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Non-commutative logic II: sequent calculus and phase semantics

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Dedicated to Jim Lambek on the occasion of his 75th birthday

Non-commutative logic, which is a unification of commutative linear logic and cyclic linear logic, is extended to all linear connectives: additives, exponentials and constants. We give two equivalent versions of the sequent calculus (directly with the structure of order varieties, and with their presentations as partial orders), phase semantics and a cut-elimination theorem. This involves, in particular, the study of the entropy relation between partial orders, and the introduction of a special class of order varieties: the series–parallel order varieties.

1. Introduction

Non-commutative logic is a unification of:

— commutative linear logic (Girard 1987) and

— cyclic linear logic (Girard 1989; Yetter 1990), which is a classical conservative extension of the Lambek calculus (Lambek 1958).

In a previous paper with Abrusci (Abrusci and Ruet 2000) we presented the multiplicative fragment of non-commutative logic, with proof nets and a sequent calculus based on the structure of order varieties, and a sequentialization theorem. Here we consider full propositional non-commutative logic.

Non-commutative logic

Let us first review the basic ideas. Consider the purely non-commutative fragment of linear logic, obtained by completely removing the exchange rule

$$\frac{\vdash \Gamma, \Delta, \Sigma, \Pi}{\vdash \Gamma, \Sigma, \Delta, \Pi},$$

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and say we want to introduce commutative connectives. First, we cannot just remove exchange entirely, because we need to be able to distinguish one formula, sometimes two, in a sequent, and separate it from the context; a reasonable solution is to admit cyclic permutations[†]:

$$\frac{\vdash \Delta, \Sigma}{\vdash \Sigma, \Delta},$$

with the well-known consequence that there is a single negation.

In order to combine a commutative conjunction and a non-commutative conjunction, we are naturally led to the idea of a single conjunction on a partial order: A *times* B can therefore have three different meanings, depending on the order between A and B. We are then faced with the following problems:

1 *Entropy*. We must be able to replace a partial order by a weaker one (or a stronger one, depending on the connective considered) – for instance a totally ordered sequent by an unordered one.
2 *Contexts*. There is a difficulty in isolating a formula from its context: should we write a sequent as $\vdash \Gamma, A$ or $\vdash A, \Gamma$, not to mention A in parallel with Γ, and other configurations. . . ?
3 *Cyclicity*. Since the non-commutative fragment should be cyclic, our system has to allow some kind of cyclicity, to move a formula A from left to right in a sequent, *etc.* The number of negations however, remains open: there could be a commutative one and a non-commutative one. . . .

Order varieties

The solution is based on:

— A syntactic idea – the seesaw rule.
— Its semantic counterpart – the structure of order variety.

Order varieties – see Ruet (1997) and Abrusci and Ruet (2000), and Section 3 – are structures that can be presented by *partial* orders in several ways, a good analogy being the oriented circle, which becomes a total order as soon as an origin is fixed. An essential property of order varieties (Proposition 2.5) is that in a sequent $\vdash \Gamma$ structured by an order variety, any formula can be isolated. For instance, if we focus on $A \in \Gamma$, the context Γ' becomes a partial order, and $\vdash \Gamma$ can be presented as a partial order in three different ways[‡]: $\vdash \Gamma'; A$, $\vdash A; \Gamma'$ or $\vdash \Gamma', A$. The point is that the three presentations are equivalent (Proposition 3.6): we can forget about the choice of A (Proposition 3.7) and this process of 'swinging' is completely reversible.

This solves Problem 2, but also Problem 3 at the same time: indeed, the restriction of the sequent calculus to the class of *total* order varieties is precisely cyclic LL. This is easily visualized in the calculus on presentations (Section 2), where the reversible structural rules (seesaw, and its inverse, which is a particular case of entropy) enable one to change a

[†] See Appendix F in Girard (1999) for the associativity problems in the absence of cyclicity.
[‡] From now on, commas and semicolons, respectively, denote the parallel and serial compositions of orders.

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parallel composition into a serial composition and *vice-versa*. A significant consequence is that there is only one negation. Another consequence of the seesaw rule is that there is a single unit **1** for both multiplicative conjunctions (and, dually, a single unit $\perp$ for both multiplicative disjunctions).

By the way, we are only dealing with binary connectives here, so another class of order varieties will play an important role, namely the *series–parallel* order varieties, introduced and characterized in Sections 3.2 and 3.3. They are the order varieties that can be presented by a series–parallel order.

Entropy

Finally, Problem 1: the good notion of entropy is the inclusion of *order varieties* – see Section 3.5. At the level of series–parallel orders, that is, the presentations of series–parallel order varieties, this corresponds to the ability to move arbitrarily from serial to parallel composition. Note that this is not quite what one could expect: for instance one might have expected that the right notion be something like the inclusion of orders, but this is too strong, since the two conjunctions become commutative, and even equivalent. Even other intermediate notions of refinement, such as '$\vdash \Gamma;(\Delta,\Sigma)$ implies $\vdash(\Gamma;\Delta),\Sigma$', lead to the same problem.

Exponentials

For the extension to exponentials, there are several possible choices:

1. ?ed formulas do not commute in a non-commutative situation: from $\vdash \Gamma[?A;B]$, we may not infer $\vdash \Gamma[B;?A]$, even if B is itself a ?ed formula. This has been considered by Demaille in Demaille (1999).
2. Bags of ?ed formulas commute. This has been considered in Ruet (1997), and it is consistent with the intuition that there is basically a single par – and a single tensor – and the isomorphisms: $(?A;?B)\cong ?(A\oplus B)\cong(?A,?B)$.
3. ?ed formulas are central: they commute with everyone. This is the choice we make here, as it is simpler than 2, while preserving the above isomorphisms.

Sequent calculus

There are two natural ways of describing the sequent calculus: either directly on order varieties or on the presentations (partial orders). We shall give both descriptions (Sections 2 and 4) and prove that they are equivalent, because the calculus on orders is closer to the phase semantics, and, on the other hand, order varieties give a better understanding of the structural rules and are necessary for the connection with proof nets.

The present sequent calculus on order varieties differs from the one given in Abrusci and Ruet (2000) essentially by the fact that it contains an explicit rule for entropy. The absence of rule for entropy in Abrusci and Ruet (2000) was motivated by the desired connection with proof nets, which indeed do not have entropy links. In Section 4.2, we show that the two presentations are equivalent, relying on non-trivial results from Sections 3.6 and 3.7, where we develop a bit further the theory of order varieties by studying:

— the appropriate notion of inf of order varieties,
— the operation of identification of two points in an order variety.

The crucial point is that it is always possible to identify two points in an order variety, and that this modifies the context in a way that can be simulated by entropy.

Phase semantics and cut elimination

Finally, in Section 5 we give a phase semantics. At this stage, we think it lacks a simple and natural construction; still it enables one to prove (weak) cut elimination (Section 6), using a technique due to Okada (Okada 1994).

2. Sequent calculus: on partial orders

2.1. *Language*

Definition 2.1. The *formulas (of NL)* are built from atoms $p, q, \ldots, p^\perp, q^\perp, \ldots$, constants $\mathbf{1}$, $\perp$ (multiplicative), $\top$, $\mathbf{0}$ (additive), and the following multiplicative connectives:

— non-commutative conjunction $\odot$ *(next)* and disjunction $\triangledown$ *(sequential)*,
— commutative multiplicative conjunction $\otimes$ *(times)* and disjunction $\parr$ *(par)*,
— additive conjunction & *(with)* and disjunction $\oplus$ *(plus)*,
— exponentials: ! *(of course)* and ? *(why not)*.

Definition 2.2 (Negation). *Negation* is defined by De Morgan rules:

$$
\begin{array}{ll}
(p)^\perp = p^\perp & (p^\perp)^\perp = p \\
(A \odot B)^\perp = B^\perp \triangledown A^\perp & (A \triangledown B)^\perp = B^\perp \odot A^\perp \\
(A \otimes B)^\perp = B^\perp \parr A^\perp & 1(A \parr B)^\perp = B^\perp \otimes A^\perp \\
(A \,\&\, B)^\perp = B^\perp \oplus A^\perp & (A \oplus B)^\perp = B^\perp \,\&\, A^\perp \\
(!A)^\perp = ?A^\perp & (?A)^\perp = !A^\perp \\
\mathbf{1}^\perp = \perp & \perp^\perp = \mathbf{1} \\
\top^\perp = \mathbf{0} & \mathbf{0}^\perp = \top
\end{array}
$$

Negation is then an involution: for any formula A, $A^{\perp\perp} = A$.

Definition 2.3 (Implications).

$$A \multimap B = A^\perp \parr B, \quad A \mathbin{-\!\bullet} B = A^\perp \triangledown B, \quad B \mathbin{\bullet\!-} A = B \triangledown A^\perp.$$

2.2. *Rules*

Definition 2.4. *Sequents* are of the form $\vdash \Gamma$, where Γ is an expression built from formulas and binary constructors $(-,-)$ and $(-;-)$.

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Table 1. *Sequent calculus I.*

Identity - Cut

$$\frac{}{\vdash A^{\perp}, A} \qquad \frac{\vdash \Gamma, A \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\ \text{cut}$$

Associativity - Commutativity

$$\frac{\vdash \Pi[\Gamma;(\Delta;\Sigma)]}{\vdash \Pi[(\Gamma;\Delta);\Sigma]}\ \text{a1} \qquad \frac{\vdash \Pi[(\Gamma;\Delta);\Sigma]}{\vdash \Pi[\Gamma;(\Delta;\Sigma)]}\ \text{1a}$$

$$\frac{\vdash \Pi[\Gamma,(\Delta,\Sigma)]}{\vdash \Pi[(\Gamma,\Delta),\Sigma]}\ \text{a2} \qquad \frac{\vdash \Pi[(\Gamma,\Delta),\Sigma]}{\vdash \Pi[\Gamma,(\Delta,\Sigma)]}\ \text{2a} \qquad \frac{\vdash \Pi[\Gamma,\Delta]}{\vdash \Pi[\Delta,\Gamma]}\ \text{com}$$

Structural rules

$$\frac{\vdash \Gamma;\Delta;\Sigma}{\vdash \Gamma,\Delta,\Sigma}\ \text{entropy} \qquad \frac{\vdash \Gamma,\Delta}{\vdash \Gamma;\Delta}\ \text{seesaw} \qquad \frac{\vdash ?\Gamma,\Delta,\Sigma}{\vdash ?\Gamma;\Delta;\Sigma}\ \text{centre}$$

Multiplicatives

$$\frac{\vdash \Gamma;A \qquad \vdash B;\Delta}{\vdash \Gamma;A \odot B;\Delta}\ \odot \qquad \frac{\vdash \Gamma;A;B}{\vdash \Gamma;A \triangledown B}\ \triangledown$$

$$\frac{\vdash \Gamma,A \qquad \vdash B,\Delta}{\vdash \Gamma,A \otimes B,\Delta}\ \otimes \qquad \frac{\vdash \Gamma,A,B}{\vdash \Gamma,A \parr B}\ \parr$$

Additives

$$\frac{\vdash \Gamma,A \qquad \vdash \Gamma,B}{\vdash \Gamma,A \& B}\ \& \qquad \frac{\vdash \Gamma,A}{\vdash \Gamma,A \oplus B}\ \oplus 1 \qquad \frac{\vdash \Gamma,B}{\vdash \Gamma,A \oplus B}\ \oplus 2$$

Exponentials

$$\frac{\vdash \Gamma,A}{\vdash \Gamma,?A}\ \text{d} \qquad \frac{\vdash ?\Gamma,A}{\vdash ?\Gamma,!A}\ ! \qquad \frac{\vdash \Gamma,?A,?A}{\vdash \Gamma,?A}\ \text{c} \qquad \frac{\vdash \Gamma}{\vdash \Gamma,?A}\ \text{w}$$

Constants

$$\frac{}{\vdash \mathbf{1}} \qquad \frac{\vdash \Gamma}{\vdash \Gamma,\perp}\ \perp \qquad \text{(no rule for } \mathbf{0}\text{)} \qquad \frac{}{\vdash \Gamma,\top}$$

The rules of the sequent calculus are given in Table 1. In the following, $\Gamma[\,]$ denotes an expression with a hole (a leaf of the expression Γ), and $\Gamma[\Delta]$ is the expression obtained by filling the hole with Δ. We also use the notation $?\Gamma$ for *any* expression whose formulas are all ?ed.

Remarks.

► *About associativity and commutativity*. There is a bijective correspondence between the equivalence classes of sequents under the rules for the associativity of $(-,-)$ and $(-;-)$

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and the commutativity of $(-,-)$, and the *series–parallel ordered sets* of occurrences of formulas – see Mohring (1989) for a survey on series–parallel orders. We now recall the definition of serial and parallel compositions of orders. Let ω_1 and ω_2 be orders on disjoint sets X and Y, respectively. Then their *serial* and *parallel* compositions $\omega_1 < \omega_2$ and $\omega_1 \parallel \omega_2$ are, respectively, two orders on $X \cup Y$ defined by

— $(\omega_1 < \omega_2)(x, y)$ iff $x <_{\omega_1} y$ or $x <_{\omega_2} y$ or $(x \in X$ and $y \in Y)$,
— $(\omega_1 \parallel \omega_2)(x, y)$ iff $x <_{\omega_1} y$ or $x <_{\omega_2} y$.

In the bijective correspondence, the constructors $(-;-)$ and $(-,-)$ of the sequent calculus are interpreted, respectively, by *serial* and *parallel* compositions of partial orders, $(- < -)$ and $(- \parallel -)$. We have included the rules for associativity and commutativity for the sake of clarity, but we shall refer explicitly to these rules as sparingly as possible. In particular, we systematically avoid useless parentheses (see the rules for ⅋, ▿, contraction, centre and entropy in Table 1) and we shall freely consider – even though this is not strictly true – that the present version of the sequent calculus relies on (series–parallel) partial orders. This is why they are separated from the more interesting structural rules: seesaw, entropy and centre.

► *About seesaw*. This is the key to the system. It is reversible and its inverse is a particular case of entropy:

$$\frac{\vdash \Gamma; \Delta}{\vdash \Gamma, \Delta}\ \text{co-seesaw.}$$

We shall call *co-seesaw* the inverse of seesaw. Together, they imply cyclic exchange in the usual sense:

$$\cfrac{\cfrac{\cfrac{\vdash \Gamma; \Delta}{\vdash \Gamma, \Delta}\ \text{co-seesaw}}{\vdash \Delta, \Gamma}\ \text{com}}{\vdash \Delta; \Gamma}\ \text{seesaw.}$$

Intuitively, the seesaw rule can be read as follows: proofs with at most 2 conclusions can freely pivot, which means that for such subproofs of a larger proof, commutative and non-commutative compositions should be indistinguishable.

Major consequences of the combination of seesaw and co-seesaw are:

— a single negation,
— a single unit, $\mathbf{1}$ (respectively, $\bot$), for both multiplicative conjunctions (respectively, disjunctions): $A \odot \mathbf{1} \dashv\vdash \mathbf{1} \odot A \dashv\vdash A \dashv\vdash A \otimes \mathbf{1}$ and $A \triangledown \bot \dashv\vdash \bot \triangledown A \dashv\vdash A \dashv\vdash A ⅋ \bot$,
— a single introduction rule for each connective,
— the following focusing[†] property.

Proposition 2.5 (Focusing). Let $\vdash \Gamma$ be a sequent and A be any formula of Γ. Then there is a sequence of seesaw and co-seesaw, the application of which leads from $\vdash \Gamma$ to a sequent of the form $\vdash \Gamma', A$ with the same formulas.

[†] The terminology should not be confused with Andreoli's homonymic property for linear logic (Andreoli 1992), which is related but not at all identical.

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Proof. The expression Γ is a binary tree with nodes either $(-,-)$ or $(-;-)$. We proceed by induction on the length l of the path from the root of Γ to A. If $l = 0$ or 1, that is, $\Gamma = A$, (Δ, A), (A, Δ), $(\Delta; A)$ or $(A; \Delta)$, the result is obtained after at most one application of the seesaw rule and/or commutativity. If $l \geqslant 2$, we have the following cases:

— $\Gamma = (\Delta; \Sigma); \Pi$. If A is in Σ, first apply the following rules:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash (\Delta; \Sigma[A]); \Pi}{\vdash (\Delta; \Sigma[A]), \Pi}\text{ co-seesaw}}{\vdash \Pi, (\Delta; \Sigma[A])}\text{ com}}{\vdash \Pi; (\Delta; \Sigma[A])}\text{ seesaw}}{\vdash (\Pi; \Delta); \Sigma[A]}\text{ a1,}$$

and then the induction hypothesis. If A is in Δ, apply one step of associativity and then the induction hypothesis.

— $\Gamma = \Pi; (\Delta; \Sigma)$ and A is in Δ or Σ. This follows by an identical argument.

