_0, induce a canonical homomorphism

$$\psi : \pi_1(\mathbb{H}, b_0) \to \check{\pi}_1(\mathbb{H}, b_0), \quad \text{where} \quad \psi([\alpha]) = ([r_1 \circ \alpha], [r_2 \circ \alpha], \dots).$$

It is known that ψ is injective [23]; see also [13]. Thus, a homotopy class $[\alpha] \in \pi_1(\mathbb{H}, b_0)$ is trivial if and only if for every $n \in \mathbb{N}$, the projection $(r_n)_\#([\alpha]) \in F_n$ reduces to the trivial word as a word in the letters $[\ell_1], [\ell_2], \dots, [\ell_n]$. Based on the injectivity of ψ, we also note that for every $n \geq 2$, the inclusions $\mathbb{H}_{\leq n} \to \mathbb{H}$ and $\mathbb{H}_{\geq n+1} \to \mathbb{H}$ induce an isomorphism $F_n * \pi_1(\mathbb{H}_{\geq n+1}, b_0) \to \pi_1(\mathbb{H}, b_0)$ on the free product. Therefore, every nontrivial $[\alpha] \in \pi_1(\mathbb{H}, b_0)$ factors uniquely as a finite product of homotopy classes of loops alternating between $\mathbb{H}_{\leq n}$ and $\mathbb{H}_{\geq n+1}$.

REMARK 2.2. If $f : (\mathbb{H}, b_0) \to (X, x)$ is a map, then the sequence $\{f \circ \ell_n\}$ is null at x. Conversely, if $\{\alpha_n\}$ is a null sequence of loops based at x, then we may define a continuous map $f : \mathbb{H} \to X$ by $f \circ \ell_n = \alpha_n$. Hence, null sequences of loops based at $x \in X$ are in bijective correspondence with maps $(\mathbb{H}, b_0) \to (X, x)$.

DEFINITION 2.3. Suppose that $\alpha_n \in \Omega(X, x)$ is a null sequence and $f : \mathbb{H} \to X$ is the map with $f \circ \ell_n = \alpha_n$.

- We write $\prod_{n=1}^{\infty} \alpha_n$ for the loop $f \circ \ell_\infty$, which we call the *infinite concatenation* of the sequence $\{\alpha_n\}$.
- We write $\prod_\tau \alpha_n$ for the loop $f \circ \ell_\tau$, which we call the *transfinite concatenation* of the sequence $\{\alpha_n\}$.

Since ℓ_∞ and ℓ_τ are fixed, permuting the terms of a null sequence $\{\alpha_n\}$ will generally change the homotopy class of both $\prod_{n=1}^{\infty} \alpha_n$ and $\prod_\tau \alpha_n$ even if $\pi_1(X, x)$ is abelian.

3. Defining and characterizing transfinite π_1-commutativity

Let $\mathbb{T} = \prod_{n \in \mathbb{N}} S^1$ be the infinite torus. Using the canonical embedding $\eta : \mathbb{H} \to \mathbb{T}$ onto the subspace $\bigvee_{n=1}^{\infty} S^1$ of $\mathbb{T}$, we may identify $\mathbb{H}$ as a subspace of $\mathbb{T}$ so that b_0 is the distinguished point of $\mathbb{T}$. The fundamental group $\pi_1(\mathbb{T}, b_0)$ is isomorphic to the Specker group $\mathbb{Z}^{\mathbb{N}}$, where $[\eta \circ \ell_n]$ is identified with the unit vector $\mathbf{e}_n \in \mathbb{Z}^{\mathbb{N}}$, which has 1 in the nth coordinate and 0's elsewhere. It is well known that the induced homomorphism $\eta_\# : \pi_1(\mathbb{H}, b_0) \to \pi_1(\mathbb{T}, b_0)$ is surjective and $[\alpha] \in \ker(\eta_\#)$ if and only if α has winding number 0 around C_n for all $n \in \mathbb{N}$. In [5, 9], $\pi_1(\mathbb{T}, b_0) \cong \mathbb{Z}^{\mathbb{N}}$ is described as an 'infinite abelianization' of $\pi_1(\mathbb{H}, b_0)$. To put our notion of transfinite

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commutativity into context, we first recall the following purely group-theoretic notions due to Eda.

DEFINITION 3.1 [9]. The *σ-abelianization* of a group G is the quotient $G/G^{\sigma'}$, where $G^{\sigma'}$ is the normal subgroup of G generated by the images $f(\ker(\eta_{\#}))$ for all homomorphisms $f : \pi_1(\mathbb{H}, b_0) \to G$. We refer to $G^{\sigma'}$ as the *σ-commutator subgroup* and we define a group G to be *σ-commutative* if and only if $G^{\sigma'} = 1$.

Clearly, the σ-commutative property is equivalent to the property that every homomorphism $f : \pi_1(\mathbb{H}, b_0) \to G$ factors as $f = g \circ \eta_{\#}$ for some homomorphism $g : \pi_1(\mathbb{T}, b_0) \to G$. While certainly relevant for group-theoretic purposes, when applied to fundamental groups $G = \pi_1(X, x)$, this purely algebraic property may fail to relate to relevant homotopy-invariant properties of X. Our approach to infinite commutativity will also be founded upon the difference between the highly noncommutative group $\pi_1(\mathbb{H}, b_0)$ and the highly commutative group $\mathbb{Z}^{\mathbb{N}}$. However, to maintain a topological viewpoint, we work at the level of loops and consider only spatial, that is, induced, homomorphisms $f_{\#} : \pi_1(\mathbb{H}, b_0) \to \pi_1(X, x)$.

DEFINITION 3.2. A space X is:

(1) *infinitely π_1-commutative at $x \in X$* if for every null sequence $\alpha_n \in \Omega(X, x)$ and bijection $\phi : \mathbb{N} \to \mathbb{N}$, we have $[\prod_{n=1}^{\infty} \alpha_n] = [\prod_{n=1}^{\infty} \alpha_{\phi(n)}]$ in $\pi_1(X, x)$;
(2) *transfinitely π_1-commutative at $x \in X$* if for every null sequence $\alpha_n \in \Omega(X, x)$ and bijection $\phi : \mathbb{N} \to \mathbb{N}$, we have $[\prod_{\tau} \alpha_n] = [\prod_{\tau} \alpha_{\phi(n)}]$ in $\pi_1(X, x)$.

A convergent series $\sum_{n=1}^{\infty} a_n$ of real numbers is absolutely convergent if and only if for every bijection $\phi : \mathbb{N} \to \mathbb{N}$, we have $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\phi(n)}$. In this sense, the property of being infinitely π_1-commutative states that all infinite products of loops are 'absolutely convergent' up to homotopy. The transfinitely π_1-commutative property extends this invariance to products indexed by countable linear orders other than the naturals.

REMARK 3.3. If X is infinitely or transfinitely π_1-commutative at any point $x \in X$, then $\pi_1(X, x)$ is commutative in the usual sense. For example, if X is transfinitely π_1-commutative at x and $\alpha, \beta \in \Omega(X, x)$, we may define $f : \mathbb{H} \to X$ by $f \circ \ell_1 = \alpha$, $f \circ \ell_3 = \beta$, and $f \circ \ell_n = c_x$ if $n \notin \{1, 3\}$. Any bijection ϕ that permutes 1 and 3 gives $[\alpha \cdot \beta] = [\prod_{\tau} f \circ \ell_n] = [\prod_{\tau} f \circ \ell_{\phi(n)}] = [\beta \cdot \alpha]$. Any bijection ϕ that permutes 1 and 2 gives the analogous equality for infinite concatenations. In particular, this means that any space with a noncommutative fundamental group cannot be transfinitely π_1-commutative at any point. Outside of considering local variants of these two properties, the transfinitely π_1-commutative property is only intended to be considered in the context of spaces with abelian fundamental groups, including topological monoids, groups, and other H-spaces.

The proof of the following lemma is sketched in [4, Example 3.10]. Since it is important for the proof of Theorem 3.5, we give a more detailed proof. In short, it states

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that any nontrivial element $[\alpha] \in \pi_1(\mathbb{H}, b_0)$, where α has winding number 0 around C_n for all $n \in \mathbb{N}$, may be factored, up to homotopy, as an infinite concatenation of a null sequence of commutators.

LEMMA 3.4. *If* $1 \neq [\alpha] \in \ker(\eta_\#)$, *then there is a null sequence* $\gamma_n \in \Omega(\mathbb{H}, b_0)$ *such that* $[\alpha] = [\prod_{n=1}^{\infty} \gamma_n]$ *and* $\gamma_n \equiv \prod_{i=1}^{k_n} (\alpha_{n,i} \cdot \beta_{n,i} \cdot \alpha_{n,i}^{-} \cdot \beta_{n,i}^{-})$ *for loops* $\alpha_{n,i}, \beta_{n,i} \in \Omega(\mathbb{H}_{\geq n}, b_0)$, $1 \leq i \leq k_n$.

PROOF. Set $G = \pi_1(\mathbb{H}, b_0)$, $G_n = \pi_1(\mathbb{H}_{\geq n}, b_0)$, and let $[K, K]$ denote the commutator subgroup of a group K. Consider a non-null-homotopic loop $\alpha \in \Omega(\mathbb{H}, b_0)$ such that $[\alpha] \in \ker(\eta_\#)$. Set $\gamma_0 = c_{b_0}$ and $\beta_0 = \alpha$ so that $[\alpha] = [\gamma_0][\beta_0]$ and let $\mathbb{H}_{\geq 0} = \mathbb{H}$. Proceeding by induction, suppose that we have constructed loops $\gamma_i \in \Omega(\mathbb{H}_{\geq i}, b_0)$, $0 \leq i \leq n-1$, and $\beta_{n-1} \in \Omega(\mathbb{H}_{\geq n}, b_0)$ such that:

- $[\alpha] = (\prod_{i=0}^{n-1} [\gamma_i])[\beta_{n-1}]$;
- $[\gamma_i] \in [G_i, G_i]$ for $0 \leq i \leq n-1$;
- $[\beta_{n-1}] \in \ker(\eta_\#)$.

Since $[\beta_{n-1}] \in \pi_1(\mathbb{H}_{\geq n}, b_0) \cap \ker(\eta_\#)$, we may factorize $[\beta_{n-1}]$ as $\prod_{j=1}^{m}([\delta_j][\ell_n]^{\epsilon_j})$, where the loop δ_j has image in $\mathbb{H}_{\geq n+1}$ and $\sum_{j=1}^{m} \epsilon_j = 0$. Since β_{n-1} and $\beta_n = \prod_{j=1}^{m} \delta_j$ are homologous in $\mathbb{H}_{\geq n}$, there is a loop $\gamma_n \in \Omega(\mathbb{H}_{\geq n}, b_0)$ such that $[\gamma_n] \in [G_n, G_n]$ and $[\beta_{n-1}] = [\gamma_n][\beta_n]$. Since $[\beta_{n-1}] \in \ker(\eta_\#)$ and $[\gamma_n] \in [G_n, G_n] \leq [G, G] \leq \ker(\eta_\#)$, we have $[\beta_n] \in \ker(\eta_\#)$. This completes the induction.

