*Journal of the History of Economic Thought, Volume 24, Number 4, 2002*

# SOME PROPERTIES OF R. G. D. ALLEN'S TREATMENT OF KALECKI'S 1935 MODEL OF BUSINESS CYCLES

BY

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*The aim of business cycle theory is to* explain *certain movements of economic variables* Jan Tinbergen (1935, p. 241).

## I. INTRODUCTION

More than sixty-five years have passed since Michal Kalecki (1935) published one of the first formal mathematical models of business cycles. His paper presents a closed-form analytic solution. This characteristic, among others, sets Kalecki's work apart from that of contemporary literary business cycle theorists such as Friedrich A. Hayek (1935) and John Maynard Keynes (1936).

This paper examines Roy George Douglas Allen's treatment of Kalecki's 1935 model. We focus on this model because it represents a seminal work that has proven important in the history of economic thought and influential in the development of macro-dynamics. In his book on economic dynamics, Giancarlo Gandolfo declares that the 1935 model "is by now a 'classic' in macrodynamics and cannot be ignored by any dynamicist" (1980, p. 527). Indeed, much of Kalecki's early work represent refinements of the 1935 model (Kalecki, 1971).[1] Stanislaw Gomulka, Adam Ostaszewski, and Roy Davies write, "Kalecki's *early work on unemployment and the business cycle* established him as a co-founder of modern macroeconomic theory. As in Keynes's *The General Theory of Employment, Interest and Money* (1936), so, too, in Kalecki's model (1933, 1935), it is

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[1] See Josef Steindl's useful essay on the evolution of Kalecki's thinking about trade cycles. Kalecki developed three models of the trade cycle between 1933 and 1968. The later models incorporate different investment functions and allow for economic growth (1990, pp. 139–48). Stanislaw Gomulka, Adam Ostaszewski, and Roy Davies offer a detailed analysis of Kalecki's post-1943 versions of his theory of the cycle and trend of a capitalist economy (1990). The paper extends Kalecki's model and corrects an error in his analysis of the model's stability. The collected works of Kalecki, edited by Jerzy Osiatynski, in English, have been available since 1996 in six volumes (1990–1996).

ISSN 1042-7716 print; ISSN 1469-9656 online/02/040463-12 © 2002 The History of Economics Society
DOI: 10.1080/1042771022000029896

the aggregate investment demand rather than the flexibility of prices or wages that plays a key role in determining the aggregate level of output and employment" (1990, p. 525, emphasis added). The significance these authors attribute to Kalecki's early model is obvious. In assessing the overall influence of Kalecki, Geoffrey C. Harcourt, writing in 1977, observes, "Kalecki is a most important patron saint of the post-Keynesians ..." (1977, p. 93). Why then examine Allen's version of Kalecki's model? The answer lies in the relative neglect of Kalecki's original paper. By contrast, Allen introduced a postwar generation of economists to Kalecki's early macrodynamics. The questions we raise are: How faithful to Kalecki is Allen's interpretation, and does Allen offer any insight into the mathematical limitations of Kalecki's seminal model? On certain points, we by necessity compare and contrast Kalecki's and Allen's versions of the 1935 model.

Malcolm C. Sawyer observes that when Kalecki first presented his theory of the business cycle at the Econometrics Society Conference in Leyden, Holland, in 1933, "There was some attention paid to this paper amongst the relatively small group of mathematically-inclined economists interested in business cycles, but little outside of that circle" (1985, p. 5). George R. Feiwel opines that while Kalecki was "First and foremost an essentially original thinker ... his work is not as well known as it deserves to be" (1975, pp. viii, ix). After arguing the superiority of Kalecki's model because "it is explicitly dynamic; it takes income distribution as well as level into account; and it makes the important distinction between investment orders and investment outlays," Lawrence R. Klein observes, "his achievement is relatively unnoticed" (1951, pp. 447, 448).