— $\Gamma = (\Delta, \Sigma); \Pi$. If A is in Σ, apply co-seesaw, commutativity and associativity of $(-,-)$ map $\vdash \Gamma$ to $\vdash (\Pi, \Delta), \Sigma$, and then the induction hypothesis applies. If A is in Δ, start with co-seesaw and associativity and commutativity of $(-,-)$.

— $\Gamma = \Pi; (\Delta, \Sigma)$. This is similar.

— $\Gamma = (\Delta; \Sigma), \Pi$ and A is in Σ. This follows by commutativity, seesaw and associativity of $(-;-)$, $\vdash \Gamma$ is mapped to $\vdash (\Pi; \Delta); \Sigma$, and then the induction hypothesis applies.

— $\Gamma = \Pi, (\Delta; \Sigma)$. This follows by an identical argument.

— $\Gamma = (\Delta, \Sigma), \Pi$. This is similar.

— $\Gamma = \Pi, (\Delta, \Sigma)$. This follows by an identical argument.

Note also that the result is unique modulo the associativity and commutativity rules. That is, all the expressions Γ' we obtain from a given sequent $\vdash \Gamma$ have the same associated partial order. □

► *About exponentials and the centre rule.* As we shall verify below (see the examples, or the phase semantics: Proposition 5.13 (viii)), the sequent calculus enjoys the essential properties of exponentials:

$$!A \odot !B \dashv\vdash !(A \& B) \dashv\vdash !A \otimes !B,$$

which express the fact that commutativity constraints are irrelevant between ?ed formulas. Besides, for cut elimination, it is necessary to add at least the rule

$$\frac{\vdash \Gamma[?\Delta, ?\Sigma]}{\vdash \Gamma[?\Delta; ?\Sigma]},$$

which, in the presence of seesaw, is equivalent to the simpler

$$\frac{\vdash \Gamma, ?\Delta, ?\Sigma}{\vdash \Gamma; ?\Delta; ?\Sigma}.$$

(For instance in the proof of $!A \odot !B \dashv\vdash !B \odot !A$ obtained by composing the proofs of $!A \odot !B \dashv\vdash !(A \& B)$ and $!(A \& B) \dashv\vdash !B \odot !A$ given below, the final cut indeed cannot be eliminated without this rule.)

Note that the above two equations do not imply that a ?ed formula commutes with any context, only with a ?ed context. However, it is cheap – and much simpler when we

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shall move to order varieties – to make this additionnal assumption, which is expressed by the *centre* rule. Hence, as in Yetter (1990), ?ed formulas are central (Proposition 2.6). In order to make the terminology as clear as possible, and to avoid the word entropy in the reversible cases, we shall adopt the same convention as for the seesaw rule and use *co-centre* rule for the inverse of the centre rule (a particular case of entropy).

Proposition 2.6 (Central exponentials). Let $\vdash \Gamma[?A]$ be a sequent where $?A$ is any formula in the expression Γ. There is a sequence of seesaw, co-seesaw, centre and co-centre, the application of which leads from $\vdash \Gamma[?A]$ to the sequent $\vdash \Gamma[\,], ?A$ with the same formulas.

Proof. As in the proof of Proposition 2.5, we proceed by induction on the length l of the path from the root of Γ to $?A$. If $l = 0$ or 1, $\Gamma = ?A$, $(\Delta, ?A)$, $(?A, \Delta)$, $(\Delta; ?A)$ or $(A; \Delta)$, and the result is obtained after at most one application of the centre rule and/or commutativity. If $l \geqslant 2$, we have the same cases as in the proof of Proposition 2.5. Let us consider the case $\Gamma = (\Delta; \Sigma[?A]); \Pi$, (the others being similar):

$$\begin{array}{c l}
\vdash (\Delta;\Sigma[?A]);\Pi & \\
\hline
\vdash (\Pi;\Delta);\Sigma[?A] & \text{cyclicity} \\
\hline
\vdash ((\Pi;\Delta);\Sigma[\,]), ?A & \text{induction hypothesis} \\
\hline
\vdash \Pi;(\Delta;\Sigma[\,]);?A & \text{seesaw} \\
\hline
\vdash \Pi,(\Delta;\Sigma[\,]), ?A & \text{co-centre} \\
\hline
\vdash (\Delta;\Sigma[\,]),\Pi, ?A & \text{com} \\
\hline
\vdash (\Delta;\Sigma[\,]);\Pi;?A & \text{centre} \\
\hline
\vdash ((\Delta;\Sigma[\,]);\Pi), ?A & \text{co-seesaw.}
\end{array}$$

□

Corollary 2.7. Let $\vdash \Pi$ be a sequent and $?A_1, \ldots, ?A_n$ be all occurrences of ?ed formulas in the set $|\Pi|$. Let $X = |\Pi| \setminus \{?A_1, \ldots, ?A_n\}$. Then given any expression Π' on the set $|\Pi|$ such that $\Pi' \restriction_X = \Pi \restriction_X$, there is a sequence of seesaw, co-seesaw, centre and co-centre, the application of which leads from $\vdash \Pi$ to $\vdash \Pi'$. In particular, taking $\Pi' = \Pi \restriction_X, ?A_1, \ldots, ?A_n$ amounts to extract all ?ed formulas from Π.

Note that a formula of MANL (the multiplicative additive fragment of NL) is provable iff it is provable without the centre rule: this is an immediate consequence of the cut elimination theorem (Theorem 6.1) and the subformula property (Proposition 2.8).

Examples.

► In the first two examples, we still mention explicitly the commutativity rule for $(-,-)$. Here is a proof of $A \otimes B \vdash A \odot B$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash A^\perp, A}{\vdash A^\perp; A}\,\text{seesaw} \qquad \dfrac{\vdash B, B^\perp}{\vdash B; B^\perp}\,\text{seesaw}}{\vdash A^\perp; A \odot B; B^\perp}\,\odot}{\vdash A^\perp, A \odot B, B^\perp}\,\text{entropy}}{\vdash A \odot B, B^\perp, A^\perp}\,\text{com}}{\vdash A \odot B, B^\perp ⅋ A^\perp}\,⅋}{\vdash B^\perp ⅋ A^\perp, A \odot B}\,\text{com}$$

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► Here is a proof of $A \otimes (A \multimap B) \vdash B$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\vdash B, B^\perp}}{\vdash B; B^\perp}\;\text{seesaw} \qquad \dfrac{\overline{\vdash A, A^\perp}}{\vdash A; A^\perp}\;\text{seesaw}}{\vdash B; B^\perp \odot A; A^\perp}\;\odot}{\vdash B, B^\perp \odot A, A^\perp}\;\text{entropy}}{\vdash B, (B^\perp \odot A) \parr A^\perp}\;\parr}{\vdash (B^\perp \odot A) \parr A^\perp, B}\;\text{com}$$

► Here are proofs of $!(A\&B) \dashv\vdash !A \odot !B$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\vdash A, A^\perp}}{\vdash ?A^\perp, A}\;\text{d}}{\vdash ?A^\perp; A}\;\text{seesaw}}{\vdash ?B^\perp, (?A^\perp; A)}\;\text{w}}{\vdash ?B^\perp; ?A^\perp; A}\;\text{seesaw}}{\vdash (?B^\perp; ?A^\perp), A}\;\text{co-seesaw} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\vdash B, B^\perp}}{\vdash B, ?B^\perp}\;\text{d}}{\vdash B; ?B^\perp}\;\text{seesaw}}{\vdash (B; ?B^\perp), ?A^\perp}\;\text{w}}{\vdash B; ?B^\perp; ?A^\perp}\;\text{seesaw}}{\vdash (?B^\perp; ?A^\perp), B}\;\text{co-seesaw}}{\vdash (?B^\perp; ?A^\perp), A\&B}\;\&}{\vdash (?B^\perp; ?A^\perp), !(A\&B)}\;!}{\vdash !(A\&B); ?B^\perp; ?A^\perp}\;\text{seesaw}}{\vdash !(A\&B); ?B^\perp \triangledown ?A^\perp}\;\triangledown}{\vdash ?B^\perp \triangledown ?A^\perp, !(A\&B)}\;\text{co-seesaw}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\vdash A^\perp, A}}{\vdash B^\perp \oplus A^\perp, A}\;\oplus 2}{\vdash ?(B^\perp \oplus A^\perp), A}\;\text{d}}{\vdash ?(B^\perp \oplus A^\perp), !A}\;!}{\vdash ?(B^\perp \oplus A^\perp); !A}\;\text{seesaw} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overline{\vdash B, B^\perp}}{\vdash B, B^\perp \oplus A^\perp}\;\oplus 1}{\vdash B, ?(B^\perp \oplus A^\perp)}\;\text{d}}{\vdash !B, ?(B^\perp \oplus A^\perp)}\;!}{\vdash !B; ?(B^\perp \oplus A^\perp)}\;\text{seesaw}}{\vdash ?(B^\perp \oplus A^\perp); !A \odot !B; ?(B^\perp \oplus A^\perp)}\;\odot}{\vdash ?(B^\perp \oplus A^\perp), !A \odot !B, ?(B^\perp \oplus A^\perp)}\;\text{entropy}}{\vdash ?(B^\perp \oplus A^\perp), !A \odot !B}\;\text{c}$$

► Here are proofs of $A \odot \mathbf{1} \dashv\vdash A$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\vdash A, A^\perp}}{\vdash A^\perp; A}\;\text{seesaw}}{\vdash \bot, (A^\perp; A)}\;\bot}{\vdash \bot; A^\perp; A}\;\text{seesaw}}{\vdash A, (\bot; A^\perp)}\;\text{co-seesaw}}{\vdash A; \bot; A^\perp}\;\text{seesaw}}{\vdash A; \bot \triangledown A^\perp}\;\triangledown}{\vdash \bot \triangledown A^\perp, A}\;\text{co-seesaw} \qquad \dfrac{\dfrac{\dfrac{\overline{\vdash A^\perp, A}}{\vdash A^\perp; A}\;\text{seesaw} \qquad \overline{\vdash \mathbf{1}}}{\vdash A^\perp; A \odot \mathbf{1}}\;\odot}{\vdash A^\perp, A \odot \mathbf{1}}\;\text{co-seesaw}$$

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► Here is a proof of $!A \odot B \vdash B \odot !A$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash B^\perp, B}}{\vdash B^\perp ; B}\,\text{seesaw} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A, A^\perp}}{\vdash A, ?A^\perp}\,\text{d}}{\vdash !A, ?A^\perp}\,!}{\vdash !A ; ?A^\perp}\,\text{seesaw}}{\vdash B^\perp ; B \odot !A ; ?A^\perp}\,\odot}{\vdash B^\perp ; (?A^\perp, B \odot !A)}\,\text{co-centre}}{\vdash B^\perp ; ?A^\perp ; B \odot !A}\,\text{centre}}{\vdash B^\perp \triangledown ?A^\perp, B \odot !A}\,\triangledown$$

Clearly, the sequent calculus enjoys the subformula property.

Proposition 2.8 (Subformula property). If $\mathscr{D}$ is a cut-free proof of $\vdash \Gamma$, then the formulas occurring in $\mathscr{D}$ are subformulas of the formulas in Γ.

Cut elimination will be proved using the phase semantics (Theorem 6.1).

2.3. *Invariants*

The sequent calculus with all the structural rules explicit (previous section) is not entirely satisfactory. This is essentially because:

— A proof of a sequent containing only non-commutative connectives may use the commutative composition $(-,-)$. For instance

$$\dfrac{\dfrac{\dfrac{}{\vdash A^\perp, A}}{\vdash A^\perp ; A}\,\text{seesaw}}{\vdash A^\perp \triangledown A}\,\triangledown,$$

— For the sake of stability by cut elimination (in particular commutation with seesaw), the cut rule has to appear in 4 different forms (premisses with $(-,-)$ or $(-;-)$), which are clearly equivalent modulo seesaw and co-seesaw:

$$\dfrac{\dfrac{\vdash \Gamma ; A}{\vdash \Gamma, A}\,\text{seesaw} \qquad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}\,\text{cut} \quad \rightarrow \quad \dfrac{\vdash \Gamma ; A \qquad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}\,\text{cut.}$$

This raises the question of determining the invariant of sequents under seesaw and co-seesaw. In other words, we are looking for the sequent calculus without seesaw and co-seesaw, corresponding to the calculus of the present section. One might expect to solve the problem by adding $\vdash A^\perp ; A$ as an axiom, and other logical rules, with nested contexts, such as

$$\dfrac{\vdash \Gamma, A \qquad \vdash \Delta[B]}{\vdash \Delta[\Gamma ; A \odot B]} \quad \ldots$$

and by eliminating the seesaw rule, having again in mind a calculus relying on orders (reminiscent of the intuitionistic calculus, see de Groote (1996)). But, in fact, to prove the associativity of multiplicative connectives, the seesaw rule is essential, which means that the mathematical structure underlying a sequent, invariant by the rules seesaw and co-seesaw, is not an order.

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This is our reason for introducing order varieties. Order varieties (Section 3) are structures that can be presented by *partial* orders in several ways, a good analogy being the oriented circle that becomes a total order as soon as an origin is fixed: provided a point of view (an element x in the base set), an order variety can be seen as a partial order on the complement of $\{x\}$. This reflects Proposition 2.5 precisely, which enables one to change the presentation of a sequent (the associated order) in a completely reversible way, for example, by 'pulling out' any formula A. Order varieties can therefore be presented in different ways by changing the viewpoint – of course they are invariant under the change of presentation.

In Section 4 we shall present the version of the sequent calculus without seesaw and co-seesaw, using order varieties. We first present the definition and some properties of order varieties (some of them are just recalled from Abrusci and Ruet (2000)), and introduce *series–parallel* order varieties.

3. Order varieties

3.1. *Order varieties and orders*

Definition 3.1 (Order varieties). Let X be a set. An *order variety* on X is a ternary relation α that is

$$\begin{cases} \textit{cyclic}: & \forall x,y,z \in X, \alpha(x,y,z) \Rightarrow \alpha(y,z,x), \\ \textit{anti-reflexive}: & \forall x,y \in X, \neg\alpha(x,x,y) \\ \textit{transitive}: & \forall x,y,z,t \in X, \alpha(x,y,z) \text{ and } \alpha(z,t,x) \Rightarrow \alpha(y,z,t), \\ \textit{spreading}: & \forall x,y,z,t \in X, \alpha(x,y,z) \Rightarrow \alpha(t,y,z) \text{ or } \alpha(x,t,z) \text{ or } \alpha(x,y,t). \end{cases}$$

An order variety α on X is said to be *total* when $\forall x,y,z \in X$, $x \neq y \neq z \neq x \Rightarrow \alpha(x,y,z)$ or $\alpha(z,y,x)$.

Ternary relations satisfying the first three axioms have been studied by Novák (Novák 1982) and called cyclic orders.

A few elementary properties and examples of order varieties are collected in the following remarks (for proofs, see Abrusci and Ruet (2000)).

Remarks.
► If α is a total order variety, $\alpha(x,y,z)$ can be read as 'y is between x and z'.

Definition 3.2. An order variety α on X induces a binary relation $\rightarrow_\alpha$ on X by $x \rightarrow_\alpha y$ iff $\forall z \in X$:

$$z \neq x \text{ and } z \neq y \Rightarrow \alpha(x,y,z).$$

One verifies easily that, when α is total, $\rightarrow_\alpha$ is an oriented cycle. In Section 4, the relation $\rightarrow_\alpha$ will be used in the introduction rule for $\triangledown$, for arbitrary order varieties.

Conversely, any oriented cycle G induces a ternary $r(G)$ on $|G|$ by: $r(G)(x,y,z)$ iff y is between x and z in G; then the set of finite oriented cycles is isomorphic to the set of finite total order varieties, by $\rightarrow_{r(G)}= G$ and $r(\rightarrow_\alpha) = \alpha$.

► The empty ternary relation on any set X is an order variety on X, called the *empty order variety* on X and denoted by $\varnothing_X$, or simply $\varnothing$ if there is no ambiguity.

► The cyclic closure of $\{(a, b, c)\}$ is not an order variety on $\{a, b, c, d\}$, as it does not enjoy the spreading condition; it is an order variety on $\{a, b, c\}$.

Notation. The finite total order variety corresponding to the oriented cycle $a_1 \to \cdots \to a_n \to a_1$ will simply be denoted $(a_1 \ldots a_n)$.

Definition 3.3.

(i) Let $\boldsymbol{\alpha}$ be an order variety on X and $x \in X$. Define the binary relation $\boldsymbol{\alpha}_x$ on $X \setminus \{x\}$ by: $\boldsymbol{\alpha}_x(y, z)$ iff $\boldsymbol{\alpha}(x, y, z)$.

(ii) Let $\omega = (X, <)$ be a (strict) partial order on X and $z \in X$. Define the binary relation $\overset{z}{<}$ by:

— $x \overset{z}{<} y$ iff $x < y$ and z is comparable with neither x nor y,

and the ternary relation $\overline{\omega}$ on X by:

— $\overline{\omega}(x, y, z)$ iff $x < y < z$ or $y < z < x$ or $z < x < y$ or $x \overset{z}{<} y$ or $y \overset{x}{<} z$ or $z \overset{y}{<} x$.

