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# SUPERCONGRUENCES INVOLVING $p$-ADIC GAMMA FUNCTIONS

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**Abstract**

We establish some supercongruences for the truncated ${}_2F_1$ and ${}_3F_2$ hypergeometric series involving the $p$-adic gamma functions. Some of these results extend the four Rodriguez-Villegas supercongruences on the truncated ${}_3F_2$ hypergeometric series. Related supercongruences modulo $p^3$ are proposed as conjectures.

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## 1. Introduction

Rodriguez-Villegas [9] observed the relationship between the number of points over $\mathbb{F}_p$ on hypergeometric Calabi–Yau manifolds and the truncated hypergeometric series. To state these results, we first define the truncated hypergeometric series. For complex numbers $a_i, b_j$ and $z$, with none of the $b_j$ being negative integers or zero, the truncated hypergeometric series are defined by

$$ {}_rF_s\left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix}; z\right]_n = \sum_{k=0}^{n} \frac{(a_1)_k (a_2)_k \cdots (a_r)_k}{(b_1)_k (b_2)_k \cdots (b_s)_k} \cdot \frac{z^k}{k!}, $$

where $(a)_0 = 1$ and $(a)_k = a(a+1)\cdots(a+k-1)$ for $k \geq 1$.

Throughout this paper, $p$ is a prime with $p \geq 5$. Rodriguez-Villegas [9] proposed four conjectural supercongruences associated to certain modular K3 surfaces. These were all of the form

$$ {}_3F_2\left[\begin{matrix} \frac{1}{2}, & -a, & a+1 \\ & 1, & 1 \end{matrix}; 1\right]_{p-1} \equiv c_p \pmod{p^2}, \tag{1.1} $$

where $a = -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}$ and $c_p$ is the $p$th Fourier coefficient of a weight-three modular form on a congruence subgroup of $\mathrm{SL}(2, \mathbb{Z})$. The case with $a = -\frac{1}{2}$ was confirmed by van Hamme [18], Ishikawa [5] and Ahlgren [1]. The other cases with $a = -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}$ were partially proved by Mortenson [8], and finally proved by Sun [15].

Let $\langle a\rangle_p$ denote the least nonnegative integer $r$ with $a \equiv r \pmod p$. Sun [12, Theorem 2.5] showed that for any $p$-adic integer $a$ with $\langle a\rangle_p \equiv 1 \pmod 2$,

$$ {}_3F_2\begin{bmatrix} \frac{1}{2}, & -a, & a+1 \\ & 1, & 1 \end{bmatrix}; 1\Bigg]_{p-1} \equiv 0 \pmod{p^2}, \tag{1.2}$$

which partially extends (1.1). Guo and Zeng [4, Theorem 1.3] obtained an interesting $q$-analogue of (1.2). Using the same idea, Sun [12, Corollary 2.2] also proved that for $\langle a\rangle_p \equiv 1 \pmod 2$,

$$ {}_2F_1\left[\begin{matrix} -a, & a+1 \\ & 1 \end{matrix}; \frac{1}{2}\right]_{p-1} \equiv 0 \pmod{p^2}. \tag{1.3}$$

The cases when $a = -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}$ on the left-hand side of (1.3) have been dealt with by Sun [11, 12], Sun [14, 16] and Tauraso [17].

In this paper, we will prove some supercongruences for the truncated ${}_2F_1$ and ${}_3F_2$ hypergeometric series involving $p$-adic gamma functions. Some of these results extend the four Rodriguez-Villegas supercongruences on the truncated ${}_3F_2$ hypergeometric series. Our proof is based on some combinatorial identities involving harmonic numbers and properties of the $p$-adic gamma functions.

