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ON THE ALGEBRAIC PROPERTIES OF THE HUMAN ABO-BLOOD GROUP INHERITANCE PATTERN

J. M. CASAS[1], M. LADRA[✉2], B. A. OMIROV[3] and R. TURDIBAEV[2]

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Abstract

We generate an algebra on blood phenotypes with multiplication based on the human ABO-blood group inheritance pattern. We assume that gametes are not chosen randomly during meiosis. We investigate some of the properties of this algebra, namely, the set of idempotents, lattice of ideals and the associative enveloping algebra.

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1. Introduction

Before the discovery of blood groups more than a century ago by Landsteiner [9], all human blood was assumed to be the same. A blood group system is a classification of blood based on the presence or absence of antigenic substances on the surface of red blood cells. Although there are numerous blood group systems, the ABO-blood group system is one of the two most important systems in human blood transfusion. Presence and absence of two different types of agglutinogens, type "A" and type "B", determines four major ABO-blood groups. Group A (respectively, group B) admits only the A (respectively, B) antigen on red cells and group AB has both A and B antigens. The last group that lacks both A and B antigens is called group O after the German word "Ohne", which means "without".

Establishing the genetics of the ABO-blood group system was one of the first breakthroughs in Mendelian genetics. There are three alleles or versions of the ABO-blood group genes – A, B and O. The allele O is recessive to A and B, and alleles A and

[1]Dpto. Matemática Aplicada I, Universidad de Vigo, E. E. Forestal, Campus Universitario A Xunqueira, 36005 Pontevedra, Spain; e-mail: jmcasas@uvigo.es.
[2]Department of Algebra, University of Santiago de Compostela 15782, Spain;
e-mail: manuel.ladra@usc.es, rustamtm@yahoo.com.
[3]Institute of Mathematics, National University of Uzbekistan, Tashkent, 100125, Uzbekistan;
e-mail: omirovb@mail.ru.

B are co-dominant. It is known that humans are diploid organisms, which means that they carry a double set of chromosomes. Therefore, blood genotypes are determined by two alleles with six possible combinations: AA, BB, OO, AB, OA and OB. Since A and B dominate over O, the genotypes AO and AA express blood group A (phenotype A) and BO together with BB corresponding to group B (phenotype B).

A number of papers are devoted to the study of distribution of blood group frequencies in different countries and ethnicities [3, 4, 8]. Some methods for estimating phenotype probabilities for ABO groups are developed and compared by Greenwood and Seber [7]. Assuming the allele probabilities to be p, q and r for the genes A, B, and O, respectively, they obtain some estimates on the probabilities that a person has a corresponding phenotype.

The algebraic relation $1 + \sqrt{O} = \sqrt{A} + \sqrt{B}$ for the equilibrium blood group frequencies was established by Bernstein [1]. The question of how the frequencies of human blood genotypes evolve after several generations in a population, is considered by Sadykov [12]. He establishes a complete list of all polynomial relations between blood genotype frequencies depending on both ABO and RhD blood group systems.

The evolution (or dynamics) of a population comprises a determined change of state in the next generations as a result of reproduction and selection. This evolution of a population can be studied by a dynamical system (iterations) of a quadratic stochastic (a so-called evolutionary) operator [2].

Yamaguchi et al. [13, 14] describe some instances when two alleles A and B are inherited from one parent. Therefore, generally speaking, the pattern of heredity of blood groups are unpredictable. Using specific models of heredity and specific collected data which contains these mutations, a limiting distribution of blood groups is studied by Ganikhodjaev et al. [5]. Throughout this article, we assume that a child obtains exactly one allele from each parent.

Most of the papers [1–3, 7, 9, 12] on this topic are dedicated to cases in which, during the fertilization, parents' gametes are chosen randomly and in an independent way. Mendel's first law allows us to quantify the types of gametes an individual can produce. For example, a person with genotype OA during meiosis produces gametes O and A with equal probability $1/2$, while an individual with blood group A during meiosis produces gamete O with probability $1/4$.

In this paper, we consider the case which deviates from Mendelian rules and allows some competition for gametes during meioses. We assume that all parents of blood group A and B contribute gamete O to the child with a constant probability α and that the allele A is selected with probability β from parents of group AB during meiosis. In the case of Mendelian genetics, $\beta = 1/2$ and $\alpha = 1/4$. Further, considering the blood groups as independent basis vectors, we generate a four-dimensional vector space over $\mathbb{R}$ and introduce a commutative and nonassociative multiplication, assigning to basis vectors a linear combination of the possible phenotypes of progeny with corresponding probabilities.

After applying some linear basis-transformations, we obtain an algebra that admits a simpler table of multiplication. We describe the lattice of ideals of this algebra in Theorem 5.4 and observe that it changes depending on the values of the initial parameters. Finally, in Section 7, we establish that two such distinct algebras are not isomorphic, unless the second parameters of these algebras add up to one and the first parameters are equal.

2. Algebras of ABO-blood group

Consider the blood groups O, A, B and AB as basis elements of a four-dimensional vector space and a bilinear operation $\circ$ as the result of meiosis.

In this paper, we assume that all parents of blood groups A and B have equal probabilities to contribute the allele O to a child's genotype and we denote this probability by $p_{O|A} = p_{O|B} = \alpha$. Furthermore, we assume that all parents with group AB contribute the allele A during meiosis with equal probability, and we denote this probability by $p_{A|AB} = \beta$. Under these assumptions, we have the following:

(i) $O \circ O = O$;

(ii) $O \circ A = p_{O|A}O + (1 - p_{O|A})A = \alpha O + (1 - \alpha)A$;

(iii) $O \circ B = p_{O|B}O + (1 - p_{O|B})B = \alpha O + (1 - \alpha)B$;

(iv) $O \circ AB = p_{A|AB}A + p_{B|AB}B = \beta A + (1 - \beta)B$;

(v) $A \circ A = p^2_{O|A}O + (1 - p^2_{O|A})A = \alpha^2 O + (1 - \alpha^2)A$;

(vi) $$A \circ B = p_{O|A}p_{O|B}O + p_{A|A}p_{O|B}A + p_{O|A}p_{B|B}B + p_{A|A}p_{B|B}AB = \alpha^2 O + \alpha(1 - \alpha)A + \alpha(1 - \alpha)B + (1 - \alpha)^2 AB;$$

(vii) $$A \circ AB = p_{A|AB}A + p_{O|A}p_{B|AB}B + p_{A|A}p_{B|AB}AB = \beta A + \alpha(1 - \beta)B + (1 - \alpha)(1 - \beta)AB;$$

(viii) $B \circ B = p^2_{O|B}O + (1 - p^2_{O|B})B = \alpha^2 O + (1 - \alpha^2)B$;

(ix) $$B \circ AB = p_{O|B}p_{A|AB}A + p_{B|AB}B + p_{B|B}p_{A|AB}AB = \alpha\beta A + (1 - \beta)B + (1 - \alpha)\beta AB;\ \textit{and}$$

(x) $$AB \circ AB = p^2_{A|AB}A + p^2_{B|AB}B + 2p_{A|AB}p_{B|AB}AB = \beta^2 A + (1 - \beta)^2 B + 2\beta(1 - \beta)AB.$$

Definition 2.1. A commutative four-dimensional $\mathbb{R}$-algebra with basis $\{O, A, B, AB\}$ and with multiplication $\circ$ satisfying equalities (i)–(x), is called a generalized ABO-blood group algebra (GBGA) and is denoted by $\mathcal{B}(\alpha, \beta)$.