There are several reasons for this neglect. First, Feiwel comments on Kalecki's "austere and laconic" style (1975, p. ix). By way of explanation Feiwel observes, "He wrote like a mathematician. Perhaps it is the very taciturnity of Kalecki's mode of expression, the restraint in the language, and the utmost concentration of thought that make his writings so difficult to understand and rather unpopular among his fellow economists (1975, p. 13). Second, "In a sense [Kalecki] was arrogant, for he made little pretence of relating his work to the existing literature" (Harcourt, 1977, p. 93). Feiwel adds, "In Kalecki's writings one finds few references to the literature on the subject ... Clearly, such an approach has both advantages and disadvantages." One disadvantage is that "it fails to place the argument within the stream of thought" (1975, p. 14). Apparently, Kalecki was not all that familiar with "Western" economics in the 1930s, during which time the work of interest here was written (Feiwel 1975, p. 14). Third, although many scholars, such as Joan Robinson (1972, p. 4), point to the significance of Kalecki's contributions to macrodynamics, it is clear that Keynes carries the day with the publication of *The General Theory* in 1936. Here then we have a scholar doing significant work that does not garner much attention because of the austerity of his writing style, the difficulty of the higher mathematics employed in his modeling, a tendency to avoid placing his work in the context of other work in the field, a lack of contact with western economists during the prewar period, and the sweeping success of the Keynesian Revolution.

Instead of reading Kalecki's original and relatively dense 1935 paper in *Econometrica*, many in the postwar audience of graduate students learned Kalecki's macrodynamics by reading Allen's *Mathematical Economics* (1956,

1959) or his text *Macro-Economic Theory*: *A Mathematical Treatment* (1967). Among other things, Allen provides context for Kalecki's 1935 model by presenting it alongside other, more familiar, formal macrodynamic models. In addition, by using what had become standard macro notation, Allen's presentation of Kalecki's model is more accessible to the postwar reader. Finally, available to Allen during the time he was writing in the 1950s and 1960s were mathematical advances in the understanding of the solution properties of mixed difference-differential equations. Of interest is the question: Does Allen use that knowledge to offer insights into the mathematical limitations of Kalecki's analysis?

Allen introduces Kalecki's model in his text by observing, "It is worthwhile investigating the effects of these complications [arising from investment lags], largely assumed away in the multiplier-accelerator models we have been using" (1967, p. 369). Likewise, Gandolfo explains inclusion of Kalecki's model in his book on economic dynamics by arguing, "The fundamental reason is that we think that mixed difference-differential equations are much more suitable than differential equations alone, or difference equations alone, for the adequate treatment of dynamic economic phenomena" (1980, p. 527, italics deleted). Here Kalecki's contribution is seen as erecting a model around a mixed difference-differential equation that allows for a fixed delay between investment orders and expenditures.

Like his contemporaries, Kalecki's goal was to explain. He thought his model revealed something meaningful about the short-run dynamics of capitalist economies, for when it was observed that his model gives rise to paradoxical results, e.g., profitable investments produce prosperity but also sow the seeds of future economic crisis, Kalecki writes, "But it is not the theory which is paradoxical but its subject—the capitalist economy" (Kalecki 1937, p. 96).

To render his model tractable, Kalecki assumes a stationary, closed economy. Moreover, the model has no government sector, no technological change and, typical of such dynamic macro models of the time, no relative prices, no factor substitution, a fixed capital/output ratio, and a constant MPC.[2] Unique to Kalecki's model is the assumption that the level of depreciation is constant, i.e., independent of the size of the capital stock.[3] This assumption is crucial to solving the model, and Kalecki justifies it because the model addresses only short-run economic fluctuations. Such simplifications are often necessary if the analyst is to make any headway in modeling complex phenomena. Tinbergen speaks to the strength of this approach to macrodynamics and places Kalecki's early business cycle modeling in prospective:

> As a rule, the analytical form of the equations is simplified as much as possible, otherwise no explicit solution would ever be possible. Such a system admits of one or more solutions and leads to a definite movement, initial conditions given. This deserves the name of business cycle theory, although it may be a very simplified one, but "open" systems cannot represent theories, since they

[2] These latter assumptions are listed because they explain the (in)stability of the model.