Theorem 1.1. *Let $p \geq 5$ be a prime. For any $p$-adic integer $a$ with $\langle a\rangle_p \equiv 0 \pmod 2$,*

$$ {}_2F_1\left[\begin{matrix} -a, & a+1 \\ & 1 \end{matrix}; \frac{1}{2}\right]_{p-1} \equiv (-1)^{(p+1)/2}\Gamma_p\Big(\frac{1}{2}\Big)\Gamma_p\Big(-\frac{a}{2}\Big)\Gamma_p\Big(\frac{a+1}{2}\Big) \pmod{p^2}, \tag{1.4}$$

*where $\Gamma_p(\cdot)$ denotes the $p$-adic gamma function recalled in the next section.*

Theorem 1.2. *Let $p \geq 5$ be a prime. For any $p$-adic integer $a$ with $\langle a\rangle_p \equiv 0 \pmod 2$,*

$$ {}_3F_2\left[\begin{matrix} \frac{1}{2}, & -a, & a+1 \\ & 1, & 1 \end{matrix}; 1\right]_{p-1} \equiv (-1)^{(p+1)/2}\Gamma_p\Big(-\frac{a}{2}\Big)^2\Gamma_p\Big(\frac{a+1}{2}\Big)^2 \pmod{p^2}. \tag{1.5}$$

In order to prove Theorem 1.2, we need the following supercongruence which is a special case of a result due to Sun ([13], Theorem 2.2).

Theorem 1.3. *Suppose $p \geq 5$ is a prime. For any $p$-adic integer $a$,*

$$ {}_3F_2\left[\begin{matrix} \frac{1}{2}, & -a, & a+1 \\ & 1, & 1 \end{matrix}; 1\right]_{p-1} \equiv {}_2F_1\left[\begin{matrix} -a, & a+1 \\ & 1 \end{matrix}; \frac{1}{2}\right]_{p-1}^2 \pmod{p^2}. \tag{1.6}$$

Supercongruence (1.6) is a $p$-adic analogue of the identity

$$ {}_3F_2\left[\begin{matrix} \frac{1}{2}, & -a, & a+1 \\ & 1, & 1 \end{matrix}; 1\right] = {}_2F_1\left[\begin{matrix} -a, & a+1 \\ & 1 \end{matrix}; \frac{1}{2}\right]^2, \tag{1.7}$$

which can be deduced from Clausen's formula. We shall give an alternative proof of (1.6) by using some combinatorial identities.

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The rest of this paper is organised as follows. In the next section we recall some properties of the $p$-adic gamma functions and establish some combinatorial identities involving harmonic numbers. We prove Theorem 1.1 in Section 3, and Theorems 1.2 and 1.3 in Section 4. Related supercongruences modulo $p^3$ are proposed as conjectures in the final section.

## 2. Some lemmas

Let $p$ be an odd prime and let $\mathbb{Z}_p$ denote the set of all $p$-adic integers. For $x \in \mathbb{Z}_p$, Morita's $p$-adic gamma function [3, Definition 11.6.5] is defined by

$$\Gamma_p(x) = \lim_{m \to x} (-1)^m \prod_{\substack{0<k<m \\ (k,p)=1}} k,$$

where the limit is for $m$ tending to $x$ $p$-adically in $\mathbb{Z}_{\geq 0}$. We recall some basic properties of the $p$-adic gamma function (see [3, Section 11.6] for more details). For $x \in \mathbb{Z}_p$,

$$\Gamma_p(1) = -1, \tag{2.1}$$

$$\Gamma_p(x)\Gamma_p(1-x) = (-1)^{s_p(x)}, \tag{2.2}$$

$$\frac{\Gamma_p(x+1)}{\Gamma_p(x)} = \begin{cases} -x & \text{if } |x|_p = 1, \\ -1 & \text{if } |x|_p < 1, \end{cases} \tag{2.3}$$

where $s_p(x) \in \{1, 2, \ldots, p\}$ with $s_p(x) \equiv x \pmod{p}$ and $|\cdot|_p$ denotes the $p$-adic norm.

For $a \in \mathbb{Z}_p$, set $G_1(a) = \Gamma_p'(a)/\Gamma_p(a)$. Then $G_1(a) \in \mathbb{Z}_p$ (see [6, Proposition 2.3]).