The induction provides null sequences $\{\beta_n\}$ and $\{\gamma_n\}$ such that $[\gamma_n] \in [G_n, G_n]$ and $[\alpha] = (\prod_{i=1}^{n} [\gamma_i])[\beta_n]$ for all $n \in \mathbb{N}$. Notice that $[\prod_{n=1}^{\infty} \gamma_n]^{-1}[\alpha]$ is represented by each loop in the null sequence $\{(\prod_{i=n+1}^{\infty} \gamma_i)^{-} \cdot \beta_n\}_{n \in \mathbb{N}}$. Thus, since $\pi_1(\mathbb{H}, b_0)$ canonically injects into $\varprojlim_n F_n$ (or, more precisely, since $\mathbb{H}$ has the much weaker property of being homotopically Hausdorff at b_0 [7]), we have $[\prod_{n=1}^{\infty} \gamma_n]^{-1}[\alpha] = 1$ in $\pi_1(\mathbb{H}, b_0)$. We conclude that $[\alpha] = [\prod_{n=1}^{\infty} \gamma_n]$, where γ_n has the form described in the statement of the lemma. □

THEOREM 3.5. *For any space* X *and* $x \in X$, *the following are equivalent:*

(1) *for every map* $f : (\mathbb{H}, b_0) \to (X, x)$, *there exists a homomorphism* $g : \pi_1(\mathbb{T}, b_0) \to \pi_1(X, x)$ *such that* $f_\# = g \circ \eta_\#$;
(2) X *is transfinitely* π_1*-commutative at* $x \in X$;
(3) X *is infinitely* π_1*-commutative at* $x \in X$.

PROOF. (1) ⇒ (2) Suppose that X satisfies (1), $f : (\mathbb{H}, b_0) \to (X, x)$ is a map, and $\phi : \mathbb{N} \to \mathbb{N}$ is a bijection. By assumption, there is a homomorphism $g : \pi_1(\mathbb{T}, b_0) \to \pi_1(X, x)$ such that $g \circ \eta_\# = f_\#$. Set $\alpha_n = f \circ \ell_n$ for $n \in \mathbb{N}$ and recall that $\ell_\tau = \prod_\tau \ell_n$. If we identify $\pi_1(\mathbb{T}, b_0)$ with $\mathbb{Z}^{\mathbb{N}}$ in the natural way, then $\eta_\#([\ell_\tau]) = (1, 1, 1, \ldots)$. Since each ℓ_n appears exactly once in the concatenation $\prod_\tau \ell_{\phi(n)}$, we also have $\eta_\#([\prod_\tau \ell_{\phi(n)}]) = (1, 1, 1, \ldots)$. Therefore, $[\prod_\tau \alpha_n] = f_\#([\prod_\tau \ell_n]) = g(\eta_\#([\prod_\tau \ell_n])) = g(1, 1, 1, \ldots)$ and, similarly, we have $[\prod_\tau \alpha_{\phi(n)}] = g(1, 1, 1, \ldots)$. Thus, $[\prod_\tau \alpha_n] = [\prod_\tau \alpha_{\phi(n)}]$, proving transfinite π_1-commutativity.

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(2) ⇒ (3) Suppose that X is transfinitely π_1-commutative at $x \in X$, $\alpha_n \in \Omega(X, x)$ is a null sequence, and $\phi : \mathbb{N} \to \mathbb{N}$ is a bijection. Define maps $f, g : \mathbb{H} \to X$ so that $f \circ \ell_{2^k - 1} = \alpha_k$, $g \circ \ell_{2^k - 1} = \alpha_{\phi(k)}$ for all $k \in \mathbb{N}$, and $f \circ \ell_n = g \circ \ell_n = c_x$ otherwise. By this choice, we have $\prod_\tau f \circ \ell_n \simeq \prod_{k=1}^{\infty} \alpha_k$ and $\prod_\tau g \circ \ell_n \simeq \prod_{k=1}^{\infty} \alpha_{\phi(k)}$ by collapsing constant subpaths. Choose any bijection $\psi : \mathbb{N} \to \mathbb{N}$ satisfying $\psi(2^k - 1) = 2^{\phi(k)} - 1$ for all $k \in \mathbb{N}$. Then we have $[\prod_{k=1}^{\infty} \alpha_k] = [\prod_\tau f \circ \ell_n] = [\prod_\tau f \circ \ell_{\psi(n)}] = [\prod_\tau g \circ \ell_n] = [\prod_{k=1}^{\infty} \alpha_{\phi(k)}]$, where the second equality follows from transfinite π_1-commutativity.

(3) ⇒ (1) Suppose that X is infinitely π_1-commutative at $x \in X$ and $f : (\mathbb{H}, b_0) \to (X, x)$ is a map. To show that such a homomorphism g exists, it suffices to show that $f_\#(\ker(\eta_\#)) = 1$. Let $1 \neq [\alpha] \in \ker(\eta_\#)$. By Lemma 3.4, we may assume that

$$\alpha \equiv \prod_{n=1}^{\infty} \Big(\prod_{i=1}^{k_n} \alpha_{n,i} \cdot \beta_{n,i} \cdot \alpha_{n,i}^{-} \cdot \beta_{n,i}^{-} \Big), \tag{*}$$

where $\alpha_{n,i}, \beta_{n,i} \in \Omega(\mathbb{H}_{\geq n}, b_0)$, $k_n \in \mathbb{N}$. Since this concatenation has the order type of $\mathbb{N}$, we may write $\alpha \equiv \prod_{m=1}^{\infty} \delta_m$, where the null sequence $\{\delta_m\}$ consists of the factors $\alpha_{n,i}$, $\beta_{n,i}$, $\alpha_{n,i}^{-}$, and $\beta_{n,i}^{-}$ appearing in the same order as the concatenation in (*). Let $\phi : \mathbb{N} \to \mathbb{N}$ be the bijection defined so that for all $j \in \mathbb{N}$, we have $\phi(4j - 3) = 4j - 3$, $\phi(4j - 2) = 4j - 1$, $\phi(4j - 1) = 4j - 2$, and $\phi(4j) = 4j$. By infinite π_1-commutativity, we have $[f \circ \alpha] = [\prod_{m=1}^{\infty} f \circ \delta_m] = [\prod_{m=1}^{\infty} f \circ \delta_{\phi(m)}]$. The bijection ϕ was defined so that

$$\prod_{m=1}^{\infty} \delta_{\phi(m)} \equiv \prod_{n=1}^{\infty} \Big(\prod_{i=1}^{k_n} \alpha_{n,i} \cdot \alpha_{n,i}^{-} \cdot \beta_{n,i} \cdot \beta_{n,i}^{-} \Big).$$

The concatenation on the right is a reparameterization of an infinite concatenation of consecutive inverse pairs and therefore is null-homotopic in $\mathbb{H}$. Since $[\prod_{m=1}^{\infty} \delta_{\phi(m)}] = 1$ in $\pi_1(\mathbb{H}, b_0)$,

$$f_\#([\alpha]) = \Big[\prod_{m=1}^{\infty} f \circ \delta_{\phi(m)} \Big] = f_\# \Big(\Big[\prod_{m=1}^{\infty} \delta_{\phi(m)} \Big] \Big) = 1$$

in $\pi_1(X, x)$. □

In Condition (1) of Theorem 3.5, note that if such a g exists, it is necessarily unique. For the remainder of this section, we consider some immediate consequences of Theorem 3.5. Note that if $\pi_1(X, x)$ is σ-commutative in the sense of Definition 3.1, then we have $f_\#(\ker(\eta_\#)) = 1$ for all maps $f : (\mathbb{H}, b_0) \to (X, x)$. Hence, we have the following corollary.

Corollary 3.6. *If $\pi_1(X, x_0)$ is σ-commutative, then X is transfinitely π_1-commutative at all of its points.*

Applying, once again, the fact that $\eta_\#([\ell_\tau]) = \eta_\#([\ell_\infty])$, the next result follows immediately.

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COROLLARY 3.7. *If X has transfinite π_1-commutativity at $x \in X$, then for every null sequence $\alpha_n \in \Omega(X, x)$, we have $[\prod_\tau \alpha_n] = [\prod_{n=1}^{\infty} \alpha_n]$.*

For a compact nowhere-dense subset $A \subseteq I$, let $\mathcal{I}(A)$ denote the set of connected components of $[\min(A), \max(A)] \backslash A$ with the natural linear ordering inherited from I. For example, the set $\mathcal{I}(C)$ of components of the complement of the Cantor set C in I has the order type of the rationals. Let $\theta : \mathbb{N} \to \mathcal{I}(C)$ denote the bijection corresponding to the loop ℓ_τ, that is, so that $\ell_\tau|_{\overline{\theta(n)}} \equiv \ell_n$ for $n \in \mathbb{N}$.

LEMMA 3.8. *Suppose that X is transfinitely π_1-commutative at $x \in X$, $\alpha, \beta \in \Omega(X, x)$ are nonconstant paths, and $K_\alpha \subseteq \alpha^{-1}(x)$ and $K_\beta \subseteq \beta^{-1}(x)$ are closed nowhere-dense sets each containing $\{0, 1\}$. If there is a bijection $\psi : \mathcal{I}(K_\alpha) \to \mathcal{I}(K_\beta)$ such that for every $J \in \mathcal{I}(K_\alpha)$, we have $\alpha|_{\overline{J}} \equiv \beta|_{\overline{\psi(J)}}$, then $[\alpha] = [\beta]$ in $\pi_1(X, x)$.*

PROOF. If $\mathcal{I}(K_\alpha)$ is finite, then the conclusion follows from the fact that $\pi_1(X, x)$ is abelian. Suppose that $\mathcal{I}(K_\alpha)$ is infinite. Since $\mathcal{I}(C)$ is a dense countable order, there exist order embeddings $\mu : \mathcal{I}(K_\alpha) \to \mathcal{I}(C)$ and $\nu : \mathcal{I}(K_\beta) \to \mathcal{I}(C)$. Find a bijection $\Psi : \mathcal{I}(C) \to \mathcal{I}(C)$, which extends ψ in the sense that $\Psi \circ \mu = \nu \circ \psi$. Define a bijection $\phi : \mathbb{N} \to \mathbb{N}$ by $\phi = \theta^{-1} \circ \Psi \circ \theta$. Hence, we have the following commutative diagram of linear orders, where μ and ν are order preserving and all other morphisms are set-bijections.

$$\begin{array}{ccccc} \mathcal{I}(K_\alpha) & \xrightarrow{\mu} & \mathcal{I}(C) & \xleftarrow{\theta} & \mathbb{N} \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\Psi} & & \downarrow{\scriptstyle\phi} \\ \mathcal{I}(K_\beta) & \xrightarrow[\nu]{} & \mathcal{I}(C) & \xleftarrow[\theta]{} & \mathbb{N} \end{array}$$

Define null sequences $\gamma_n, \delta_n \in \Omega(X, x)$ as follows. For $n \in \mathbb{N}$, let $\gamma_n \equiv \alpha|_{\overline{J}}$ if $J \in \mathcal{I}(K_\alpha)$ and $\theta(n) = \mu(J)$ and let $\gamma_n = c_x$ be constant otherwise. We have $\prod_\tau \gamma_n \simeq \alpha$ by collapsing constant loops. Similarly, let $\delta_m \equiv \beta|_{\overline{J}}$ if $J \in \mathcal{I}(K_\beta)$ and $\theta(m) = \nu(J)$ and let $\delta_m = c_x$ be constant otherwise. From this choice, we have $\prod_\tau \delta_m \simeq \beta$ by collapsing constant loops.