[3] If depreciation were proportional to last period's capital stock, as in $D = dK(t-1)$, it would introduce another lagged term for capital in addition to $K(t-\theta)$, thus making finding a solution for the model extremely difficult, if not impossible, using analytic methods.

> do not give a complete system of hypotheses sufficient to determine the movement of variables. Many literary theories seem to fall into this category, in that they do not clearly state all relations which are necessary or are tacitly included.... The use of mathematics is of peculiar value in this field ... the question is how to find the happy medium between the complexity of the real world and the simplicity of an amenable model, (Tinbergen 1935, pp. 242–43).

Kalecki makes the same argument, "... to approach the dynamic process in all its complexity is certainly a hopeless task" (quoted in Feiwel 1975, p. 159). We accept the *economic* assumptions that Kalecki makes, which are incorporated into Allen's version of the model, without further discussion.

## II. ALLEN'S VERSION OF KALECKI'S 1935 MODEL

Let us now turn to Allen's work. In his *Mathematical Economics* (1956) and *Macro-Economic Dynamics* (1967), Allen presents an interpretation of Kalecki's short-run macrodynamic model. The structure of Kalecki's 1935 model is of special interest for two reasons. *Economically*, the model's cyclic behavior arises in the investment sector. Kalecki introduces a fixed delay between investment orders and the delivery of producers' goods. This implies a gap between investment expenditures, which increase aggregate demand, and the delivery of equipment, which increases productive capacity or aggregate supply. Investment orders are a function of the difference between the desired and actual capital stock. The desired capital stock, which is unobservable, is proportional to current income. The factor of proportionality is the constant capital/output ratio $v$. Symbolically, the desired capital stock equals $vY(t)$.

*Mathematically*, much of the interest in Kalecki's seminal work stems from the fact that the model reduces to a mixed difference-differential equation. Allen observes:

> [The model] has a number of distinctive features ... Kalecki has chosen to introduce a fixed-time delay (between decisions to invest and deliveries of equipment) as well as continuous variation represented by derivatives and integrals. The resulting equation of the model ... is of the mixed difference-differential type ... The analysis of the present section is basically that of Kalecki (1935) with only minor variations (1959, pp. 242, 251).

Allen's version differs from that of the original in several respects. First, Allen drops the distributional dimensions of Kalecki's model wherein only capitalists earn profits and engage in savings. Secondly, in principle, the model can be solved for any of five endogenous variables. Allen solves for $K(t)$, the capital stock, and $Y(t)$, national income, while Kalecki finds $B(t)$, investment orders. Third, Kalecki eliminates the constant terms for autonomous consumption ($C_0$) and capital depreciation ($U$), during his derivation of the characteristic equation for $B(t)$. By contrast, Allen simply drops the terms from the model at the start. In effect he sets $C_0 = U = 0$. The structural equations of the model, a mixed integral-differential system, are presented in Table 1.

Before offering a critique of Allen's version of Kalecki's model, we want to discuss the mathematical technique available to Allen for addressing the model

**Table 1.** The Structural Equations of Kalecki's 1935 Model

| | | |
|---|---|---|
| $Y(t) = C(t) + I(t)$ | National Income | (1) |
| $C(t) = C_0 + C_1 Y(t)$ | Consumption Function | (2) |
| $I(t) = (1/\theta)\int_{t-\theta}^{t} B(\tau)d\tau$ | Investment Expenditures | (3) |
| $B(t) = \lambda[vY(t) - K(t)]$ | Investment Orders | (4) |
| $K'(t) = B(t-\theta) - U$ | Change in the Capital Stock | (5) |

We use standard notation. For this particular model $B(t)$ is investment orders; $\theta$ is the delivery lag for investment goods, and $U$ is the constant rate of capital depreciation. Note: If a window on the economy is opened at $t = 0$, then equations (1)–(5) must hold, with constant parameters, for $t \geqslant -\theta$.