Lemma 2.1. *Let $p$ be an odd prime. For any $x \in \mathbb{Z}_p$,*

$$G_1(x) \equiv G_1(1) + H_{s_p(x)-1} \pmod{p}, \tag{2.4}$$

*where $H_n$ denotes the nth harmonic number $H_n = \sum_{k=1}^{n} (1/k)$.*

Proof. The $p$-adic logarithm is defined by

$$\log_p(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}.$$

It converges for $x \in \mathbb{C}_p$ with $|x|_p < 1$. Taking the $\log_p$ derivative on both sides of (2.3),

$$G_1(x+1) - G_1(x) = \begin{cases} 1/x & \text{if } |x|_p = 1, \\ 0 & \text{if } |x|_p < 1. \end{cases} \tag{2.5}$$

For any $p$-adic integers $a$ and $b$ with $a \equiv b \pmod{p}$, by [6, (2.2) and (2.3)], we have $\Gamma_p(a) \equiv \Gamma_p(b) \pmod{p}$ and $\Gamma_p'(a) \equiv \Gamma_p'(b) \pmod{p}$, and so $G_1(a) \equiv G_1(b) \pmod{p}$. By repeatedly applying (2.5), we obtain

$$\begin{aligned} G_1(x) &\equiv G_1(s_p(x)) \pmod{p} \\ &= G_1(s_p(x)-1) + \frac{1}{s_p(x)-1} \\ &= G_1(1) + H_{s_p(x)-1} \end{aligned}$$

which is the desired result. □

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We also need some combinatorial identities.

LEMMA 2.2. *For any even integer n,*

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{n+k}{2k}\left(-\frac{1}{2}\right)^k=\frac{\binom{n}{n/2}}{(-4)^{n/2}}, \tag{2.6}$$

$$\sum_{k=0}^{n}\binom{2k}{k}^2\binom{n+k}{2k}\left(-\frac{1}{4}\right)^k=\frac{\binom{n}{n/2}^2}{4^n}, \tag{2.7}$$

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{n+k}{2k}\left(-\frac{1}{2}\right)^k\sum_{i=1}^{k}\frac{1}{n+i}=\frac{\binom{n}{n/2}}{(-4)^{n/2}}\left(\frac{1}{2}H_n-\frac{1}{2}H_{n/2}\right), \tag{2.8}$$

$$\sum_{k=0}^{n}\binom{2k}{k}^2\binom{n+k}{2k}\left(-\frac{1}{4}\right)^k\sum_{i=1}^{k}\frac{1}{n+i}=\frac{\binom{n}{n/2}^2}{4^n}\left(\frac{3}{2}H_n-H_{n/2}\right). \tag{2.9}$$

PROOF. Identities (2.6) and (2.7) are deduced directly from (1.7) and the identity

$${}_2F_1\left[\begin{matrix}-a, & a+1\\ & 1\end{matrix};\frac{1}{2}\right]=\frac{\Gamma(1/2)}{\Gamma((1-a)/2)\Gamma(1+a/2)} \tag{2.10}$$

(see [2, (2), page 11]), by setting $a=n$.

Note that

$$\sum_{i=1}^{k}\frac{1}{n+i}=H_{n+k}-H_n.$$

In order to prove (2.8) and (2.9), by (2.6) and (2.7), it suffices to show that

$$\sum_{k=0}^{2n}\binom{2k}{k}\binom{2n+k}{2k}\left(-\frac{1}{2}\right)^k(2H_{2n+k}-3H_{2n}+H_n)=0, \tag{2.11}$$

$$\sum_{k=0}^{2n}\binom{2k}{k}^2\binom{2n+k}{2k}\left(-\frac{1}{4}\right)^k(2H_{2n+k}-5H_{2n}+2H_n)=0. \tag{2.12}$$