Fix $n \in \mathbb{N}$. If $\theta(n) = \mu(J)$, then we have $\theta(\phi(n)) = \nu(\psi(J))$ and thus $\gamma_n \equiv \alpha|_{\overline{J}} \equiv \beta|_{\overline{\psi(J)}} \equiv \delta_{\phi(n)}$. If $\theta(n) \notin \mathrm{Im}(\mu)$, then $\theta(\phi(n)) \notin \mathrm{Im}(\mu)$ and we have $\gamma_n = \delta_{\phi(n)} = c_x$. Hence, $\gamma_n \equiv \delta_{\phi(n)}$ for all $n \in \mathbb{N}$. This gives

$$[\alpha] = \Big[\prod_\tau \gamma_n\Big] = \Big[\prod_\tau \delta_{\phi(n)}\Big] = \Big[\prod_\tau \delta_n\Big] = [\beta],$$

where the third equality is given by transfinite π_1-commutativity. □

As a specific example, and analogue of infinite double series, the next corollary applies Lemma 3.8 to infinite concatenations of order type ω^2.

COROLLARY 3.9. *If X has transfinite π_1-commutativity at $x \in X$ and $\alpha_{m,n} \in \Omega(X, x)$, $m, n \in \mathbb{N}$, is a doubly indexed sequence such that every neighborhood of x contains all*

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but finitely many of the sets Im $(\alpha_{m,n})$, then

$$\left[\prod_{m=1}^{\infty}\prod_{n=1}^{\infty}\alpha_{m,n}\right]=\left[\prod_{n=1}^{\infty}\prod_{m=1}^{\infty}\alpha_{m,n}\right]$$

in $\pi_1(X,x)$.

COROLLARY 3.10. *Let* $[(\mathbb{H},b_0),(X,x)]$ *denote the set of homotopy rel. basepoint classes of based maps* $(\mathbb{H},b_0)\to(X,x)$ *equipped with the natural group structure from the co-H-group structure of* $\mathbb{H}$. *If X is transfinitely* π_1*-commutative at x, then the two functions* $[(\mathbb{H},b_0),(X,x)]\to\pi_1(X,x)$ *given by* $[f]\mapsto[f\circ\ell_\tau]$ *and* $[f]\mapsto[f\circ\ell_\infty]$ *are homomorphisms that agree with each other.*

PROOF. For maps $f,g\in(X,x)^{(\mathbb{H},b_0)}$, if we take $f\circ\ell_n=\alpha_n$ and $g\circ\ell_n=\beta_n$, then verifying the two desired homomorphism equalities amounts to checking that $\prod_\tau\alpha_n\cdot\prod_\tau\beta_n\simeq\prod_\tau(\alpha_n\cdot\beta_n)$ and $\prod_{n=1}^{\infty}\alpha_n\cdot\prod_{n=1}^{\infty}\beta_n\simeq\prod_{n=1}^{\infty}(\alpha_n\cdot\beta_n)$. Each of these homotopies may be obtained by an appropriate application of Lemma 3.8. According to Corollary 3.7, these homomorphisms are equal. □

In our analysis and anticipated applications, it is useful to know that the property of being transfinitely π_1-commutative at a given basepoint is invariant under basepoint-preserving homotopy equivalence. Let $\mathbf{tc}(X)$ denote the set of points at which X is transfinitely π_1-commutative and $\mathbf{ntc}(X)=X\backslash\mathbf{tc}(X)$. A space X is, in a sense, 'wild' at the points of $\mathbf{ntc}(X)$: either $\pi_1(X,x_0)$ is nonabelian (in which case $\mathbf{ntc}(X)=X$) or $\pi_1(X,x_0)$ is abelian and $\mathbf{ntc}(X)$ consists of those points at which a nontrivial and noninfinitely commuting infinite product exists. Thus, it is the subspace $\mathbf{ntc}(X)$, rather than $\mathbf{tc}(X)$, which gives rise to a homotopy invariant of X.

LEMMA 3.11. *If a map* $f:X\to Y$ *induces an injection on* π_1, *then* $f(\mathbf{ntc}(X))\subseteq\mathbf{ntc}(Y)$. *Moreover, the homotopy type of* $\mathbf{ntc}(X)$ *is a homotopy invariant of X.*

PROOF. Suppose that $f(x)=y$ with $x\in\mathbf{ntc}(X)$. Then there exist a null sequence $\alpha_n\in\Omega(X,x)$ and a bijection $\phi:\mathbb{N}\to\mathbb{N}$ such that $\prod_\tau\alpha_n$ and $\prod_\tau\alpha_{\phi(n)}$ are not homotopic in X. Since $f_\#:\pi_1(X,x)\to\pi_1(Y,y)$ is injective, $f\circ\alpha_n\in\Omega(Y,y)$ is a null sequence, where $\prod_\tau(f\circ\alpha_n)$ and $\prod_\tau(f\circ\alpha_{\phi(n)})$ are not homotopic in Y. Thus, $y\in\mathbf{ntc}(X)$, proving that $f(\mathbf{ntc}(X))\subseteq\mathbf{ntc}(Y)$.

For the second statement, suppose that $f:X\to Y$ and $g:Y\to X$ are homotopy inverses. Then $f(\mathbf{ntc}(X))\subseteq\mathbf{ntc}(Y)$ and $g(\mathbf{ntc}(Y))\subseteq\mathbf{ntc}(X)$ by the first paragraph. Therefore, the restrictions $f|_{\mathbf{ntc}(X)}:\mathbf{ntc}(X)\to\mathbf{ntc}(Y)$ and $g|_{\mathbf{ntc}(Y)}:\mathbf{ntc}(Y)\to\mathbf{ntc}(X)$ are well defined. Let $H:X\times I\to X$ be a homotopy with $H(x,0)=x$ and $H(x,1)=g(f(x))$. Since the projection $X\times I\to X$ and inclusions $X\times\{t\}\to X\times I$ all induce injections on π_1, we have $\mathbf{ntc}(X\times I)=\mathbf{ntc}(X)\times I$. Since H also induces an injection on π_1, we have $H(\mathbf{ntc}(X)\times I)=H(\mathbf{ntc}(X\times I))\subseteq\mathbf{ntc}(X)$. Thus, H restricted to $\mathbf{ntc}(X)\times I$ is a homotopy in $\mathbf{ntc}(X)$ from $id_{\mathbf{ntc}(X)}$ to $g|_{\mathbf{ntc}(Y)}\circ f|_{\mathbf{ntc}(X)}$. The symmetric argument shows that $id_{\mathbf{ntc}(Y)}$ is homotopic to $f|_{\mathbf{ntc}(X)}\circ g|_{\mathbf{ntc}(Y)}$. □

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COROLLARY 3.12. *If $f : (X, x) \to (Y, y)$ is a basepoint-relative homotopy equivalence, then X is transfinitely π_1-commutative at x if and only if Y is transfinitely π_1-commutative at y.*

4. Examples of transfinite π_1-commutativity

4.1. Cotorsion-free groups. Recall from Corollary 3.6 that if $\pi_1(X, x)$ is σ-commutative in the sense of Definition 3.1, then X is transfinitely π_1-commutative at all of its points. We combine this observation with Eda and Kawamura's characterization of $H_1(\mathbb{H})$ in [15]. Recall that an abelian group A is *cotorsion* provided that whenever $A \leq G$ with G abelian and G/A torsion-free, we have $G = A \oplus B$ for some $B \leq G$. The abelian group A is called *cotorsion-free* if it does not contain a nontrivial cotorsion subgroup.

REMARK 4.1. The cotorsion-free group $H_1(\mathbb{T}) \cong \mathbb{Z}^{\mathbb{N}}$ may be naturally identified with the first Čech homology group $\check{H}_1(\mathbb{H})$. It is shown in [15] that the kernel of the induced homomorphism $\eta_* : H_1(\mathbb{H}) \to \check{H}_1(\mathbb{H})$ is cotorsion and hence $H_1(\mathbb{H}) \cong \check{H}_1(\mathbb{H}) \oplus \ker(\eta_*)$. Applying the decomposition theory of infinite abelian groups (as in [16]), it is known that $\ker(\eta_*)$ is abstractly (that is, not naturally) isomorphic to the group $\mathbb{Z}^{\mathbb{N}} / \oplus_{\mathbb{N}} \mathbb{Z}$.

Let $h_X : \pi_1(X, x) \to H_1(X)$ denote the Hurewicz homomorphism for a space X.

PROPOSITION 4.2. *If $\pi_1(X, x)$ is a cotorsion-free abelian group for some $x \in X$, then X is transfinitely π_1-commutative at all of its points.*

PROOF. If $\pi_1(X, x)$ is cotorsion-free, then $\pi_1(X, y)$ is cotorsion-free for all $y \in X$. Hence, it suffices to show that X is transfinitely π_1-commutative at the given point $x \in X$. Let $f : (\mathbb{H}, b_0) \to (X, x)$ be a map. Since $\pi_1(X, x)$ is abelian, there is a unique homomorphism $g' : H_1(\mathbb{H}) \to \pi_1(X, x)$ such that $g' \circ h_{\mathbb{H}} = f_\#$. Since $\ker(\eta_*)$ is cotorsion and any homomorphic image of a cotorsion group is cotorsion, we must have $g'(\ker(\eta_*)) = 0$. Hence, there is a unique homomorphism $g'' : H_1(\mathbb{H})/\ker(\eta_*) \to \pi_1(X, x)$ such that if $p : H_1(\mathbb{H}) \to H_1(\mathbb{H})/\ker(\eta_*)$ is the projection, then $g'' \circ p = g'$. Let $k : H_1(\mathbb{T}) \to H_1(\mathbb{H})/\ker(\eta_*)$ be the canonical isomorphism and set $g = g'' \circ k \circ h_{\mathbb{T}}$. From the naturality of the Hurewicz homomorphisms, it follows directly that $g \circ \eta_\# = f_\#$, verifying Condition (1) in Theorem 3.5. □

Since the Specker group is cotorsion-free, it follows that the infinite torus $\mathbb{T}$ is transfinitely π_1-commutative at all of its points.

4.2. Semilocally simply connected spaces. Here, we consider when transfinite π_1-commutativity holds trivially for topological reasons in a refinement of the semilocally simply connected property.

DEFINITION 4.3 [2]. A space X is π_1-*finitary at* $x \in X$ if for every map $f : (\mathbb{H}, b_0) \to (X, x)$, there exists $N \in \mathbb{N}$ such that $f \circ \ell_n$ is null-homotopic in X for all $n \geq N$.