Proposition 3.4.

(i) If $\boldsymbol{\alpha}$ is an order variety on X and $x \in X$, then $\boldsymbol{\alpha}_x$ is a partial order on $X \setminus \{x\}$. It is called the order *induced by* $\boldsymbol{\alpha}$ *and* x.

(ii) If (X, ω) is a partial order, then $\overline{\omega}$ is an order variety on X.

Proposition 3.4 expresses the possibility of focusing on an arbitrary element x in an order variety ($\boldsymbol{\alpha} \mapsto \boldsymbol{\alpha}_x \parallel x$) to perform operations (the usual operations on binary orders) and then come back to an order variety ($\omega \mapsto \overline{\omega}$). Note the following properties of commutation with restriction.

Facts 3.5.

(i) Let ω be an order on X and $Y \subseteq X$. Then $(\overline{\omega}) \restriction_Y = \overline{\omega \restriction_Y}$.

(ii) Let $\boldsymbol{\alpha}$ be an order variety on X, $x \in X$ and $Y \subseteq X \setminus \{x\}$. Then $\boldsymbol{\alpha}_x \restriction_Y = (\boldsymbol{\alpha} \restriction_{Y \uplus \{x\}})_x$.

Proposition and Definition 3.6 (Glueing). If ω and τ are two partial orders on disjoints sets, then $\overline{\omega < \tau} = \overline{\omega \parallel \tau} = \overline{\tau < \omega}$. Define

$$\boxed{\omega * \tau \; = \; \overline{\omega < \tau} \; = \; \overline{\omega \parallel \tau} \; = \; \overline{\tau < \omega}.}$$

For instance, one may easily check as an exercise that if $\omega * \tau$ is a total order variety, with $|\omega|$ and $|\tau|$ non-empty, then both ω and τ are total orders. The following proposition establishes a reversible relation between order varieties and partial orders:

Proposition 3.7. Let $\boldsymbol{\alpha}$ be an order variety on a set X, $x \in X$ and ω a partial order on $X \setminus \{x\}$. Then

$$\boxed{\boldsymbol{\alpha}_x * x \; = \; \boldsymbol{\alpha}} \quad \text{and} \quad \boxed{(\omega * x)_x \; = \; \omega.}$$

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An order variety is indeed a glueing of order structures, as its name implies, but a very strict kind of glueing, more like the one-point compactification of the plane than general manifold glueing.

3.2. *Series–parallel order varieties*

Recall that the class of so-called *series–parallel* orders is the least class of finite orders containing empty orders on singletons and closed by serial and parallel compositions. See Mohring (1989) for a survey on series–parallel orders.

In this section, we define the corresponding class of order varieties, called *series–parallel order varieties*. In practice the sequent calculus of Section 4 will rely on series–parallel order varieties.

Definition 3.8. Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be order varieties on the sets X and Y, respectively, with $X \cap Y = \{x\}$. Define

$$\begin{cases} \boldsymbol{\alpha} \odot_x \boldsymbol{\beta} &= \overline{\boldsymbol{\alpha}_x < x < \boldsymbol{\beta}_x} &= (\boldsymbol{\beta}_x < \boldsymbol{\alpha}_x) * x \\ \boldsymbol{\alpha} \otimes_x \boldsymbol{\beta} &= \overline{\boldsymbol{\alpha}_x \parallel x \parallel \boldsymbol{\beta}_x} &= (\boldsymbol{\alpha}_x \parallel \boldsymbol{\beta}_x) * x. \end{cases}$$

Clearly, if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are order varieties on the sets X and Y, respectively, with $X \cap Y = \{x\}$, then $\boldsymbol{\alpha} \odot_x \boldsymbol{\beta}$ and $\boldsymbol{\alpha} \otimes_x \boldsymbol{\beta}$ are order varieties on $X \cup Y$.

Example. If $X \cap Y = \{x\}$, $\varnothing_X \otimes_x \varnothing_Y = \varnothing_{X \cup Y}$, but $\varnothing_X \odot_x \varnothing_Y \neq \varnothing_{X \cup Y}$.

Definition 3.9 (Series–parallel order varieties). Given a set X, the class of *series–parallel order varieties* on X is the least class of order varieties containing the empty order varieties $\varnothing_{\{a\}}$ and $\varnothing_{\{a,b\}}$ on singletons and pairs ($a, b \in X$), and closed by $\odot_x$ and $\otimes_x$.

For instance, total and empty order varieties are series–parallel order varieties. The following is a straightforward calculation.

Lemma 3.10. Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be order varieties on the sets X and Y, respectively, with $X \cap Y = \{x\}$, and let $y \in X \setminus \{x\}$, $z \in Y \setminus \{x\}$.

$$\begin{array}{lcllcl} (\boldsymbol{\alpha} \odot_x \boldsymbol{\beta})_x &=& \boldsymbol{\beta}_x < \boldsymbol{\alpha}_x & (\boldsymbol{\alpha} \otimes_x \boldsymbol{\beta})_x &=& \boldsymbol{\beta}_x \parallel \boldsymbol{\alpha}_x \\ (\boldsymbol{\alpha} \odot_x \boldsymbol{\beta})_y &=& \boldsymbol{\alpha}_y[(x < \boldsymbol{\beta}_x)/x] & (\boldsymbol{\alpha} \otimes_x \boldsymbol{\beta})_y &=& \boldsymbol{\alpha}_y[(x \parallel \boldsymbol{\beta}_x)/x] \\ (\boldsymbol{\alpha} \odot_x \boldsymbol{\beta})_z &=& \boldsymbol{\beta}_z[(\boldsymbol{\alpha}_x < x)/x] & (\boldsymbol{\alpha} \otimes_x \boldsymbol{\beta})_z &=& \boldsymbol{\beta}_z[(\boldsymbol{\alpha}_x \parallel x)/x] \end{array}$$

Lemma 3.11. If $\boldsymbol{\alpha}$ is a series–parallel order variety on a non-empty set X, and $x \in X$, then $\boldsymbol{\alpha}_x$ is a series–parallel order on $X \setminus \{x\}$.

Proof. By Proposition 3.4 (i), $\boldsymbol{\alpha}_x$ is an order on $X \setminus \{x\}$. To show that it is series–parallel, we proceed by induction on the construction of series–parallel order varieties. For singletons and pairs, it is obvious. Let $\boldsymbol{\gamma} = \boldsymbol{\alpha} \odot_t \boldsymbol{\beta}$ or $\boldsymbol{\alpha} \otimes_t \boldsymbol{\beta}$, by the induction hypothesis $\boldsymbol{\alpha}_x$, $\boldsymbol{\alpha}_t$, $\boldsymbol{\beta}_x$ and $\boldsymbol{\beta}_t$ are series–parallel orders, so by Lemma 3.10, $(\boldsymbol{\alpha} \odot_t \boldsymbol{\beta})_x$ and $(\boldsymbol{\alpha} \otimes_t \boldsymbol{\beta})_x$ are series–parallel. □

Proposition 3.12. Let $\boldsymbol{\alpha}$ be an order variety on a non-empty set X. Then $\boldsymbol{\alpha}$ is series–parallel iff there exists a series–parallel order ω on X such that $\overline{\omega} = \boldsymbol{\alpha}$.

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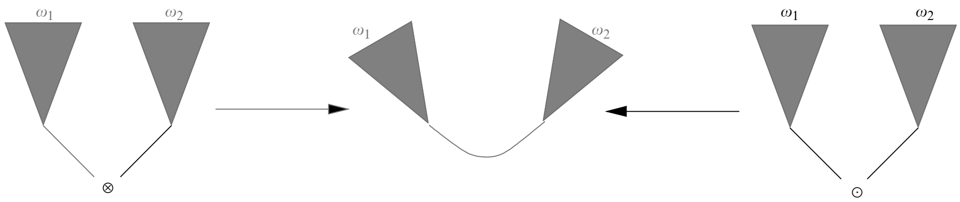
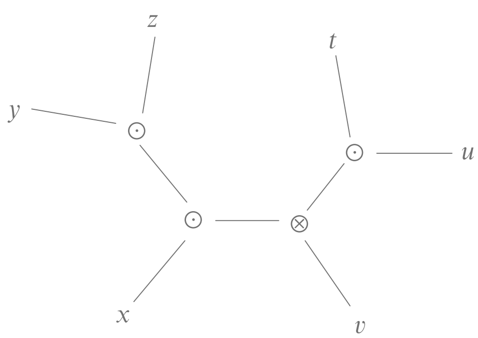
Proof. If $\boldsymbol{\alpha}$ is series–parallel, let $x \in X$. Then, by Lemma 3.11, $(x \parallel \boldsymbol{\alpha}_x)$ is a series–parallel order, and by Proposition 3.7, $\boldsymbol{\alpha} = \overline{x \parallel \boldsymbol{\alpha}_x}$.

Conversely, if X has one or two elements, it is obvious. Otherwise, take $x \in X$. By Lemma 3.11, $\boldsymbol{\alpha}_x$ is a series–parallel order on the set $X \setminus \{x\}$ that has at least two elements, so $\boldsymbol{\alpha}_x = (\omega_1 \parallel \omega_2)$ or $(\omega_1 < \omega_2)$. By Proposition 3.7, $\boldsymbol{\alpha} = x * \boldsymbol{\alpha}_x$, and hence $\boldsymbol{\alpha} = x * (\omega_1 \parallel \omega_2)$ or $x * (\omega_1 < \omega_2)$. In the first case, $\boldsymbol{\alpha} = (x * \omega_1) \otimes_x (x * \omega_2)$ is series–parallel, and in the second case, $\boldsymbol{\alpha} = (x * \omega_1) \odot_x (x * \omega_2)$ is series–parallel as well. □

Propositions 3.12, 3.6 and 3.7 suggest the possibility of visualizing series–parallel order varieties on a set X as rootless trees with leaves labelled by elements of X and ternary nodes labelled by $\otimes$ or $\odot$. Given an order variety $\boldsymbol{\alpha}$ on X ($\sharp X \geqslant 2$), $\boldsymbol{\alpha} = \overline{\omega}$ for some (non-unique) series–parallel order ω. Write ω as a (non-unique) binary tree t with leaves labelled by elements of X, and root and nodes labelled by $\otimes$ (in the case of parallel composition) or $\odot$ (serial composition); then remove the root of t.



For instance $\overline{(x < y < z) \parallel v \parallel (t < u)}$ can be represented by



To read the tree, take three leaves a, b, c. Then (a, b, c) is in the order variety iff:

— the node $\bullet$ at the intersection of the three paths ab, bc and ca is labelled by $\odot$ and

— the paths $a\bullet, b\bullet$ and $c\bullet$ are in this cyclic order while moving clockwise around $\bullet$.

Note that the law of spreading, in particular, is easy to check in the tree representation. Note also how the removal of the root corresponds to the glueing operation of Proposition 3.6.

The resulting tree is obviously not unique (change $x \odot (y \odot z)$ for $(x \odot y) \odot z$ in the above example). However, if we quotient the set of such trees by associativity of $\otimes$ and $\odot$ and commutativity of $\otimes$, then the result is unique (in particular it is independent from the choice of ω) – see Proposition 3.17.

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3.3. *Characterization of series–parallel order varieties*

There is a characterization of series–parallel orders as those orders on finite sets X whose restriction to any 4-elements subset $\{a, b, c, d\}$ is different from the order $N(a, b, c, d) = \{(a, b), (c, b), (c, d)\}$. There is a similar characterization of series–parallel order varieties.

Definition and Lemma 3.13. $G(a, b, c, d, e)$ denotes the ternary relation

$$(a, b, d) \cup (a, c, d) \cup (c, b, e) \cup (c, d, e) \cup (a, b, e).$$

It is an order variety on $\{a, b, c, d, e\}$.

Note that $G(a, b, c, d, e)_e = \{(a, b), (c, b), (c, d)\} = N(a, b, c, d)$ is not a series–parallel order, so according to Lemma 3.11, $G(a, b, c, d, e)$ is not a series–parallel order variety. Moreover, we have the following proposition.

Proposition 3.14. Let α be an order variety on a non-empty finite set X. The following are equivalent:

(i) α is series–parallel,
(ii) there exists a series–parallel order ω on X such that $\overline{\omega} = \alpha$,
(iii) the restriction of α to every 5-element subset $\{a, b, c, d, e\}$ of X is different from $G(a, b, c, d, e)$.

Proof. (i $\Leftrightarrow$ ii) *Cf.* Proposition 3.12.
(ii $\Rightarrow$ iii) Assume that $\overline{\omega} = \alpha$ and $\alpha \restriction_Y = G(a, b, c, d, e)$ for some 5-element subset $Y = \{a, b, c, d, e\}$ of X. Then $\overline{\omega \restriction_Y} = (\overline{\omega}) \restriction_Y = G(a, b, c, d, e)$, so $\overline{\omega \restriction_Y}$ is not a series–parallel order variety and $\omega \restriction_Y$ is not a series–parallel order by Proposition 3.12. Hence ω is not a series–parallel order either, which is a contradiction.

(iii $\Rightarrow$ ii) Let $e \in X$. We can concentrate on sets of at least 5 elements (for smaller sets the result is obvious). If (iii), then α_e is a series–parallel order: indeed, otherwise $\alpha_e \restriction_{\{a,b,c,d\}} = (\alpha \restriction_{\{a,b,c,d,e\}})_e = N(a, b, c, d)$, thus $\alpha \restriction_{\{a,b,c,d,e\}} = N(a, b, c, d) * e = G(a, b, c, d, e)$, which is a contradiction. Therefore there exists a series–parallel order ω on X such that $\overline{\omega} = \alpha$, namely $\omega = (e \parallel \alpha_e)$. □

In particular, we have the following consequence.

Proposition 3.15 (Restriction).

(i) Let ω be an order on X and $Y \subseteq X$. Then $(\overline{\omega}) \restriction_Y = \overline{\omega \restriction_Y}$.
If α is an order variety on a set X and $Y \subseteq X$, then the restriction $\alpha \restriction_Y$ of α to Y (as a set of triples) is an order variety on Y. Moreover if $Y \neq \emptyset$ and α is series–parallel, then so is $\alpha \restriction_Y$.
(ii) If α is an order variety on $X \cup \{x\}$ with $x \notin X$, then $\alpha \restriction_X = \overline{\alpha_x}$.

Proof. (i) Immediate consequences of (i), Proposition 3.7 and Proposition 3.14.
(ii) Let α be an order variety on $X \cup \{x\}$ with $x \notin X$. $\alpha \restriction_X$ and $\overline{\alpha_x}$ are both order varieties on X, and by Proposition 3.7, $\alpha \restriction_X = \overline{(\alpha_x \parallel x)} \restriction_X = \overline{\alpha_x}$. □

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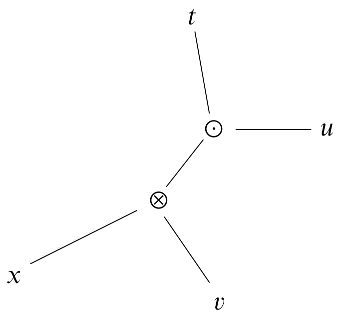
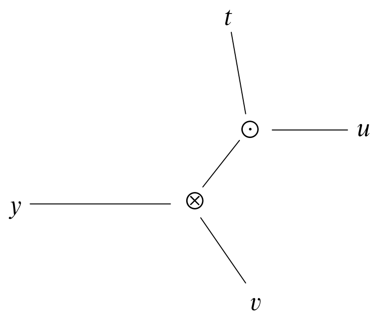
In the tree representation for series–parallel order varieties, the restriction to a subset Y is obtained by cancelling all nodes and edges that are not on a simple path between two leaves of Y: in particular, the leaves outside Y and the adjacent edges are cancelled, and so on. For instance, the restrictions of $\overline{(x < y < z) \parallel v \parallel (t < u)}$ to the sets $\{x, t, u, v\}$ and $\{y, t, u, v\}$ are, respectively,





3.4. *Seesaw*

The equivalence of series–parallel orders induced by the seesaw rule of Section 2 is the same as the equality of the associated order varieties.

Definition 3.16 (Seesaw). $\sim$ is the equivalence relation between partial orders on the same set, defined by

$$\boxed{\omega \sim \sigma \ \text{ iff } \ \overline{\omega} = \overline{\sigma}.}$$

Proposition 3.17. The restriction of $\sim$ to series–parallel orders is the least equivalence relation between series–parallel orders on the same set such that

$$(\omega_1 \parallel \omega_2) \sim (\omega_1 < \omega_2).$$

Proof. One direction is just Proposition 3.6.

Conversely, let $\approx$ be the equivalence relation defined by $(\omega_1 \parallel \omega_2) \approx (\omega_1 < \omega_2)$. The result is obvious if $X = \emptyset$. If $\sharp X > 0$, let $x \in X$. One proves that $\omega \approx (x \parallel (\overline{\omega})_x)$ by induction on $h(x, \omega) = \min\{$length of the path from x to the root of $t \mid t$ tree representing $\omega\}$:

— If $h(x, \omega) = 0$ or 1, it is clear.