Remark 2.2. The algebra $\mathcal{B}(\alpha, \beta)$ is not associative for any parameters $0 \le \alpha, \beta \le 1$. Indeed, assuming associativity, one obtains $\alpha = 0$ from the equation $O \circ AB = O \circ (O \circ AB)$. Moreover, $(A \circ B) \circ B = A \circ (B \circ B) = AB$ and $(B \circ A) \circ A = B \circ (A \circ A) = AB$ imply that $\beta A + (1 - \beta)AB = AB = (1 - \beta)B + \beta AB$, which does not hold for any value of β.

If, during meiosis, we assume that parents' gametes are chosen randomly and independently, then $\alpha = 1/4$ and $\beta = 1/2$.

Definition 2.3. A GBGA $\mathcal{B}(1/4, 1/2)$ is called an ABO-blood group algebra (BGA).

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From now on, we assume that $0 < \alpha, \beta < 1$. Note that if we interchange A and B and β to $1 - \beta = 1 - p_{A|AB} = p_{B|AB}$, we obtain the same products as above, that is, we have $\mathcal{B}(\alpha, \beta) \cong \mathcal{B}(\alpha, 1 - \beta)$. Later, in Section 7, we establish that no other isomorphisms between two GBGAs exist for different values of the parameters α and β.

We consider the algebraic relations defining a GBGA from a different perspective. Let x_1, x_2, x_3, x_4 be the corresponding proportions of O, A, B, AB phenotypes, respectively, in one population. Then, for the underlying allele frequencies,

$$\begin{aligned} p_O &= x_1 + \alpha x_2 + \alpha x_3, \\ p_A &= (1 - \alpha)x_2 + \beta x_4, \\ p_B &= (1 - \alpha)x_3 + (1 - \beta)x_4. \end{aligned}$$

Straightforward computation of the frequencies of O, A, B and AB phenotypes in zygotes of the next generation (state) yields an extension of the Hardy–Weinberg Law: that is,

$$\begin{cases} x_1' = p_O^2, \\ x_2' = p_A^2 + 2p_A p_O, \\ x_3' = p_B^2 + 2p_B p_O, \\ x_4' = 2p_A p_B. \end{cases}$$

Consider $S^3 = \{\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 + x_2 + x_3 + x_4 = 1, x_i \geq 0, 1 \leq i \leq 4\}$, a three-dimensional canonical simplex. Following Lyubich [11], we have a so-called evolutionary (quadratic stochastic) operator, $V\colon S^3 \to S^3$, describing an evolution of the population mapping a state $\mathbf{x} = (x_1, x_2, x_3, x_4)$ to the next state $V(\mathbf{x}) = (x_1', x_2', x_3', x_4')$. By linearity, V can be extended to $\mathbb{R}^4$, if necessary.

The relation that establishes a connection between the evolutionary operator V and the multiplication operator $\circ$ of a GBGA is $\mathbf{x} \circ \mathbf{x} = V(\mathbf{x})$ and, consequently,

$$\mathbf{x} \circ \mathbf{y} = \tfrac{1}{4}\{V(\mathbf{x} + \mathbf{y}) - V(\mathbf{x} - \mathbf{y})\}.$$

In order to simplify our investigation of the structure of a GBGA, we make the linear basis-transformation

$$\begin{cases} o = O, \\ a = \dfrac{1}{(1 - \alpha)^2}(O - A), \\ b = \dfrac{1}{(1 - \alpha)^2}(O - B), \\ ab = \dfrac{1}{(1 - \alpha)^3}(\alpha O - \beta A - (1 - \beta)B + (1 - \alpha)AB), \end{cases}$$

and obtain a simpler table of multiplication of a GBGA given by

$$\mathcal{B}'(\lambda,\beta):\begin{cases} o\circ o = o,\\ o\circ a = a\circ o = \lambda a,\\ o\circ b = b\circ o = \lambda b,\\ a\circ a = a,\\ b\circ b = b,\\ a\circ b = b\circ a = \dfrac{\lambda-\beta}{\lambda}a + \dfrac{\lambda-(1-\beta)}{\lambda}b + ab,\end{cases}$$

where $\lambda = 1-\alpha$, and the omitted products are assumed to be zero.

Due to the convenience of the above products, we now investigate the algebraic properties of the algebra $\mathcal{B}'(\lambda,\beta)$. Note that $O\in\mathcal{B}(\alpha,\beta)$ and $o\in\mathcal{B}'(\lambda,\beta)$ are nonzero elements of the algebras.

3. Absolute nilpotent and idempotent elements

Taking an initial point $\mathbf{x}\in S^3$, one can consider its trajectory $\{V^k(\mathbf{x}) \mid k\ge 1\}$. The study of limit behaviour of trajectories of quadratic stochastic operators play an important role in several questions of population genetics. Trajectories of genotype frequencies are studied by Ganikhodjaev et al. [6] and Lyubich [10]. Note that if the limit point of a trajectory exists, then it is a fixed point: that is, $\mathbf{x} = V(\mathbf{x}) = \mathbf{x}\circ\mathbf{x}$.

Definition 3.1. An element x of an algebra $(A,\circ)$, with $x\circ x = \mu x$, is called an absolute nilpotent if $\mu = 0$ and idempotent if $\mu = 1$.

We see that the set of absolute nilpotent elements of a GBGA constitutes the kernel of V, and idempotent elements are fixed points of V. From the table of multiplication of GBGA, it follows that ab is annihilated in the algebra $\mathcal{B}'(\lambda,\beta)$ and is an absolute nilpotent element, while o, a and b are idempotent elements.

Theorem 3.2. *The set of absolute nilpotent elements of $\mathcal{B}'(\lambda,\beta)$ is $\langle ab\rangle$.*

Proof. If $\xi o + n$, where n belongs to the ideal $\langle a,b,ab\rangle$, is an absolute nilpotent element, then $(\xi o+n)^2 = \xi^2 o + 2\xi o\circ n + q\circ n \equiv \xi^2 o \mod \langle a,b,ab\rangle$, which implies that $\xi = 0$.