Let $A_n$ and $B_n$ denote the left-hand sides of (2.11) and (2.12), respectively. Using the software package `Sigma` developed by Schneider [10], we find that $A_n$ and $B_n$ satisfy the recurrences

$$(2n+1)A_n+2(n+1)A_{n+1}=0$$

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and

$$4(n+1)^2(2n+1)^2(4n+7)B_n-(4n+5)(32n^4+160n^3+296n^2+240n+71)B_{n+1}$$
$$+4(n+2)^2(2n+3)^2(4n+3)B_{n+2}=0,$$

respectively. It is easy to verify that $A_0=0$ and $B_0=B_1=0$, and so $A_n=B_n=0$ for all $n\geq 0$. □

Remark 2.3. The combinatorial identities (2.8) and (2.9) can also be automatically discovered and proved by Schneider's computer algebra package `Sigma`. We refer to [10, Section 3.1] for an interesting approach to finding and proving combinatorial identities of this type.

## 3. Proof of Theorem 1.1

We can rewrite (1.4) as

$$\sum_{k=0}^{p-1}\binom{2k}{k}\binom{a+k}{2k}\Big(-\frac{1}{2}\Big)^k\equiv(-1)^{(p+1)/2}\Gamma_p\Big(\frac{1}{2}\Big)\Gamma_p\Big(-\frac{a}{2}\Big)\Gamma_p\Big(\frac{a+1}{2}\Big)\pmod{p^2}. \tag{3.1}$$

Let $\delta=(a-\langle a\rangle_p)/p$. It is clear that $\delta$ is a $p$-adic integer and $a=\langle a\rangle_p+\delta p$. Since

$$\prod_{i=1}^{k}(C+x\pm i)=\Big(\prod_{i=1}^{k}(C\pm i)\Big)\Big(1+x\sum_{i=1}^{k}\frac{1}{C\pm i}\Big)+O(x^2),$$

it follows that

$$\begin{aligned}\binom{2k}{k}\binom{a+k}{2k}&=\binom{2k}{k}\binom{\langle a\rangle_p+\delta p+k}{2k}\\&=\binom{2k}{k}\prod_{i=1}^{k}(\langle a\rangle_p+\delta p+i)\prod_{i=1}^{k}(\langle a\rangle_p+\delta p+1-i)\prod_{i=1}^{2k}i^{-1}\\&\equiv\binom{2k}{k}\binom{\langle a\rangle_p+k}{2k}\Big(1+\delta p\Big(\sum_{i=1}^{k}\frac{1}{\langle a\rangle_p+i}+\sum_{i=1}^{k}\frac{1}{\langle a\rangle_p+1-i}\Big)\Big)\pmod{p^2},\end{aligned} \tag{3.2}$$

where we have used the fact that $\binom{2k}{k}\prod_{i=1}^{2k}(1/i)\in\mathbb{Z}_p$ for $0\leq k\leq p-1$. Now

$$\begin{aligned}\text{LHS (3.1)}\equiv&\sum_{k=0}^{p-1}\binom{2k}{k}\binom{\langle a\rangle_p+k}{2k}\Big(-\frac{1}{2}\Big)^k\\&\times\Big(1+\delta p\Big(\sum_{i=1}^{k}\frac{1}{\langle a\rangle_p+i}+\sum_{i=1}^{k}\frac{1}{\langle a\rangle_p+1-i}\Big)\Big)\pmod{p^2}.\end{aligned} \tag{3.3}$$

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Let $b = p - \langle a\rangle_p$. It is clear that $\langle a\rangle_p \equiv -b \pmod p$ and $0 \le b-1 \le p-1$ is an even integer. Note that $\binom{-b+k}{2k} = \binom{b-1+k}{2k}$. Thus,

$$\begin{aligned}\sum_{k=0}^{p-1}&\binom{2k}{k}\binom{\langle a\rangle_p+k}{2k}\Big(-\frac{1}{2}\Big)^k\sum_{i=1}^{k}\frac{1}{\langle a\rangle_p+1-i}\\&\equiv-\sum_{k=0}^{p-1}\binom{2k}{k}\binom{-b+k}{2k}\Big(-\frac{1}{2}\Big)^k\sum_{i=1}^{k}\frac{1}{b-1+i}\pmod p\\&=-\sum_{k=0}^{p-1}\binom{2k}{k}\binom{b-1+k}{2k}\Big(-\frac{1}{2}\Big)^k\sum_{i=1}^{k}\frac{1}{b-1+i}\\&\overset{(2.8)}{=}\frac{\binom{b]-1}{(b-1)/2}}{(-4)^{(b-1)/2}}\Big(\frac{1}{2}H_{(b-1)/2}-\frac{1}{2}H_{b-1}\Big).\end{aligned}\tag{3.4}$$

