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COPULA REPRESENTATIONS FOR THE SUM OF DEPENDENT RISKS: MODELS AND COMPARISONS

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The study of the distributions of sums of dependent risks is a key topic in actuarial sciences, risk management, reliability and in many branches of applied and theoretical probability. However, there are few results where the distribution of the sum of dependent random variables is available in a closed form. In this paper, we obtain several analytical expressions for the distribution of the aggregated risks under dependence in terms of copulas. We provide several representations based on the underlying copula and the marginal distribution functions under general hypotheses and in any dimension. Then, we study stochastic comparisons between sums of dependent risks. Finally, we illustrate our theoretical results by studying some specific models obtained from Clayton, Ali-Mikhail-Haq and Farlie-Gumbel-Morgenstern copulas. Extensions to more general copulas are also included. Bounds and the limiting behavior of the hazard rate function for the aggregated distribution of some copulas are studied as well.

Keywords: convolution, c-convolution, distorted distributions, hazard rate, stochastic orders

1. INTRODUCTION

Risk theory has become nowadays a crucial theory in actuarial and financial sciences. The *aggregated risks* are functions representing the total amount of risk for a company or a portfolio. Internationally banks are required to set aside capital to offset various types of risks, such as market, credit and operational risks. The study of the distribution of the sum of random variables is a key topic in this science. The representations for the distribution of a sum of independent random variables (i.e., a convolution) are well known in probability. However, in many applications in actuarial sciences and risk analysis, these random variables are dependent because the risks share the same environment. For example, in Herrmann [27], they are used to determine the optimal Expected-Shortfall in a portfolio selection. The same happen in the reliability theory where the sums of random lifetimes represent standby mechanisms in systems.

This topic has had an increasing interest from the 2000s and several distributions have been studied obtaining models for the sums of dependent and independent random variables.

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Some recent references include: [2,5,9,16,21,25,26,34,36,41,43] , among others. Recently, some results have been obtained for the limiting behavior of the hazard rate functions of these sums in Block *et al.* [7,8] .

In the case of aggregation of risks assuming dependence, we have available some results using different copula structures (see, e.g., [2,15,24,45]). For Farlie-Gumbel-Morgenstern (FGM) copulas and mixed Erlang marginal distributions, Cossette *et al.* [16] have obtained closed expressions for the distribution of the aggregated risk and for capital allocation problems. Following and extending this research, Hashorva and Ratovomirija [26] have considered an extension of the previous model using the Sarmanov distribution to represent the dependence structure, demonstrating that the aggregated distribution belongs to the class of Erlang mixtures. More recently, Vernic [48] uses the Sarmanov's distribution to define the dependence structure, obtaining some formulas assuming marginals with exponential distributions. For the case of a Clayton copula with classical Pareto marginals, Sarabia *et al.* [44] have studied aggregation in multivariate-dependent Pareto distributions, providing analytical formulas in the cases of individual risk models and collective risk models, assuming several usual distributions as primary distributions. As an extension to previous results, Sarabia *et al.* [45] have considered aggregation of risks in the case of Archimedean copulas modeled in terms of mixtures of exponential distributions. For the case of a Pareto copula and log-normal marginals, Bølviken and Guillén [9] also considered the risk aggregation problem, improving the accuracy of the model by updating the skewness recursively. A flexible methodology in terms of tree structures was recently proposed by Côté and Genest [19]. Several copula-based representations for the sum of two-dependent random variables were obtained in Refs. [12]. There some applications to econometrics are provided as well.

On the other hand, bounds for the distribution and the Value-at-Risk (VaR) of such sums were obtained (under different assumptions) in Refs. [23,30,49] and references therein. Bounds for the hazard rate and reversed hazard rate functions of dependent sums were given recently in Belzunce and Martínez-Riquelme [6].

In this paper, we obtain several analytical expressions for the distribution of the sum under dependence by using copulas. We provide several representations in terms of the underlying copula and the marginal distributions under general hypotheses. Then, we study stochastic comparisons between the distributions of aggregations of dependent risks. Specifically, we study the following orderings: the stochastic order, the hazard rate order, the reversed order and the likelihood ratio order. Finally, we study some specific models with copulas of the type Clayton, Ali-Mikhail-Haq and Farlie-Gumbel-Morgenstern. Extensions to more general copulas are also included. We also study stochastic comparisons between different sums and the limiting behavior of the hazard rate functions of the sums for some copulas. In particular, we obtain sharp bounds for the distribution of dependent aggregated risks in terms of independent aggregated risks.

The rest of the paper is organized as follows. In Section 2, we include the representations for the distributions of the sums of two and n dependent random variables. The comparison results are placed in Section 3. Some models with relevant copulas and specific marginals are studied in Section 4 and an application to real data is included in Section 5. The conclusions are given in Section 6.