Let $n = xa + yb + zab$ be an absolute nilpotent, where $x,y,z\in\mathbb{R}$. Then

$$\begin{aligned} n\circ n &= x^2a + y^2b + 2xya\circ b\\ &= \Big\{x^2 + \frac{2xy}{\lambda}(\lambda-\beta)\Big\}a + \Big\{y^2 + \frac{2xy}{\lambda}(\lambda+\beta-1)\Big\}b + 2xyab = 0.\end{aligned}$$

Hence, we need to solve the system of equations

$$\begin{cases} 0 = x^2 + \dfrac{2xy}{\lambda}(\lambda-\beta),\\ 0 = y^2 + \dfrac{2xy}{\lambda}(\lambda+\beta-1),\\ 0 = 2xy.\end{cases}$$

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Clearly, the only solutions are the triples $(0, 0, z)$. This completes the proof of the theorem. □

DEFINITION 3.3. Denote by $P = \{(\lambda, \beta) \mid 0 < \lambda \leq 1/3, \beta = (1 \pm \sqrt{(1-\lambda)(1-3\lambda)})/2\}$.

Next, we describe the idempotents of $\mathcal{B}'(\lambda, \beta)$.

THEOREM 3.4. *For the algebra $\mathcal{B}'(\lambda, \beta)$, the set $\mathcal{I}$ of idempotents depending on the parameters λ, β is as follows:*

- $\mathcal{I} = \{o, a, b\}$ *if* $(\lambda, \beta) \in \{(1/2, 1/4), (1/2, 3/4)\}$;
- $\mathcal{I} = \{o, a, b, j_0\}$ *if* $\lambda = 1/2, \beta \neq 1/4, 3/4$;
- $\mathcal{I} = \{o, a, b, o + (1-2\lambda)a, o + (1-2\lambda)b\}$ *if* $(\lambda, \beta) \in P \cup \{(2\beta, \beta) \mid \beta \neq 1/4\} \cup \{(2-2\beta, \beta) \mid \beta \neq 3/4\}$; *and*
- $\mathcal{I} = \{o, a, b, o + (1-2\lambda)a, o + (1-2\lambda)b, j_0, j_1\}$ *otherwise;*

where, for $\xi = 0, 1$,

$$j_\xi = \xi o + \rho_\xi(2\beta - \lambda)a + \rho_\xi(2 - 2\beta - \lambda)b + 2\rho_\xi^2(2\beta - \lambda)(2 - 2\beta - \lambda)ab$$

and

$$\rho_\xi = \frac{\lambda(1 - 2\xi\lambda)}{-3\lambda^2 + 4\beta^2 + 4\lambda - 4\beta}.$$

PROOF. As in the proof of Theorem 3.2, we deduce that an idempotent admits the form $i = \xi o + xa + yb + zab$ with $\xi = \xi^2$. In particular, $\xi = 0$ or $\xi = 1$.

The equation $i \circ i = i$ yields the system of equations

$$\begin{cases} x = x^2 + \dfrac{2xy}{\lambda}(\lambda - \beta) + 2\xi\lambda x, \\ y = y^2 + \dfrac{2xy}{\lambda}(\lambda + \beta - 1) + 2\xi\lambda y, \\ z = 2xy. \end{cases} \tag{3.1}$$

The first two equations of (3.1) transform into

$$0 = x\Big\{x + 2y\frac{\lambda - \beta}{\lambda} + (2\xi\lambda - 1)\Big\},$$
$$0 = y\Big\{y + 2x\frac{\lambda + \beta - 1}{\lambda} + (2\xi\lambda - 1)\Big\},$$

respectively. If $x = 0$, then $z = 0$, and we obtain $y\{y + (2\xi\lambda - 1)\} = 0$. Further, either $y = 0$ or $y = 1 - 2\xi\lambda$. This yields two idempotents ξo and $\xi o + (1 - 2\xi\lambda)b$. Taking into account the possible values for ξ, we conclude that $o, b, o + (1 - 2\lambda)b$ are idempotents.

If $y = 0$, then $z = 0$, and we obtain $x\{x + 2y(\lambda - \beta)/\lambda + (2\xi\lambda - 1)\} = 0$. Further, either $x = 0$ or $x = 1 - 2\xi\lambda$. This yields the idempotents ξo and $\xi o + (1 - 2\xi\lambda)a$. Since $\xi = 0, 1$, we get that a and $o + (1 - 2\lambda)a$ are idempotents.

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Now we consider the case when $xy \neq 0$. We obtain

$$\begin{cases} x + 2\dfrac{\lambda - \beta}{\lambda} y = 1 - 2\xi\lambda, \\ 2\dfrac{\lambda + \beta - 1}{\lambda} x + y = 1 - 2\xi\lambda. \end{cases} \tag{3.2}$$

In order to solve this system, we consider the following two cases.

Case 1. Let $\det\begin{pmatrix} 1 & 2(\lambda-\beta)/\lambda \\ 2(\lambda+\beta-1)/\lambda & 1 \end{pmatrix} \neq 0$.

That is, $1 - (4/\lambda^2)(\lambda - \beta)(\lambda + \beta - 1) \neq 0$. Then we have a unique solution of (3.2), namely,

$$x = \rho_\xi(2\beta - \lambda) \quad \text{and} \quad y = \rho_\xi(2 - 2\beta - \lambda),$$

where

$$\rho_\xi = \frac{\lambda(1 - 2\xi\lambda)}{-3\lambda^2 + 4\beta^2 + 4\lambda - 4\beta}.$$

Hence, $z = 2xy = 2\rho_\xi^2(2\beta - \lambda)(2 - 2\beta - \lambda)$. Thus the desired idempotents are $j_\xi = \xi o + xa + yb + zab$, where $\xi = 0, 1$.

If $\lambda = 2\beta$, then $j_\xi = \xi o + (1 - 2\xi\lambda)b$, which, for possible values of ξ, yields the already listed idempotents b and $o + (1 - 2\lambda)b$. Similarly, $\lambda = 2 - 2\beta$ does not permit $j_\xi = \xi o + (1 - 2\xi\lambda)a$ to be distinct idempotents from the already listed a and $o + (1 - 2\lambda)a$. If $\lambda = 1/2$, then $j_1 = o$, while

$$j_0 = \frac{4\beta - 1}{1 + 4(1 - 2\beta)^2} a + \frac{3 - 4\beta}{1 + 4(1 - 2\beta)^2} b + 2\frac{4\beta - 1}{1 + 4(1 - 2\beta)^2} \cdot \frac{3 - 4\beta}{1 + 4(1 - 2\beta)^2} ab.$$

Moreover, $j_0 = b$ if $\beta = 1/4$, and $j_0 = a$ if $\beta = 3/4$.

Case 2. Let $\det\begin{pmatrix} 1 & 2(\lambda-\beta)/\lambda \\ 2(\lambda+\beta-1)/\lambda & 1 \end{pmatrix} = 0$.

This condition is equivalent to $(1 - 3\lambda)(1 - \lambda) = (2\beta - 1)^2$, for which it is necessary and sufficient that $0 < \lambda \leq 1/3$ and $\beta = \{1 \pm \sqrt{(3\lambda - 1)(\lambda - 1)}\}/2$. So the determinant is zero if and only if $(\lambda, \beta) \in P$.