Since $\binom{2n}{n}(-\frac{1}{4})^n = \binom{-1/2}{n}$ and $b + \langle a\rangle_p = p$,

$$\frac{\binom{b-1}{(b-1)/2}}{(-4)^{(b-1)/2}}=\binom{-\frac{1}{2}}{\frac{b-1}{2}}\equiv\binom{\frac{p-1}{2}}{\frac{b-1}{2}}=\binom{\frac{p-1}{2}}{\frac{\langle a\rangle_p}{2}}\equiv\binom{-\frac{1}{2}}{\frac{\langle a\rangle_p}{2}}=\frac{\binom{\langle a\rangle_p}{\langle a\rangle_p/2}}{(-4)^{\langle a\rangle_p/2}}\pmod p.\tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\begin{aligned}\sum_{k=0}^{p-1}&\binom{2k}{k}\binom{\langle a\rangle_p+k}{2k}\Big(-\frac{1}{2}\Big)^k\sum_{i=1}^{k}\frac{1}{\langle a\rangle_p+1-i}\\&\equiv\frac{\binom{\langle a\rangle_p}{\langle a\rangle_p/2}}{(-4)^{\langle a\rangle_p/2}}\Big(\frac{1}{2}H_{(p-\langle a\rangle_p-1)/2}-\frac{1}{2}H_{p-\langle a\rangle_p-1}\Big)\pmod p.\end{aligned}\tag{3.6}$$

Furthermore,

$$\sum_{k=0}^{p-1}\binom{2k}{k}\binom{\langle a\rangle_p+k}{2k}\Big(-\frac{1}{2}\Big)^k\overset{(2.6)}{=}\frac{\binom{\langle a\rangle_p}{\langle a\rangle_p/2}}{(-4)^{\langle a\rangle_p/2}}\tag{3.7}$$

and

$$\sum_{k=0}^{p-1}\binom{2k}{k}\binom{\langle a\rangle_p+k}{2k}\Big(-\frac{1}{2}\Big)^k\sum_{i=1}^{k}\frac{1}{\langle a\rangle_p+i}\overset{(2.8)}{=}\frac{\binom{\langle a\rangle_p}{\langle a\rangle_p/2}}{(-4)^{\langle a\rangle_p/2}}\Big(\frac{1}{2}H_{\langle a\rangle_p}-\frac{1}{2}H_{\langle a\rangle_p/2}\Big).\tag{3.8}$$

Combining (3.3) and (3.6)–(3.8) gives

$$\text{LHS (3.1)}\equiv\Big(-\frac{1}{4}\Big)^{\langle a\rangle_p/2}\binom{\langle a\rangle_p}{\langle a\rangle_p/2}\Big(1+\frac{\delta p}{2}(H_{(p-\langle a\rangle_p-1)/2}-H_{\langle a\rangle_p/2})\Big)\pmod{p^2},\tag{3.9}$$

where we have used the fact that $H_{p-1-k} \equiv H_k \pmod p$ for $0 \le k \le p-1$.