2. COPULA-BASED REPRESENTATIONS

In this section, we obtain and discuss the distribution of the sum of dependent random variables (risks) in terms of the corresponding copula. We begin with some definitions and preliminary results (next subsection) and with the general case (sums of n dependent random variables). Then, we study in detail the particular case of two dependent risks.

2.1. Some Previous Definitions and Basic Results

Throughout the paper, we say that a function g is *increasing* (resp. *decreasing*) if $g(x) \leq g(y)$ ($\geq$) for all $x \leq y$. The partial derivative of a real-valued function G with respect to its ith variable will be represented by $\partial_i G$ (assuming that it exists). Moreover, $\partial_i \partial_j G$ will be shortly represented as $\partial_{i,j} G$, and so on. Whenever we consider a conditional random variable, we are tacitly assuming that it exists. If its distribution is not unique, then we just consider one of them.

We say that a function F is a *distorted distribution* if $F(t) = q(G(t))$ for all t, where G is a distribution function and $q : [0,1] \to [0,1]$ is a *distortion function* (i.e., q is increasing, continuous and satisfies $q(0) = 0$ and $q(1) = 1$). In this case, a similar representation holds for the respective survival (or reliability) functions $\bar{F}(t) = \bar{q}(\bar{G}(t))$, where $\bar{q}(u) = 1 - q(1 - u)$ is another distortion function, called *dual distortion function*. For further results on distorted distributions, see [32,38] and the references therein.

Let $X_1, \ldots, X_n$ be n random variables with the absolutely continuous joint distribution function (cdf)

$$\boldsymbol{F}(x_1, \ldots, x_n) = \Pr(X_1 \leq x_1, \ldots, X_n \leq x_n)$$

and marginal distributions $F_i(x) = \Pr(X_i \leq x_i)$ for $i = 1, \ldots, n$. It is well known from copula theory (see, e.g., [42] p. 18) that $\boldsymbol{F}$ can be written as

$$\boldsymbol{F}(x_1, \ldots, x_n) = C(F_1(x), \ldots, F_n(x_n)),$$

where C is a *copula* (i.e., a multivariate distribution function with uniform marginals over the interval $(0, 1)$). Then, the joint probability density function (pdf) $\boldsymbol{f}$ of $(X_1, \ldots, X_n)$ can be written as

$$\boldsymbol{f}(x_1, \ldots, x_n) = f_1(x_1) \cdots f_n(x_n) \partial_{1,\ldots,n} C(F_1(x_1), \ldots, F_n(x_n)),$$

where $f_i = F_i'$ is the pdf of X_i for $i = 1, \ldots, n$ and $\partial_{1,\ldots,n} C$ is the pdf of C.

Analogously, the joint survival function can be written as

$$\bar{\mathbf{F}}(x_1, \ldots, x_n) = \Pr(X_1 > x_1, \ldots, X_n > x_n) = \hat{C}(\bar{F}_1(x_1), \ldots, \bar{F}_n(x_n)), \qquad \textbf{(2.1)}$$

where $\bar{F}_i$ is the survival function of X_i for $i = 1, \ldots, n$ and $\hat{C}$ is a copula called the *survival copula*.

Finally, we introduce some distributions that will be used in the following sections. A random variable X is said to have a Pareto (type II) distribution with parameters $a, \sigma > 0$ (shortly written as $X \sim P(a, \sigma)$) if its pdf is given by

$$f_P(t; a, \sigma) = \frac{a}{\sigma(1 + t/\sigma)^{a+1}}, \quad t \geq 0. \qquad \textbf{(2.2)}$$

A random variable X is said to have a Gamma distribution with parameters $b, \lambda > 0$ (shortly written as $X \sim G(b, \lambda)$) if its pdf is

$$f_G(t; b, \lambda) = \frac{t^{b-1}\, e^{-t/\lambda}}{\lambda^b \Gamma(b)}, \quad t \geq 0.$$

The Erlang distribution corresponds to the case $G(k, 1/\beta)$ with $\beta > 0$, $k \in \mathbb{N}$. It is denoted as $X \sim E(k, \beta)$ and its density is given by

$$f_E(t; k, \beta) = \frac{\beta^k t^{k-1}\, e^{-\beta t}}{(k-1)!}, \quad t > 0.$$

A random variable X has a mixed Erlang distribution if its pdf can be written as

$$f_{\mathrm{ME}}(t; \underline{p}, \beta) = \sum_{k=1}^{\infty} p_k f(t; k, \beta), \tag{2.3}$$

where $\underline{p} = \{p_k,\ k = 1, 2, \ldots\}$ are nonnegative weights with $\sum_{k=1}^{\infty} p_k = 1$. This case is denoted as $X \sim \mathrm{ME}(\underline{p}, \beta)$.

2.2. Sum of n Random Variables

First, we obtain the general expression for the distribution of a sum of n possibly dependent random variables.