Note that one obtains the first equality by multiplying the second one with $2(\lambda - \beta)/\lambda$. Thus we have $2(\lambda - \beta)(1 - 2\xi\lambda)/\lambda = 1 - 2\xi\lambda$; consequently, either $2(\lambda - \beta)/\lambda = 1$ or $1 - 2\xi\lambda = 0$. Simple observations lead to a contradiction with $(\lambda, \beta) \in P$. This completes the proof of the theorem. □

4. Plenary powers

In this section, we investigate which states, $\mathbf{x} \in \mathbb{R}^4$, admit zero as a limit point after a finite number of iterations, that is, $V^k(\mathbf{x}) = 0$ for some $k \geq 1$. Recall from Section 3 that the kernel of V consists of the absolute nilpotent elements.

Definition 4.1. The plenary powers of an arbitrary element m in $\mathcal{B}'(\lambda, \beta)$ are defined, recursively, as

$$m^{[1]} = m, \quad m^{[n+1]} = m^{[n]} \circ m^{[n]}, \quad n \geq 1.$$

An element m in $\mathcal{B}'(\lambda, \beta)$ is called solvable if there exists $n \in \mathbb{N}$ such that $m^{[n]} = 0$, and the least such number n is called its solvability index.

Note that absolute nilpotent elements are solvable with solvability index two.

THEOREM 4.2. *For an algebra of ABO-blood group $\mathcal{B}'(\lambda,\beta)$ to admit a solvable element of index $n \geq 3$, it is necessary and sufficient that $(\lambda,\beta) \in P$, where P is the set introduced in Definition 3.3. Moreover, solvable elements of degree n are*

$$-2^{n-4}\left(\frac{\lambda+\beta-1}{\lambda}\right)^{n-4} ta + tb + sab, \quad \text{where } t, s \in \mathbb{R},\ t \neq 0.$$

PROOF. Let m be a solvable element in $\mathcal{B}'(\lambda,\beta)$ with solvability index n. Similarly, as in the proof of Theorem 3.2, we assume that a solvable element does not contain the component o. We set $m^{[k]} = X_k a + Y_k b + Z_k ab$ for all $1 \leq k \leq n$.

In order to obtain recursive relations between the pairs (X_{k+1}, Y_{k+1}) and (X_k, Y_k) for $1 \leq k \leq n-1$, we consider the following equivalences modulo the ideal $\langle ab\rangle$, given by

$$\begin{aligned} X_{k+1}a + Y_{k+1}b &\equiv m^{[k+1]} = m^{[k]} \circ m^{[k]} \\ &\equiv (X_k a + Y_k b) \circ (X_k a + Y_k b) \\ &\equiv X_k\left(X_k + 2\frac{\lambda-\beta}{\lambda}Y_k\right)a + Y_k\left(2\frac{\lambda+\beta-1}{\lambda}X_k + Y_k\right)b \mod \langle ab\rangle. \end{aligned}$$

Therefore we obtain a system of equations, for $1 \leq k \leq n-1$,

$$S_k : \begin{cases} X_{k+1} = X_k\left(X_k + 2\dfrac{\lambda-\beta}{\lambda}Y_k\right), \\ Y_{k+1} = Y_k\left(2\dfrac{\lambda+\beta-1}{\lambda}X_k + Y_k\right). \end{cases}$$

Since we have described all absolute nilpotent elements, we assume that $n \geq 3$. Due to $m^{[n-1]}$ being an absolute nilpotent element, we know that $m^{[n-1]} \equiv 0 \mod \langle ab\rangle$, so $X_{n-1} = Y_{n-1} = 0$. Therefore, the system S_{n-2} has the form

$$\begin{cases} 0 = X_{n-2}\left(X_{n-2} + 2\dfrac{\lambda-\beta}{\lambda}Y_{n-2}\right), \\ 0 = Y_{n-2}\left(2\dfrac{\lambda+\beta-1}{\lambda}X_{n-2} + Y_{n-2}\right). \end{cases}$$

Obviously, either $X_{n-2}Y_{n-2} \neq 0$ or $X_{n-2} = Y_{n-2} = 0$. But, in the latter case, $m^{[n-2]} = Z_{n-2}ab$ and $m^{[n-1]} = 0$, which is a contradiction. Therefore, $X_{n-2}Y_{n-2} \neq 0$ and we have the linear system of equations

$$\begin{cases} 0 = X_{n-2} + 2\dfrac{\lambda-\beta}{\lambda}Y_{n-2}, \\ 0 = 2\dfrac{\lambda+\beta-1}{\lambda}X_{n-2} + Y_{n-2}. \end{cases}$$

If the determinant of this system is not zero, then we obtain a trivial solution which contradicts our assumption. Therefore, the determinant is zero, that is, $(\lambda,\beta) \in P$, and we get a solution $(X_{n-2}, Y_{n-2}) = (-2(\lambda-\beta)t/\lambda, t)$ for some $t \in \mathbb{R}^*$.

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Let $n = 3$. Then an element $m = X_1a + Y_1b + Z_1ab$ is solvable if and only if the following hold:

(1) $(\lambda,\beta) \in P$; and
(2) $(X_1, Y_1, Z_1) = (-2(\lambda-\beta)t/\lambda, t, s)$ for free parameters t and s, where $t \neq 0$.

Now let us assume that $n \geq 4$. Using the singularity of the determinant, we transform the last equation in the system S_k to the form

$$\begin{cases} X_{k+1} = X_k\Big(X_k + 2\dfrac{\lambda-\beta}{\lambda}Y_k\Big), \\ Y_{k+1} = 2\dfrac{\lambda+\beta-1}{\lambda}Y_k\Big(X_k + 2\dfrac{\lambda-\beta}{\lambda}Y_k\Big), \end{cases}$$

for any $1 \leq k \leq n-1$. Recall that here we consider the case $X_kY_k \neq 0$ for $1 \leq k \leq n-2$ (otherwise, the solvability index is less than n). Since both sides of each of the equation in the above system are assumed to be nonzero for $1 \leq k \leq n-3$, we obtain

$$\frac{X_k}{Y_k} = 2 \cdot \frac{\lambda+\beta-1}{\lambda} \cdot \frac{X_{k+1}}{Y_{k+1}} \quad \text{for any } 1 \leq k \leq n-3.$$

Therefore,

$$\frac{X_1}{Y_1} = \Big(2\frac{\lambda+\beta-1}{\lambda}\Big)^{n-3}\frac{X_{n-2}}{Y_{n-2}} = \Big(2\frac{\lambda+\beta-1}{\lambda}\Big)^{n-3}\Big(-2\frac{\lambda-\beta}{\lambda}\Big) = -\Big(2\frac{\lambda+\beta-1}{\lambda}\Big)^{n-4}.$$

Hence, for an element $m = Xa + Yb + Zab$ to be solvable with solvability index $n \geq 4$, it is necessary and sufficient for the following to hold:

(1) $(\lambda,\beta) \in P$; and
(2) $(X, Y, Z) = \Big(-2^{n-4}\Big(\dfrac{\lambda+\beta-1}{\lambda}\Big)^{n-4}t, t, s\Big)$, where $t, s \in \mathbb{R}$, $t \neq 0$.