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Note that

$$\frac{(\frac{1}{2})_k}{(1)_k} = \frac{\binom{2k}{k}}{4^k}, \tag{3.10}$$

and for $a \in \mathbb{Z}_p$, $n \in \mathbb{N}$ such that none of $a, a+1, \ldots, a+n-1$ are in $p\mathbb{Z}_p$ (see [7, Lemma 17, (4)]),

$$(a)_n = (-1)^n \frac{\Gamma_p(a+n)}{\Gamma_p(a)}. \tag{3.11}$$

From (3.10) and (3.11), we deduce that

$$\Big(-\frac{1}{4}\Big)^{\langle a\rangle_p/2} \binom{\langle a\rangle_p}{\langle a\rangle_p/2} \overset{(3.10)}{=} (-1)^{\langle a\rangle_p/2} \frac{(1/2)_{\langle a\rangle_p/2}}{(1)_{\langle a\rangle_p/2}} \overset{(3.11)}{=} (-1)^{\langle a\rangle_p/2} \frac{\Gamma_p(1)\Gamma_p((1+\langle a\rangle_p)/2)}{\Gamma_p(1/2)\Gamma_p(1+\langle a\rangle_p/2)}. \tag{3.12}$$

By (2.2),

$$\Gamma_p\Big(\frac{1}{2}\Big)^2 = (-1)^{(p+1)/2}, \tag{3.13}$$

$$\Gamma_p\Big(1+\frac{\langle a\rangle_p}{2}\Big)\Gamma_p\Big(-\frac{\langle a\rangle_p}{2}\Big) = (-1)^{1+\langle a\rangle_p/2}. \tag{3.14}$$

Applying (2.1), (3.13) and (3.14) to the right-hand side of (3.12) and then using $\langle a\rangle_p = a - \delta p$,

$$\begin{aligned}\Big(-\frac{1}{4}\Big)^{\langle a\rangle_p/2} \binom{\langle a\rangle_p}{\langle a\rangle_p/2} &= (-1)^{(p+1)/2}\Gamma_p\Big(\frac{1}{2}\Big)\Gamma_p\Big(\frac{1+\langle a\rangle_p}{2}\Big)\Gamma_p\Big(-\frac{\langle a\rangle_p}{2}\Big)\\ &= (-1)^{(p+1)/2}\Gamma_p\Big(\frac{1}{2}\Big)\Gamma_p\Big(\frac{1+a-\delta p}{2}\Big)\Gamma_p\Big(\frac{-a+\delta p}{2}\Big).\end{aligned} \tag{3.15}$$

Note that for $a, b \in \mathbb{Z}_p$ (see [7, Theorem 14]),

$$\Gamma_p(a+bp) \equiv \Gamma_p(a)(1+G_1(a)bp) \pmod{p^2}. \tag{3.16}$$

Furthermore, applying (3.16) to the right-hand side of (3.15),

$$\begin{aligned}\Big(-\frac{1}{4}\Big)^{\langle a\rangle_p/2} \binom{\langle a\rangle_p}{\langle a\rangle_p/2} &\equiv (-1)^{(p+1)/2}\Gamma_p\Big(\frac{1}{2}\Big)\Gamma_p\Big(\frac{1+a}{2}\Big)\Gamma_p\Big(-\frac{a}{2}\Big)\\ &\quad\times\Big(1+\frac{\delta p}{2}\Big(G_1\Big(-\frac{a}{2}\Big)-G_1\Big(\frac{1+a}{2}\Big)\Big)\Big) \pmod{p^2}.\end{aligned} \tag{3.17}$$

It follows from (3.9) and (3.17) that

$$\begin{aligned}\text{LHS (3.1)} &\equiv (-1)^{(p+1)/2}\Gamma_p\Big(\frac{1}{2}\Big)\Gamma_p\Big(\frac{1+a}{2}\Big)\Gamma_p\Big(-\frac{a}{2}\Big)\\ &\quad\times\Big(1+\frac{\delta p}{2}\Big(H_{(p-\langle a\rangle_p-1)/2}-H_{\langle a\rangle_p/2}+G_1\Big(-\frac{a}{2}\Big)-G_1\Big(\frac{1+a}{2}\Big)\Big)\Big) \pmod{p^2}.\end{aligned}$$

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In order to prove (3.1), it suffices to show that

$$H_{(p-\langle a\rangle_p-1)/2} - H_{\langle a\rangle_p/2} + G_1\Big(-\frac{a}{2}\Big) - G_1\Big(\frac{1+a}{2}\Big) \equiv 0 \pmod p. \tag{3.18}$$