THEOREM 2.1: *If $X_1, \ldots, X_n$ have an absolutely continuous joint distribution with marginal distribution functions $F_1, \ldots, F_n$ with pdf $f_1, \ldots, f_n$, and copula C, then the distribution function H_n of $S_n = X_1 + \cdots + X_n$ is equal to*

$$\begin{aligned} H_n(t) = {} & \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_1(x_1) \cdots f_{n-1}(x_{n-1}) \\ & \cdot \partial_{1,\ldots,n-1} C(F_1(x_1), \ldots, F_{n-1}(x_{n-1}), F_n(t - x_1 - \cdots - x_{n-1}))\, dx_{n-1} \ldots dx_1, \end{aligned} \tag{2.4}$$

when $\lim_{u \to 0^+} \partial_{1,\ldots,n-1} C(u_1, \ldots, u_{n-1}, u) = 0$ *for all* $u_1, \ldots, u_{n-1} \in (0, 1)$.

PROOF: The pdf of $(X_n \,|\, X_1 = x_1, \ldots, X_{n-1} = x_{n-1})$ is

$$\mathbf{f}_{n|1,\ldots,n-1}(x_n \,|\, x_1, \ldots, x_{n-1}) = \frac{\mathbf{f}(x_1, \ldots, x_n)}{\mathbf{f}_{1,\ldots,n-1}(x_1, \ldots, x_{n-1})},$$

where

$$\mathbf{f}(x_1, \ldots, x_n) = f_1(x_1) \ldots f_n(x_n) \partial_{1,\ldots,n} C(F_1(x_1), \ldots, F_n(x_n))$$

is the pdf of $(X_1, \ldots, X_n)$ and $\mathbf{f}_{1,\ldots,n-1}$ is the pdf of $(X_1, \ldots, X_{n-1})$. The copula of $(X_1, \ldots, X_{n-1})$ is $C(u_1, \ldots, u_{n-1}, 1)$. So, its pdf is

$$\mathbf{f}_{1,\ldots,n-1}(x_1, \ldots, x_{n-1}) = f_1(x_1) \cdots f_{n-1}(x_{n-1}) \partial_{1,\ldots,n-1} C(F_1(x_1), \ldots, F_{n-1}(x_{n-1}), 1) \tag{2.5}$$

and we get

$$\mathbf{f}_{n\,|\,1,\ldots,n-1}(x_n \,|\, x_1, \ldots, x_{n-1}) = f_n(x_n) \frac{\partial_{1,\ldots,n} C(F_1(x_1), \ldots, F_n(x_n))}{\partial_{1,\ldots,n-1} C(F_1(x_1), \ldots, F_{n-1}(x_{n-1}), 1)}.$$

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Hence, the distribution function of $(X_n \mid X_1 = x_1, \ldots, X_{n-1} = x_{n-1})$ is

$$\begin{aligned}\mathbf{F}_{n|1,\ldots,n-1}(x_n \mid x_1, \ldots, x_{n-1}) &= \int_{-\infty}^{x_n} f_n(t) \frac{\partial_{1,\ldots,n} C(F_1(x_1), \ldots, F_{n-1}(x_{n-1}), F_n(t))}{\partial_{1,\ldots,n-1} C(F_1(x_1), \ldots, F_{n-1}(x_{n-1}), 1)}\, dt \\ &= \left[\frac{\partial_{1,\ldots,n-1} C(F_1(x_1), \ldots, F_{n-1}(x_{n-1}), F_n(t))}{\partial_{1,\ldots,n-1} C(F_1(x_1), \ldots, F_{n-1}(x_{n-1}), 1)}\right]_{t=-\infty}^{x_n} \\ &= \frac{\partial_{1,\ldots,n-1} C(F_1(x_1), \ldots, F_{n-1}(x_{n-1}), F_n(x_n))}{\partial_{1,\ldots,n-1} C(F_1(x_1), \ldots, F_{n-1}(x_{n-1}), 1)} \qquad \mathbf{(2.6)}\end{aligned}$$

when $\lim_{u\to 0^+} \partial_{1,\ldots,n-1} C(F_1(x_1), \ldots, F_{n-1}(x_{n-1}), u) = 0$.

The distribution function $H_n(t) = \Pr(S_n \le t)$ of $S_n = X_1 + \cdots + X_n$ is equal to

$$H_n(t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathbf{F}_{n|1,\ldots,n-1}(t - x_1 - \cdots - x_{n-1} \mid x_1, \ldots, x_{n-1}) \mathbf{f}_{1,\ldots,n-1}(x_1, \ldots, x_{n-1})\, dx_{n-1} \ldots dx_1.$$

Therefore, by using (2.5) and (2.6) in the above expression for H_n, we obtain (2.4). ■

A similar representation can be stated from the survival copula (see the following subsections). The exponential distribution plays a central role in reliability theory and survival studies (representing units without aging). If one marginal distribution is an exponential distribution, then we have the following explicit representation for H_n as a distorted distribution.