and how the limitations of that technique shaped the goals of his analysis. Like Kalecki before him, Allen can analyze the model only by offering a closed form solution. This may explain, for example, why it was mathematically desirable to eliminate $C_0$ and $U$ from the analysis in order to express the reduced form equations of the model as *homogeneous* equations. The latter are easier to solve in closed form. Again, like Kalecki, Allen does not explore the time paths of the model's endogenous variables. He is content to set forth the conditions necessary for the model to generate cyclic behavior for $K(t)$ and $Y(t)$. This allows Allen to offer at least a partial description of the model's business cycle. By contrast, we use numerical methods to explore fully the properties of the model. Thus, we are able to uncover several significant limitations of the model of which neither Kalecki nor Allen seem to have been aware.

To facilitate our critique, we reproduce Allen's reduction of the model to a mixed equation for $K(t)$, except that we *do not discard* $C_0$ and $U$. Shifting (5) to $B(\tau) = K'(\tau + \theta) + U$, and substituting into (3) gives:

$$I(t) = (1/\theta)\int_{t-\theta}^{t} [K'(\tau+\theta) + U]d\tau = [K(t+\theta) - K(t)]/\theta + U. \qquad (6)$$

With no lags in consumption or output (Allen 1967, p. 371), combining (1)–(2) and (6), where $s = 1 - C_1$, yields:

$$sY(t) = I(t) + C_0 = [K(t+\theta) - K(t)]/\theta + U + C_0. \qquad (7)$$

Combining (4) and (5), after shifting periods in (4), gives:

$$K'(t) = B(t-\theta) - U = \lambda[vY(t-\theta) - K(t-\theta)] - U. \qquad (8)$$

Finally, shifting periods in (7), substituting into (8), and collecting terms yields:

$$K'(t) = (\lambda v/s\theta)K(t) - [(\lambda v/s\theta) + \lambda]K(t-\theta) + [\lambda v/s(C_0 + U) - U]. \qquad (9)$$

Equation (9) is Allen's difference-differential equation for $K(t)$.

In addition, by differentiating (7) and substituting from (9) we obtain:

$$Y'(t) = (\lambda v/s\theta)Y(t) - [(\lambda v/s\theta) + \lambda]Y(t-\theta) + (\lambda/s)(U + C_0). \qquad (10)$$

https://doi.org/10.1080/104277102200029896 Published online by Cambridge University Press

Finally, differentiating (4) and using (9) and (10), gives:

$$B'(t) = (\lambda v/s\theta)B(t) - [(\lambda v/s\theta) + \lambda]B(t - \theta) + \lambda U. \tag{11}$$

Equations (10) and (11) are the difference-differential equations for $Y(t)$ and $B(t)$.

This paper identifies four shortcomings of Allen's treatment of the Kalecki model. Two arise from the elimination of $C_0$ and $U$; one from the acute sensitivity of the model to parameter values and, the last, from the failure of Allen to discuss initial conditions with any specificity. When Allen was writing *Mathematical Economics* in the mid-1950s, much more was known about the solution properties of mixed difference-differential equations than when Kalecki was writing twenty years earlier. This was even more true when Allen published *Macro-Economic Theory* in 1967.[4] Yet Allen's treatment of Kalecki's model is virtually unchanged in his two works. Moreover, he fails to address all the issues with which Kalecki deals. Much the same can be said of Gandolfo's presentation in 1980 of Kalecki's model.

## *Implications of Eliminating* C$_0$ *and* U

Two consequences flow directly from the elimination of $C_0$ and $U$ from the analysis. The first is that solutions to equations (9), (10), and (11) mean that $K(t)$, $Y(t)$, and $B(t)$ oscillate around intertemporal equilibrium values of zero e.g., $\bar{K} = \bar{Y} = \bar{B} = 0$. This may seem a minor simplification and, in one sense it is, but discarding $C_0$ and $U$ unnecessarily sacrifices some economic insights. For example, if $C_0$, $U > 0$, the mixed equation for $Y(t)$, equation (10), oscillates about the equilibrium: $\bar{Y} = (1/s(U + C_0))$, which is, of course, the textbook Keynesian income multiplier. This is an interesting result because it relates Allen's version of the model to a whole class of macro models. In any event, it is certainly more insightful than a result that intertemporal equilibrium income equals zero, $\bar{Y} = 0$.