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The *algebraic* 1-*wild set* of X is the subspace $\mathbf{aw}(X) \subseteq X$ consisting of all $x \in X$ at which X is not π_1-finitary at x. We say that X is π_1-*finitary* if $\mathbf{aw}(X) = \emptyset$.

Note that X is π_1-finitary at x if and only if for every map $f : (\mathbb{H}, b_0) \to (X, x)$, there exists N such that $f_\#(\pi_1(\mathbb{H}_{\geq N}, b_0)) = 1$. If X is semilocally simply connected at x, then X is π_1-finitary at x and the converse holds if X is first countable at x. Additionally, if $\pi_1(X, x)$ is noncommutatively slender in the sense of [9], X is π_1-finitary at all of its points. Much like $\mathbf{ntc}(X)$, the homotopy type of $\mathbf{aw}(X)$ is a homotopy invariant of X; we refer to [2] for more on $\mathbf{aw}(X)$.

PROPOSITION 4.4. *If $\pi_1(X, x)$ is abelian, then $\mathbf{ntc}(X) \subseteq \mathbf{aw}(X)$.*

PROOF. Suppose that $x \notin \mathbf{aw}(X)$. It suffices to show that X is transfinitely π_1-commutative at x. Suppose that $f : (\mathbb{H}, b_0) \to (X, x)$ is a map and $[\alpha] \in \ker(\eta_\#)$. By Lemma 3.4, we may assume that $\alpha \equiv \prod_{n=1}^{\infty} \gamma_n$, where γ_n is a finite concatenation of commutators in $\mathbb{H}_{\geq n}$. Find $N \in \mathbb{N}$ such that $f_\#(\pi_1(\mathbb{H}_{\geq N}, b_0)) = 0$. Then

$$f_\#([\alpha]) = \left(\prod_{n=1}^{N-1} [f \circ \gamma_n] \right) \left[f \circ \prod_{n=N}^{\infty} \gamma_n \right] = \prod_{n=1}^{N-1} [f \circ \gamma_n] = 0$$

since finite products of commutators in $\pi_1(X, x)$ are trivial. Thus, $f_\#(\ker(\eta_\#)) = 0$. By Theorem 3.5, x is transfinitely π_1-commutative at x, that is, $x \notin \mathbf{ntc}(X)$. □

COROLLARY 4.5. *If $\pi_1(X, x)$ is abelian and X is π_1-finitary at all of its points, then X is transfinitely π_1-commutative at all of its points.*

EXAMPLE 4.6. If X is any CW-complex or manifold with abelian, but non-cotorsion-free, fundamental group, then X is transfinitely π_1-commutative at all of its points. In particular, one may construct a CW-complex X, with fundamental group isomorphic to $\mathbb{Z}^{\mathbb{N}} / \oplus_{\mathbb{N}} \mathbb{Z}$, which is a torsion-free, cotorsion group. However, according to Proposition 4.4, X is (trivially) transfinitely π_1-commutative at all of its points since no geometrically represented infinite π_1-products exist. Hence, the converse of Proposition 4.2 is far from being true. This example emphasizes the fact that the transfinitely π_1-commutative property is an invariant property of the infinitary structure of fundamental groups inherited from the loop space and not the underlying (finitary) group structure.

PROPOSITION 4.7. *If $\{X_j \mid j \in J\}$ is a set of spaces, then $\mathbf{tc}(\prod_{j \in J} X_j) = \prod_{j \in J} \mathbf{tc}(X_j)$.*

PROOF. We prove that $\prod_{j \in J} X_j$ is transfinitely π_1-commutative at $(x_j) \in \prod_{j \in J} X_j$ if and only if for every $j \in J$, X_j is transfinitely π_1-commutative at $x_j \in X_j$. First, if $\prod_{j \in J} X_j$ is transfinitely π_1-commutative at J-tuple (x_j), then for fixed j_0 the embedding $X_{j_0} \to \prod_{j \in J} X_j$ onto $X_{j_0} \times \prod_{j \neq j_0} \{x_j\}$ induces an injection on π_1. Lemma 3.11 then implies that X_j is transfinitely π_1-commutative at x_j. For the converse, suppose that X_j is transfinitely π_1-commutative at x_j for all $j \in J$ and let $p_j : \prod_{j \in J} X_j \to X_j$ denote the projection map. Suppose that $f : (\mathbb{H}, b_0) \to (\prod_{j \in J} X_j, (x_j))$ is a map. By assumption, we have $(p_j \circ f)_\#(\ker(\eta_\#)) = 0$ for all $j \in J$ and so we have an

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induced homomorphism $g_j : \pi_1(\mathbb{T}, b_0) \to \pi_1(X_j, x_j)$ such that $g_j \circ \eta_\# = p_j \circ f$. Let $\phi : \pi_1(\prod_{j\in J} X_j, (x_j)) \to \prod_{j\in J} \pi_1(X_j, x_j)$ be the canonical isomorphism and $(g_j) : \pi_1(\mathbb{T}, b_0) \to \prod_{j\in J} \pi_1(X_j, x_j)$ be the induced homomorphism to the product. Then $g = \phi^{-1} \circ (g_j) : \pi_1(\mathbb{T}, b_0) \to \pi_1(\prod_{j\in J} X_j, (x_j))$ is the desired homomorphism satisfying $g \circ \eta_\# = f$ and confirming Condition (1) in Theorem 3.5. □

Example 4.8. If $\{X_j \mid j \in J\}$ is any infinite set of CW-complexes or manifolds with nontrivial, abelian fundamental groups, then $\prod_{j\in J} X_j$ is not π_1-finitary but is transfinitely π_1-commutative at all of its points. This includes spaces, such as the infinite product $\prod_{n=1}^{\infty} \mathbb{RP}^2$ of projective planes, whose fundamental groups are not torsion-free or, more generally, do not satisfy the cotorsion-freeness condition.

We also note that even though the transfinitely π_1-commutative property is defined ‘at a point’, it is not a purely local property since it is possible for a space to fail to be transfinitely π_1-commutative at a point due to the global structure of the space.

Example 4.9. Let $\mathbb{T}'$ be a homeomorphic copy of $\mathbb{T}$ with basepoint b_0'. Consider the Peano continuum $X = (\mathbb{T} \cup I \cup \mathbb{T}')/\sim$, where $0 \sim b_0$ and $1 \sim b_0'$. Then $\pi_1(X, b_0)$ is isomorphic to the free product $\pi_1(\mathbb{T}, b_0) * \pi_1(\mathbb{T}', b_0')$ by the van Kampen theorem. Since $\pi_1(X, b_0)$ is not abelian, X is not transfinitely π_1-commutative at any point. However, the open sets $U, V \subseteq X$, which are the respective images of $[0, 2/3) \cup \mathbb{T}$ and $(1/3, 1] \cup \mathbb{T}'$ in X, cover X and have cotorsion-free fundamental groups isomorphic to $\mathbb{Z}^{\mathbb{N}}$. Hence, by Proposition 4.2, U and V are each transfinitely π_1-commutative at all of their points. Similar observations show that every point $x \in X$ has a neighborhood basis consisting of open sets, which are transfinitely π_1-commutative at x.

4.3. Loop spaces. For $n \in \mathbb{N}$ and a based space (X, x), the relative mapping space $\Omega^n(X, x) = (X, x)^{(I^n, \partial I^n)}$ is transfinitely π_1-commutative at the constant map $c_x \in \Omega^n(X, x)$. To show this, we employ the k-dimensional Hawaiian earring $\mathbb{H}_k$, which we construct as the $(k-1)$th reduced suspension $\Sigma^{k-1}\mathbb{H}$ of $\mathbb{H}$. It is known that for $k \geq 2$, $\mathbb{H}_k$ is $(k-1)$-connected and the natural embedding $\zeta : \mathbb{H}_k \to \prod_{m\in\mathbb{N}} S^k$ induces an isomorphism on π_k. Hence, $\pi_k(\mathbb{H}_k, b_0) \cong \mathbb{Z}^{\mathbb{N}}$ [14].

Consider a map $f : (\mathbb{H}, b_0) \to (\Omega^n(X, x), c_x)$. By the loop-suspension adjunction, there is a unique based map $\widetilde{f} : \mathbb{H}_{n+1} \to X$ such that $f = \Omega^n(\widetilde{f}) \circ \sigma$, where $\sigma : \mathbb{H} \to \Omega^n(\mathbb{H}_{n+1}, b_0)$ is the unit map of the adjunction. The induced homomorphism $f_\# : \pi_1(\mathbb{H}, b_0) \to \pi_1(\Omega^n(X, x), c_x) = \pi_n(X, x)$ factors as in the diagram below, where the bottom isomorphism is induced by the product of the suspension isomorphism $\pi_1(S^1) \to \pi_{n+1}(S^{n+1})$.

$$\begin{array}{ccccc} & & \overset{f_\#}{\frown\!\!\!\longrightarrow} & & \\ \pi_1(\mathbb{H}, b_0) & \xrightarrow{\sigma_\#} & \pi_{n+1}(\mathbb{H}_{n+1}, b_0) & \xrightarrow[\Omega^n(\widetilde{f})_\#]{} & \pi_n(X, x) \\ \downarrow{\scriptstyle \eta_\#} & & {\scriptstyle \cong}\downarrow{\scriptstyle \zeta_\#} & & \\ \pi_1(\mathbb{T}, b_0) & \xrightarrow[\cong]{} & \pi_{n+1}(\prod_{m\in\mathbb{N}} S^{n+1}, b_0) & & \end{array}$$

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Since $f_\#$ factors through $\eta_\#$, Condition (1) of Theorem 3.5 is satisfied and we obtain the following result.

THEOREM 4.10. *For any based space (X,x) and $n \in \mathbb{N}$, $\Omega^n(X,x)$ is transfinitely π_1-commutative at the constant loop c_x.*

Theorem 4.10 suggests that there is no need to define a point-wise notion of 'transfinitely π_n-commutative' for $n \geq 2$ (for example, using the n-dimensional Hawaiian earring) since this theorem states that higher dimensional homotopy groups are infinitely commutative at the basepoint. Additionally, using Theorem 4.10, one can directly prove that the higher-dimensional analogue $[(\mathbb{H}_k, b_0),(X,x)] \to \pi_k(X,x)$, $[f] \mapsto [f \circ \Sigma^{k-1}\ell_\infty]$ of the function in Corollary 3.10 is a group homomorphism. That this function is a homomorphism is precisely the main technical hurdle in the proof of [21, Theorem 2.1], which verifies the splitting of the natural homomorphism $\pi_k(\widetilde{\bigvee}_{n\in\mathbb{N}} X_n) \to \prod_{n\in\mathbb{N}} \pi_k(X_n)$ for any shrinking wedge $\widetilde{\bigvee}_{n\in\mathbb{N}} X_n$.

5. Infinitary abelianization

In light of Theorem 3.5, we are motivated to give the following definition.