By (2.4),

$$G_1\Big(-\frac{a}{2}\Big) - G_1\Big(\frac{1+a}{2}\Big) \equiv H_{s_p(-a/2)-1} - H_{s_p((1+a)/2)-1} \pmod p. \tag{3.19}$$

Since $\langle a\rangle_p$ is an even integer,

$$s_p\Big(-\frac{a}{2}\Big) - 1 = p - \frac{\langle a\rangle_p}{2} - 1, \tag{3.20}$$

$$s_p\Big(\frac{1+a}{2}\Big) - 1 = \frac{p+\langle a\rangle_p+1}{2} - 1. \tag{3.21}$$

Substituting (3.19) into the left-hand side of (3.18) and then using (3.20) and (3.21),

$$\text{LHS } (3.18) \equiv H_{(p-\langle a\rangle_p-1)/2} - H_{\langle a\rangle_p/2} + H_{p-(\langle a\rangle_p/2)-1} - H_{(p+\langle a\rangle_p-1)/2} \equiv 0 \pmod p,$$

where we have utilised the fact that $H_{p-k-1} \equiv H_k \pmod p$ for $0 \le k \le p-1$.

## 4. Proofs of Theorems 1.2 and 1.3

The proof of Theorem 1.2 directly follows from (1.4), (1.6) and (3.13). It remains to prove Theorem 1.3. We distinguish two cases to show (1.6).

If $\langle a\rangle_p \equiv 1 \pmod 2$, by (1.2) and (1.3), then (1.6) clearly holds.

If $\langle a\rangle_p \equiv 0 \pmod 2$, by (3.9) and (3.10), it suffices to show that

$$\sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{a+k}{2k} \Big(-\frac{1}{4}\Big)^k \equiv \Big(\frac{1}{4}\Big)^{\langle a\rangle_p} \binom{\langle a\rangle_p}{\langle a\rangle_p/2}^2 (1 + \delta p(H_{(p-\langle a\rangle_p-1)/2} - H_{\langle a\rangle_p/2})) \pmod{p^2}. \tag{4.1}$$

Applying (3.2) to the left-hand side of (4.1) yields

$$\text{LHS } (4.1) \equiv \sum_{k=0}^{p-1} \binom{2k}{k}^2 \binom{\langle a\rangle_p+k}{2k} \Big(-\frac{1}{4}\Big)^k \times \Big(1 + \delta p\Big(\sum_{i=1}^{k} \frac{1}{\langle a\rangle_p+i} + \sum_{i=1}^{k} \frac{1}{\langle a\rangle_p+1-i}\Big)\Big) \pmod{p^2}. \tag{4.2}$$

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Using the same idea as in the previous section and the identities (2.7) and (2.9),

$$\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{\langle a\rangle_p+k}{2k}\Big(-\frac{1}{4}\Big)^k\sum_{i=1}^{k}\frac{1}{\langle a\rangle_p+1-i} \equiv \frac{\binom{\langle a\rangle_p}{\langle a\rangle_p/2}^2}{4^{\langle a\rangle_p}}\Big(H_{(p-\langle a\rangle_p-1)/2}-\frac{3}{2}H_{p-\langle a\rangle_p-1}\Big) \pmod p, \tag{4.3}$$

$$\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{\langle a\rangle_p+k}{2k}\Big(-\frac{1}{4}\Big)^k=\frac{\binom{\langle a\rangle_p}{\langle a\rangle_p/2}^2}{4^{\langle a\rangle_p}}, \tag{4.4}$$

and

$$\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{\langle a\rangle_p+k}{2k}\Big(-\frac{1}{4}\Big)^k\sum_{i=1}^{k}\frac{1}{\langle a\rangle_p+i}=\frac{\binom{\langle a\rangle_p}{\langle a\rangle_p/2}^2}{4^{\langle a\rangle_p}}\Big(\frac{3}{2}H_{\langle a\rangle_p}-H_{\langle a\rangle_p/2}\Big). \tag{4.5}$$