THEOREM 2.2: *If $X_1, \ldots, X_n$ are random variables with copula C and marginal distribution functions $F_1, \ldots, F_n$, where F_n is an exponential distribution with mean μ, then the distribution function H_n of $X_1 + \cdots + X_n$ can be written as $H_n(t) = q_n(F_n(t))$ for all t, where*

$$\begin{aligned}q_n(u) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{-\mu \ln(1-u) - x_1 - \cdots - x_{n-2}} & f_1(x_1) \cdots f_{n-1}(x_{n-1}) \\ & \cdot \partial_{1,\ldots,n-1} C(F_1(x_1), \ldots, F_{n-1}(x_{n-1}), 1 - (1-u)\, e^{x_1/\mu} \ldots e^{x_{n-1}/\mu}) \\ & dx_{n-1}\, dx_{n-2} \ldots dx_1. \qquad \mathbf{(2.7)}\end{aligned}$$

PROOF: The exponential distribution with mean μ is given by $F_n(x) = 1 - \exp(-x/\mu)$ for $x \ge 0$ (0 otherwise). Hence,

$$F_n(t - x_1 - \cdots - x_{n-1}) = 1 - e^{-t/\mu}\, e^{x_1/\mu} \cdots e^{x_{n-1}/\mu}$$

for $t - x_1 - \cdots - x_{n-1} \ge 0$, that is, for $x_{n-1} \le t - x_1 - \cdots - x_{n-2}$.

Moreover, if $u := F_n(t) = 1 - \exp(-t/\mu)$, then $t = -\mu \ln(1-u)$. So,

$$F_n(t - x_1 - \cdots - x_{n-1}) = 1 - (1-u)\, e^{x_1/\mu} \cdots e^{x_{n-1}/\mu}$$

for $x_{n-1} \le -\mu \ln(1-u) - x_1 - \cdots - x_{n-2}$. Finally, if we use this expression in (2.4), we get (2.7). ■

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If $X_1,\dots,X_{n-1}$ are nonnegative random variables, then (2.7) can be written as

$$q_n(u)=\int_0^{-\mu\ln(1-u)}\int_0^{-\mu\ln(1-u)-x_1}\cdots\int_0^{-\mu\ln(1-u)-x_1-\cdots-x_{n-2}} f_1(x_1)\cdots f_{n-1}(x_{n-1}) \cdot\partial_{1,\dots,n-1}C(F_1(x_1),\dots,F_{n-1}(x_{n-1}),1-(1-u)\,e^{x_1/\mu}\cdots e^{x_{n-1}/\mu})\, dx_{n-1}\dots dx_2\,dx_1. \quad \mathbf{(2.8)}$$

The explicit expression of q_n depends on C and on the other marginals $F_1,\dots,F_{n-1}$. Some examples are given in Section 4.

2.3. Sums of Two Random Variables

In this subsection, we consider $S=X+Y$, where X and Y have distribution functions F and G, pdf f and g and a bivariate copula C. Note that the acronym pdf is used both for singular and plural cases. The main theorem can be stated as follows (it is a consequence of the result for the general case).

Theorem 2.3: *The distribution function H of $S=X+Y$ can be written as*

$$H(t)=\int_{-\infty}^{\infty} f(x)\partial_1 C(F(x),G(t-x))\,dx, \quad \mathbf{(2.9)}$$

provided that $\lim_{v\to 0^+}\partial_1 C(u,v)=0$ *for all* $u\in(0,1)$.

If F^{-1} is the inverse function of F, then (2.9) can be written as

$$H(t)=\int_0^1 \partial_1 C(u,G(t-F^{-1}(u)))\,du, \quad \mathbf{(2.10)}$$

which is expression (3) in Cherubini *et al.* [12] (see also (4.2) in [14]). However, in some models, F^{-1} does not have an explicit expression or it is quite complicated. In these cases, it is better to use (2.9) instead of (2.10). In that paper, this distribution is called the *C-convolution* and it is represented as $H=F\overset{C}{*}G$. We can do the same in (2.9) with the following definition for the C-convolution

$$F\overset{C}{*}G(t):=\int_{-\infty}^{\infty}\partial_1 C(F(x),G(t-x))\,dF(x).$$

Note that the distribution function H in (2.9) is a mixture of the distorted distribution functions $H_x(t):=\partial_1 C(F(x),G(t-x))=q_x(G(t-x))$, where $q_x(u):=\partial_1 C(F(x),u)$, with mixing pdf $f(x)$. Both the distortion function q_x and the baseline distribution $G(t-x)$ depend on x. Here, H_x is the distribution function of $(Y+x\,|\,X=x)$ (we will use this representation later to obtain comparison results). Hence, the pdf of $(Y+x\,|\,X=x)$ is $h_x(t)=H_x'(t)=g(t-x)\partial_{1,2}C(F(x),G(t-x))$. Therefore, the pdf h of S is

$$h(t)=f\overset{C}{*}g(t)=\int_{-\infty}^{\infty} f(x)g(t-x)\partial_{1,2}C(F(x),G(t-x))\,dx. \quad \mathbf{(2.11)}$$

Now, we consider some particular cases of interest.