— If $h(x, \omega) > 1$, then ω is a ternary combination of series–parallel orders with $\parallel$ and $<$. Consider the case $\omega = (\omega_1[x] \parallel \omega_2) < \omega_3$, the other cases being similar. Let $\omega'' = \omega_1[x] \parallel (\omega_2 \parallel \omega_3)$. On the one hand $\omega \approx \omega''$, so $\omega \sim \omega''$ and $\overline{\omega} = \overline{\omega''}$, and on the other hand $\omega'' \approx (x \parallel (\overline{\omega''})_x)$ because $h(x, \omega'') = h(x, \omega) - 1$, and hence $\omega \approx \omega'' \approx (x \parallel (\overline{\omega''})_x) = (x \parallel (\overline{\omega})_x)$.

Therefore, if $\omega \sim \sigma$, that is, $\overline{\omega} = \overline{\sigma}$, then $\omega \approx (x \parallel (\overline{\omega})_x) = (x \parallel (\overline{\sigma})_x) \approx \sigma$. □

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3.5. *Entropy*

Definition 3.18 (Entropy). $\trianglelefteq$ is the relation defined on the set of all partial orders for a fixed set by

$$\boxed{\omega \trianglelefteq \sigma \ \text{ iff } \ \omega \subseteq \sigma \ \text{ and } \ \overline{\omega} \subseteq \overline{\sigma}.}$$

The following facts are obvious.

Facts 3.19. (i) $\trianglelefteq$ is a partial order.
(ii) $\trianglelefteq$ is compatible with restriction.

Lemma 3.20. $\trianglelefteq$ is compatible with the compositions $\parallel$ and $<$ of partial orders.

Proof. If $\omega \trianglelefteq \omega'$ and $\sigma \trianglelefteq \sigma'$, then $(\omega \parallel \sigma) \subseteq (\omega' \parallel \sigma')$, and, furthermore, $\overline{\omega \parallel \sigma} \subseteq \overline{\omega' \parallel \sigma'}$. Indeed:

— $\overline{\omega \parallel \sigma} \restriction_{|\omega|} = \overline{\omega} = \overline{\omega' \parallel \sigma'} \restriction_{|\omega|}$ and $\overline{\omega \parallel \sigma} \restriction_{|\sigma|} = \overline{\sigma} = \overline{\omega' \parallel \sigma'} \restriction_{|\sigma|}$.
— If $a, b \in |\omega|$ and $c \in |\sigma|$, then $\overline{\omega \parallel \sigma}(a, b, c) \Leftrightarrow \omega(a, b) \Rightarrow \omega'(a, b) \Leftrightarrow \overline{\omega' \parallel \sigma'}(a, b, c)$; *idem* for $a, b \in |\sigma|$ and $c \in |\omega|$.

Idem for $<$. □

Corollary 3.21. If $\omega \trianglelefteq \omega'$ and $\sigma \trianglelefteq \sigma'$, then $\omega * \sigma \subseteq \omega' * \sigma'$.

In Proposition 3.24, we will see that $\trianglelefteq$ is a generalization to arbitrary partial orders of the relation considered in the sequent calculus of Section 2 in the series–parallel case: moving from serial composition to parallel composition. The fact that $\trianglelefteq$ corresponds to the *inclusion* of order varieties (Proposition 3.22) confirms that it is indeed a good choice, and it will serve as a basis for defining a calculus on order varieties in Section 4.

Proposition 3.22. Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be order varieties on X and $x \in X$. Then the following are equivalent:

(i) $\boldsymbol{\alpha} \subseteq \boldsymbol{\beta}$.
(ii) $\boldsymbol{\alpha}_x \trianglelefteq \boldsymbol{\beta}_x$.
(iii) There exist partial orders ω and σ on X such that $\boldsymbol{\alpha} = \overline{\omega}$, $\boldsymbol{\beta} = \overline{\sigma}$ and $\omega \trianglelefteq \sigma$.

Proof. (i) $\Rightarrow$ (ii). If $\boldsymbol{\alpha} \subseteq \boldsymbol{\beta}$, then $\boldsymbol{\alpha}_x \subseteq \boldsymbol{\beta}_x$, and on the other hand $\overline{\boldsymbol{\alpha}_x} = \boldsymbol{\alpha} \restriction_{X \setminus \{x\}} \subseteq \boldsymbol{\beta} \restriction_{X \setminus \{x\}} = \overline{\boldsymbol{\beta}_x}$ by Proposition 3.15, therefore $\boldsymbol{\alpha}_x \trianglelefteq \boldsymbol{\beta}_x$.

(ii) $\Rightarrow$ (iii). $\boldsymbol{\alpha}_x \trianglelefteq \boldsymbol{\beta}_x$ implies the existence of the required partial orders: take $\omega = (\boldsymbol{\alpha}_x \parallel x)$ and $\sigma = (\boldsymbol{\beta}_x \parallel x)$; by Lemma 3.20, $\omega \trianglelefteq \sigma$.

(iii) $\Rightarrow$ (i). If there exist partial orders ω and σ on X such that $\boldsymbol{\alpha} = \overline{\omega}$, $\boldsymbol{\beta} = \overline{\sigma}$ and $\omega \trianglelefteq \sigma$, then by definition of $\trianglelefteq$, $\overline{\omega} \subseteq \overline{\sigma}$, that is, $\boldsymbol{\alpha} \subseteq \boldsymbol{\beta}$. □

Lemma 3.23. $\omega[\omega_1 \parallel \omega_2] \trianglelefteq \omega[\omega_1 < \omega_2]$.

Proof. If $\omega = \omega[\omega_1 \parallel \omega_2]$ and $\sigma = \omega[\omega_1 < \omega_2]$, then:

— Clearly, $\omega \subseteq \sigma$.

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— $\overline{\varpi} \subseteq \overline{\sigma}$. Indeed, consider $a, b, c \in X$ such that $\overline{\varpi}(a, b, c)$. Let $X = |\omega|$, $X_1 = |\omega_1|, X_2 = |\omega_2|$, $X_3 = X \setminus (X_1 \cup X_2)$ and I be the least set of indices $\subseteq \{1, 2, 3\}$ such that $\{a, b, c\} \subseteq \bigcup_{i \in I} X_i$. If $I = \{i\}$, then $\overline{\sigma}(a, b, c)$ holds trivially. If $I = \{1, 2\}$, then say $a, b \in X_1$ and $c \in X_2$, whence $\overline{\varpi}(a, b, c)$ iff $\omega_1(a, b)$ iff $\overline{\sigma}(a, b, c)$. If $I = \{1, 3\}$, then $\overline{\varpi}(a, b, c)$ iff $\overline{\omega \restriction_{X_1 \cup X_3}}(a, b, c)$ iff $\overline{\sigma \restriction_{X_1 \cup X_3}}(a, b, c)$ iff $\overline{\sigma}(a, b, c)$. The case $I = \{2, 3\}$ is similar. Finally, $I = \{1, 2, 3\}$ simply contradicts $\overline{\varpi}(a, b, c)$.

Hence $\omega \trianglelefteq \sigma$. □

Proposition 3.24. The restriction of $\trianglelefteq$ to series–parallel orders is the least reflexive transitive relation between series–parallel orders on the same set such that

$$\omega[\omega_1 \parallel \omega_2] \trianglelefteq \omega[\omega_1 < \omega_2].$$

Proof. By Lemma 3.23, the above rule is sound. Conversely, let ω and σ be two series–parallel orders on the same set such that $\omega \trianglelefteq \sigma$. We use the axiomatisation of the *inclusion* between series–parallel orders given by Bechet *et al.* (1997); the result stated in Propositions 3.2 and 4.1 of Bechet *et al.* (1997) is $\omega \subseteq \sigma$ iff ω is obtained from σ by means of the following reflexive and transitive congruence:

$$\begin{array}{llcl}
(1) & \tau < \theta & \rightarrow & \tau \parallel \theta \\
(2) & \tau < (\theta_1 \parallel \theta_2) & \rightarrow & (\tau < \theta_1) \parallel \theta_2 \\
(3) & (\theta_1 \parallel \theta_2) < \tau & \rightarrow & \theta_1 \parallel (\theta_2 < \tau) \\
(4) & (\tau_1 \parallel \tau_2) < (\theta_1 \parallel \theta_2) & \rightarrow & (\tau_1 < \theta_1) \parallel (\tau_2 < \theta_2).
\end{array}$$

Now $\omega \trianglelefteq \sigma$, thus $\omega \subseteq \sigma$. We have to use condition $\overline{\varpi} \subseteq \overline{\sigma}$ to restrict ourselves to the rewrite rule (1). We shall write $\xrightarrow{1}$ for the congruence induced by (1). We proceed by induction on the cardinality of $\sigma \setminus \omega$. If $\sigma \setminus \omega = \emptyset$, the result is trivial. If $\sigma \setminus \omega \neq \emptyset$, there is a derivation $\omega \leftarrow \sigma$ of $\omega \subseteq \sigma$. There are four cases:

— The first rule applied is (1), that is, $\omega \leftarrow \rho[\tau \parallel \theta] \leftarrow \rho[\tau < \theta] = \sigma$. Then clearly $\rho[\tau \parallel \theta] \xleftarrow{1} \sigma$ and $\omega \subseteq \rho[\tau \parallel \theta]$. On the other hand, $\overline{\varpi} \subseteq \overline{\rho[\tau \parallel \theta]}$. Indeed,

$$\begin{array}{rcl}
\overline{\varpi} \restriction_{|\rho|} & \subseteq & \overline{\sigma} \restriction_{|\rho|} \\
& = & \overline{\rho[\tau \parallel \theta]} \restriction_{|\rho|} \;\text{ and} \\
\overline{\varpi} \restriction_{X \setminus |\rho|} & \subseteq & \overline{\sigma} \restriction_{X \setminus |\rho|} \\
& = & \overline{\tau < \theta} \\
& = & \overline{\tau \parallel \theta} \\
& = & \overline{\rho[\tau \parallel \theta]} \restriction_{X \setminus |\rho|} .
\end{array}$$

Besides, $\overline{\varpi}(x, a, b)$ for some $x \in |\rho|$ and $a, b \in X \setminus |\rho|$ implies $\overline{\sigma}(x, a, b)$, and then $a, b \in |\tau|$ or $a, b \in |\theta|$ because $\omega \subseteq \rho[\tau \parallel \theta]$, whence $\overline{\rho[\tau \parallel \theta]}(x, a, b)$ as well. Finally, $\overline{\varpi}(x, y, a)$ for some $x, y \in |\rho|$ and $a \in X \setminus |\rho|$ implies $\overline{\sigma}(x, y, a)$, and clearly $\overline{\rho[\tau \parallel \theta]}(x, y, a)$ holds. Hence $\omega \trianglelefteq \rho(\tau \parallel \theta)$, and by induction $\omega \xleftarrow{1} \rho(\tau \parallel \theta)$, therefore $\omega \xleftarrow{1} \sigma$ by transitivity of $\xleftarrow{1}$.

— The first rule applied is (2): $\omega \leftarrow \rho[(\tau < \theta_1) \parallel \theta_2] \leftarrow \rho[\tau < (\theta_1 \parallel \theta_2)] = \sigma$. Since $\overline{\varpi} \subseteq \overline{\sigma}$, for all $a \in |\tau|, b_1 \in |\theta_1|, b_2 \in |\theta_2|$, we have $\neg\overline{\varpi}(a, b_1, b_2)$ and $\neg\overline{\varpi}(a, b_2, b_1)$. By $\omega \subseteq$

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$\rho[(\tau < \theta_1) \parallel \theta_2]$, we conclude that a and b_1 are incomparable in ω, hence

$$\omega \subseteq \rho[\tau \parallel \theta_1 \parallel \theta_2].$$

We prove $\overline{\omega} \subseteq \overline{\rho[\tau \parallel \theta_1 \parallel \theta_2]}$ as above, by replacing θ by $\theta_1 \parallel \theta_2$. Therefore $\omega \trianglelefteq \rho[\tau \parallel \theta_1 \parallel \theta_2]$, and by induction, $\omega \overset{1}{\leftarrow} \rho[\tau \parallel \theta_1 \parallel \theta_2]$. Now, obviously, $\rho[\tau \parallel \theta_1 \parallel \theta_2] \overset{1}{\leftarrow} \sigma$, therefore $\omega \overset{1}{\leftarrow} \sigma$ by transitivity of $\overset{1}{\leftarrow}$.

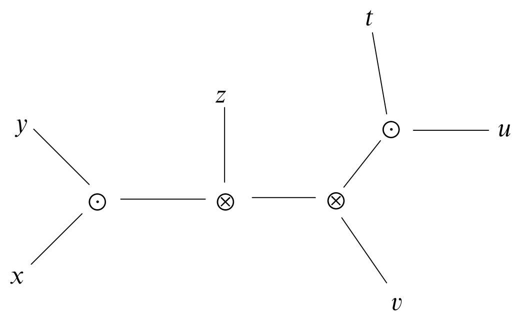
— The first rule applied is (3). This case is similar.

— The first rule applied is (4):

$$\omega \leftarrow \rho[(\tau_1 < \theta_1) \parallel (\tau_2 < \theta_2)] \leftarrow \rho[(\tau_1 \parallel \tau_2) < (\theta_1 \parallel \theta_2)] = \sigma.$$

Since $\overline{\omega} \subseteq \overline{\sigma}$ and $\omega \subseteq \rho[(\tau_1 < \theta_1) \parallel (\tau_2 < \theta_2)]$, we can prove, as in the case of Rule (2), that $\omega \subseteq \rho[\tau_1 \parallel \theta_1 \parallel \tau_2 \parallel \theta_2]$. Then by replacing τ by $\tau_1 \parallel \tau_2$, and θ by $\theta_1 \parallel \theta_2$ in the argument used for Rule (1), we get $\overline{\omega} \subseteq \overline{\rho[\tau_1 \parallel \theta_1 \parallel \tau_2 \parallel \theta_2]}$, and we conclude by using $\rho[\tau_1 \parallel \theta_1 \parallel \tau_2 \parallel \theta_2] \overset{1}{\leftarrow} \sigma$. □

In the tree representation for series–parallel order varieties, entropy is performed by changing some $\odot$-nodes into $\otimes$-nodes. For instance, the following order variety is obtained by entropy from $\overline{(x < y < z) \parallel v \parallel (t < u)}$:



Lemma 3.25.

(i) The partial orders ω such that $\omega \trianglerighteq N(a, b, c, d)$ are $N(a, b, c, d)$, $a < c < b < d$ and $c < d < a < b$.

(ii) The order varieties α such that $\alpha \supseteq G(a, b, c, d, e)$ are $G(a, b, c, d, e)$ and the total order varieties (a, c, b, d, e) and (c, d, a, b, e).

(iii) The partial orders ω such that $\omega \trianglelefteq N(a, b, c, d)$ are $N(a, b, c, d)$ and $a \parallel b \parallel c \parallel d$.

(iv) The order varieties α such that $\alpha \subseteq G(a, b, c, d, e)$ are $G(a, b, c, d, e)$ and $\varnothing_{\{a,b,c,d,e\}}$.

Proof. (i) Let ω be a partial order such that $\omega \trianglerighteq N(a, b, c, d)$, and assume $\omega \neq N(a, b, c, d)$. Then, in particular, $\omega \supset N(a, b, c, d)$, and is therefore one of the following 12

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non-trivial extensions of $N(a,b,c,d)$:

$$\begin{array}{ll} (a \parallel c) < (b \parallel d) & (a \parallel c) < b < d \\ a < c < b < d & (a \parallel c) < d < b \\ a < c < d < b & (a \parallel (c < d)) < b \\ c < a < b < d & a < c < (b \parallel d) \\ c < a < d < b & c < a < (b \parallel d) \\ c < d < a < b & c < ((a < b) \parallel d). \end{array}$$

It is easy to check that the additional condition $\overline{\varpi} \supseteq \overline{N(a,b,c,d)} = (a,b,d) \cup (a,c,d)$ implies the result.

(ii) By Proposition 3.22, $\boldsymbol{\alpha} \supseteq G(a,b,c,d,e)$ iff $\boldsymbol{\alpha}_e \trianglerighteq G(a,b,c,d,e)_e = N(a,b,c,d)$, whence the result.

(iii) Among the 8 partial orders $\omega \subseteq N(a,b,c,d)$, only $N(a,b,c,d)$ and $a \parallel b \parallel c \parallel d$ enjoy $\overline{\varpi} \subseteq \overline{N(a,b,c,d)}$.

(iv) We use the same argument as in (ii). □

3.6. *Interior and wedges*

Before turning to the sequent calculus on order varieties, we develop the theory of order varieties a bit further by studying:

— the appropriate notion of inf of order varieties,
— the operation of identification of two points in an order variety (see the next section).