The second limitation that stems from setting $C_0 = U = 0$ is that (9), (10), and (11) reduce to the same homogenous equation. Thus the time paths of $K(t)$, $Y(t)$, and $B(t)$ have the same periods and damping factors. This is a model of synchronized swimming. There can be no leads or lags among the five endogenous variables; they all move together. Moreover, they are all equally stable, stationary, or unstable. Yet there is no theoretically satisfying reason for believing that the cyclic behavior of national income, consumption, the aggregate capital stock, and investment orders share these characteristics. Allen hints that he makes this assumption because "the paths of the other variables ($Y(t)$ and $B(t)$) follow once [9] is solved" (Allen 1967, p. 372). It is easier to solve one homogeneous equation than to solve a set of non-homogeneous equations. Allen must have been aware of these two limitations of the model but, perhaps, his primary interest in including Kalecki's model in his texts was heuristic, and so he did not think it important to discuss them.

[4] Bellman and Cooke published the standard text on mixed difference-differential equations in 1963. Of course, much work in the area had been done in the immediate postwar period before the publication of their text.

**Table 2.** Kalecki's Original Values for the Parameters of the Model

| Parameter | Value |
|---|---|
| $C_0$ | 16.0 |
| $C_1$ | 0.950 |
| $K(0)$ | 120.0 |
| $U$ | 6.0 |
| $\theta$ | 0.600 |
| $\lambda$ | 0.121 |
| $v$ | 0.392562 |

## *Two Additional Limitations*

By using numerical methods we are able to bring to light two additional limitations of the model not explored by Allen. Thus the third limitation of Allen's analysis is his failure to recognize the inherent instability of the model. Unlike Allen, Kalecki chooses values for the model's parameters. He does this for two reasons. First, Kalecki wants to find a "reasonable" combination of parameter values that will yield complex roots for the analytic solution of the model. Complex roots imply cyclic behavior for investment orders $B(t)$, the endogenous variable of interest. Second, Kalecki seeks to calculate the period ($T$) and damping factor ($d$) for the model. For those calculations the model must be calibrated. Table 2 contains Kalecki's assumed value for the model's parameters (Kalecki 1935, pp. 338–39).

Allen's analysis merely discusses the general conditions under which the model would exhibit cyclic behavior. Using Kalecki's parameter values as a benchmark, we perform sensitivity analysis. Table 3 offers an example of our sensitivity test.

Given the values of all the other parameters, a *MPC* ($C_1$) equal to .95, as assumed by Kalecki, generates a cycle period $T = 9.9506$ and a damping factor $d = 0.9724$. The latter implies a slightly damped or stable oscillatory path for

**Table 3.** Sensitivity Analysis of Kalecki's $T$ and $d$

| Parameter | Deviation in Parameter | Parameter Value | Period $T$ | Damping Factor $d$ |
|---|---|---|---|---|
| $C_1$ | + 1.0% | 0.95950 | 12.3743 | 67.1328 |
| $C_1$ | + 0.5% | 0.95475 | 10.4398 | 5.0944 |
| $C_1$ | 0.0% | 0.95000* | 9.9506 | 0.9724 |
| $C_1$ | − 0.5% | 0.94525 | 10.0625 | 0.2396 |
| $C_1$ | − 1.0% | 0.94050 | 10.6181 | 0.0597 |

*Kalecki's value for $C_1$.

https://doi.org/10.1080/1042771022000029896 Published online by Cambridge University Press

$B(t)$. Increasing the value of $C_1$ by one half of one percent (0.5 percent) renders the path of $B(t)$ highly unstable, $d = 5.0944$. A one-percent increase raises $d$ to 67.1328. By any measure this is a high degree of sensitivity. Checking other parameter values, under *ceteris paribus* conditions, yields the broad result that varying *any* parameter from a minimal 0.5 percent to a maximum of just 5.0 percent destabilizes the model. Kalecki's model and Allen's version of it belong to a class of macro models, including, for example, the Harrod-Domar model, that have knife-edge stability conditions. This is not a desirable trait in a dynamic model of a decentralized economy.[5]

Returning to Allen's variant of Kalecki's model, the fourth and final limitation becomes apparent. Stated briefly: Although the mixed difference-differential equation model holds out some promise of providing richer insights into dynamic behavior, such sophistication comes at a high price. It involves a complication never mentioned by Kalecki or Allen. This higher-order dynamic system requires a startup period before it can track an economy.