REMARK 2.4: *Of course, if X and Y are independent, then* $C(u,v)=uv$, $\partial_1 C(u,v)=v$ *and*

$$H(t)=F*G(t)=\int_{-\infty}^{\infty} f(x)G(t-x)\,dx,$$

which is the very well-known formula for the convolution of F and G.

REMARK 2.5: *If X and Y have a copula C and are nonnegative (a usual assumption in many economic and actuarial random variables), then (2.9) can be written as*

$$H(t)=F\overset{C}{*}G(t)=\int_0^t f(x)\partial_1 C(F(x),G(t-x))\,dx$$

and its pdf as

$$h(t)=H'(t)=\int_0^t f(x)g(t-x)\partial_{1,2}C(F(x),G(t-x))\,dx. \quad \textbf{(2.12)}$$

REMARK 2.6: *Similar expressions can be obtained from the survival copula (see Corollary 2.1 in [13]). Thus, by using representation (2.1), the survival function* $\bar{H}(t):=\Pr(S>t)$ *of S can be expressed as*

$$\bar{H}(t)=\int_{-\infty}^{\infty} f(x)\partial_1\hat{C}(\bar{F}(x),\bar{G}(t-x))\,dx. \quad \textbf{(2.13)}$$

In particular, for nonnegative random variables, we have

$$\bar{H}(t)=\bar{F}(t)+\int_0^t f(x)\partial_1\hat{C}(\bar{F}(x),\bar{G}(t-x))\,dx.$$

In the case of exponential marginals, we can obtain an explicit expression. Thus, if $F(t)=G(t)=1-e^{-t}$ *for* $t\geq 0$, *then*

$$\bar{H}(t)=\bar{F}(t)+\int_0^t f(x)\partial_1\hat{C}(\bar{F}(x),\bar{F}(t)/\bar{F}(x))\,dx=\bar{F}(t)+\int_{\bar{F}(t)}^1 \partial_1\hat{C}(v,\bar{F}(t)/v)\,dv=\bar{q}(\bar{F}(t)),$$

where $\bar{q}(u)=u+\int_u^1\partial_1\hat{C}(v,u/v)\,dv$ *for* $u\in[0,1]$, *that is, S has a distorted distribution from F.*

3. COMPARISON RESULTS

We will study the following orderings: the stochastic order ($\leq_{\rm st}$), the hazard rate order ($\leq_{\rm hr}$), the reversed hazard rate ($\leq_{\rm rh}$) and the likelihood ratio order ($\leq_{\rm lr}$). For their definitions, basic properties and applications we refer the reader to [4,47] . It is well known that

$$X\leq_{\rm st}Y\Leftarrow X\leq_{\rm rh}Y\Leftarrow X\leq_{\rm lr}Y\Rightarrow X\leq_{\rm hr}Y\Rightarrow X\leq_{\rm st}Y.$$

For the sums of n dependent random variables with a common copula, we have the following result (see [47] Thm. 6.B.14). Its proof can also be obtained from (2.4).

THEOREM 3.1: *If* $(X_1,\dots,X_n)$ *and* $(X_1^*,\dots,X_n^*)$ *have a common copula C and* $X_i\leq_{st}X_i^*$ *for* $i=1,\dots,n$, *then* $X_1+\cdots+X_n\leq_{st}X_1^*+\cdots+X_n^*$.

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This result can be extended to get $\phi_1(X_1) + \cdots + \phi_n(X_n) \leq_{\text{st}} \phi_1(X_1^*) + \cdots + \phi_n(X_n^*)$ for all strictly increasing real-valued functions $\phi_1, \ldots, \phi_n$ since $(\phi_1(X_1), \ldots, \phi_n(X_n))$ and $(\phi_1(X_1^*), \ldots, \phi_n(X_n^*))$ also have the same copula C.

Remark 3.2: *If we assume common marginals, that is, $X_i =_{st} X_i^*$ for $i = 1, \ldots, n$ and different copulas satisfying $\partial_{1,\ldots,n-1}C \geq \partial_{1,\ldots,n-1}C^*$, from (2.4), we obtain $X_1 + \cdots + X_n \leq_{st} X_1^* + \cdots + X_n^*$. However, as $E(X_1 + \cdots + X_n) = E(X_1^* + \cdots + X_n^*)$, then $X_1 + \cdots + X_n =_{st} X_1^* + \cdots + X_n^*$ from Theorem 1.A.8 in [47]. Hence, this condition implies $C = C^*$ and so it is not useful, that is, the sums of random variables with the same marginal distributions and different copulas cannot be strictly st-ordered. Results for the weaker stop-loss (icx) order were obtained in Baüerle and Müller [5].*

For sums of n dependent random variables with different copulas and different distributions, we can state the following theorem which is the main result of this section.