The reason for these two sections is to show (Section 4) that the forthcoming calculus on order varieties is consistent with the multiplicative sequent calculus given in Abrusci and Ruet (2000). They can therefore be skipped in a first reading.

Intersections of order varieties are obviously cyclic orders but not necessarily order varieties. As we are definitely dealing with order varieties, we need a way to transform cyclic orders into order varieties. This motivates the following definitions.

Definition 3.26 (Interior). Let $\boldsymbol{\alpha}$ be a cyclic order on X (that is, an order variety without the spreading condition). Define its *interior* $\natural\boldsymbol{\alpha}$ by:

$$\boxed{\natural\boldsymbol{\alpha} \;=\; \textstyle\bigcap_{x \in X} (\boldsymbol{\alpha}_x * x).}$$

(The definition of the partial order $\boldsymbol{\alpha}_x$ on $X \setminus \{x\}$ is the same as for order varieties.) For instance if $\boldsymbol{\alpha} = (x,y,z,t) \cup (x,y,u)$, then $\natural\boldsymbol{\alpha} = (x,y,z) \cup (x,y,t) \cup (x,y,u)$. As shown in Abrusci and Ruet (2000), the interior of a cyclic order satisfies the following properties.

Proposition 3.27. Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be cyclic orders on X.

(i) $\natural\boldsymbol{\alpha}$ is an order variety on X.
(ii) $\natural\boldsymbol{\alpha} \subseteq \boldsymbol{\alpha}$.
(iii) $\natural\natural\boldsymbol{\alpha} = \natural\boldsymbol{\alpha}$.
(iv) $\natural\boldsymbol{\alpha}$ is the largest order variety included in $\boldsymbol{\alpha}$.

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(v) $\boldsymbol{\alpha} \subseteq \boldsymbol{\beta} \Rightarrow \natural\boldsymbol{\alpha} \subseteq \natural\boldsymbol{\beta}$.
(vi) If $Y \subseteq X$ then $(\natural\boldsymbol{\alpha})\restriction_Y \subseteq \natural(\boldsymbol{\alpha}\restriction_Y)$.
(vii) $\natural(\boldsymbol{\alpha} \cap \boldsymbol{\beta}) \subseteq \natural\boldsymbol{\alpha} \cap \natural\boldsymbol{\beta}$.
(viii) $\natural(\natural\boldsymbol{\alpha} \cap \natural\boldsymbol{\beta}) = \natural(\boldsymbol{\alpha} \cap \boldsymbol{\beta})$.
(ix) $\rightarrow_{\natural\boldsymbol{\alpha}} = \rightarrow_{\boldsymbol{\alpha}}$.

Definition 3.28 (Wedge of order varieties). Let $\boldsymbol{\alpha}_i, i \in I$, be order varieties on X. Define

$$\boxed{\bigwedge \boldsymbol{\alpha}_i \;=\; \natural \bigcap \boldsymbol{\alpha}_i.}$$

If I has cardinality 2, we write $\boldsymbol{\alpha}_1 \wedge \boldsymbol{\alpha}_2$.

Proposition 3.29. (i) $\wedge$ is commutative and associative.
(ii) $\bigwedge \boldsymbol{\alpha}_i$ is the largest order variety included in all the $\boldsymbol{\alpha}_i$.

Proof. (i) Commutativity is clear. For associativity,

$$\begin{aligned}(\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) \wedge \boldsymbol{\gamma} &= \natural(\natural(\boldsymbol{\alpha} \cap \boldsymbol{\beta}) \cap \boldsymbol{\gamma}) \\ &= \natural(\natural(\boldsymbol{\alpha} \cap \boldsymbol{\beta}) \cap \natural\boldsymbol{\gamma}) \\ &= \natural(\boldsymbol{\alpha} \cap \boldsymbol{\beta} \cap \boldsymbol{\gamma}).\end{aligned}$$

The second equality holds because $\boldsymbol{\gamma}$ is an order variety and the third is a consequence of Proposition 3.27 (viii). Similarly, $\boldsymbol{\alpha} \wedge (\boldsymbol{\beta} \wedge \boldsymbol{\gamma}) = \natural(\boldsymbol{\alpha} \cap \boldsymbol{\beta} \cap \boldsymbol{\gamma})$.
(ii) This follows from an obvious application of Proposition 3.27. □

Corollary 3.30. Order varieties on a given set form a complete inf-semi-lattice for inclusion and wedge.

Definition 3.31 (Wedge of orders). Let $\omega_i, i \in I$, be partial orders on X. Define

$$\boxed{\bigwedge \omega_i \;=\; (\bigwedge \omega_i * x)_x.}$$

where $x \notin X$. If I has cardinality 2, we write $\omega_1 \wedge \omega_2$.

Lemma 3.32.

(i) If $Y \subseteq |\boldsymbol{\alpha}_i|$, we have $(\bigwedge \boldsymbol{\alpha}_i)\restriction_Y \subseteq \bigwedge \boldsymbol{\alpha}_i\restriction_Y$.
(ii) If $Y \subseteq |\omega_i|$, we have $(\bigwedge \omega_i)\restriction_Y \trianglelefteq \bigwedge \omega_i\restriction_Y$.
(iii) Wedge commutes with focusing: $(\bigwedge \boldsymbol{\alpha}_i)_x = \bigwedge(\boldsymbol{\alpha}_i)_x$ for any $x \in |\boldsymbol{\alpha}_i|$.
(iv) Wedge commutes with glueing: $(\bigwedge \omega_i) * x = \bigwedge(\omega_i * x)$ for any $x \notin |\omega_i|$.
(v) More generally, $(\bigwedge_{i\in I} \omega_i) * (\bigwedge_{j \in J} \tau_j) = \bigwedge_{i\in I, j\in J}(\omega_i * \tau_j)$ when $|\omega_i| \cap |\tau_j| = \emptyset$.

Proof. (i) This follows from Proposition 3.27 (vi).
(ii) This follows from

$$\begin{aligned}(\bigwedge \omega_i)\restriction_Y &= ((\bigwedge \omega_i * x)_x)\restriction_Y \\ &\trianglelefteq (\bigwedge \omega_i\restriction_Y * x)_x \quad \text{by (i) and Proposition 3.22} \\ &= \bigwedge \omega_i\restriction_Y .\end{aligned}$$

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(iii) This follows from the equality $\bigwedge(\boldsymbol{\alpha}_i)_x = [\bigwedge((\boldsymbol{\alpha}_i)_x * x)]_x = (\bigwedge \boldsymbol{\alpha}_i)_x$, which is a consequence of Proposition 3.6.
(iv) This is analogous.
(v) It is enough to prove that $(\bigwedge_{i\in I} \omega_i) * \tau = \bigwedge_{i\in I}(\omega_i * \tau)$. The inclusion $(\bigwedge \omega_i) * \tau \subseteq \bigwedge(\omega_i * \tau)$ follows from $\bigwedge \omega_i \trianglelefteq \omega_i$ and Corollary 3.21. For the converse, let $X = |\omega_i|$, $Y = |\tau|$, and consider $a, b, c \in X \cup Y$ such that $\bigwedge(\omega_i * \tau)\ (a, b, c)$. Either the a, b, c are all in X or all in Y, or one of them is in one set and the other two are in the other, but in any case we can apply (iv). For instance, if $a, b, c \in X$, take $y \in Y$. Then by (i) and (iv) we have

$$\begin{aligned}
(\bigwedge(\omega_i * \tau)) \restriction_{X\cup\{y\}} &\subseteq \bigwedge(\omega_i * \tau)\restriction_{X\cup\{y\}} \\
&= \bigwedge(\omega_i * y) \\
&= (\bigwedge \omega_i) * y,
\end{aligned}$$

therefore $\bigwedge(\omega_i * \tau)\ (a, b, c)$ implies $((\bigwedge \omega_i) * y)\ (a, b, c)$, and hence $((\bigwedge \omega_i) * \tau)\ (a, b, c)$. If, on the other hand, $a, b \in X$ and $c \in Y$, we apply the same property with $y = c$. □

Proposition 3.33. (i) $\wedge$ is a commutative and associative operation on partial orders.
(ii) $\bigwedge \omega_i$ is the largest order ω (with respect to $\trianglelefteq$) such that $\omega \trianglelefteq \omega_i$ for all i.

Proof. (i) Commutativity is obvious. Associativity is a consequence of Propositions 3.29 and 3.6.
(ii)

$\tau \trianglelefteq$ all the ω_i	iff	$\tau * x \subseteq \bigcap(\omega_i * x)$	by Proposition 3.22
	iff	$\tau * x \subseteq \bigwedge(\omega_i * x)$	because $\tau * x$ is an order variety
	iff	$(\tau * x)_x \trianglelefteq [\bigwedge(\omega_i * x)]_x$	by Proposition 3.22 again
	iff	$(\tau * x)_x \trianglelefteq \bigwedge(\omega_i * x)_x$	by Lemma 3.32
	iff	$\tau \trianglelefteq \bigwedge \omega_i$	by Proposition 3.6.

□

Remarks.
► $\omega \wedge \tau \subset \omega \cap \tau$ but the inclusion is strict in general. Consider, for instance, $\omega = (a \parallel b) < c$ and $\tau = (a < c) \parallel b$. Since $\tau \subseteq \omega$, we have $\omega \cap \tau = \tau$. On the other hand,

$$\begin{aligned}
\omega \wedge \tau &= (\omega * x \wedge \tau * x)_x \\
&= [((a, c, x) \cup (b, c, x)) \wedge ((a, c, b) \cup (a, c, x))]_x \\
&= [\natural(a, c, x)]_x \\
&= (\varnothing_{\{a,b,c,x\}})_x \\
&= \varnothing,
\end{aligned}$$

because $\natural(a, c, x) \subseteq (a, c, x)_b * b = \varnothing_{\{a,b,c,x\}}$.

► $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ may be series–parallel and not $\boldsymbol{\alpha} \wedge \boldsymbol{\beta}$: take $\boldsymbol{\alpha} = (a, c, b, d, e)$ and $\boldsymbol{\beta} = (c, d, a, b, e)$, they are even total, but $\boldsymbol{\alpha} \wedge \boldsymbol{\beta} = G(a, b, c, d, e) = \boldsymbol{\alpha} \cap \boldsymbol{\beta}$.

► For similar reasons, ω and τ may be series–parallel but not $\omega \wedge \tau$:

$$\begin{aligned}(c<d<a<b)\wedge(a<c<b<d) &= ((c<d<a<b)*e\wedge(a<c<b<d)*e)_e\\ &= ((c,d,a,b,e)\wedge(a,c,b,d,e))_e\\ &= G(a,b,c,d,e)_e\\ &= N(a,b,c,d).\end{aligned}$$

By the way, note that the wedge of two order varieties equals their intersection as soon as this intersection is an order variety, whereas the intersection of partial orders is always an order, and we cannot conclude anything about their wedge.

3.7. *Identification*

Definition 3.34 (Identification). Let $\boldsymbol{\alpha}$ be an order variety on a set $X\cup\{x,y\}$, with $x,y\notin X$, $x\neq y$, and let $z\notin X$. Define the *identification* $\boldsymbol{\alpha}[z/x,y]$ of x and y into z in $\boldsymbol{\alpha}$ by

$$\boxed{\boldsymbol{\alpha}[z/x,y] \;=\; \boldsymbol{\alpha}\restriction_{X\cup\{x\}}[z/x] \;\wedge\; \boldsymbol{\alpha}\restriction_{X\cup\{y\}}[z/y].}$$

Clearly, if $\boldsymbol{\alpha}$ is an order variety, then so is $\boldsymbol{\alpha}[z/x,y]$.

Lemma 3.35.

(i) $\boldsymbol{\alpha}[z/x,y]_z = (\overline{\boldsymbol{\alpha}_x})_y \wedge (\overline{\boldsymbol{\alpha}_y})_x$.
(ii) $\boldsymbol{\alpha}[z/x,y]_z * (x \parallel y) \subseteq \boldsymbol{\alpha}$.
(iii) Let $\boldsymbol{\alpha}$ be an order variety on $X\cup\{x,y\}$, with x,y different and not in X, and ω be a partial order on X such that $\omega*(x\parallel y)\subseteq\boldsymbol{\alpha}$. Then $\omega*(x\parallel y)\subseteq\boldsymbol{\alpha}[z/x,y]_z*(x\parallel y)$, or, equivalently, $\omega \trianglelefteq \boldsymbol{\alpha}[z/x,y]_z$.

Proof. (i) By Lemma 3.32,

$$\begin{aligned}\boldsymbol{\alpha}[z/x,y]_z &= \boldsymbol{\alpha}\restriction_{X\cup\{x\}}[z/x]_z\wedge\boldsymbol{\alpha}\restriction_{X\cup\{y\}}[z/y]_z\\ &= (\boldsymbol{\alpha}\restriction_{X\cup\{x\}})_x\wedge(\boldsymbol{\alpha}\restriction_{X\cup\{y\}})_y\\ &= (\overline{\boldsymbol{\alpha}_y})_x\wedge(\overline{\boldsymbol{\alpha}_x})_y.\end{aligned}$$

(ii) Let $a,b,c\in X\cup\{x,y\}$ be such that $\boldsymbol{\alpha}[z/x,y]_z*(x\parallel y)\;(a,b,c)$. First, $\{a,b,c\}$ does not contain both x and y, so assume $y\notin\{a,b,c\}$, whence $(\boldsymbol{\alpha}[z/x,y]_z*(x\parallel y))\restriction_{X\cup\{x\}}(a,b,c)$, that is, $(\boldsymbol{\alpha}[z/x,y]_z*x)(a,b,c)$. By (i) and Proposition 3.33:

$$\begin{aligned}\boldsymbol{\alpha}[z/x,y]_z*x &= (\overline{\boldsymbol{\alpha}_x})_y*x\wedge(\overline{\boldsymbol{\alpha}_y})_x*x\\ &\subseteq (\overline{\boldsymbol{\alpha}_y})_x*x\\ &= \overline{\boldsymbol{\alpha}_y}\\ &= \boldsymbol{\alpha}\restriction_{X\cup\{x\}}.\end{aligned}$$

Therefore $\boldsymbol{\alpha}(a,b,c)$. The case where $x\notin\{a,b,c\}$ is similar.

(iii) Let us prove that $\omega*z\subseteq\boldsymbol{\alpha}[z/x,y]$, that is, $\omega*z\subseteq\boldsymbol{\alpha}\restriction_{X\cup\{x\}}[z/x]$ and $\omega*z\subseteq\boldsymbol{\alpha}\restriction_{X\cup\{y\}}[z/y]$. This is immediate, since by hypothesis $\omega*x\subseteq\boldsymbol{\alpha}\restriction_{X\cup\{x\}}$ and $\omega*y\subseteq\boldsymbol{\alpha}\restriction_{X\cup\{y\}}$. □

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Proposition 3.36.

(i) Inclusion is compatible with identification in order varieties.
(ii) If $\boldsymbol{\alpha}$ is a series–parallel order variety, so is $\boldsymbol{\alpha}[z/x, y]$.

Proof.
(i) We have to show that $\boldsymbol{\alpha} \subseteq \boldsymbol{\beta}$ implies $\boldsymbol{\alpha}[z/x, y] \subseteq \boldsymbol{\beta}[z/x, y]$. By Lemma 3.35 (ii), $\boldsymbol{\alpha}[z/x, y]_z * (x \parallel y) \subseteq \boldsymbol{\alpha} \subseteq \boldsymbol{\beta}$. Therefore by Lemma 3.35 (iii) applied to $\omega = \boldsymbol{\alpha}[z/x, y]_z$, we have $\boldsymbol{\alpha}[z/x, y]_z \trianglelefteq \boldsymbol{\beta}[z/x, y]_z$.