To understand the problem, consider, for example, equation (11) for $B(t)$. At time $t = 0$, (8) states that *past* investment orders, $B(-\theta)$, are a function of *past* income, $Y(-\theta)$. But from (7) we know that $Y(-\theta)$ depends on the present value of the capital stock, $K(0)$. Therefore, $B(-\theta) = f\{K(0), \cdot\}$. This functional relationship implies that *past* values of $B(t)$, over the interval $-\theta \leqslant t \leqslant 0$, must be consistent with *future* values of $K(t)$, *for* $t \geqslant 0$, which in turn are dependent on *past* values of $K(t)$. Therefore a prior consistency must exist among *past* values of $K(t)$, $Y(t)$, and $B(t)$, for $-\theta \leqslant t \leqslant 0$. Absent this consistency, the phase and amplitude of the model's business cycle are mathematically and economically arbitrary. The paradox is that the required consistency is assured only if past values of $B(t)$ had been determined by the model's investment function, but that function has no history at $t = 0$, when the model begins generating a dynamic time path.

To be more specific, consider the dynamic paths implied by equations (9), (10), and (11) for $K(t)$, $Y(t)$, and $B(t)$, respectively. If we wish to open a window at $t = 0$ and begin tracking the time path of any one of these variables, initial conditions must be specified. Analytically, the critical question is: How are these initial conditions at $t = 0$ determined? That is: How are the values of $K(0)$, $Y(0)$, and $B(0)$ to be established? There are, in fact, a number of reasonable ways of assigning these values. We demonstrate that, if one is interested only in calculating the period $T$ and damping factor $d$ of the model, it does not matter what method is used to determine the initial values. This non-intuitive result, that $T$ and $d$ are independent of initial conditions, apparently was unknown to Kalecki and to Allen after him. By contrast, we demonstrate that the phase and amplitude of the time paths of $K(t)$ or $Y(t)$ or $B(t)$ are highly dependent on the method for determining initial conditions. Again, Kalecki and Allen do not seem to have appreciated this.

Like a differential equation, a mixed difference-differential equation has an infinite number of solutions. A particular solution is selected from initial

[5] This sensitivity stems from the lack of behavioral adjustment in this class of models. They typically exhibit fixed factor proportions, no factor prices, and no induced technological change.

conditions. For example, the general solution of a first order ordinary differential equation normally involves an arbitrary constant of integration. To eliminate the arbitrary constant and obtain a unique solution requires plugging in an initial value for the endogenous variable at $t = 0$. For difference-differential equations the situation is more complicated. To specify a unique solution for equation (11), for example, requires knowledge of the solution *at every point* in the interval $[-\theta, 0]$. In other words, for a linear ordinary differential equation with constant coefficients, e.g., $y'(t) = -y(t) + 1$, with a stable equilibrium solution, here $y(t) = 1$, all solutions for $y(t) = ce^{-t} + 1$ will converge to the equilibrium path *regardless* of the initial condition, i.e., the value of $y(0)$. However, for the mixed difference-differential equation (11), differences in the solution paths persist given differing initial conditions.

We illustrate the problem in a very precise way by proposing two methods for determining $B(0)$. We then use each proposed value of $B(0)$ to solve equation (11) for the time path of $B(t)$. Using numerical methods and Kalecki's parameter values, reproduced in Table 2, we solve equation (11) twice and compare and contrast the resulting time paths for $B(t)$.