Theorem 3.3: *If $(X_1, \ldots, X_n)$ and $(X_1^*, \ldots, X_n^*)$ have the copulas C and C^*, respectively, $X_i \leq_{st} X_i^*$ for $i = 1, \ldots, n-1$, and the distribution functions F_n and F_n^* of X_n and X_n^* satisfy*

$$\partial_{1,\ldots,n-1}C(u_1, \ldots, u_{n-1}, F_n(x)) \geq \partial_{1,\ldots,n-1}C^*(u_1, \ldots, u_{n-1}, F_n^*(x)) \tag{3.1}$$

for all $u_1, \ldots, u_{n-1} \in [0,1]$ and all x, then $X_1 + \cdots + X_n \leq_{st} X_1^* + \cdots + X_n^*$.

Proof: Let us consider a random vector $(\tilde{X}_1, \ldots, \tilde{X}_n)$ with copula C and marginals $F_1^*, \ldots, F_{n-1}^*$ and F_n. Then, from Theorem 3.1, we have $X_1 + \cdots + X_n \leq_{\text{st}} \tilde{X}_1 + \cdots + \tilde{X}_n$. Now, by applying (3.1), we have

$$f_n^*(x)\partial_{1,\ldots,n-1}C(u_1, \ldots, u_{n-1}, F_n(x)) \geq f_n^*(x)\partial_{1,\ldots,n-1}C^*(u_1, \ldots, u_{n-1}, F_n^*(x))$$

for all t, x. Hence, from (2.4), the distribution function $\tilde{H}$ of $\tilde{X}_1 + \cdots + \tilde{X}_n$ satisfies $\tilde{H}(t) \geq H^*(t)$ for all t, that is, $\tilde{X}_1 + \cdots + \tilde{X}_n \leq_{\text{st}} X_1^* + \cdots + X_n^*$. Therefore,

$$X_1 + \cdots + X_n \leq_{\text{st}} \tilde{X}_1 + \cdots + \tilde{X}_n \leq_{\text{st}} X_1^* + \cdots + X_n^*.$$

■

Remark 3.4: *Note that if C^* is the product copula, then $\partial_{1,\ldots,n-1}C^*(u_1, \ldots, u_{n-1}, F_n^*(x)) = F_n^*(x)$. Hence, this theorem can be used to obtain upper sharp bounds (in the st order) for the distribution of dependent aggregated risks in terms of independent aggregated risks. To this goal, we need to determine*

$$G_0(x) := \inf_{u_1,\ldots,u_{n-1}\in[0,1]} \partial_{1,\ldots,n-1}C(u_1, \ldots, u_{n-1}, F_n(x)) \quad \textit{(resp. sup)}$$

and to assume $F_n^* \leq G_0$. If G_0 is a distribution function and the above infimum value is obtained at $u_1^*, \ldots, u_{n-1}^* \in [0,1]$, then the bound is attained in the limit with $F_n^* \leq G_0$ and $F_i^* = F_i \to u_i^*$ for $i = 1, \ldots, n-1$. In a similar way, the preceding theorem can also be used to obtain lower sharp bounds by choosing C as the product copula. Some examples are given in Section 4.

The result for the bivariate case is given as follows. If $X \leq_{\rm st} X^*$ and the distribution functions G and G^* of Y and Y^* satisfy

$$\partial_1 C(u, G(x)) \geq \partial_1 C^*(u, G^*(x)) \quad \text{for all } u \in [0,1] \text{ and all } x, \tag{3.2}$$

then $X + Y \leq_{\rm st} X^* + Y^*$. In the following section, we show how to apply this result to Clayton copulas. These conditions can also be written in terms of the survival copula from (2.13). The result can be stated as follows.

Corollary 3.5: *If X and Y have the survival copula $\hat{C}$, X^* and Y^*, the survival copula $\hat{C}^*$, $X \leq_{st} X^*$ and the survival functions $\bar{G}$ and $\bar{G}^*$ of Y and Y^* satisfy*

$$\partial_1 \hat{C}(u, \bar{G}(x)) \geq \partial_1 \hat{C}^*(u, \bar{G}^*(x)) \quad \textit{for all } u \in [0,1] \textit{ and all } x, \tag{3.3}$$

then $X + Y \geq_{st} X^* + Y^*$.

As above, the preceding result can also be used to get sharp bounds (based on $\hat{C}$) for the C-convolution in terms of the usual convolution. Some examples are provided in Section 4..

If one variable has an exponential distribution, then we have the following results as direct consequences of Theorem 2.2 and the results for distorted distributions obtained in [39,40]. Note that here we can compare sums with different numbers of addends.