DEFINITION 5.1. The *infinitary commutator subgroup of $\pi_1(X,x)$ at a subset $A \subseteq X$* is the subgroup $C_\tau(A)$ of $\pi_1(X,x)$ generated by all homotopy classes of loops of the form $\beta \cdot (\prod_\tau \alpha_n) \cdot (\prod_\tau \alpha_{\phi(n)})^- \cdot \beta^-$, where $\beta : I \to X$ is a path from x to $a \in A$, $\alpha_n \in \Omega(X,a)$ is a null sequence, and $\phi : \mathbb{N} \to \mathbb{N}$ is a bijection. The *infinitary abelianization of $\pi_1(X,x)$ at A* is the quotient group $\mathfrak{H}_A(X) = \pi_1(X,x)/C_\tau(A)$. We write $\langle\alpha\rangle$ for the coset $C_\tau(A)[\alpha]$, where $\alpha \in \Omega(X,x)$.

- In the case $A = \emptyset$, we take $C_\tau(A)$ to be the ordinary commutator subgroup.
- In the case $A = X$, we will refer to $C_\tau(X)$ and $\mathfrak{H}(X) = \mathfrak{H}_X(X)$ respectively as the *infinitary commutator* and *infinitary abelianization* of $\pi_1(X,x)$.

REMARK 5.2. If $x \in A$, then by considering constant paths β and applying the observations made in Remark 3.3, we see that $C_\tau(A)$ contains the ordinary commutator subgroup of $\pi_1(X,x)$. Applying a basepoint-change isomorphism allows one to make the same conclusion when $x \notin A$. Hence, $C_\tau(A)$ is a normal subgroup of $\pi_1(X,x)$ and the quotient $\mathfrak{H}_A(X)$ is abelian in the usual sense. Since $\mathfrak{H}_A(X)$ is naturally a quotient of $H_1(X)$, we choose to omit reference to the basepoint in our notation. Additionally, we have that X is transfinitely π_1-commutative at every point of A if and only if $C_\tau(A) = 1$.

REMARK 5.3. Since any infinite concatenation is homotopic to a transfinite concatenation by inserting constant subpaths (recall the argument (2) $\Rightarrow$ (3) in Theorem 3.5), it follows that loops of the form $\beta \cdot (\prod_{n=1}^\infty \alpha_n) \cdot (\prod_{n=1}^\infty \alpha_{\phi(n)})^- \cdot \beta^-$, where $\beta(1) \in A$, always represent elements of $C_\tau(A)$.

The next three statements lay out elementary properties of the infinitary commutator and abelianization constructions. Since the proofs are short and straightforward, we omit them.

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PROPOSITION 5.4. *Let (X,x) and (Y,y) be based spaces.*

(1) *Every basepoint-change isomorphism $\pi_1(X,x) \to \pi_1(X,x')$ maps $C_\tau(A) \leq \pi_1(X,x)$ onto the corresponding subgroup $C_\tau(A) \leq \pi_1(X,x')$.*
(2) *If $A \subseteq B \subseteq X$, then $C_\tau(A) \leq C_\tau(B)$ in $\pi_1(X,x)$.*
(3) *If X is transfinitely π_1-commutative at all points in a subset $B \subseteq X$, then $C_\tau(A) = C_\tau(A \cup B)$. In particular, $C_\tau(\emptyset) = C_\tau(\mathbf{tc}(X))$ and $C_\tau(X) = C_\tau(\mathbf{ntc}(X))$.*
(4) *If $f : (X,x) \to (Y,y)$ is a based map, then $f_\#(C_\tau(A)) \leq C_\tau(f(A))$.*
(5) *If $f : (X,x) \to (Y,y)$ is a based map, then there is an induced homomorphism $f_* : \mathfrak{H}_A(X) \to \mathfrak{H}_{f(A)}(Y)$, $f_*(\langle\alpha\rangle) = \langle f \circ \alpha\rangle$.*

PROPOSITION 5.5. *If $f : (X,x) \to (Y,y)$ is a continuous map and Y is transfinitely π_1-commutative at each point in $B \subseteq Y$, then there exists a unique homomorphism $F : \mathfrak{H}_{f^{-1}(B)}(X) \to \pi_1(Y,y)$ such that $F \circ p = f_\#$, where $p : \pi_1(X,x) \to \mathfrak{H}_{f^{-1}(B)}(X)$ is the projection.*

The next lemma is a direct consequence of the proof of Lemma 3.8.

LEMMA 5.6. *Suppose that $\alpha,\beta \in \Omega(X,x)$ are nonconstant paths and $C \subseteq \alpha^{-1}(x)$ and $D \subseteq \beta^{-1}(x)$ are closed nowhere-dense sets each containing $\{0,1\}$. If $\psi : \mathcal{I}(C) \to \mathcal{I}(D)$ is a bijection such that for every $J \in \mathcal{I}(C)$, we have $\alpha|_{\overline{J}} \equiv \beta|_{\overline{\psi(J)}}$, then $[\alpha][\beta]^{-1} \in C_\tau(\{x\})$.*

We observe how the subgroup $C_\tau(X)$ refines previously defined notions of infinite commutators appearing in the literature by showing that $C_\tau(X)$ lies in all alternative notions of 'infinite commutator' subgroups.

REMARK 5.7 (Comparison to Eda's σ-commutator). Recall Definition 3.1 and consider a generator of $C_\tau(X)$, namely an element

$$g = \Big[\beta \cdot \Big(\prod_\tau \alpha_n\Big) \cdot \Big(\prod_\tau \alpha_{\phi(n)}\Big)^- \cdot \beta^-\Big],$$

where $\beta(1) = a \in X$. Define $f : (\mathbb{H}, b_0) \to (X,a)$ by $f \circ \ell_n = \alpha_n$ and note that $[\prod_\tau \ell_n \cdot (\prod_\tau \ell_{\phi(n)})^-] \in \ker(\eta_\#)$. Let $\varphi_\beta : \pi_1(X,a) \to \pi_1(X,x)$, $\varphi_\beta([\gamma]) = [\beta][\gamma][\beta]^{-1}$ be the basepoint-change isomorphism. Now $\varphi_\beta \circ f_\# : \pi_1(\mathbb{H}, b_0) \to \pi_1(X,x)$ is a homomorphism such that $g \in \varphi_\beta \circ f_\#(\ker(\eta_\#)) \leq \pi_1(X,x)^{\sigma'}$. We conclude that $C_\tau(X)$ is a subgroup of Eda's σ-commutator subgroup $\pi_1(X,x)^{\sigma'}$. This argument also may be extended to show that $C_\tau(A)$ is precisely the subgroup of $\pi_1(X,x)$ generated by all subgroups of the form $\varphi_\beta(f_\#(\ker(\eta_\#)))$, where $\beta(1) \in A$ and $f : (\mathbb{H}, b_0) \to (X, \beta(1))$ is a map.

EXAMPLE 5.8. If $\mathbf{tc}(X) = X$ and $\pi_1(X,x)$ is not cotorsion-free, then $C_\tau(X) = 1$ and $\pi_1(X,x)^{\sigma'} \neq 1$. For instance, it is well known that there exists a surjection $\mathbb{Z}^{\mathbb{N}}/\oplus_{\mathbb{N}}\mathbb{Z} \to A$ onto any finite abelian group A. Since $\ker(\eta_* : H_1(\mathbb{H}) \to \mathbb{Z}^{\mathbb{N}}) \cong \mathbb{Z}^{\mathbb{N}}/\oplus_{\mathbb{N}}\mathbb{Z}$

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and $H_1(\mathbb{H}) \cong \mathbb{Z}^{\mathbb{N}} \oplus \ker(\eta_*)$, composition with the Hurewicz map gives the existence of a homomorphism $g : \pi_1(\mathbb{H}, b_0) \to A$ such that $g(\ker(\eta_\#)) = A$. Thus, if $X = \mathbb{RP}^2$ or if X is any other CW-complex with a finite abelian fundamental group, then $\pi_1(X,x)^{\sigma'} = \pi_1(X,x)$, giving a trivial σ-abelianization. However, X is transfinitely π_1-commutative at all of its points by Proposition 4.4. Thus, $\pi_1(X,x_0) \cong \mathfrak{H}(X)$.

The next proposition identifies an important case where the σ-commutator subgroup agrees with $C_\tau(X)$.

PROPOSITION 5.9. *If X is a one-dimensional or planar Peano continuum, then $C_\tau(X) = \pi_1(X,x)^{\sigma'}$.*

PROOF. In light of Remark 5.7, it suffices to show that $\pi_1(X,x)^{\sigma'} \leq C_\tau(X)$. Suppose that $g : \pi_1(\mathbb{H}, b_0) \to \pi_1(X,x)$ is a homomorphism. It suffices to show that $g(\ker(\eta_\#)) \leq C_\tau(X)$. It is known that there exist a path $\beta : I \to X$ with $\beta(0) = x$ and a map $f : (\mathbb{H}, b_0) \to (X, \beta(1))$ such that $g = \varphi_\beta \circ f_\#$, where, as before, φ_β denotes the path-conjugation isomorphism (see [10] for the one-dimensional case and [6] for the planar case). Then $g(\ker(\eta_\#)) = \varphi_\beta \circ f_\#(\ker(\eta_\#))$. Let $a \in \ker(\eta_\#)$. By Lemma 3.4, we may write $a = [\prod_{m=1}^{\infty} \delta_m]$, where $\delta_m \in \Omega(\mathbb{H}, b_0)$ is null, $\delta_{4k-3} \equiv \delta_{4k-1}^{-}$, and $\delta_{4k-2} \equiv \delta_{4k}^{-}$. Define infinite concatenations $\alpha_1 = \delta_1 \cdot \delta_2 \cdot \delta_5 \cdot \delta_6 \cdot \delta_9 \cdot \delta_{10} \cdots$ and $\alpha_2 = \delta_3 \cdot \delta_4 \cdot \delta_7 \cdot \delta_8 \cdot \delta_{11} \cdot \delta_{12} \cdots$ and note that $[\alpha_1 \cdot \alpha_2] \in C_\tau(\{b_0\})$ by Lemma 5.6. Another application of Lemma 5.6 gives $[\prod_{m=1}^{\infty} \delta_m][\alpha_1 \cdot \alpha_2]^{-1} \in C_\tau(\{b_0\})$. Thus, $a \in C_\tau(\{b_0\})$. We conclude that

$$g(a) = \varphi_\beta \circ f_\#(a) \in [\beta] f_\#(C_\tau(\{b_0\}))[\beta]^{-1} \leq C_\tau(\{f(b_0)\}) \leq C_\tau(X)$$

in $\pi_1(X,x)$. □

In [8], Corson defined the 'strong abelianization' of a fundamental group $G = \pi_1(X,x)$ to be the quotient group $G/\overline{[G,G]}$, where $\overline{[G,G]}$ denotes the closure of the commutator subgroup in G equipped with the natural quotient topology inherited from $\Omega(X,x)$. We refer to [3] for the basic theory of the quotient topology on π_1.