(ii) Let $|\boldsymbol{\alpha}| = X \uplus \{x, y\}$, and assume for a contradiction that $\boldsymbol{\alpha}[z/x, y]$ is not series–parallel, that is, that $\boldsymbol{\alpha}[z/x, y]_z$ is not a series–parallel order: $\boldsymbol{\alpha}[z/x, y]_z \upharpoonright_Y = N(a, b, c, d)$ for some $Y = \{a, b, c, d\} \subseteq X$. By Lemma 3.35 (i), $\boldsymbol{\alpha}[z/x, y]_z = (\overline{\boldsymbol{\alpha}_x})_y \wedge (\overline{\boldsymbol{\alpha}_y})_x$, hence by Lemma 3.32 (ii),

$$\begin{aligned} \boldsymbol{\alpha}[z/x, y]_z \upharpoonright_Y &\trianglelefteq (\overline{\boldsymbol{\alpha}_x})_y \upharpoonright_Y \wedge (\overline{\boldsymbol{\alpha}_y})_x \upharpoonright_Y \\ &= (\overline{\boldsymbol{\beta}_x})_y \wedge (\overline{\boldsymbol{\beta}_y})_x, \end{aligned}$$

with $\boldsymbol{\beta} = \boldsymbol{\alpha} \upharpoonright_{Y \uplus \{x,y\}} = \boldsymbol{\alpha} \upharpoonright_{\{a,b,c,d,x,y\}}$. Therefore $N(a, b, c, d) \trianglelefteq (\overline{\boldsymbol{\beta}_x})_y$ and $N(a, b, c, d) \trianglelefteq (\overline{\boldsymbol{\beta}_y})_x$. Moreover, $(\overline{\boldsymbol{\beta}_x})_y$ and $(\overline{\boldsymbol{\beta}_y})_x$ are series–parallel order varieties because so is $\boldsymbol{\beta}$. Hence by Lemma 3.25, $(\overline{\boldsymbol{\beta}_x})_y$ and $(\overline{\boldsymbol{\beta}_y})_x$ are either $(a < c < b < d)$ or $(c < d < a < b)$. They cannot be equal, because otherwise their wedge, $\boldsymbol{\alpha}[z/x, y]_z \upharpoonright_Y$, would equal their common value. So we may assume without loss of generality that

$$\begin{aligned} (\overline{\boldsymbol{\beta}_x})_y &= (a < c < b < d) \\ (\overline{\boldsymbol{\beta}_y})_x &= (c < d < a < b), \end{aligned}$$

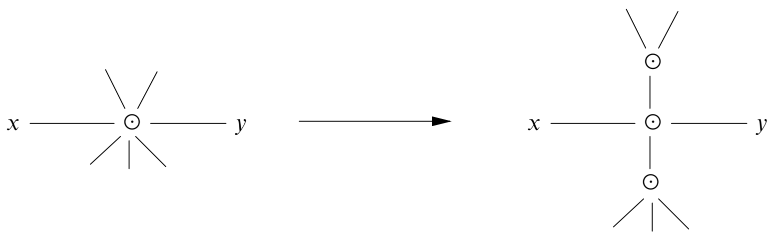
that is,

$$\begin{aligned} \boldsymbol{\alpha} \upharpoonright_{\{a,b,c,d,y\}} &= \overline{\boldsymbol{\beta}_x} = (a, c, b, d, y) \\ \boldsymbol{\alpha} \upharpoonright_{\{a,b,c,d,x\}} &= \overline{\boldsymbol{\beta}_y} = (c, d, a, b, x). \end{aligned}$$

We now have a contradiction because the first equality implies $\boldsymbol{\alpha}(a, c, b)$ whereas the second implies $\boldsymbol{\alpha}(a, b, c)$. □

In the series–parallel case, the identification can be performed directly on the tree representation introduced in Section 3.2. Take a representation T of $\boldsymbol{\alpha}$, an order variety on $X \cup \{x, y\}$, and let p be the path between x and y in T. Then 'normalize' T as follows:

— Collapse neighbouring nodes of the same type ($\odot$ or $\otimes$) in p, so that in the resulting tree T', nodes of the path p' between x and y have alternating types.
— Transform every $\odot$-node in p' into 2 or 3 adjacent $\odot$-nodes as follows:



It is obvious that the normal tree still represents $\boldsymbol{\alpha}$, and the identification $\boldsymbol{\alpha}[z/x, y]$ is obtained by changing all the $\odot$-nodes into $\otimes$-nodes and by identifying x and y.

To see that this is the largest series–parallel order variety obtained by identifying x and y (in fact the largest order variety thanks to Proposition 3.36), first recall that, in general, entropy between series–parallel order varieties is indeed achieved in the tree representation by changing some $\odot$-nodes into $\otimes$-nodes. Of course, there is some choice in where to perform entropies. In the case of identification, however, we know from Lemma 3.35 that x and y have to be in parallel in the end. The above 'fusions' and 'fissions' are then performed to minimize the number of $\odot$-nodes between x and y to which entropies are applied: indeed these operations commute with entropy.

4. Sequent calculus: on order varieties

4.1. *Rules*

Definition 4.1. A *sequent* $\vdash \boldsymbol{\alpha} \mid \Gamma$ consists of an order variety $\boldsymbol{\alpha}$ of formula occurrences, and a set Γ of formula occurrences (with no additional structure). Γ is required to be disjoint from the support of $\boldsymbol{\alpha}$.

The rules of the sequent calculus are given in Table 2. When there is no ambiguity, that is, when $\boldsymbol{\alpha}$ or Γ is empty and the notation enables one to decide whether it is an order variety or a set, we avoid the stoup $\mid$. If ω is an order on X, we denote X by $|\omega|$.

Example. Here is a proof of $!A \odot !B \vdash !(A\&B)$:

$$
\begin{array}{c}
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash A * A^{\perp}}}{\vdash A \mid A^{\perp}}\,\text{centre}}{\vdash A \mid A^{\perp}, B^{\perp}}\,\text{w} \qquad \dfrac{\dfrac{\dfrac{}{\vdash B * B^{\perp}}}{\vdash B \mid B^{\perp}}\,\text{centre}}{\vdash B \mid B^{\perp}, A^{\perp}}\,\text{w}}{\vdash A\&B \mid B^{\perp}, A^{\perp}}\,\&}{\vdash !(A\&B) \mid B^{\perp}, A^{\perp}}\,!}{\vdash \overline{!(A\&B) < ?B^{\perp}} \mid A^{\perp}}\,\text{d}}{\vdash (!(A\&B) < ?B^{\perp}) * ?A^{\perp}}\,\text{d} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\dfrac{\vdash !(A\&B) * (?B^{\perp} < ?A^{\perp})}{\vdash !(A\&B) * (?B^{\perp} \triangledown ?A^{\perp})}\,\triangledown
\end{array}
$$

The dotted line corresponds to a change in the presentation of the series–parallel order variety.

Remarks.

► By Proposition 3.7, there is no loss of generality in taking order varieties of the form $\omega * A$, since any order variety $\boldsymbol{\alpha}$ can be written as $\boldsymbol{\alpha}_A * A$.

Note that the condition of series–parallelism in entropy, dereliction and the $\perp$-rule suffices to preserve series–parallelism along the construction of a proof.

Lemma 4.2. If $\vdash \boldsymbol{\alpha} \mid \Gamma$ is a provable sequent, then $\boldsymbol{\alpha}$ is series–parallel.

On the other hand, these conditions are necessary since a partial order ω may not be series–parallel, even when $\overline{\omega}$ is (for instance, $\overline{N(a,b,c,d)} = (a,b,d) \cup (a,c,d)$), and a

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Table 2. *Sequent calculus II.*
All order varieties and partial orders here are series–parallel. In particular, in dereliction and the $\perp$-rule, ω is series–parallel, and in the entropy rule, $\boldsymbol{\alpha}$ is series–parallel.

Identity - Cut

$$\frac{}{\vdash A^{\perp} * A} \qquad \left\{ \begin{array}{l} \dfrac{\vdash \omega * A \mid \Gamma \qquad \vdash \omega' * A^{\perp} \mid \Gamma'}{\vdash \omega * \omega' \mid \Gamma, \Gamma'}\ \text{cut} \\[2ex] \dfrac{\vdash \boldsymbol{\alpha} \mid A, \Gamma \qquad \vdash !A^{\perp} \mid \Gamma'}{\vdash \boldsymbol{\alpha} \mid \Gamma, \Gamma'}\ \text{cut}_{!} \end{array} \right.$$

Structural rules

$$\frac{\vdash \omega * A \mid \Gamma}{\vdash \varpi \mid A, \Gamma}\ \text{centre} \qquad \frac{\vdash \boldsymbol{\beta} \mid \Gamma}{\vdash \boldsymbol{\alpha} \mid \Gamma}\ \text{entropy, } \boldsymbol{\alpha} \subseteq \boldsymbol{\beta}$$

Multiplicatives

$$\frac{\vdash \omega * A \mid \Gamma \qquad \vdash \omega' * B \mid \Gamma'}{\vdash (\omega' < \omega) * A \odot B \mid \Gamma, \Gamma'}\ \odot \qquad \frac{\vdash \omega * (A < B) \mid \Gamma}{\vdash \omega * A \triangledown B \mid \Gamma}\ \triangledown$$

$$\frac{\vdash \omega * A \mid \Gamma \qquad \vdash \omega' * B \mid \Gamma'}{\vdash (\omega \parallel \omega') * A \otimes B \mid \Gamma, \Gamma'}\ \otimes \qquad \frac{\vdash \omega * (A \parallel B) \mid \Gamma}{\vdash \omega * A ⅋ B \mid \Gamma}\ ⅋$$

Additives

$$\frac{\vdash \omega * A \mid \Gamma \qquad \vdash \omega * B \mid \Gamma}{\vdash \omega * A \& B \mid \Gamma}\ \& \qquad \left\{ \begin{array}{l} \dfrac{\vdash \omega * A \mid \Gamma}{\vdash \omega * A \oplus B \mid \Gamma}\ \oplus 1 \\[2ex] \dfrac{\vdash \omega * B \mid \Gamma}{\vdash \omega * A \oplus B \mid \Gamma}\ \oplus 2 \end{array} \right.$$

Exponentials

$$\frac{\vdash \varpi \mid A, \Gamma}{\vdash \omega * ?A \mid \Gamma}\ \text{d} \qquad \frac{\vdash A \mid \Gamma}{\vdash !A \mid \Gamma}\ ! \qquad \frac{\vdash \boldsymbol{\alpha} \mid A, A, \Gamma}{\vdash \boldsymbol{\alpha} \mid A, \Gamma}\ \text{c} \qquad \frac{\vdash \boldsymbol{\alpha} \mid \Gamma}{\vdash \boldsymbol{\alpha} \mid A, \Gamma}\ \text{w}$$

Constants

$$\frac{}{\vdash \mathbf{1}} \qquad \frac{\vdash \varpi \mid \Gamma}{\vdash \omega * \perp \mid \Gamma}\ \perp \qquad \text{(no rule for } \mathbf{0}\text{)} \qquad \frac{}{\vdash \omega * \top \mid \Gamma}$$

non-series–parallel order variety may be included in a series–parallel one (for instance, $G(a, b, c, d, e) \subseteq (a, c, b, d, e)$ by Lemma 3.25).

► *About the stoup.* The centre rule and the rule for weakening take formulas out of the scope of the order variety. Conversely, on its way back to earth (the scope of the order variety), a formula can be placed in any position that does not affect the current structure $\boldsymbol{\alpha}$ on X, that is, the extended order variety $\boldsymbol{\alpha}'$ on $X + x$ is *any* series–parallel

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order variety enjoying $\alpha' \restriction_X = \alpha$. Indeed, if α' is such an order variety, take $\omega = \alpha'_x$. Then by Proposition 3.15, we have $\alpha = \alpha' \restriction_X = \overline{\alpha'_x} = \overline{\omega}$.

This corresponds to the previous centre rule in the calculus of Section 2. Note that in the absence of exponentials, the 'set' part of sequents remains empty (a consequence of the subformula property) and can henceforth be forgotten.

► *About the* $\triangledown$*-rule.* The rule for $\triangledown$ corresponding to the one given in Abrusci and Ruet (2000) would be

$$\frac{\vdash \alpha[A, B] \mid \Gamma}{\vdash \alpha[A \triangledown B/A, B] \mid \Gamma} \triangledown\star, \text{ if } A \rightarrow_\alpha B,$$

where the notation $\alpha[A, B, \ldots]$ stands for an order variety on a set X such that $A, B, \ldots \in X$. In Table 2, we prefered to replace it by another one, where the relation $\rightarrow_\alpha$ is expressed more simply. They are equivalent. Indeed, if $A \rightarrow_\alpha B$, consider the partial order α_B, where every C is less than or equal to A, so $\alpha_B = (\omega < A)$, and hence $\alpha = \omega * (A < B)$. Conversely, it is clear that $A \rightarrow_{\omega*(A<B)} B$.

In order to simplify the notation, we assume that the sequent calculus of Section 2 relies on partial orders.

Theorem 4.3 (Equivalence between the two sequent calculi).

1 Given a proof of $\vdash \omega$ in the calculus on orders, let $?X$ be the set of $?$ed formulas in $|\omega|$ and $Y = |\omega| \backslash ?X$. We construct, by forgetting the seesaw rule, a proof of $\vdash \alpha \mid X$ in the calculus on order varieties for $\alpha = \overline{\omega} \restriction_Y$. The notation X means that the main $?$ has been removed in all formulas of $?X$.
2 Conversely, given a proof of $\vdash \alpha \mid X$, where the support of α is Y, we construct, by making some structural steps explicit, a proof of $\vdash \omega$ for any series–parallel order ω on $Y \cup ?X$ such that $\overline{\omega} \restriction_Y = \alpha$.
3 The mappings (1) and (2) preserve the absence of cuts.

Proof.

1 We proceed by induction on a proof of $\vdash \omega$. We consider the last rule of the proof:

— Axiom: At most one of the formulas is $?$ed, so the translation is either just an axiom or an axiom followed by centre.

— Cut rule:

$$\frac{\vdash \omega, A \qquad \vdash A^\perp, \omega'}{\vdash \omega, \omega'} \text{cut}$$

If neither A nor $A^\perp$ is $?$ed, then the induction hypothesis gives proofs of $\vdash \tau * A \mid X$ and $\vdash \tau' * A^\perp \mid X'$, where $|\omega| = ?X \uplus Y$, $|\omega'| = ?X' \uplus Y'$, $\tau * A = (\omega * A)\restriction_Y$ and $\tau' * A^\perp = (\omega' * A^\perp)\restriction_{Y'}$. An application of cut then gives a proof of $\vdash \tau * \tau' \mid X, X'$. We have $\tau = \omega\restriction_Y$ and $\tau' = \omega'\restriction_{Y'}$ so $\tau * \tau' = (\omega * \omega')\restriction_{Y \cup Y'}$.
When A or $A^\perp$ is $?$ed, say A, then the induction hypothesis gives proofs of $\vdash \alpha \mid A, X$ and $\vdash \tau' * !A^\perp \mid X'$ with $\alpha = \overline{\omega} \restriction_Y$ and $\tau' * !A^\perp = (\omega' * !A^\perp)\restriction_{Y'}$. Taking $\tau = \omega\restriction_Y$, we have $\alpha = \overline{\tau}$, so by an application of dereliction and a cut, we get a

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proof of $\vdash \tau * \tau' \mid X, X'$ enjoying $\tau * \tau' = (\omega * \omega') \upharpoonright_{Y \cup Y'}$:

$$\frac{\vdash \omega, ?A \qquad \vdash !A^\perp, \omega'}{\vdash \omega, \omega'}\,\text{cut} \quad \mapsto \quad \cfrac{\cfrac{\cfrac{\vdash \boldsymbol{\alpha} \mid A, X}{\vdash \overline{\tau} \mid A, X}}{\vdash \tau * ?A \mid X}\,\text{d} \qquad \vdash \tau' * !A^\perp \mid X'}{\vdash \tau * \tau' \mid X, X'}\,\text{cut.}$$

— Entropy: By Lemma 3.23, $\overline{\omega \parallel \tau \parallel \sigma} \subseteq \overline{\omega < \tau < \sigma}$.

— Seesaw and centre: By Proposition 3.6, $\overline{\omega < \tau} = \overline{\omega \parallel \tau}$.

— The cases of the rules for $\odot$, $\otimes$, $\triangledown$, $⅋$, $\&$, $\oplus$ and ! are handled in a similar way to the cut rule, by using the dereliction rule when an active formula is ?ed.

— For dereliction, either the active formula A is not ?ed and the translation of the rule is a centre rule, or it is already ?ed and we apply dereliction followed by centre.

— The cases of contraction, weakening and the rules for **1**, $\perp$ and $\top$ are trivial.

2 We do not need to prove that much: it is enough to exhibit a proof of $\vdash \omega$ for *some* series–parallel order ω on $Y \cup ?X$ such that $\overline{\omega} \upharpoonright_Y = \boldsymbol{\alpha}$. Indeed, by Corollary 2.7, we can then arbitrarily move the formulas in $?X$ without changing the order on Y, and by Proposition 3.17, we can change ω for another arbitrary series–parallel order ω' such that $\overline{\omega' \upharpoonright_Y} = \boldsymbol{\alpha}$.
Proceed by induction on a proof of $\vdash \boldsymbol{\alpha} \mid X$. The above remark makes the result trivial, except for the following cases:

— The centre rule is translated by a step of dereliction.

— In the case of entropy, the induction hypothesis gives, in particular, a proof of $\vdash \omega \parallel A$ for some formula $A \in |\boldsymbol{\beta}| = Y$ and $\omega \upharpoonright_Y = \boldsymbol{\beta}_A$. Now $\boldsymbol{\alpha}_A \trianglelefteq \boldsymbol{\beta}_A$ by Proposition 3.22, so by Proposition 3.24, we get, by applications of entropy, a proof of $\vdash \tau \parallel A$ where $\tau \upharpoonright_Y = \boldsymbol{\alpha}_A$.

— In the cases of dereliction and the $\perp$-rule, the point is to choose a proof where the restriction of the order to $|\omega|$ is precisely ω.