In fact, if $n = 3$, then

$$-2^{n-4}\Big(\frac{\lambda+\beta-1}{\lambda}\Big)^{n-4} = \Big(-2 \cdot \frac{\lambda+\beta-1}{\lambda}\Big)^{-1} = -2 \cdot \frac{\lambda-\beta}{\lambda}.$$

This completes the proof of the theorem. □

5. Ideals of $\mathcal{B}'(\lambda,\beta)$

In this section, we determine all ideals of $\mathcal{B}'(\lambda,\beta)$. The lattice of ideals depends on values that the parameters λ and β take.

Proposition 5.1. *The ideal $\langle a, b, ab\rangle$ is the only maximal ideal of $\mathcal{B}'(\lambda,\beta)$.*

Proof. Let $X = x_1o + x_2a + x_3b + x_4ab$ be an element of an ideal I of the algebra $\mathcal{B}'(\lambda,\beta)$. Then $X \circ o \in I$ implies that $x_1o + \lambda x_2a + \lambda x_3b \in I$.

Considering $\lambda X - X \circ o \in I$ yields $(\lambda-1)x_1o + x_4ab \in I$. Multiplying the last element by o, we obtain $(\lambda-1)x_1o \in I$. Since $\lambda \neq 1$, we get $x_1o \in I$ and, therefore,

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$x_1a = x_1o \circ (a/\lambda) \in I$ and $x_1b = x_1o \circ (b/\lambda) \in I$. If $x_1 \neq 0$, then $o, a, b \in I$ and $ab \in I$, which yields $I = \mathcal{B}'(\lambda,\beta)$. Therefore $X = x_2a + x_3b + x_4ab$ and $I \subseteq \langle a, b, ab\rangle$, which is a maximal ideal. □

Next, let us focus our attention on the two-dimensional ideals.

Proposition 5.2. *The algebra $\mathcal{B}'(\lambda,\beta)$, where $\beta \neq \lambda, \beta \neq 1-\lambda$, does not admit two-dimensional ideals. For the remaining possible values of the parameters, two-dimensional ideals exist, all of which are presented as follows:*

- $\langle a, ab\rangle \trianglelefteq \mathcal{B}'(\lambda, 1-\lambda)$, *where* $\lambda \neq 1/2$;
- $\langle b, ab\rangle \trianglelefteq \mathcal{B}'(\lambda, \lambda)$, *where* $\lambda \neq 1/2$; *and*
- $\langle a, ab\rangle, \langle b, ab\rangle \trianglelefteq \mathcal{B}'(1/2, 1/2)$.

Proof. Let I be a two-dimensional ideal of the algebra. Proposition 5.1 shows that any element $X \in I$ is of the form $X = x_2a + x_3b + x_4ab$. Since $X - (X/\lambda) \circ o \in I$, $x_2a + x_3b, x_4ab \in I$.

The following equalities and memberships hold

$$x_2^2a - x_3^2b = (x_2a + x_3b) \circ (x_2a - x_3b) \in I,$$
$$(x_2 + x_3)a \circ b = (x_2a + x_3b) \circ (a + b) - (x_2a + x_3b) \in I,$$
$$(x_2 + x_3)\frac{\lambda-\beta}{\lambda}a = (x_2 + x_3)(a \circ b) \circ a - (x_2 + x_3)\frac{\lambda-\beta+1}{\lambda}a \circ b \in I,$$
$$(x_2 + x_3)\frac{\lambda-\beta+1}{\lambda}b = (x_2 + x_3)(a \circ b) \circ b - (x_2 + x_3)\frac{\lambda-\beta}{\lambda}a \circ b \in I,$$
$$(x_2 + x_3)ab = (x_2 + x_3)a \circ b - (x_2 + x_3)\frac{\lambda-\beta}{\lambda}a - (x_2 + x_3)\frac{\lambda-\beta+1}{\lambda}b \in I.$$

In order to complete the description of two-dimensional ideals we consider distinctive cases.

Case 1. Let $\lambda \neq \beta$ and $\lambda \neq 1-\beta$.

The above inclusions imply that $(x_2 + x_3)a$, $(x_2 + x_3)b, (x_2 + x_3)ab \in I$. Consequently, $x_2 + x_3 = 0$ and $X = x_2(a - b) + x_4ab$. Consider $(\lambda - 1)b = \lambda(a - b) \circ b - \lambda ab - (\lambda - \beta)(a - b) \in I$. Thus, $b \in I$ which gives $a \in I$ and we derive a contradiction. Therefore, in this case, there are no two-dimensional ideals.

Case 2. Let $\lambda = \beta$.

Then $a \circ b = (2\lambda - 1)b/\lambda + ab$. Hence, $(2\lambda - 1)(x_2 + x_3)b = (x_2 + x_3)(a \circ b) \circ o \in I$ and we continue by considering the following subcases.

Case 2.1. Let $\lambda \neq 1/2$. Then $(x_2 + x_3)b \in I$ and $x_2(a - b) = x_2a + x_3b - (x_2 + x_3)b \in I$. Together with $x_4ab \in I$, we need to have $(x_2 + x_3)x_2x_4 = 0$. Assuming that $x_4 = 0$, any element in the ideal is in the form $X = x_2a + x_3b$. But $ab = a \circ b - (2\lambda - 1)b/\lambda \in I$, which is a contradiction.

Next, let us suppose that $x_2 + x_3 = 0$. Then $a - b, ab \in I$. However, $(\lambda - 1)b/\lambda = (a - b) \circ b - ab \in I$ and we obtain $a, b \in I$, which is a contradiction.

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Thus, finally suppose that $x_2 = 0$. In this case, every element in the ideal is in the form $X = x_3 b + x_4 ab$ and $I = \langle b, ab\rangle$ is an ideal.

Case 2.2. Let $\lambda = 1/2$. Then $\beta = 1 - \beta = 1/2$ and $a \circ b = ab$. Suppose that $x_2 + x_3 = 0$. Then $a - b, ab \in I$, while $a = (a - b) \circ a + ab \in I$ and, therefore, $b \in I$, which is a contradiction. Hence, we assume that $x_2 + x_3 \neq 0$. Then $ab \in I$. Note that $x_2(x_2 + x_3)a = x_2^2 a - x_3^2 b + x_3(x_2 a + x_3 b) \in I$ and, similarly, that $x_3(x_2 + x_3)b \in I$. Therefore, $x_2 a, x_3 b \in I$. Since I is two-dimensional, then $x_2 x_3 = 0$. If $x_3 = 0$, then $I = \langle a, ab\rangle$ is an ideal. Otherwise, $\langle b, ab\rangle$ forms an ideal.

Case 3. Let $\lambda = 1 - \beta$.

The study of this case is carried out analogously to the second case and gives the same results up to substitution of a for b and vice versa. This completes the proof of the proposition. □

Next, we analyse the one-dimensional ideals.

PROPOSITION 5.3. *The ideal $\langle ab\rangle$ is the only one-dimensional ideal of $\mathcal{B}'(\lambda, \beta)$.*

PROOF. Let $I = \langle x\rangle$ be an ideal of $\mathcal{B}'(\lambda, \beta)$. If $x \circ x = 0$, then x is an absolute nilpotent element and we know that, in this case, $I = \langle ab\rangle$ (see Theorem 3.2).