REMARK 5.10 (Comparison to Corson's strong abelianization). We observe that $C_\tau(X)$ is contained in the closure $\overline{[G,G]}$ of the commutator subgroup. Consider a generator g of $C_\tau(X)$, a map f, and a conjugation homomorphism φ_β as described in Remark 5.7. Since $h = [\prod_\tau \ell_n \cdot (\prod_\tau \ell_{\phi(n)})^{-}] \in \ker(\eta_\#)$, Lemma 3.4 permits us to write $h = [\prod_{n=1}^{\infty} \gamma_n]$, where γ_n is a commutator in $\mathbb{H}_{\geq n}$. If $h_n = [\prod_{i=1}^{n} \gamma_i]$, then $\{h_n\} \to h$ in $\pi_1(\mathbb{H}, b_0)$. Therefore, $h \in \overline{[\pi_1(\mathbb{H}, b_0), \pi_1(\mathbb{H}, b_0)]}$. Note that $f_\#$ is continuous by functoriality of the quotient topology and φ_β is continuous since G is a quasitopological group. Therefore, $\{\varphi_\beta \circ f_\#(h_n)\} \to \varphi_\beta \circ f_\#(h) = g$ in $\pi_1(X,x)$. Since homomorphisms send commutators to commutators, $\varphi_\beta \circ f_\#(h_n) \in [G,G]$ for all $n \in \mathbb{N}$ and thus $g \in \overline{[G,G]}$. We conclude that $C_\tau(X) \leq \overline{[G,G]}$.

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EXAMPLE 5.11. We construct a space Y which is semilocally simply connected at all of its points and thus $C_\tau(Y) = [\pi_1(Y,x), \pi_1(Y,x)]$. However, the space Y is constructed so that

$$\overline{[\pi_1(Y,x), \pi_1(Y,x)]} \neq [\pi_1(Y,x), \pi_1(Y,x)].$$

Let $D_a \subseteq \mathbb{R}^2$ denote the circle with diameter $[0,a] \times \{0\}$ and let $\mu_a : [0,1] \to X$ be the origin-based loop traversing D_a once in the counterclockwise direction. Consider the compact metric space $X = D_1 \cup \bigcup\{D_{(n+1)/n} \mid n \in \mathbb{N}\}$ with basepoint $x = (0,0)$. Let $A = D_2 \cup D_{3/2}$ and note that $H = \pi_1(A,x)$ is the free group on two generators. It is well known that the commutator subgroup $[H,H]$ is an infinitely generated free group. Let $\{[\alpha_1], [\alpha_2], [\alpha_3], \ldots\}$ be a set of free generators for $[H,H]$. Let Y be the space obtained by attaching a 2-cell to X along the attaching loops $\mu_{(n+1)/n} \cdot \alpha_n^-$ for $n \in \mathbb{N}$. If $G = \pi_1(Y,x)$, then $[\mu_{(n+1)/n}] \in [G,G]$ for all $n \in \mathbb{N}$. Since $\{\mu_{(n+1)/n}\} \to \mu_1$ uniformly, we have $\{[\mu_{(n+1)/n}]\} \to [\mu_1]$ in the quotient topology and thus $[\mu_1] \in \overline{[G,G]}$. Hence, $\overline{[G,G]}$ contains nontrivial elements of G, which are neither commutators nor infinite products of commutators. One can construct a locally path-connected separable metric space with the same phenomenon embedded in the fundamental group by attaching a countable sequence of arcs to X whose closure includes D_1 so that X becomes a Peano continuum, and requiring Y to inherit the quotient metric topology.

LEMMA 5.12. *For subsets $A \subseteq \mathbb{H}$,*

$$C_\tau(A) = \begin{cases} \ker(\eta_\#) & \text{if } b_0 \in A, \\ [\pi_1(\mathbb{H}, b_0), \pi_1(\mathbb{H}, b_0)] & \text{if } b_0 \notin A. \end{cases}$$

PROOF. First, suppose that $b_0 \in A$. The inclusion $C_\tau(A) \leq \ker(\eta_\#)$ holds since any loop $\beta \cdot (\prod_\tau \alpha_n) \cdot (\prod_\tau \alpha_{\phi(n)})^- \cdot \beta^-$ representing a generator of $C_\tau(A)$ must have winding number 0 around each circle of $\mathbb{H}$. Let $1 \neq [\alpha] \in \ker(\eta_\#)$. By Lemma 3.4, we may assume that $\alpha \equiv \prod_{m=1}^{\infty} \delta_m$, where $\delta_m \in \Omega(\mathbb{H}, b_0)$ is null, $\delta_{4k-3} \equiv \delta_{4k-1}^-$, and $\delta_{4k-2} \equiv \delta_{4k}^-$. Note that there is a bijection ϕ such that $[\prod_{m=1}^{\infty} \delta_{\phi(m)}] = 1$. Therefore, $[\alpha]$ may be represented as

$$\Big[\prod_{m=1}^{\infty} \delta_m\Big] = \Big[\prod_{m=1}^{\infty} \delta_m\Big]\Big[\prod_{m=1}^{\infty} \delta_{\phi(m)}\Big]^{-1} = \Big[\Big(\prod_{m=1}^{\infty} \delta_m\Big) \cdot \Big(\prod_{m=1}^{\infty} \delta_{\phi(m)}\Big)^-\Big].$$

According to Remark 5.3, the right-most representation indicates that $[\alpha]$ is an element of $C_\tau(\{b_0\})$. Since $C_\tau(\{b_0\}) \leq C_\tau(A)$, it follows that $[\alpha] \in C_\tau(A)$. This verifies the inclusion $\ker(\eta_\#) \leq C_\tau(A)$.

When $b_0 \notin A$, we first note that $[\pi_1(\mathbb{H}, b_0), \pi_1(\mathbb{H}, b_0)] \leq C_\tau(A)$ always holds. Consider any loop $\gamma = \beta \cdot (\prod_\tau \alpha_n) \cdot (\prod_\tau \alpha_{\phi(n)})^- \cdot \beta^-$ representing a generator of $C_\tau(A)$ and where $\alpha_n \in \Omega(\mathbb{H}, a)$. Since $\mathbb{H}$ is locally contractible at a, this loop is homotopic to a path of the form $\beta \cdot (\prod_{i=1}^{m} \delta_i) \cdot (\prod_{i=1}^{m} \delta_{\psi(i)})^- \cdot \beta^-$ for some bijection $\psi : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, m\}$. Such a loop is clearly null-homologous in $\mathbb{H}$. Therefore, $[\gamma]$ lies in the commutator subgroup of $\pi_1(\mathbb{H}, b_0)$. □

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COROLLARY 5.13. *For subsets $A \subseteq \mathbb{H}$, we have the natural isomorphism*

$$\mathfrak{H}_A(\mathbb{H}) \cong \begin{cases} \check{H}_1(\mathbb{H}) & \text{if } b_0 \in A, \\ H_1(\mathbb{H}) & \text{if } b_0 \notin A. \end{cases}$$

LEMMA 5.14. *Given a based space X and a set $A \subseteq X$, let Y be the space obtained by attaching 2-cells to X along all loops of the form $\prod_\tau \alpha_n \cdot (\prod_\tau \alpha_{\phi(n)})^-$, where $\{\alpha_n\}$ is a null sequence of loops based at a point $a \in A$ and $\phi : \mathbb{N} \to \mathbb{N}$ is a bijection. Then Y is transfinitely π_1-commutative at each point in $A \subseteq Y$ and the inclusion $i : X \to Y$ induces an isomorphism $\mathfrak{H}_A(X) \to \pi_1(Y, x)$.*

PROOF. Suppose that $f : (\mathbb{H}, b_0) \to (Y, a)$ is a map based at $a \in A$. Employing Theorem 3.5, we show that $f_\#$ factors through $\eta_\#$. Since $f(\mathbb{H})$ is compact, it may intersect at most finitely many of the attached 2-cells. Call these cells $e_1, e_2, \ldots, e_m$. Let d_j be a closed disk in int (e_j) so that $U_j = e_j \backslash d_j$ deformation retracts onto ∂e_j for all j. Then $X' = X \cup U_1 \cup U_2 \cup \cdots \cup U_m$ is an open neighborhood of a in Y equipped with a deformation retraction $r : X' \times [0, 1] \to X'$ with $r(x', 0) = x'$, $r(x, t) = x$ for $x \in X$, and $r(x', 1) \in X$ for $x' \in X'$. Find $N \in \mathbb{N}$ such that $f(\mathbb{H}_{\geq N}) \subseteq X'$. Define $r_1 : X' \to X$ by $r_1(x') = r(x', 1)$ and $f' : \mathbb{H} \to X$ so that $f'|_{\mathbb{H}_{\geq N}} = r_1 \circ f|_{\mathbb{H}_{\geq N}}$ and, if $n \in \{1, 2, \ldots, N-1\}$, let $f' \circ \ell_n$ be any based loop in X that is homotopic to $f \circ \ell_n$ in Y. We have $f \simeq i \circ f'$ rel. basepoint and thus $f_\# = i_\# \circ (f')_\#$ as homomorphisms $\pi_1(\mathbb{H}, b_0) \to \pi_1(Y, a)$. Recall from Lemma 5.12 that $C_\tau(\{b_0\}) = \ker(\eta_\#)$ in $\pi_1(\mathbb{H}, b_0)$ and note that, by construction, we have $C_\tau(A) = \ker(i_\#)$ in $\pi_1(X, a)$. Thus, $f_\#(\ker(\eta_\#)) = i_\# \circ (f')_\#(C_\tau(\{b_0\})) \leq i_\#(C_\tau(A)) = 0$. The last statement of the lemma follows from the fact that $\ker(i_\#) = C_\tau(A)$ in $\pi_1(X, x)$. □

EXAMPLE 5.15 (Harmonic archipelago). The harmonic archipelago is the space $\mathbb{HA}$ obtained from $\mathbb{H}$ by attaching 2-cells along the loops $\ell_n \cdot \ell_{n+1}^-$, $n \in \mathbb{N}$. Let $i : \mathbb{H} \to \mathbb{HA}$ denote the inclusion. Note that $\mathbb{HA}$ is locally contractible at all points except for b_0. Thus, if $b_0 \notin A$, then $\mathfrak{H}_A(\mathbb{HA}) \cong H_1(\mathbb{HA})$ (see [20] for more on this homology group). If $b_0 \in A$, we have $C_\tau(\{b_0\}) = C_\tau(A)$ and only need to consider the case $A = \{b_0\}$. Recall from Example 5.12 that we may identify $\mathfrak{H}_{\{b_0\}}(\mathbb{H})$ with $\mathbb{Z}^{\mathbb{N}}$ and the projection $\pi_1(\mathbb{H}, b_0) \to \mathfrak{H}_{\{b_0\}}(\mathbb{H})$ with $\eta_\#$. Let Y be the space obtained from $\mathbb{H}$ by attaching 2-cells along all of the loops $\prod_\tau \alpha_n \cdot (\prod_\tau \alpha_{\phi(n)})^-$ in $\mathbb{H}$, where $\alpha_n \in \Omega(\mathbb{H}, b_0)$ is a null sequence and $\phi : \mathbb{N} \to \mathbb{N}$ is a bijection. By Lemma 5.14, Y is transfinitely π_1-commutative at b_0. Applying Example 5.12 and the second statement of Lemma 5.14 gives that the inclusion $\mathbb{H} \to Y$ induces an isomorphism $\mathfrak{H}_{\{b_0\}}(\mathbb{H}) \to \mathfrak{H}_{\{b_0\}}(Y)$.