For the first method of finding $B(0)$, we return to the structural equations of the model. Substituting (3) into (7) gives:

$$sY(0) = (1/\theta) \int_{-\theta}^{0} B(\tau) d\tau + C_0.$$

Evaluating (4) at $t = 0$ gives:

$$B(0) = \lambda[vY(0) - K(0)].$$

Combining the last two equations yields:

$$B(0) = \lambda[(v/s\theta) \int_{-\theta}^{0} B(\tau) d\tau + (vC_0/s) - K(0)]. \tag{12}$$

Equation (12) provides one reasonable way of establishing a value for $B(0)$. For this purpose, we construct a history function that generates values of $B(t)$ for the interval $-\theta < t < 0$. The function is given by:

$$g(t) = \lambda[51 - 2\cos t]. \tag{13}$$

$g(t)$ oscillates about the value $51\lambda$, approximately 6.0. Recall that the steady state equilibrium value of $B(t)$ is $U$ and that Kalecki assigns $U$ a value of 6.0. It seems natural to propose a function that is fairly close to $U$. A second reasonable way to determine the value of $B(0)$ is to use equation (13): $B(0) = g(0)$. This latter alternative preserves continuity at $t = 0$, while using (12) to compute $B(0)$ generates a discontinuity at $t = 0$.[6] In either case, equation (11) determines the subsequent evolution of $B(t)$.

[6] Both time paths are generated by numerical methods, in particular, the integral in (12) is calculated using Simpson's rule and (11) is integrated with the method usually associated with the name of Heun, but sometimes referred to as the improved Euler method.

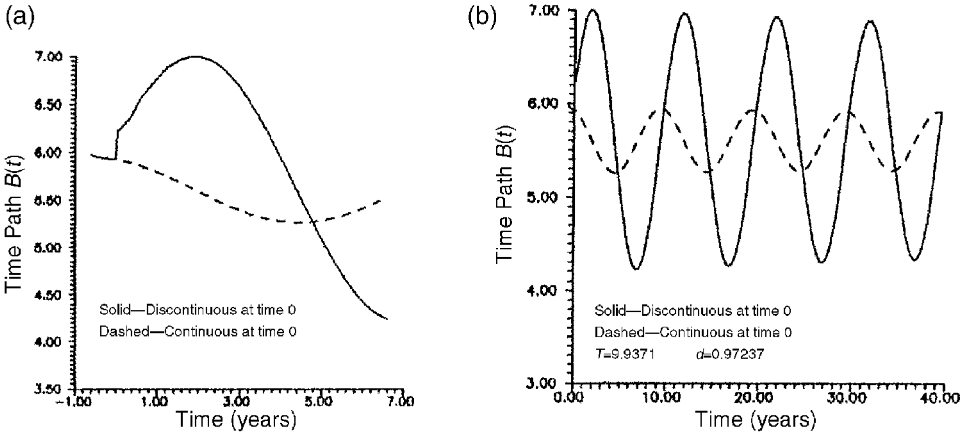


**Figure 1.** The results of the simulation.

The computed period $T$ and damping factor $d$ for the discontinuous case, using equation (12), are $T = 9.9371$ and $d = .097237$. The values, using equation (13), are $T = 9.95058$ and $d = .097245$.[7] The periods and damping factors are remarkably close for both series. We suspect that the slight differences are attributable to rounding errors in computation.

Figure 1 presents the results of the simulation. In order to show detail, Panel (a) enlarges and replicates the time path of $B(t)$ for the first seven years of the forty years covered in Panel (b). The solid line represents the time path for the discontinuous $B(t)$ arising from the use of (12) to compute $B(0)$. The dashed line portrays the time path using the continuous $B(t)$, i.e., assuming that $B(0)$ is given by $g(0)$ from the history function. Panel (a) shows that the two paths diverge. At $t = 0$, the solid line depicts an upturn in investment orders while the dashed line shows a decline. Panel (b) demonstrates that this fundamental difference in the predicted course of investment orders persists. Over time, the two paths remain out of phase. Eventually the two series will converge along different paths, to the same intertemporal equilibrium because the cycles are slightly damped, $d = .097237$. However, for $d > 1$, differences between the two paths increase as $t \to \infty$. Amplitudes of the two paths differ because $C_0$ and $U$ are lost in Kalecki's and Allen's versions of the model that underlie the solid line. By contrast, we retain all elements of the model. It is not surprising that inclusion of $C_0$ and $U$ influences the amplitudes of the two paths.