If $x \circ x = \delta x$ for a nonzero δ, then denoting $y = x/\delta$ yields $y \circ y = y$, that is, I is generated by an idempotent. The proof of the proposition is completed by checking which idempotents from Theorem 3.4 generate a one-dimensional ideal. □

Summarizing the above results, we state the following theorem.

THEOREM 5.4. *The lattices of ideals of corresponding algebras are*

$$\mathcal{B}'(\lambda, \lambda), \lambda \neq \tfrac{1}{2}: \quad \begin{tikzcd} \langle a, b, ab\rangle \arrow[d, dash] \\ \langle b, ab\rangle \arrow[d, dash] \\ \langle ab\rangle \end{tikzcd} \qquad \mathcal{B}'(\lambda, 1-\lambda), \lambda \neq \tfrac{1}{2}: \quad \begin{tikzcd} \langle a, b, ab\rangle \arrow[d, dash] \\ \langle a, ab\rangle \arrow[d, dash] \\ \langle ab\rangle \end{tikzcd}$$

$$\mathcal{B}'(\lambda, \beta), \beta \neq \lambda, \beta \neq 1-\lambda: \quad \begin{tikzcd} \langle a, b, ab\rangle \arrow[d, dash] \\ \langle ab\rangle \end{tikzcd} \qquad \mathcal{B}'(\tfrac{1}{2}, \tfrac{1}{2}): \quad \begin{tikzcd} & \langle a, b, ab\rangle \arrow[dl, dash] \arrow[dr, dash] & \\ \langle a, ab\rangle \arrow[dr, dash] & & \langle b, ab\rangle \arrow[dl, dash] \\ & \langle ab\rangle & \end{tikzcd}$$

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6. Associative enveloping algebra of $\mathcal{B}'(\lambda,\beta)$

For an arbitrary algebra A, we can consider its embedding in the associative algebra $\mathrm{End}(A)$, via left and right actions of A on A.

Consider the operators of left multiplication by basis elements of the algebra $\mathcal{B}'(\lambda,\beta)$. Their matrix forms are

$$l_o = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = E_{11} + \lambda E_{22} + \lambda E_{33},$$

$$l_a = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \lambda & 1 & (\lambda-\beta)/\lambda & 0 \\ 0 & 0 & (\lambda+\beta-1)/\lambda & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \lambda E_{21} + E_{22} + \frac{\lambda-\beta}{\lambda}E_{23} + \frac{\lambda+\beta-1}{\lambda}E_{33} + E_{43},$$

$$l_b = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & (\lambda-\beta)/\lambda & 0 & 0 \\ \lambda & (\lambda+\beta-1)/\lambda & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \frac{\lambda-\beta}{\lambda}E_{22} + \lambda E_{31} + \frac{\lambda+\beta-1}{\lambda}E_{32} + E_{33} + E_{42},$$

$$l_{ab} = O_4,$$

where E_{ij} is the matrix with entry 1 at the crossing of ith row and jth column and zero otherwise.

Let $\mathcal{A}$ be the associative subalgebra of the algebra $\mathrm{End}(\mathcal{B}'(\lambda,\beta))$ with the generating set $\{l_o, l_a, l_b\}$. We denote some subalgebras of the matrix algebra $M_4(\mathbb{R})$ as

$$M_0 = \left\langle \begin{pmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & 0 & * & 0 \\ * & * & * & 0 \end{pmatrix} \right\rangle, \quad M_1 = \left\langle \begin{pmatrix} * & 0 & 0 & 0 \\ * & * & * & 0 \\ * & 0 & * & 0 \\ * & * & * & 0 \end{pmatrix} \right\rangle,$$

$$M_2 = \left\langle \begin{pmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & 0 \end{pmatrix} \right\rangle, \quad M_3 = \left\langle \begin{pmatrix} * & 0 & 0 & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \end{pmatrix} \right\rangle,$$

where $*$ is an arbitrary real number.

Note that the generators of $\mathcal{A}$ are contained in M_3. Below, we establish a result on the associative enveloping algebra of $\mathcal{A}$ depending on the values of λ and β.

Theorem 6.1. *Let $\mathcal{A}$ be the associative subalgebra of* $\mathrm{End}(\mathcal{B}'(\lambda,\beta))$ *with generators* $\{l_o, l_a, l_b\}$. *Then the following statements hold:*

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(1) *if* $\lambda = \beta = 1/2$, *then* $\mathcal{A} = M_0$;
(2) *if* $\lambda \neq \beta, \lambda = 1 - \beta$, *then* $\mathcal{A} = M_1$;
(3) *if* $\lambda = \beta, \lambda \neq 1 - \beta$, *then* $\mathcal{A} = M_2$; *and*
(4) *if* $\lambda \neq \beta, \lambda \neq 1 - \beta$, *then* $\mathcal{A} = M_3$.

Proof. Since $\lambda(1-\lambda)(E_{22}+E_{33}) = l_o - l_o^2 \in \mathcal{A}$, $E_{11} = l_o - \lambda(E_{22}+E_{33}) \in \mathcal{A}$ and $\lambda E_{21} = l_a - l_a \cdot (E_{22}+E_{33}) \in \mathcal{A}$. These yield $E_{22}+E_{33}, E_{11}, E_{21} \in \mathcal{A}$.

Consider

$$x_1 = \frac{\lambda-\beta}{\lambda}E_{23} + \frac{\beta-1}{\lambda}E_{33} + E_{43} = l_a \cdot (E_{22}+E_{33}) - (E_{22}+E_{33}) \in \mathcal{A}.$$

Then

$$x_2 = \frac{\lambda-\beta}{\lambda}E_{23} + \frac{\beta-1}{\lambda}E_{33} = (E_{22}+E_{33}) \cdot \left(\frac{\lambda-\beta}{\lambda}E_{23} + \frac{\beta-1}{\lambda}E_{33} + E_{43}\right) \in \mathcal{A},$$

and we obtain $E_{43} = x_1 - x_2 \in \mathcal{A}$. Next, consider

$$x_3 = \left\{l_b - \frac{\lambda-\beta}{\lambda}(E_{22}+E_{33})\right\} \cdot (E_{22}+E_{33}) \in \mathcal{A}$$

and

$$\lambda E_{31} = l_b - \frac{\lambda-\beta}{\lambda}(E_{22}+E_{33}) - x_3 \in \mathcal{A}.$$

So we obtain $E_{31} \in \mathcal{A}$ and $E_{41} = E_{43} \cdot E_{31} \in \mathcal{A}$. Moreover, $E_{42} = x_3 - (E_{22} + E_{33})x_3 \in \mathcal{A}$. So $E_{33} = \{\lambda/(\lambda-1)\}(x_3 - E_{42}) \cdot x_2 \in \mathcal{A}$ and $E_{22} = (E_{22}+E_{33}) - E_{33} \in \mathcal{A}$. Summarizing, we get $M_0 \subseteq \mathcal{A}$.