Let Z be the space obtained from Y by attaching 2-cells along the same set of loops $\ell_n \cdot \ell_{n+1}^-$ in $\mathbb{H}$. Since every loop in $\Omega(\mathbb{HA}, b_0)$ is homotopic rel. basepoint to a loop in $\mathbb{H}$ and every map $(\mathbb{H}, b_0) \to (\mathbb{HA}, b_0)$ is homotopic rel. basepoint to a map $(\mathbb{H}, b_0) \to (\mathbb{H}, b_0)$ (using the same deformation-retraction argument used to prove Lemma 5.14), we may apply Lemma 5.14 to see that Z is transfinitely π_1-commutative at b_0 and that the inclusion $\mathbb{HA} \to Z$ induces an isomorphism $\mathfrak{H}_{\{b_0\}}(\mathbb{HA}) \to \mathfrak{H}_{\{b_0\}}(Z)$. Consider the following diagram of surjective homomorphisms, where the top square and bottom horizontal map are induced by the natural inclusions. The bottom pair of vertical maps are the projections, which are isomorphisms since both Y and Z are transfinitely π_1 commutative at b_0.

$$\begin{array}{ccc}
\mathfrak{H}_{\{b_0\}}(\mathbb{H}) = \mathbb{Z}^{\mathbb{N}} & \longrightarrow & \mathfrak{H}_{\{b_0\}}(\mathbb{HA}) \\
\cong\downarrow & & \downarrow\cong \\
\mathfrak{H}_{\{b_0\}}(Y) & \longrightarrow & \mathfrak{H}_{\{b_0\}}(Z) \\
\cong\uparrow & & \uparrow\cong \\
\pi_1(Y, b_0) & \longrightarrow & \pi_1(Z, b_0)
\end{array}$$

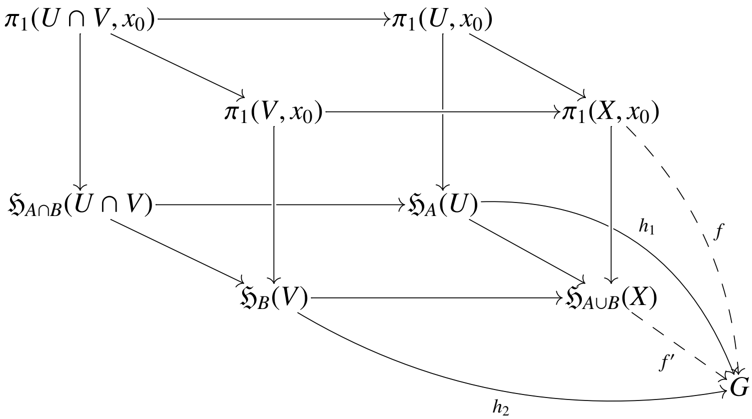
Since we may identify the middle and bottom horizontal maps and Z is obtained from Y by attaching 2-cells along the loops $\ell_n \cdot \ell_{n+1}^{-}$, we have that $\mathfrak{H}_{\{b_0\}}(Z) \cong \mathbb{Z}^{\mathbb{N}}/N$, where N is the subgroup generated by $\{\mathbf{e}_n - \mathbf{e}_{n+1} \mid n \in \mathbb{N}\}$. We conclude that $\mathfrak{H}_{\{b_0\}}(\mathbb{HA})$ is naturally isomorphic to $\mathbb{Z}^{\mathbb{N}}/N$; it is a straightforward exercise to show that $\mathbb{Z}^{\mathbb{N}}/N$ is abstractly isomorphic to $\mathbb{Z}^{\mathbb{N}}/\oplus_{\mathbb{N}}\mathbb{Z}$.

EXAMPLE 5.16. In [19], Karimov constructed a space $\widehat{X}$ which is the one-point compactification of a countable CW-complex and for which every loop at the canonical basepoint $\widehat{x}$ is homotopic to an infinite concatenation of commutator loops. Hence, the group $G = \pi_1(\widehat{X}, \widehat{x})$ is perfect in an infinitary sense, namely $G = C_\tau(\widehat{X})$ holds in a strong way. We remark that the space $\widehat{X}$ is nicely described in [1], where it is also shown that $H_1(\widehat{X})$ is uncountable.

THEOREM 5.17. *If $U, V, U \cap V$ are path-connected open sets with $X = U \cup V$ and $x_0 \in U \cap V$, and $A \subseteq U$, $B \subseteq V$, then the following diagram induced by the inclusion maps is a pushout square in the category of abelian groups.*

$$\begin{array}{ccc}
\mathfrak{H}_{A\cap B}(U \cap V) & \longrightarrow & \mathfrak{H}_A(U) \\
\downarrow & & \downarrow \\
\mathfrak{H}_B(V) & \longrightarrow & \mathfrak{H}_{A\cup B}(X)
\end{array}$$

PROOF. Let G be an abelian group and consider homomorphisms $h_1 : \mathfrak{H}_A(U) \to G$ and $h_2 : \mathfrak{H}_B(V) \to G$ making the bottom portion of the diagram below commute. The vertical morphisms are the natural projection maps. The top portion of the diagram is a pushout diagram by the van Kampen theorem. Hence, there is a unique homomorphism $f : \pi_1(X, x_0) \to G$ making the diagram commute in the obvious way. Since the vertical morphisms are surjective, in order to show that f induces the desired homomorphism $f' : \mathfrak{H}_{A\cup B}(X) \to G$, it suffices to show that $f(C_\tau(A \cup B)) = 0$.



Consider a loop $\gamma = \beta \cdot (\prod_\tau \alpha_n) \cdot (\prod_\tau \alpha_{\phi(n)})^- \cdot \beta^-$, where $\beta : I \to X$ is a path from x_0 to $a \in A \cup B$, $\alpha_n \in \Omega(X, a)$ is a null sequence, and $\phi : \mathbb{N} \to \mathbb{N}$ is a bijection. We prove that $f([\gamma]) = 0$ when $a \in A$. The proof when $a \in B$ is identical. Find $N \in \mathbb{N}$ such that Im $(\alpha_n) \subseteq U$ for all $n \geq N$.

For any $k < N$, we may write $\prod_\tau \alpha_n \equiv \delta_1 \cdot \alpha_k \cdot \delta_2$ and $\prod_\tau \alpha_\phi(n) \equiv \delta_3 \cdot \alpha_k \cdot \delta_4$ for subloops $\delta_1, \delta_2, \delta_3, \delta_4$. Then we have that $[\gamma]$ is equal to

$$[\beta \cdot \delta_1 \cdot \beta^-][\beta \cdot \alpha_k \cdot \beta^-][\beta \cdot \delta_2 \cdot \beta^-][\beta \cdot \delta_4^- \cdot \beta^-][\beta \cdot \alpha_k^- \cdot \beta^-][\beta \cdot \delta_3^- \cdot \beta^-]$$

by inserting inverse pairs. Since G is abelian, we may apply f to this product and rearrange the factors to see that

$$f([\gamma]) = f([\beta \cdot (\delta_1 \cdot \delta_2) \cdot (\delta_3 \cdot \delta_4)^- \cdot \beta^-]),$$

where the loop on the right is obtained from γ by deleting the appearances of α_k and α_k^-. In other words, we may delete α_k from the null sequence $\{\alpha_n\}$ in the definition of γ. Applying this deletion procedure for all $k \in \{1, 2, \ldots, N-1\}$, we may replace $\{\alpha_n\}$ with a null sequence for which $\mathrm{Im}(\alpha_n) \subseteq U$ for all $n \in \mathbb{N}$. If β does not have image in U, we may write $\beta \equiv \beta_1 \cdot \beta_2$, where $\beta_1(1) \in U \cap V$ and $\mathrm{Im}(\beta_2) \subseteq U$. Let $\beta_3 : I \to U \cap V$ be a path from x_0 to $\beta_1(1)$. Now

$$[\gamma] = [\beta_1 \cdot \beta_3^-]\Big[(\beta_3 \cdot \beta_2) \cdot \prod_\tau \alpha_n \cdot \Big(\prod_\tau \alpha_{\phi(n)}\Big)^- \cdot (\beta_3 \cdot \beta_2)^-\Big][\beta_1 \cdot \beta_3^-]^{-1}.$$

Once again applying the fact that G is abelian, we may delete the conjugating element $[\beta_1 \cdot \beta_3^-] \in \pi_1(X, x_0)$ without changing $f([\gamma])$. Thus, by replacing β with $\beta_3 \cdot \beta_2$, we may assume that $\mathrm{Im}(\beta) \subseteq U$.

Since we may assume that β and each α_n lies in U, it is clear that $[\gamma] \in C_\tau(A) \leq \pi_1(U, x_0)$. Therefore, $\langle\gamma\rangle = 0 \in \mathfrak{H}_A(U)$. Using the above diagram, it now follows that $f([\gamma]) = 0$. □

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EXAMPLE 5.18 (Double Hawaiian earring). Consider the space X consisting of two copies of the Hawaiian earring adjoined by an arc. More precisely, take $\mathbb{H}'$ to be a homeomorphic copy of $\mathbb{H}$ with $x \in \mathbb{H}$ corresponding to $x' \in \mathbb{H}'$ and let $X = \mathbb{H} \cup I \cup \mathbb{H}'$, where $0 \sim b_0$ and $1 \sim b_0'$. Take U and V to be the images of $\mathbb{H} \cup [0, 2/3)$ and $(1/3, 1] \cup \mathbb{H}'$ respectively and $x_0 = 1/2$ to be the basepoint. Applying Theorem 5.17 and Corollary 5.13, we have natural isomorphisms

$$\mathfrak{H}_A(X) \cong \begin{cases} \check{H}_1(\mathbb{H}) \oplus \check{H}_1(\mathbb{H}') & \text{if } \{b_0, b_0'\} \subseteq A, \\ \check{H}_1(\mathbb{H}) \oplus H_1(\mathbb{H}') & \text{if } \{b_0, b_0'\} \cap A = \{b_0\}, \\ H_1(\mathbb{H}) \oplus \check{H}_1(\mathbb{H}') & \text{if } \{b_0, b_0'\} \cap A = \{b_0'\}, \\ H_1(\mathbb{H}) \oplus H_1(\mathbb{H}') & \text{if } \{b_0, b_0'\} \cap A = \emptyset. \end{cases}$$

According to Lemma 5.14, one may attach 2-cells to X to create a space Y with abelian fundamental group, which is transfinitely π_1-commutative at either b_0, b_0' or at both b_0 and b_0'.