3 This is just a straighforward remark. □

4.2. *Discussion about entropy and the par-rule*

The $⅋$-rule corresponding to the one given in Abrusci and Ruet (2000) would be:

$$\frac{\vdash \boldsymbol{\alpha}[A, B] \mid \Gamma}{\vdash \boldsymbol{\alpha}[A⅋B/A, B] \mid \Gamma}\,⅋\star$$

where $\boldsymbol{\alpha}[A⅋B/A, B]$ is the identification defined in Section 3.7. In Abrusci and Ruet (2000) there is no explicit rule for entropy: it is hidden in the $⅋\star$-rule through the absence of a condition on the order variety. In the multiplicative fragment, the two versions are equivalent (hence our presentation is consistent with Abrusci and Ruet (2000)) in the following sense:

— By Lemma 3.35 (ii), $\boldsymbol{\alpha}[A⅋B/A, B]_{A⅋B} * (A \parallel B) \subseteq \boldsymbol{\alpha}$, so our new pair of rules, entropy and the $⅋$-rule, can mimic the $⅋\star$-rule.

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— By Lemma 3.35 (iii), $\omega * (A \parallel B) \subseteq \boldsymbol{\alpha}$ implies $\omega * (A \parallel B) \subseteq \boldsymbol{\alpha}[A⅋B/A, B]_{A⅋B} * (A \parallel B)$, so ⅋$\star$ is an optimized version of ⅋-rule, where entropy has been minimized.

Removing the rule of entropy from the sequent calculus would carry us closer to proof nets since proof nets do not have entropy links, but it is not clear how to do it in full NL. In the multiplicative–additive fragment MANL, it is possible to remove the rule of entropy and define an optimized rule for & as well. Because entropy is implicit, it is then necessary to allow different order varieties $\omega * x$ and $\tau * x$ (of equal support, of course) in the two premisses:

$$\frac{\vdash \omega * A \mid \Gamma \qquad \vdash \tau * B \mid \Gamma}{\vdash (\omega \wedge \tau) * A\&B \mid \Gamma}\ \&\star.$$

Note that $\omega \wedge \tau$ may not be series–parallel, even if ω and τ are (see Section 3.6), so the situation becomes subtler.

In the presence of exponentials, the problem is with entropy, dereliction and the $\perp$-rule, where points move into or out of the order variety. The point is that it is possible to have a partial order ω on X such that $\overline{\omega} \subseteq \boldsymbol{\alpha}$ and no partial order τ satisfying both $\overline{\tau} = \boldsymbol{\alpha}$ and $\omega \trianglelefteq \tau$: in other words, it becomes more difficult to relate the calculus on order varieties to the one on partial orders because the inclusion of order varieties may not be simulated by entropy for any given choice of presentation: this is also true in the series–parallel case. For instance, take $\omega = (a \parallel b \parallel d) < (c \parallel e)$ and $\boldsymbol{\alpha} = \underline{a * [(b \parallel c)} < (d \parallel e)]$: $\overline{\omega} = \emptyset$, but the two conditions $(b \parallel d) < (c \parallel e) = \omega \restriction_{b,c,d,e} \subseteq \tau \restriction_{b,c,d,e}$ and $\overline{\tau \restriction_{b,c,d,e}} = \boldsymbol{\alpha} \restriction_{b,c,d,e} = \emptyset$ are incompatible.

So we leave this study – essentially general proof nets for NL – to further work and keep explicit entropy in our calculus.

5. Phase semantics

5.1. *Phase spaces*

Definition 5.1. A *phase space* is a sextuplet $P = (P, \cdot, \star, 1, \leqslant, \perp)$ such that:

1 $(P, \cdot, 1)$ is a monoid.
2 $(P, \star, 1)$ is a commutative monoid.
3 $\leqslant$ is a partial order on P, compatible with both monoidal structures and such that $x \star y \leqslant x \cdot y, \forall x, y \in P$.
4 $\perp \subseteq P$ is an order ideal such that $\forall x, y \in P, x \cdot y \in \perp \Leftrightarrow y \cdot x \in \perp \Leftrightarrow x \star y \in \perp$.

The elements of P are called *phases*, the elements of $\perp$ are called the *antiphases*.

The compatibility condition for $\leqslant$ means that $x \leqslant x'$ and $y \leqslant y' \Rightarrow x \cdot y \leqslant x' \cdot y'$ and $x \star y \leqslant x' \star y'$. As an order ideal, $\perp$ satisfies $x \in \perp$ and $y \leqslant x \Rightarrow y \in \perp$.

Examples.

► If $(P, \star, 1, \perp)$ is a commutative phase space in the sense of Girard (1987), then $(P, \star, \star, 1, =, \perp)$ is a phase space.

► Let $(P, \cdot, 1, \leqslant, \wedge, \vee)$ be a lattice-ordered monoid (a good reference is Fuchs (1963)), that is, a monoid $(P, \cdot, 1)$ together with a lattice structure $(\leqslant, \wedge, \vee)$ compatible with multiplication. The product $x \star y = xy \wedge yx$ is obviously commutative with unit 1, and satisfies $x \star y \leqslant xy$. If for all $x, y, z \in P$

$$xyz \wedge zyx = yzx \wedge xzy,$$

that is, the quantity $xyz \wedge zyx$ is invariant by any permutation of x, y, z, then $x \star y$ is associative.

Definition 5.2. If G is a subset of P, its *dual* is defined by

$$G^{\perp} = \{p \in P \mid \forall q \in G, p \star q \in \perp\}.$$

Alternatively, $G^{\perp} = \{p \in P \mid \forall q \in G, p \cdot q \in \perp\} = \{p \in P \mid \forall q \in G, q \cdot p \in \perp\}$. For G, H subsets of P, define

$$\begin{array}{rcl} G \cdot H &=& \{p \cdot q \mid p \in G, q \in H\} \\ G \star H &=& \{p \star q \mid p \in G, q \in H\}. \end{array}$$

Definition 5.3 (Fact). A *fact* is a subset A of P such that $A^{\perp\perp} = A$.

As is the case for the usual commutative and cyclic linear logic, we have the following items (i)–(v) and (vii). Item (vi) is specific to NL.

Proposition 5.4.

(i) For any $G \subseteq P$, $G \subseteq G^{\perp\perp}$.
(ii) For any $G, H \subseteq P$, $G \subseteq H \Rightarrow H^{\perp} \subseteq G^{\perp}$.
(iii) $G \subseteq P$ is a fact iff it is of the form $H^{\perp}$ for some $H \subseteq P$.
(iv) If G is any subset of P, then $G^{\perp\perp}$ is the smallest fact containing G.
(v) $\perp$ is a fact since $\perp = \{1\}^{\perp}$.
(vi) Facts are order ideals.
(vii) Facts are closed under arbitrary intersections.

Proof. (i) to (v) are immediate.

For (vi), let G be a fact, and take $x \in G$ and $y \leqslant x$. If $z \in G^{\perp}$, then $x \cdot z \in \perp$, so $y \cdot z \in \perp$ (monotonicity of $\cdot$). Therefore $y \in G^{\perp\perp} = G$.

For (vii) it suffices to verify that if $(G_i)_{i \in I}$ is a family of facts, then $\bigcap G_i = (\bigcup G_i^{\perp})^{\perp}$. If $x \in \bigcap G_i$ then for all $i \in I$, $x \in G_i$. Now if $y \in \bigcup G_i^{\perp}$, then $y \in G_{i_0}^{\perp}$ for some $i_0 \in I$, so $x \cdot y \in \perp$. Conversely, if $x \in (\bigcup G_i^{\perp})^{\perp}$, then for all $i \in I$ and all $y \in G_i^{\perp}$, $x \cdot y \in \perp$, so $x \in G_i^{\perp\perp} = G_i$. □

Definition 5.5. A few notable facts: the largest one $\top = \varnothing^{\perp} = P$ (with respect to inclusion); the smallest one $\mathbf{0} = \top^{\perp}$, and $\mathbf{1} = \perp^{\perp}$.

Definition 5.6. Define the following operations on facts A, B:

$$\begin{array}{rcl} A \odot B &=& (A \cdot B)^{\perp\perp} \\ A \triangledown B &=& (B^{\perp} \cdot A^{\perp})^{\perp} \\ A \otimes B &=& (A \star B)^{\perp\perp} \\ A \parr B &=& (B^{\perp} \star A^{\perp})^{\perp} \\ A \,\&\, B &=& A \cap B \\ A \oplus B &=& (A \cup B)^{\perp\perp}. \end{array}$$

Phase spaces enjoy the following fundamental property.

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Lemma 5.7. For any subsets F and G of P, $F^{\perp\perp} \cdot G^{\perp\perp} \subseteq (F \cdot G)^{\perp\perp}$ and $F^{\perp\perp} \star G^{\perp\perp} \subseteq (F \star G)^{\perp\perp}$.

Proof. Consider the case of $\cdot$. Let $p \in F^{\perp\perp}$ and $q \in G^{\perp\perp}$. If $v \in (F \cdot G)^{\perp}$, then for all $f \in F$ and $g \in G$, $v \star (f \cdot g) \in \perp$, so $v \cdot (f \cdot g) = (v \cdot f) \cdot g \in \perp$ (because $v \star v' \in \perp$ iff $v \cdot v' \in \perp$), so for all $f \in F$, $v \cdot f \in G^{\perp} = G^{\perp\perp\perp}$, and $q \cdot (v \cdot f) = (q \cdot v) \cdot f \in \perp$, whence $q \cdot v \in F^{\perp} = F^{\perp\perp\perp}$. Therefore $p \cdot q \cdot v \in \perp$.

For $\star$, apply a similar argument: if $v \in (F \star G)^{\perp}$, then for all $f \in F$ and $g \in G$, $v \cdot (f \star g) \in \perp$, so $v \star (f \star g) = (v \star f) \star g \in \perp$, and $(v \star f) \cdot g \in \perp$. Therefore for all $f \in F$, $v \star f \in G^{\perp} = G^{\perp\perp\perp}$, so $q \star (v \star f) = (q \star v) \star f \in \perp$, whence $q \star v \in F^{\perp} = F^{\perp\perp\perp}$. Therefore $p \star q \star v \in \perp$. □

Proposition 5.8.

(i) De Morgan laws hold for $\odot$ and $\triangledown$, $\otimes$ and $⅋$, & and $\oplus$. Moreover, these 6 operations are associative; $\otimes$, $⅋$, & and $\oplus$ are commutative; **1** is neutral for $\odot$ and $\otimes$; $\perp$ is neutral for $\triangledown$; and $⅋$, $\top$ and **0** are, respectively, neutral for & and $\oplus$. Distributivity properties hold for $\odot$ and $\oplus$, $\otimes$ and $\oplus$, $\triangledown$ and &, and $⅋$ and &.

(ii) With A and B any facts,

$$A \otimes B \subseteq A \odot B \text{ (dually } A \triangledown B \subseteq A ⅋ B).$$

Proof. Only the following deserve attention:

— Associativity of the multiplicatives (by duality, we just consider the conjunctions): Using Lemma 5.7, we have $(A \odot B) \odot C = ((A \cdot B)^{\perp\perp} \cdot C)^{\perp\perp} = ((A \cdot B)^{\perp\perp} \cdot C^{\perp\perp})^{\perp\perp} \subseteq (A \cdot B \cdot C)^{\perp\perp}$, and $(A \cdot B \cdot C)^{\perp\perp} \subseteq (A \odot B) \odot C$ is immediate. Hence $(A \odot B) \odot C = (A \cdot B \cdot C)^{\perp\perp}$, as required. The case of $\otimes$ is similar.

— Neutrality: These properties rely on the neutrality of $1 \in \mathbf{1}$ for both $\cdot$ and $\star$.

— Distributivities: The proofs are exactly the same as for commutative LL.

— $A \otimes B \subseteq A \odot B$: It is enough to show that $A \star B \subseteq A \odot B = (A \cdot B)^{\perp\perp}$. If $a \in A$ and $b \in B$, then $a \cdot b \in A \cdot B \subseteq (A \cdot B)^{\perp\perp}$. $(A \cdot B)^{\perp\perp}$ is a fact and $a \star b \leqslant a \cdot b$, so by Proposition 5.4 (vi), $a \star b \in (A \cdot B)^{\perp\perp}$. □

Definition 5.9.

$$\begin{aligned} A \multimap B &= \{x \in P \mid \forall a \in A, a \star x \in B\} \\ A \rightarrow B &= \{x \in P \mid \forall a \in A, a \cdot x \in B\} \\ B \leftarrow A &= \{x \in P \mid \forall a \in A, x \cdot a \in B\}. \end{aligned}$$

Proposition 5.10. Let A and B be any facts. Then

$$\begin{gathered} A \rightarrow B = A^{\perp} \triangledown B \\ B \leftarrow A = B \triangledown A^{\perp} \\ A \multimap B = A^{\perp} ⅋ B \\ A \multimap \perp = A \rightarrow \perp = \perp \leftarrow A = A^{\perp}. \end{gathered}$$

Hence, in particular, $A \rightarrow B$, $B \leftarrow A$ and $A \multimap B$ are facts.

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Proof. Here we use again the fact that for all $x, y \in P$, $x \cdot y \in \bot$ iff $x \star y \in \bot$. Let us just consider the case of $\multimap$ (the others are similar). Assume $x \in A \multimap B$. Let $a \in A$ and $y \in B^\bot$. Then $a \cdot x \in B$, so $y \cdot a \cdot x \in \bot$, and thus $x \in A^\bot \triangledown B = (B^\bot \cdot A)^\bot$. Conversely, assume $x \in A^\bot \triangledown B$, and take $a \in A$. For all $y \in B^\bot$, $y \cdot a \cdot x \in \bot$, and thus $a \cdot x \in B^{\bot\bot} = B$, whence $x \in A \multimap B$. □

As in Girard (1995) and Lafont (1997), we extend the semantics to exponential connectives. If P is a phase space, define $J(P) = \{x \in \mathbf{1} \mid x \in \{x \star x\}^{\bot\bot}\}$. Note that $x \in J(P) \Rightarrow x \in \{x \cdot x\}^{\bot\bot}$, because $\{x \star x\}^{\bot\bot} \subseteq \{x \cdot x\}^{\bot\bot}$.

Definition 5.11. An *enriched phase space* consists of a phase space P and a subset K of $J(P)$ such that:

— For any $x \in K$ and $y \in P$, we have $\{x \cdot y\}^{\bot\bot} = \{y \cdot x\}^{\bot\bot} = \{x \star y\}^{\bot\bot}$.
— Both $(K, \star, 1)$ and $(K, \cdot, 1)$ are monoids.

The enriched phase space will still be denoted by P.

Definition 5.12. With A a fact, we define

$$\begin{aligned} ?A &= (A^\bot \cap K)^\bot \\ !A &= (A \cap K)^{\bot\bot}. \end{aligned}$$

Proposition 5.13.

(i) For every fact A, $?A$ and $!A$ are facts.
(ii) ! and ? satisfy the De Morgan laws.
(iii) For every facts A and B, $A \subseteq B \Rightarrow\ !A \subseteq\ !B$,
(iv) $!!A =\ !A \subseteq A$,
(v) $!A \otimes !B \subseteq !(A \otimes B)$ and $!A \odot !B \subseteq !(A \odot B)$,
(vi) $!A \subseteq \mathbf{1}$, $!A = !A \otimes !A$ and $!A = !A \odot !A$,
(vii) $!A \otimes B =\ !A \odot B = B \odot !A$,
(viii) $!A \otimes !B =\ !(A \& B) =\ !A \odot !B =\ !B \odot !A$.

Proof. (i), (ii) and (iii) are immediate.

(iv) For a fact A, one clearly has $!A \subseteq A^{\bot\bot} = A$. In particular, $!!A \subseteq !A$, and, moreover, $A \cap K \subseteq (A \cap K)^{\bot\bot} = !A$ and $A \cap K \subseteq K$, thus $A \cap K \subseteq !A \cap K \subseteq (!A \cap K)^{\bot\bot}$, whence $!A = (A \cap K)^{\bot\bot} \subseteq (!A \cap K)^{\bot\bot} = !!A$.

(v) $!A \otimes !B = ((A \cap K)^{\bot\bot} \star (B \cap K)^{\bot\bot})^{\bot\bot} = ((A \cap K) \star (B \cap K))^{\bot\bot} \subseteq ((A \star B) \cap K)^{\bot\bot}$ since $(K, \star)$ is a monoid, therefore $!A \otimes !B \subseteq ((A \star B)^{\bot\bot} \cap K)^{\bot\bot} = !(A \otimes B)$. Using the fact that $(K, \cdot)$ is a monoid, one proves that $!A \odot !B \subseteq !(A \odot B)$.

(vi) The first inclusion is obvious. If $x \in A \cap K$, then $x \in K$ thus $x \in \{x \star x\}^{\bot\bot} \subseteq ((A \cap K) \star (A \cap K))^{\bot\bot}$, therefore $A \cap K \subseteq ((A \cap K) \star (A \cap K))^{\bot\bot} = (!A \star !A)^{\bot\bot}$ according to Lemma 5.7, whence $!A = (A \cap K)^{\bot\bot} \subseteq !A \otimes !A$. One proceeds similarly for $\odot$.