Proposition 3.6: *Let $X_1, \ldots, X_n$ and $X_1^*, \ldots, X_m^*$ be random variables with copulas C and C^* and marginal distributions $F_1, \ldots, F_n$ and $F_1^*, \ldots, F_m^*$, respectively, where F_n and F_m^* are exponential distributions with mean μ. Let q_n and q_m^* be the respective distortion functions obtained from (2.7) and let $\bar{q}_n$ and $\bar{q}_m^*$ be the respective dual distortion functions. Then, the following properties hold:*

(i) $X_1 + \cdots + X_n \leq_{st} X_1^* + \cdots + X_m^*$ iff $q_n \geq q_m^*$.
(ii) $X_1 + \cdots + X_n \leq_{hr} X_1^* + \cdots + X_m^*$ iff $\bar{q}_m^*/\bar{q}_n$ is decreasing.
(iii) $X_1 + \cdots + X_n \leq_{rh} X_1^* + \cdots + X_m^*$ iff q_m^*/q_n is increasing.
(iv) $X_1 + \cdots + X_n \leq_{lr} X_1^* + \cdots + X_m^*$ iff $(\bar{q}_m^*)'/\bar{q}_n'$ is decreasing.

Moreover, we can obtain preservation results for the IFR (Increasing Failure Rate), DFR (Decreasing Failure Rate) and DRFR (Decreasing Reversed Failure Rate) aging classes. Their definitions and main properties can be seen in, e.g., [47].

Proposition 3.7: *Let $X_1, \ldots, X_n$ be random variables with copula C and marginal distributions $F_1, \ldots, F_n$, where F_n is an exponential distribution with mean μ. Let q_n be the distortion function obtained from (2.7) and let $\bar{q}_n$ be the respective dual distortion function. Let $\alpha(u) = u\bar{q}_n'(u)/\bar{q}_n(u)$ and $\beta(u) = uq_n'(u)/q_n(u)$. Then, the following properties hold:*

(i) $X_1 + \cdots + X_n$ is IFR iff α is decreasing.
(ii) $X_1 + \cdots + X_n$ is DFR iff α is increasing.
(iii) $X_1 + \cdots + X_n$ is DRFR iff β is decreasing.

The two preceding propositions are applied to a Clayton copula in the following section.

4. MODELS

The aim of this section is to show how to use the theoretical results obtained above to study aggregations of models defined by different copulas. In particular, we consider the copulas of Clayton, Ali-Mikhail-Haq and Farlie-Gumbel-Morgenstern. Some extensions to polynomials copulas are also discussed. In some models, the limiting behavior of the hazard rate function of the sum is studied as well. Other copula models can be studied in a similar way.

4.1. Clayton Copulas

The Clayton copula is defined as

$$C(u_1,\ldots,u_n)=\left(\sum_{i=1}^{n}u_i^{-1/\alpha}-n+1\right)^{-\alpha},\quad u_1,\ldots,u_n\in[0,1],\tag{4.1}$$

where $\alpha>0$. The dependence increases with the parameter α and the independence case is obtained when $\alpha\to 0$. The Fréchet upper bound copula is attained when $\alpha\to\infty$. This copula is specially relevant in risk analysis and actuarial sciences since it is the copula associated with the multivariate Pareto of type II (see [3]). If we consider the distribution of S_n, with copula (4.1) and identically distributed marginals of Pareto type II, it can be proved (see [44]) that the distribution of the aggregated risk is a second kind beta distribution. As well, Sarabia *et al.* [44] obtained closed expressions for the cdf and pdf of the aggregated distribution in the collective model, assuming dependence among risks (see also [25]).

First, we show how to calculate the distribution of a sum of two-dependent random variables with a fixed Clayton copula and exponential marginal distributions. This model shows that the sums of ordered random variables with different copulas (dependence structures) are not necessarily ordered.

Example 4.1: *Let us consider the following Clayton copula*

$$C(u,v)=\frac{uv}{u+v-uv},\quad u,v\in[0,1]$$

($n=2$ and $\alpha=1$) and exponential marginals with mean 1, that is, $F(t)=G(t)=1-e^{-t}$ for $t\geq 0$. Then,

$$\partial_1 C(u,v)=\frac{v^2}{(u+v-uv)^2}$$

and, from (2.9), the distribution function H of $S=X+Y$ is

$$H(t)=\int_0^t e^{-x}\partial_1 C(1-e^{-x},1-e^{x-t})\,dx=\int_0^t e^{-x}\frac{(1-e^{x-t})^2}{(1-e^{-t})^2}\,dx=\frac{1-2te^{-t}-e^{-2t}}{1-2e^{-t}+e^{-2t}}$$

for $t\geq 0$. If X^* and Y^* are independent random variables (i.e., $C^*(u,v)=uv$ for $u,v\in[0,1]$) with exponential distributions with mean 1, then the distribution function H^* of $S^*=X^*+Y^*$ is

$$H^*(t)=\int_0^t e^{-x}\partial_1 C^*(1-e^{-x},1-e^{x-t})\,dx=\int_0^t e^{-x}(1-e^{x-t})\,dx=1-e^{-t}-te^{-t}$$

for $t\geq 0$ (a well-known result). By plotting the distribution functions H and H^* (see Figure 1, left), we see that they are not ordered and so $X+Y$ and X^*+Y^* are not st-ordered