In short, a dynamic macro model that can be described mathematically by a mixed difference-differential equation cannot open a window at $t = 0$ and begin tracking the economy. In the instant case explored above, the only meaningful time path for investment orders results when $B(0)$ is consistent with *prior* values

[7] We also note that Frisch and Holme calculated that $T = 9.95$ and $d = .09725$ for Kalecki's model (1935, p. 237).

of $B(t)$ generated by the model's investment function. *Every other method* of assigning values for $B(t)$ for $-\theta \leqslant t \leqslant 0$ will yield arbitrary paths for $B(t)$ that can persist through time. Allen is silent on the path dependence of $K(t)$, $Y(t)$, and $B(t)$ on their respective historical values for the interval $-\theta \leqslant t \leqslant 0$.

## III. SUMMARY AND CONCLUSION

Allen's version of Kalecki's 1935 model has four significant limitations as a model of business cycles. Two of those limitations result directly from discarding autonomous consumption and fixed-capital depreciation. First, interesting inter-temporal equilibrium solutions are lost. For example it can be shown that Allen's version of the model belongs to a class of Keynesian models. Second, loss of $C_0$ and $U$ means that the model gives rise to identical homogeneous equations of motion for all of the model's five endogenous variables. Thus the time paths of $C(t)$, $Y(t)$, $K(t)$, $I(t)$, and $B(t)$ are dynamically synchronous.

Two additional limitations are revealed by numerical analysis of the model employing Kalecki's original parameter values. Numerical analysis is a technique not available to either Kalecki or Allen. The third limitation is that the model exhibits knife-edge stability. Slight deviations in parameter values from those specified by Kalecki render the model highly unstable. Allen does not explore the stability properties of the model. Four, the requirements of the model for generating economically and mathematically meaningful time paths are stringent. The model cannot begin tracking the economy at $t = 0$ absent consistency among the *prior* values of the relevant endogenous variables. Allen ignores these initial conditions necessary to the existence and uniqueness of his solution. The proposed business cycle has no roots in the history of the economy as required by the lag structure of the investment function.

Several final observations are in order. Kalecki's 1935 model is all the more remarkable when it is realized that not much was known at that time about the solution properties of mixed difference-differential equations. Twenty-eight years passed before Richard Bellman and Kenneth L. Cooke published the standard text in the field (1963). In one sense, Kalecki (1935) and Ragnar Frisch and Harald Holme (1935) were pioneers in exploring the characteristics of mixed difference-differential equations.

The analyses of the 1935 model by Kalecki and Allen are correct as far as they go. Kalecki chooses parameter values such that the model's solution has complex roots, thereby generating cyclical time paths for the endogenous variables. Specifying the model's parameters was a tedious chore, not to be under-estimated, given the computational techniques available in the 1930s. Given the assumed parameters, Kalecki's calculations of the period and damping factor are accurate. And that is all he claimed for his model. Frisch and Holme (1935, pp. 238–39) criticize Kalecki for choosing parameter values so as to obtain a particular result rather than providing a sound statistical basis for the magnitudes of the parameters. But that is an entirely different issue.

Allen rests content to specify the necessary and sufficient conditions for the model's solution having complex roots. Again, his mathematical analysis is correct as far as it goes. Allen truncates the model, thereby sacrificing economic insights, and he fails to explore many properties of the model, perhaps, because his interpretation of the model is conditioned by the requirements of textbook presentation. Making the model mathematically transparent has pedagogical value. Neither Kalecki nor Allen explored the time paths generated by the 1935 model, both were content deriving a few qualitative properties of the model, and both implicitly assumed that the model held, with constant parameters, for the time period of interest.

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