Furthermore, from x_2 and $x_3 - E_{42}$, we obtain $((\lambda-\beta)/\lambda)E_{23} \in \mathcal{A}$ and $((\lambda+\beta-1)/\lambda)E_{32} \in \mathcal{A}$. Thus the following cases occur.

Case 1. Let $\lambda \neq \beta$ and $\lambda \neq 1-\beta$. Then $E_{23}, E_{32} \in \mathcal{A}$ and $\mathcal{A} = M_3$.

Case 2. Let $\lambda = \beta$ and $\lambda \neq 1-\beta$. Then we obtain $E_{32} \in \mathcal{A}$ and $M_2 \subseteq \mathcal{A}$. However, $l_o, l_a, l_b \in M_2$ and, therefore, $\mathcal{A} = M_2$.

Case 3. Let $\lambda \neq \beta$ and $\lambda = 1-\beta$. Then $E_{23} \in \mathcal{A}$ and $M_1 \subseteq \mathcal{A}$. Moreover, $l_o, l_a, l_b \in M_1$ and $\mathcal{A} = M_1$.

Case 4. Let $\lambda = \beta = 1-\beta = 1/2$. Then we get $l_o, l_a, l_b \in M_0$ and $\mathcal{A} = M_0$.

This completes the proof of the theorem. □

7. Isomorphisms of ABO-group blood algebras

In this section, we analyse the conditions under which the two algebras $\mathcal{B}'(\lambda',\beta')$ and $\mathcal{B}'(\lambda,\beta)$ with corresponding basis $\{o',a',b',ab'\}$ and $\{o,a,b,ab\}$ are isomorphic.

Theorem 7.1. *Two distinct ABO-BGAs* $\mathcal{B}'(\lambda,\beta)$ *and* $\mathcal{B}'(\lambda',\beta')$ *are isomorphic if and only if* $\lambda' = \lambda$ *and* $\beta' = 1-\beta$.

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Proof. Let the isomorphism $\varphi\colon \mathcal{B}'(\lambda',\beta') \to \mathcal{B}'(\lambda,\beta)$ be given by

$$\begin{aligned}\varphi(o') &= d_{11}o + d_{12}a + d_{13}b + d_{14}ab,\\ \varphi(a') &= d_{21}o + d_{22}a + d_{23}b + d_{24}ab,\\ \varphi(b') &= d_{31}o + d_{32}a + d_{33}b + d_{34}ab,\\ \varphi(ab') &= d_{41}o + d_{42}a + d_{43}b + d_{44}ab.\end{aligned}$$

Since ab' annihilates $\mathcal{B}'(\lambda',\beta')$, it is clear that $\varphi(ab')$ also annihilates $\mathcal{B}'(\lambda,\beta)$.

Therefore,

$$0 = \varphi(ab') \circ o = (d_{41}o + d_{42}a + d_{43}b + d_{44}ab) \circ o = d_{41}o + \lambda d_{42}a + \lambda d_{43}b,$$

and, since $\lambda \neq 0$, we obtain $d_{41} = d_{42} = d_{43} = 0$.

Considering the equations

$$\varphi(o') \circ \varphi(o') = \varphi(o' \circ o'), \varphi(a') \circ \varphi(a') = \varphi(a' \circ a'), \varphi(b') \circ \varphi(b') = \varphi(b' \circ b'),$$
$$\varphi(o') \circ \varphi(a') = \varphi(o' \circ a'), \varphi(o') \circ \varphi(b') = \varphi(o' \circ b'), \varphi(a') \circ \varphi(b') = \varphi(a' \circ b'),$$

and comparing the coefficients at the corresponding basis elements $\{o, a, b, ab\}$, we derive the systems of equations

$$(I) : \begin{cases} d_{11} = d_{11}^2, \\ d_{12} = d_{12}^2 + 2d_{11}d_{12}\lambda + 2d_{12}d_{13}\dfrac{\lambda-\beta}{\lambda}, \\ d_{13} = d_{13}^2 + 2d_{11}d_{13}\lambda + 2d_{12}d_{13}\dfrac{\lambda+\beta-1}{\lambda}, \\ d_{14} = 2d_{12}d_{13} \end{cases}$$

$$(II) : \begin{cases} d_{21} = d_{21}^2 \\ d_{22} = d_{22}^2 + 2d_{21}d_{22}\lambda + 2d_{22}d_{23}\dfrac{\lambda-\beta}{\lambda}, \\ d_{23} = d_{23}^2 + 2d_{21}d_{23}\lambda + 2d_{22}d_{23}\dfrac{\lambda+\beta-1}{\lambda}, \\ d_{24} = 2d_{22}d_{23}, \end{cases}$$

$$(III) : \begin{cases} d_{31} = d_{31}^2, \\ d_{32} = d_{32}^2 + 2d_{31}d_{32}\lambda + 2d_{32}d_{33}\dfrac{\lambda-\beta}{\lambda} \\ d_{33} = d_{33}^2 + 2d_{31}d_{33}\lambda + 2d_{32}d_{33}\dfrac{\lambda+\beta-1}{\lambda}, \\ d_{34} = 2d_{32}d_{33}, \end{cases}$$

$$(IV) : \begin{cases} \lambda' d_{21} = d_{11}d_{21}, \\ \lambda' d_{22} = d_{12}d_{22} + (d_{11}d_{22} + d_{12}d_{21})\lambda + (d_{12}d_{23} + d_{13}d_{22})\dfrac{\lambda-\beta}{\lambda}, \\ \lambda' d_{23} = d_{13}d_{23} + (d_{11}d_{23} + d_{13}d_{21})\lambda + (d_{12}d_{23} + d_{13}d_{22})\dfrac{\lambda+\beta-1}{\lambda}, \\ \lambda' d_{24} = d_{12}d_{23} + d_{13}d_{22}, \end{cases}$$

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$$(V):\begin{cases}\lambda' d_{31} = d_{11}d_{31},\\ \lambda' d_{32} = d_{12}d_{32} + (d_{11}d_{32} + d_{12}d_{31})\lambda + (d_{12}d_{33} + d_{13}d_{32})\dfrac{\lambda-\beta}{\lambda},\\ \lambda' d_{33} = d_{13}d_{33} + (d_{11}d_{33} + d_{13}d_{31})\lambda + (d_{12}d_{33} + d_{13}d_{32})\dfrac{\lambda+\beta-1}{\lambda},\\ \lambda' d_{34} = d_{12}d_{33} + d_{13}d_{32},\end{cases}$$

$$(VI):\begin{cases}\dfrac{\lambda'-\beta'}{\lambda'}d_{21} + \dfrac{\lambda'+\beta'-1}{\lambda'}d_{31} = d_{21}d_{31},\\ \dfrac{\lambda'-\beta'}{\lambda'}d_{22} + \dfrac{\lambda'+\beta'-1}{\lambda'}d_{32} = d_{22}d_{32} + (d_{21}d_{32} + d_{22}d_{31})\lambda\\ \quad +(d_{22}d_{33} + d_{23}d_{32})\dfrac{\lambda-\beta}{\lambda},\\ \dfrac{\lambda'-\beta'}{\lambda'}d_{23} + \dfrac{\lambda'+\beta'-1}{\lambda'}d_{33} = d_{23}d_{33} + (d_{21}d_{33} + d_{23}d_{31})\lambda,\\ \quad + (d_{22}d_{33} + d_{23}d_{32})\dfrac{\lambda+\beta-1}{\lambda},\\ \dfrac{\lambda'-\beta'}{\lambda'}d_{24} + \dfrac{\lambda'+\beta'-1}{\lambda'}d_{34} + d_{44} = d_{22}d_{33} + d_{23}d_{32}.\end{cases}$$