Combining (4.2)–(4.5) gives the desired result that

$$\text{LHS (4.1)} \equiv \Big(\frac{1}{4}\Big)^{\langle a\rangle_p}\binom{\langle a\rangle_p}{\langle a\rangle_p/2}^2(1+\delta p(H_{(p-\langle a\rangle_p-1)/2}-H_{\langle a\rangle_p/2})) \pmod{p^2}.$$

## 5. Some open conjectures

Long and Ramakrishna [7, Theorem 3] have extended the case when $a=-\frac{1}{2}$ in (1.1) to a supercongruence modulo $p^3$. Numerical calculation suggests that the other three cases when $a=-\frac{1}{3},-\frac{1}{4},-\frac{1}{6}$ in (1.1) have similar modulo $p^3$ extensions. These three conjectural supercongruences follow.

Conjecture 5.1. Let $p\geq 5$ be a prime. Then, modulo $p^3$,

$$ {}_3F_2\left[\begin{matrix}\frac{1}{2}, & \frac{1}{3}, & \frac{2}{3}\\ & 1, & 1\end{matrix};1\right]_{p-1} \equiv \begin{cases}(-1)^{(p+1)/2}\Gamma_p\Big(\frac{1}{6}\Big)^2\Gamma_p\Big(\frac{1}{3}\Big)^2 & \text{if } p\equiv 1 \pmod 6,\\ (-1)^{(p-1)/2}\dfrac{p^2}{18}\Gamma_p\Big(\frac{1}{6}\Big)^2\Gamma_p\Big(\frac{1}{3}\Big)^2 & \text{if } p\equiv 5 \pmod 6,\end{cases} \tag{5.1}$$

$$ {}_3F_2\left[\begin{matrix}\frac{1}{2}, & \frac{1}{4}, & \frac{3}{4}\\ & 1, & 1\end{matrix};1\right]_{p-1} \equiv \begin{cases}(-1)^{(p+1)/2}\Gamma_p\Big(\frac{1}{8}\Big)^2\Gamma_p\Big(\frac{3}{8}\Big)^2 & \text{if } p\equiv 1,3 \pmod 8,\\ (-1)^{(p-1)/2}\dfrac{3p^2}{64}\Gamma_p\Big(\frac{1}{8}\Big)^2\Gamma_p\Big(\frac{3}{8}\Big)^2 & \text{if } p\equiv 5,7 \pmod 8,\end{cases} \tag{5.2}$$

$$ {}_3F_2\left[\begin{matrix}\frac{1}{2}, & \frac{1}{6}, & \frac{5}{6}\\ & 1, & 1\end{matrix};1\right]_{p-1} \equiv \begin{cases}-\Gamma_p\Big(\frac{1}{12}\Big)^2\Gamma_p\Big(\frac{5}{12}\Big)^2 & \text{if } p\equiv 1 \pmod 4,\\ -\dfrac{5p^2}{144}\Gamma_p\Big(\frac{1}{12}\Big)^2\Gamma_p\Big(\frac{5}{12}\Big)^2 & \text{if } p\equiv 3 \pmod 4.\end{cases} \tag{5.3}$$

There is strong numerical evidence to suggest that supercongruence (1.5) also holds modulo $p^3$.

CONJECTURE 5.2. Let $p \geq 5$ be a prime. For any $p$-adic integer $a$ with $\langle a \rangle_p \equiv 0 \pmod 2$,

$$ {}_3F_2\left[\begin{matrix} \frac{1}{2}, & -a, & a+1 \\ & 1, & 1 \end{matrix}; 1\right]_{p-1} \equiv (-1)^{(p+1)/2}\Gamma_p\left(-\frac{a}{2}\right)^2\Gamma_p\left(\frac{a+1}{2}\right)^2 \pmod{p^3}. \tag{5.4} $$

It is clear that (5.4) reduces to the first cases of (5.1)–(5.3) when $a = -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{6}$. Unfortunately, the method in this paper is not applicable for proving these conjectures.

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