It is well known that when the basepoints of spaces X and Y lie in $\mathbf{aw}(X)$ and $\mathbf{aw}(Y)$, respectively, it is not necessarily true that $H_1(X \vee Y)$ is isomorphic to $H_1(X) \oplus H_1(Y)$. We show that such an isomorphism does hold for our notion of transfinite abelianization.

THEOREM 5.19. *Suppose that $x_0 \in A \subseteq X$, $y_0 \in B \subseteq Y$, and $X \vee Y = (X, x_0) \vee (Y, y_0)$ is the one-point union with canonical basepoint w_0. If $A \vee B$ is the image of $A \cup B$ in the quotient, then there is a canonical isomorphism $\Psi : \mathfrak{H}_{A\vee B}(X \vee Y) \to \mathfrak{H}_A(X) \oplus \mathfrak{H}_B(Y)$.*

PROOF. The natural retractions $r_1 : X \vee Y \to X$ and $r_2 : X \vee Y \to Y$ induce the horizontal homomorphisms $\psi([\alpha]) = ([r_1 \circ \alpha], [r_2 \circ \alpha])$ and $\psi(\langle\alpha\rangle) = (\langle r_1 \circ \alpha\rangle, \langle r_2 \circ \alpha\rangle)$ in the diagram below (here we are invoking Proposition 5.4(5)). The vertical maps are the canonical projections.

$$\begin{array}{ccc} \pi_1(X \vee Y, w_0) & \xrightarrow{\ \psi\ } & \pi_1(X, x_0) \times \pi_1(Y, y_0) \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle g_1 \times g_2} \\ \mathfrak{H}_{A\vee B}(X \vee Y) & \xrightarrow[\ \Psi\ ]{} & \mathfrak{H}_A(X) \oplus \mathfrak{H}_B(Y) \end{array}$$

Since the top and vertical maps are surjective, so is Ψ. For injectivity, suppose that $\alpha \in \Omega(X \vee Y, w_0)$ is nonconstant and such that $\Psi(g([\alpha])) = (\langle r_1 \circ \alpha\rangle, \langle r_2 \circ \alpha\rangle) = (0, 0)$ in $\mathfrak{H}_A(X) \oplus \mathfrak{H}_B(Y)$. Then $[r_1 \circ \alpha] \in C_\tau(A) \le \pi_1(X, x_0)$ and $[r_2 \circ \alpha] \in C_\tau(B) \le \pi_1(Y, y_0)$. It suffices to show that $[\alpha] \in C_\tau(A \vee B)$. By collapsing constant subpaths if necessary, we may assume that the closed set $\alpha^{-1}(w_0)$ containing $\{0, 1\}$ is nowhere dense. As previously defined, $\mathcal{I}(\alpha^{-1}(x_0))$ is the ordered set of components of $I \backslash \alpha^{-1}(x_0)$. Consider the suborders $\mathcal{O}_X = \{K \in \mathcal{I}(\alpha^{-1}(x_0)) \mid \alpha(\overline{K}) \subseteq X\}$ and $\mathcal{O}_Y = \{K \in \mathcal{I}(\alpha^{-1}(x_0)) \mid \alpha(\overline{K}) \subseteq Y\}$. Find a closed nowhere-dense subset C_X of $[0, 1/2]$ containing $\{0, 1/2\}$ such that $\mathcal{I}(C_X)$ has the order type of $\mathcal{O}_X$. Similarly, find a closed nowhere-dense

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subset C_Y of $[1/2,1]$ containing $\{1/2,1\}$ such that $\mathcal{I}(C_Y)$ has the order type of $\mathscr{O}_Y$. Fix order-preserving bijections $\Phi_X : \mathcal{I}(C_X) \to \mathscr{O}_X$ and $\Phi_Y : \mathcal{I}(C_Y) \to \mathscr{O}_Y$ and define a bijection $\Phi : \mathcal{I}(\alpha^{-1}(x_0)) \to \mathcal{I}(C_X \cup C_Y)$ so that $\Phi(K) = \Phi_X(K)$ if $K \in \mathscr{O}_X$ and $\Phi(K) = \Phi_Y(K)$ if $K \in \mathscr{O}_Y$. Define $\alpha' : I \to X$ to be a loop so that if $K \in \mathcal{I}(C_X \cup C_Y)$, then $\alpha'|_{\overline{K}} \equiv \alpha|_{\overline{\Phi^{-1}(K)}}$. Note that α and α' satisfy the hypotheses of Lemma 5.6 and thus $[\alpha][\alpha']^{-1} \in C_\tau(\{w_0\}) \leq C_\tau(A \vee B)$ in $\pi_1(X \vee Y, w_0)$. Thus, it suffices to show that $[\alpha'] \in C_\tau(A \vee B)$.

By construction, we have $\alpha' = \alpha'_1 \cdot \alpha'_2$, where $\alpha'_i = r_i \cdot \alpha' \simeq r_i \circ \alpha$ for $i \in \{1,2\}$. Thus, $[\alpha'_1] \in C_\tau(A)$ and $[\alpha'_2] \in C_\tau(B)$. Hence, α'_1 is homotopic in X to a finite product of loops of the form $\beta \cdot (\prod_\tau \gamma_n) \cdot (\prod_\tau \gamma_\phi(n)) \cdot \beta^-$, where $\beta : I \to X$ is a path from x_0 to a point $a \in A$, $\gamma_n \in \Omega(X, a)$ is null, and $\phi : \mathbb{N} \to \mathbb{N}$ is a bijection. Similarly, α'_2 is homotopic in Y to a finite product of loops of the same form in Y but where $\beta(1) = b \in B$ and $\gamma_n \in \Omega(Y, b)$. By concatenating these two finite products, we see that $\alpha' = \alpha'_1 \cdot \alpha'_2$ is homotopic in $X \vee Y$ to a finite product of loops of the form $\beta \cdot \prod_\tau \gamma_n \cdot (\prod_\tau \gamma_\phi(n)) \cdot \beta^-$, where $\beta(1) \in A \vee B$, $\gamma_n \in \Omega(X \vee Y, \beta(1))$ is null, and $\phi : \mathbb{N} \to \mathbb{N}$ is a bijection. Thus, $[\alpha'] \in C_\tau(A \vee B)$. □

Since a compact subset of an infinite one-point union $\bigvee_j X_j$ (with the weak topology) must lie in the union of finitely many X_j, we have the following consequence.

COROLLARY 5.20. *If $\bigvee_j X_j$ is the one-point union of a family of based path-connected spaces $\{(X_j, x_j) \mid j \in J\}$, $x_j \in A_j \subseteq X_j$ for all $j \in J$, and $A = \bigvee_j A_j$, then*

$$\mathfrak{H}_A\Big(\bigvee_{j\in J} X_j\Big) \cong \bigoplus_{j\in J} \mathfrak{H}_{A_j}(X_j).$$

REMARK 5.21. It is also true that infinitary abelianization preserves finite products in the sense that if $A \subseteq X$ and $B \subseteq Y$, then there is a canonical isomorphism $\mathfrak{H}_A(X) \oplus \mathfrak{H}_B(Y) \cong \mathfrak{H}_{A\times B}(X \times Y)$. We do not have immediate need of this result and leave the proof as an exercise.

In our final example, we observe a nontrivial application of Theorem 5.19.

EXAMPLE 5.22 (Griffiths twin cone). Let $C\mathbb{H} = \mathbb{H} \times I/\mathbb{H} \times \{1\}$ be the cone over the Hawaiian earring. Let x_0 be the image of $(b_0, 0)$ in the quotient and take this to be the basepoint of $C\mathbb{H}$. The *Griffiths twin cone* is the one-point union $\mathbb{G} = C\mathbb{H} \vee C\mathbb{H}$ [17]. Since $C\mathbb{H}$ is simply connected, we have $\mathfrak{H}_A(C\mathbb{H}, x_0) = 0$ for all $A \subseteq C\mathbb{H}$. Although $H_1(\mathbb{G})$ is uncountable and abstractly isomorphic to $\mathbb{Z}^{\mathbb{N}}/\oplus_{\mathbb{N}} \mathbb{Z}$ (as shown in [12]), we have the following description of $\mathfrak{H}_A(\mathbb{G})$ by Theorem 5.19 and the fact that $\mathbf{aw}(\mathbb{G}) = \{x_0\}$ contains only the wedge point:

$$\mathfrak{H}_A(\mathbb{G}) = \begin{cases} 0 & \text{if } x_0 \in A, \\ H_1(\mathbb{G}) & \text{if } x_0 \notin A. \end{cases}$$

More generally, we may conclude that for any spaces X and Y, we have $\mathfrak{H}_A(CX \vee CY) = 0$ when A contains the wedge point. It is known that $H_1(\mathbb{HA}) \cong H_1(\mathbb{G})$ and it remains

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a difficult open problem if the groups $\pi_1(\mathbb{HA}, b_0)$ and $\pi_1(\mathbb{G}, x_0)$ are isomorphic [20, Problem 1.1]. We have shown that $C_\tau(\{b_0\})$ is nontrivial and proper in $\pi_1(\mathbb{HA}, b_0)$ while $C_\tau(\{x_0\}) = \pi_1(\mathbb{G}, x_0)$. This distinction identifies a fundamental difference in the infinitary algebraic structure of their fundamental groups. However, it does not immediately distinguish the group-isomorphism types of $\pi_1(\mathbb{HA}, b_0)$ and $\pi_1(\mathbb{G}, x_0)$ since it remains unclear if $\pi_1(\mathbb{HA}, b_0)^{\sigma'} = \pi_1(\mathbb{HA}, b_0)$.

COROLLARY 5.23. $\pi_1(\mathbb{G}, x_0)^{\sigma'} = \pi_1(\mathbb{G}, x_0)$.

REMARK 5.24. We conclude this paper by reminding the reader that the transfinite commutator group $C_\tau(A)$ is the smallest subgroup of $\pi_1(X, x_0)$ generated by geometrically represented infinite products of commutators (based at the points of A). However, $C_\tau(A)$ might not be closed under homotopy classes of infinite concatenations of loop generators of the form $\beta \cdot (\prod_\tau \alpha_n) \cdot (\prod_\tau \alpha_{\phi(n)})^- \cdot \beta^-$, nor infinite concatenations of such infinite concatenations, etc. This is the reason why one should not expect an isomorphism $\mathfrak{H}(\widetilde{\bigvee}_n X_n) \cong \prod_{n\in\mathbb{N}} \mathfrak{H}(X_n)$ for a shrinking wedge $\widetilde{\bigvee}_n X_n$ of an arbitrary sequence of based path-connected spaces $\{(X_n, x_n)\}_{n\in\mathbb{N}}$ unless $\mathbf{aw}(X_n) \subseteq \{x_n\}$ for all $n \in \mathbb{N}$. By employing the closure-operator framework of [4], one may effectively introduce larger infinitary commutator subgroups so that such an isomorphism does hold. In particular, one may recursively construct a growing transfinite sequence of infinitary commutator subgroups (all containing $C_\tau(A)$) as well as other 'uniform' notions of commutator subgroup, more closely aligned with Corson's definition. We plan to consider and compare such constructions in future research that is beyond the scope of the current paper.

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