We denote by $(I.i)$ the ith equation of the system (I) and similarly for the other systems of equations. From the first equation of system (I), we obtain $d_{11} = 0$ or $d_{11} = 1$. If $d_{11} = 0$, then the first equations of (IV) and (V) yield $d_{21} = d_{31} = 0$, which, together with $d_{41} = 0$, gives a contradiction for φ being an isomorphism. Therefore, $d_{11} = 1$ and, from the same equations, we obtain $d_{21} = d_{31} = 0$.

Let us denote $\Delta = d_{22}d_{33} - d_{23}d_{32}$. Note that, due to the results above, $\det[\varphi] = d_{44}\cdot\Delta \neq 0$. Multiplying equations $(IV.2)$, $(V.2)$ by d_{33}, d_{23}, respectively, and subtracting, yields $\lambda'\Delta = d_{12}\Delta + \lambda\Delta + d_{13}((\lambda-\beta)/\lambda)\Delta$. Since $\Delta \neq 0$, we obtain

$$\lambda' - \lambda = d_{12} + d_{13}\frac{\lambda-\beta}{\lambda}.$$

Analogously, multiplying equations $(IV.3)$, $(V.3)$ by d_{32}, d_{22}, respectively, and subtracting the first one from the second one , we get

$$\lambda' - \lambda = d_{13} + d_{12}\frac{\lambda+\beta-1}{\lambda}.$$

Hence, $d_{13} = ((1-\beta)/\beta)d_{12}$.

Case 1. Let $d_{12} \neq 0$.

Then $d_{13} \neq 0$. Note that equations (I.2) and (I.3) give rise to the system of corresponding equations (3.2) for the value $\xi = 1$: that is,

$$\begin{cases} d_{12} + 2\dfrac{\lambda-\beta}{\lambda}d_{13} = 1 - 2\lambda,\\ 2\dfrac{\lambda+\beta-1}{\lambda}d_{12} + d_{13} = 1 - 2\lambda.\end{cases}$$

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It is known that this system does not have a solution, if the determinant of the system is equal to zero. Therefore, assuming that $(\lambda, \beta) \notin P$ (that is, the determinant is not zero), we obtain the solution

$$d_{12} = \frac{\lambda(1-2\lambda)}{-3\lambda^2 + 4\beta^2 + 4\lambda - 4\beta}(2\beta - \lambda),$$
$$d_{13} = \frac{\lambda(1-2\lambda)}{-3\lambda^2 + 4\beta^2 + 4\lambda - 4\beta}(2 - 2\beta - \lambda).$$

Taking into account $d_{13} = ((1-\beta)/\beta)d_{12}$, we obtain $\beta = 1/2$. Hence, $d_{12} = d_{13} = \lambda(1-2\lambda)/(1-3\lambda)$. By subtracting equation (IV.3) from (IV.2) and taking into account that $d_{12} = d_{13}, \beta = 1/2$, we obtain

$$\lambda'(d_{22} - d_{23}) = d_{12}(d_{22} - d_{23}) + \lambda(d_{22} - d_{23}).$$

Similarly, by subtracting equation (V.3) from (V.2), we obtain

$$\lambda'(d_{32} - d_{33}) = d_{12}(d_{32} - d_{33}) + \lambda(d_{32} - d_{33}).$$

Observe that both values $d_{22} - d_{23}$ and $d_{32} - d_{33}$ cannot be zero simultaneously, since it contradicts that $\Delta = 0$. Therefore, at least one of these values is nonzero. Then we obtain $\lambda' - \lambda = d_{12}$ and $\lambda' - \lambda = d_{12}\{1 + (\lambda - \beta)/\lambda\}$; hence $\lambda = \beta = 1/2$. However, it implies $d_{12} = 0$, which is a contradiction.

Case 2. Let $d_{12} = 0$.

Then $d_{13} = 0$ and $\lambda' = \lambda$. From (I.4), we obtain $d_{14} = 0$. Furthermore, (IV.4) and (V.4) yield $d_{24} = d_{34} = 0$. Equations (II.4) and (III.4) lead to $d_{22}d_{23} = d_{32}d_{33} = 0$. Together with condition $\Delta \neq 0$, we have the following possible cases.

Case 2.1. Let $d_{23} = d_{32} = 0$ and $d_{22} \cdot d_{33} \neq 0$. Then systems (II) and (III) yield $d_{22} = d_{33} = 1$. Substituting these values into (VI.2), we obtain $(\lambda' - \beta')/\lambda' = (\lambda - \beta)/\lambda$, which implies that $\beta' = \beta$.

Case 2.2. Let $d_{22} = d_{33} = 0$ and $d_{23} \cdot d_{32} \neq 0$. Then, from (II.3) and (III.2), we get $d_{23} = d_{32} = 1$. Substituting the obtained values into system (VI), we derive $\beta' = 1 - \beta$.

Hence, $\mathcal{B}'(\lambda, \beta) \cong \mathcal{B}'(\lambda, 1-\beta)$ via the change of basis $(o, a, b, ab) \mapsto (o, b, a, ab)$.

This completes the proof of the theorem. □

8. Conclusion

The results of this paper are obtained by assuming two deviations from Mendelian genetics: all parents of blood group A and B contribute gamete O to the child with a constant probability α and the allele A is selected with probability β from parents of blood group AB. We generate an algebra with the multiplication based on this modified inheritance pattern, and investigate some of the properties of this algebra related to the equilibrium (idempotents) and annihilation (solvable elements) of a population, the dominating subpopulations (lattice of ideals) and the behaviour due to changes of the parameters. We establish that two isolated populations with different values of initial chosen parameters (unless the probabilities that blood groups A and B contribute with allele O during meiosis are equal in both populations and the probabilities of blood group AB contributing with allele A during meiosis in both populations add up to one) have different nonisomorphic corresponding algebras that describe heredity (evolution of population).

Another problem in mathematical biology, for a given evolution operator V, is the study of the asymptotic behaviour of the trajectories $\{\mathbf{x}^{[n]}\}_{n=1}^{\infty}$ for any initial state $\mathbf{x}$ of a population. It could be interesting to investigate the trajectories in our case. One of the possible generalizations in the future is to distinguish the probability of the contribution of allele O of a parent with blood group B. This leads to a construction of an algebra with three real parameters.

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