(vii) As $B = B^{\bot\bot}$, $!A \otimes B = ((A \cap K)^{\bot\bot} \star B)^{\bot\bot} = ((A \cap K) \star B)^{\bot\bot} = ((A \cap K) \cdot B)^{\bot\bot} = (B \cdot (A \cap K))^{\bot\bot}$ by definition of K.

(viii) In view of (vii) it is enough to prove that $!A \otimes !B = !(A \& B)$. According to (vi), $!(A \& B) \subseteq !(A \& B) \otimes !(A \& B)$, now $A \& B = A \cap B \subseteq A$, whence by (iii), $!(A \& B) \subseteq !A$ and

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similarly $!(A\&B) \subseteq !B$, thus $!(A\&B) \subseteq !A \otimes !B$. Conversely, $!A \otimes !B \subseteq !A \otimes \mathbf{1} = !A \subseteq A$, and similarly $!A \otimes !B \subseteq B$, thus $!A \otimes !B \subseteq A\&B$. Hence, according to (iv) and (v), $!A \otimes !B = !!A \otimes !!B \subseteq !(!A \otimes !B) \subseteq !(A\&B)$. □

The phase semantics we have defined, when restricted to the connectives ⅋ and $\otimes$ (respectively, $\triangledown$ and $\odot$), is the phase semantics of commutative (respectively, cyclic) linear logic.

5.2. *Soundness*

Definition 5.14 (Phase structure, Validity). A *phase structure* (P, S) is an enriched phase space P, together with a *valuation* that assigns a fact $S(p)$ to any positive propositional symbol p.

Given a phase structure, one defines inductively the *interpretation* $S(A)$ of a formula A in the obvious way. The interpretation of a context Γ is defined by induction: $S(()) = \bot$, $S(\Gamma;\Delta) = S(\Gamma) \triangledown S(\Delta)$ and $S(\Gamma,\Delta) = S(\Gamma) ⅋ S(\Delta)$.

With A a formula, A is said to be *valid in S* when $1 \in S(A)$. A is a *tautology* if it is valid in every phase structure. A sequent $\vdash \Gamma$ is valid if $1 \in S(\Gamma)$ for every phase structure S.

Theorem 5.15. If a sequent is provable in the sequent calculus, it is valid.

Proof. We use the sequent calculus on orders (Section 2). Let (P, S) be a phase structure, and $\vdash \Gamma$ be a sequent provable in the sequent calculus. First note that implicit associativity of $(-,-)$ and $(-;-)$, and commutativity of $(-,-)$ in the sequent calculus are sound, because of the associativity of ⅋ and $\triangledown$ and of the commutativity of ⅋ (Proposition 5.8 (i)). Now we proceed by induction on a proof of $\vdash \Gamma$:

— The proof is an axiom $\vdash A^\perp, A$: The interpretation is $S(A^\perp, A) = S(A^\perp ⅋ A) = S(A \multimap A)$ by Proposition 5.10, thus $S(A^\perp, A) = S(A) \multimap S(A)$ and $1 \in S(A^\perp, A)$.
— The proof is an axiom $\vdash \mathbf{1}$: Here $1 \in \mathbf{1} = \bot^\perp$.
— The proof is an axiom $\vdash \Gamma, \top$: The interpretation is $S(\Gamma, \top) = S(\Gamma) ⅋ S(\top) = S(\top) ⅋ S(\Gamma) = S(\mathbf{0}) \multimap S(\Gamma) = \mathbf{0} \multimap S(\Gamma)$, and $\mathbf{0}$ is the least fact, thus $\mathbf{0} \subseteq S(\Gamma)$ and $1 \in \mathbf{0} \multimap S(\Gamma)$.
— The proof ends with a cut rule: By the induction hypothesis, $1 \in S(\Gamma) ⅋ S(A)$ and $1 \in S(A^\perp) ⅋ S(\Delta)$, which means $S(\Gamma)^\perp \subseteq S(A)$ and $S(A^\perp)^\perp = S(A) \subseteq S(\Delta)$.
— The proof ends with an entropy: This follows from Proposition 5.8 (ii).
— The proof ends with the seesaw rule: By the induction hypothesis, $1 \in S(\Gamma) ⅋ S(\Delta)$, that is, $S(\Gamma)^\perp \subseteq S(\Delta)$. As 1 is neutral for $\cdot$, it is equivalent to $1 \in S(\Gamma)^\perp \multimapdotinv S(\Delta) = S(\Gamma) \triangledown S(\Delta)$.
— The proof ends with the centre rule: This follows from Proposition 5.13 (vii).
— The proof ends with the &-rule: This is an immediate consequence of $A\&B = A \cap B$.
— The proof ends with a $\oplus$-rule: This is an immediate consequence of $A \oplus B = (A \cup B)^{\perp\perp}$ and of Proposition 5.4 (v).
— The proof ends with a $\triangledown$ or ⅋-rule: This is an immediate.
— The proof ends with the $\odot$-rule: By induction, $S(\Gamma)^\perp \subseteq S(A)$ and $S(\Delta)^\perp \subseteq S(B)$, thus $(S(\Delta) \triangledown S(\Gamma))^\perp = S(\Gamma)^\perp \odot S(\Delta)^\perp \subseteq S(A) \odot S(B)$, that is, $1 \in (S(\Delta) \triangledown S(\Gamma)) \triangledown S(A \odot B) = S(\Delta; \Gamma; A \odot B)$. Therefore by seesaw, $1 \in S(\Gamma; A \odot B; \Delta)$.
— The proof ends with the $\otimes$-rule: This follows by a similar argument, using the fact that $G^\perp \subseteq H$ iff $1 \in G ⅋ H$.

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— The proof ends with the dereliction rule: By induction, $S(\Gamma)^\perp \subseteq S(A)$, and by Proposition 5.13 (iv) (translated in dual terms of '?'), $S(A) \subseteq S(?A)$.
— The proof ends with the promotion rule: This is an immediate consequence of the monotonicity of ! (Proposition 5.13 (iii)–(v)).
— The proof ends with the contraction rule: This is immediate because $S(?A) = S(?A \parr ?A)$ (Proposition 5.13 (vi)).
— The proof ends with the weakening rule: This is an immediate consequence of $\perp \subseteq S(?A)$ (Proposition 5.13 (vi)), and $\perp$ neutral for $\parr$ (Proposition 5.8 (i)).
— The proof ends with an introduction of $\perp$: This is as for weakening. □

5.3. *Completeness*

Theorem 5.16. If a sequent is valid, then it is provable in the sequent calculus.

Proof. We define the following enriched phase space:

— P is the set of contexts, $\Gamma \cdot \Delta$ is sequential composition $(\Gamma;\Delta)$, $\Gamma \star \Delta$ is parallel composition (Γ,Δ), and the neutral is $()$.
— The order $\leqslant$ is the least order such that $(\Gamma,\Delta) \leqslant (\Gamma;\Delta)$ for every $\Gamma,\Delta \in P$, and $\Gamma \leqslant \Gamma'$ and $\Delta \leqslant \Delta'$ imply $(\Gamma;\Delta) \leqslant (\Gamma';\Delta')$ and $(\Gamma,\Delta) \leqslant (\Gamma',\Delta')$.
— $\perp$ is the set of contexts Γ such that $\vdash \Gamma$ is provable in the sequent calculus.
— K is the set of contexts of the form $?\Gamma$ (where Γ is an arbitrary context).

By entropy and seesaw, $\perp$ satisfies the axioms of Definition 5.1. Moreover, K satisfies the axioms of Definition 5.11:

— K contains $1 = ()$ and is closed by $\cdot$ and $\star$.
— By the weakening rule, $?\Gamma \in \perp^\perp = \mathbf{1}$.
— By the contraction rule, $?\Gamma \in \{?\Gamma \star ?\Gamma\}^{\perp\perp}$.
— By the centre and co-centre rules, $\{?\Gamma \star \Delta\}^{\perp\perp} = \{?\Gamma \cdot \Delta\}^{\perp\perp} = \{\Delta \cdot ?\Gamma\}^{\perp\perp}$.

Thus what we have defined is an enriched phase space.

We have $\{A\}^\perp = \{\Gamma \in P \mid \vdash \Gamma, A \text{ is provable in the sequent calculus}\}$.

The $\{A\}^\perp$'s are facts, more precisely $\{A\}^\perp = \{A^\perp\}^{\perp\perp}$ (the proof is the same as in Girard (1987)). Define a phase structure S by letting $S(p) = \{p\}^\perp$ for every positive propositional symbol p. One then easily proves, as in the commutative case, by induction on A, that $S(A) = \{A\}^\perp$: it amounts to proving that commutations of the type $\{A \otimes B\}^\perp = \{A\}^\perp \otimes \{B\}^\perp$. Let us consider the case of the exponentials:

— $?\{A\}^\perp = (\{A\}^{\perp\perp} \cap K)^\perp = (\{A^\perp\}^\perp \cap K)^\perp$. Let $\Gamma \in \{?A\}^\perp$ and $?\Delta \in \{A^\perp\}^\perp$. One has $?\Delta \in \{!A^\perp\}^\perp$ by the promotion rule, whence by the cut rule $\Gamma, ?\Delta \in \perp$, which shows that $\{?A\}^\perp \subseteq ?\{A\}^\perp$. Conversely, by the dereliction rule, $?A \in \{A^\perp\}^\perp$, and, moreover, $?A \in K$, thus if $\Gamma \in ?\{A\}^\perp$, then $\Gamma, ?A \in \perp$, that is, $\Gamma \in \{?A\}^\perp$, as required.
— $S(!A)^\perp = ?S(A^\perp) = ?\{A^\perp\}^\perp = \{?A^\perp\}^\perp = \{(!A)^\perp\}^\perp = \{!A\}^{\perp\perp}$, thus $S(!A) = \{!A\}^\perp$.

Finally, if $\vdash \Gamma$ is a valid sequent, $1 = () \in S(\Gamma) = \{\Gamma\}^\perp$ and thus $\vdash \Gamma$ is provable. □

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6. Cut elimination

As in Okada (1994), one can use the phase semantics to prove cut elimination.

Theorem 6.1. If a sequent is valid, then it is provable in the sequent calculus without cuts.

Proof. Define the following enriched phase space P'. It is defined like the phase space P in the proof of Theorem 5.16 except that here $\perp$ is the set of contexts Γ such that $\vdash \Gamma$ is provable in the sequent calculus *without cuts*. $\perp$ again satisfies the axioms of Definition 5.1, thus the above is a well-defined enriched phase space.

$$\{A\}^\perp = \{\Gamma \in P' \mid \vdash \Gamma, A \text{ is provable in the sequent calculus } \textit{without cut}\}.$$

The $\{A\}^\perp$'s are facts, thus one can define a phase structure S' by letting $S'(p) = \{p\}^\perp$ for every positive propositional symbol p. One then proves, by induction on A, that $S'(A) \subseteq \{A\}^\perp$:

- For a positive propositional symbol p, it is clear.
- For the dual $p^\perp$ of a propositional symbol, one has $S'(p^\perp) = S'(p)^\perp = \{p\}^{\perp\perp} \subseteq \{p^\perp\}^\perp$ because $p \in \{p^\perp\}^\perp$ (axiom).
- $S'(A \parr B) = S'(A) \parr S'(B) = (S'(B)^\perp \star S'(A)^\perp)^\perp \subseteq (\{B\}^{\perp\perp} \star \{A\}^{\perp\perp})^\perp \subseteq (\{B\} \star \{A\})^\perp$ by Lemma 5.7, so $S'(A \parr B) \subseteq \{A \parr B\}^\perp$ by the $\parr$-rule.
- Similarly, $S'(A \triangledown B) \subseteq \{A \triangledown B\}^\perp$ by the $\triangledown$-rule.
- $S'(A \otimes B) = (S'(A) \star S'(B))^{\perp\perp} \subseteq (\{A\}^\perp \star \{B\}^\perp)^{\perp\perp} \subseteq \{A \otimes B\}^\perp$ because $\{A\}^\perp \star \{B\}^\perp \subseteq \{A \otimes B\}^\perp$ by the $\otimes$-rule.
- Similarly, $S'(A \odot B) \subseteq \{A \odot B\}^\perp$ by the $\odot$-rule.
- $S'(A \& B) = S'(A) \cap S'(B) \subseteq \{A\}^\perp \cap \{B\}^\perp \subseteq \{A \& B\}^\perp$ by the &-rule.
- $S'(A \oplus B) = (S'(A) \cup S'(B))^{\perp\perp} \subseteq (\{A\}^\perp \cup \{B\}^\perp)^{\perp\perp} \subseteq \{A \oplus B\}^\perp$ because $\{A\}^\perp \cup \{B\}^\perp \subseteq \{A \oplus B\}^\perp$ by the $\oplus$-rules.
- $S'(\perp) = \perp \subseteq \{\perp\}^\perp$ by the $\perp$-rule.
- $S'(\mathbf{1}) = \perp^\perp \subseteq \{\mathbf{1}\}^\perp$ because $\mathbf{1} \in \perp$ by the axiom for $\mathbf{1}$.
- $S'(\top) = P' = \{\top\}^\perp$ by the axiom for $\top$.
- $S'(\mathbf{0}) = P'^\perp \subseteq \{\mathbf{0}\}^\perp$.
- $S'(?A) = (S'(A)^\perp \cap K)^\perp \subseteq (\{A\}^{\perp\perp} \cap K)^\perp \subseteq \{?A\}^\perp$ because $?A \in K$ and $\{A\}^\perp \subseteq \{?A\}^\perp$ by the dereliction rule.
- $S'(!A) = (S'(A) \cap K)^{\perp\perp} \subseteq (\{A\}^\perp \cap K)^{\perp\perp} \subseteq \{!A\}^\perp$ because $\{A\}^\perp \cap K \subseteq \{!A\}^\perp$ by the promotion rule.

Finally, if $\vdash \Gamma$ is a valid sequent, $1 = () \in S'(\Gamma) \subseteq \{\Gamma\}^\perp$, and thus $\vdash \Gamma$ is provable in the sequent calculus without cuts. □

Corollary 6.2. If a sequent is provable in the sequent calculus, it is provable without the cut rule.

Proof. This is an immediate consequence of the soundness of the phase semantics (Theorem 5.15), and of its completeness with respect to the sequent calculus without cuts (Theorem 6.1). □

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References

Abrusci, V. M. and Ruet, P. (2000) Non-commutative logic I: the multiplicative fragment. *Annals Pure Appl. Logic* **101** (1) 29–64.

Andreoli, J.-M. (1992) Logic programming with focusing proofs in linear logic. *J. Logic and Comp.* **2** (3).

Bechet, D., de Groote, Ph. and Retoré, Ch. (1997) A complete axiomatisation for the inclusion of series–parallel orders. Proc. Int. Conf. on Rewriting Techniques and Applications. *Springer-Verlag Lecture Notes in Computer Science.*

de Groote, Ph. Partially commutative linear logic : sequent calculus and phase semantics. In: Abrusci, V. M. and Casadio, C. (eds.) *Proofs and Linguistic Categories, Proc. 1996 Roma Workshop*, Cooperativa Libraria Universitaria Editrice Bologna.

Demaille, A. (1999) *Logique linéaires hybrides et leurs modalités: théories et applications*, Ph.D. thesis, Ec. Nat. Sup. Télécommunications.

Fuchs, L. (1963) *Partially ordered algebraic systems*, Pergamon Press.

Girard, J.-Y. (1987) Linear logic. *Theoret. Comp. Sci.* **50** (1) 1–102.

Girard, J.-Y. (1989) Towards a geometry of interaction. *Contemp. Math.* **92** 69–108.

Girard, J.-Y. (1995) Linear logic: its syntax and semantics. In: *Advances in Linear Logic*, London Mathematical Society Lecture Note Series, **222**, Cambridge University Press, 1–42.

Girard, J.-Y. (1999) On the meaning of logical rules I: syntax vs. semantics. In: Berger, U. and Schwichtenberg, H. (eds.) *Computational Logic*, NATO Series F 165, Springer-Verlag.

Lafont, Y. (1997) The finite model property for various fragments of linear logic. *J. Symb. Logic* **62** (4) 1202–1208.

Lambek, J. (1958) The mathematics of sentence structure. *Amer. Math. Monthly* **65** (3) 154–170.

Mohring, R. (1989) *Computationally tractable classes of ordered sets*, ASI Series 222, NATO.

Novák, V. (1982) Cyclically ordered sets. *Czech. Math. J.* **32** (107) 460–473.

Okada, M. (1994) Girard's phase semantics and a higher-order cut-elimination proof. Preprint Institut de Mathématiques de Luminy.

P. Ruet. (1997) *Logique non-commutative et programmation concurrente par (P.) contraintes*, Ph.D. thesis, Université Denis Diderot, Paris 7.

Yetter, D. N. (1990) Quantales and (non-commutative) linear logic. *J. Symb. Logic* **55** (1).