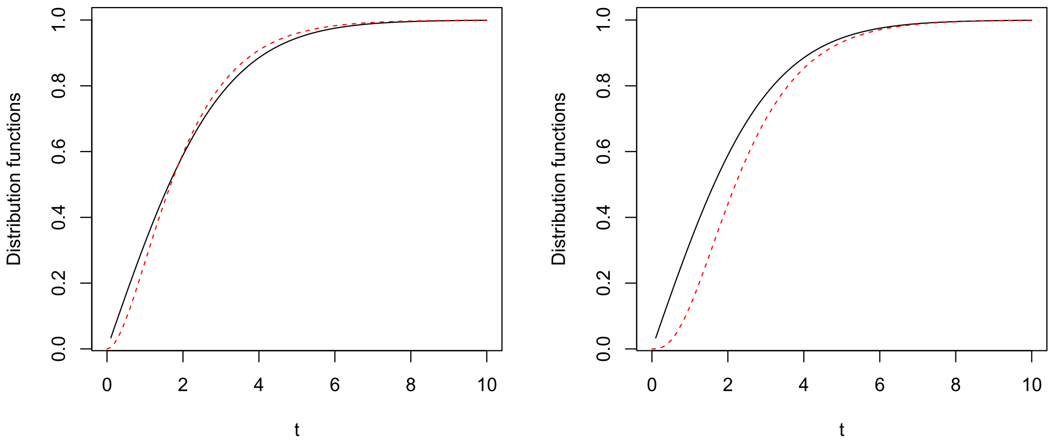

FIGURE **1.** Distribution functions for the sum of two standard exponential distributions (left) with a Clayton copula (continuous black line) and a product copula (dashed red line); see Example 4.1. Note that they are not ordered. However, they are ordered (right) if, in the product copula, we use $F^* = F$ and the distribution function G_0 given in (4.2). The bound is sharp when $F^* = F \to 1$.

(as expected from Remark 3.2 since they have the same marginals). Note that $\partial_1 C(u,v)$ *and* $\partial_1 C^*(u,v) = v$ *are not ordered since* $\partial_1 C(1,v) = v^2 < v$ *and* $\partial_1 C(0,v) = 1 > v$ *for* $v \in (0,1)$.

However, if we assume different distributions, from Theorem 3.3, $S \leq_{st} S^*$ *holds if* $X \leq_{st} X^*$ *and the distribution functions* G *and* G^* *of* Y *and* Y^**, respectively, satisfy*

$$\partial_1 C(u, G(x)) = \frac{G^2(x)}{(u + G(x) - uG(x))^2} \geq G^*(x) = \partial_1 C^*(u, G^*(x))$$

for all $u \in [0,1]$ *and all* x*. As the denominator is increasing in* u*, this condition is equivalent to*

$$\inf_{u\in[0,1]} \frac{G^2(x)}{(u + G(x) - uG(x))^2} = \partial_1 C(1, G(x)) = G^2(x) \geq G^*(x),$$

that is, $S \leq_{st} S^*$ *holds whenever* $F^* \leq F$ *and* $G^* \leq G_0$ *where* $G_0 := G^2$*. For example, if* $G(t) = 1 - e^{-t}$ *for* $t \geq 0$*, then it holds when*

$$G^*(t) \leq G_0(t) = (1 - e^{-t})^2 \qquad \textbf{(4.2)}$$

for $x \geq 0$ *(see Figure 1, right). The bound is attained when* $G^* = G_0$ *and* $F = F^* \to 1$.

If $F(t) = G(t) = 1 - e^{-t}$ *for* $t \geq 0$*, then, from Theorem 2.2,* $H(t) = q(F(t))$ *with*

$$q(u) = \frac{2\ln(1-u) - 2u\ln(1-u) + 2u - u^2}{u^2}$$

for $u \in [0,1]$*. Analogously, the survival functions satisfy* $\bar{H}(t) = \bar{q}(\bar{F}(t))$ *for all* t*, where*

$$\bar{q}(u) = 1 - q(1-u) = \frac{2u^2 - 2u - 2u\ln(u)}{1 - 2u + u^2}.$$

Hence, from Proposition 3.7, S *is IFR (DFR) iff the following function*

$$\alpha(u) = \frac{-2 + 4u - 2u^2 - \ln(u) + u^2\ln(u)}{(u - 1 - \ln(u))(1 - 2u + u^2)}$$

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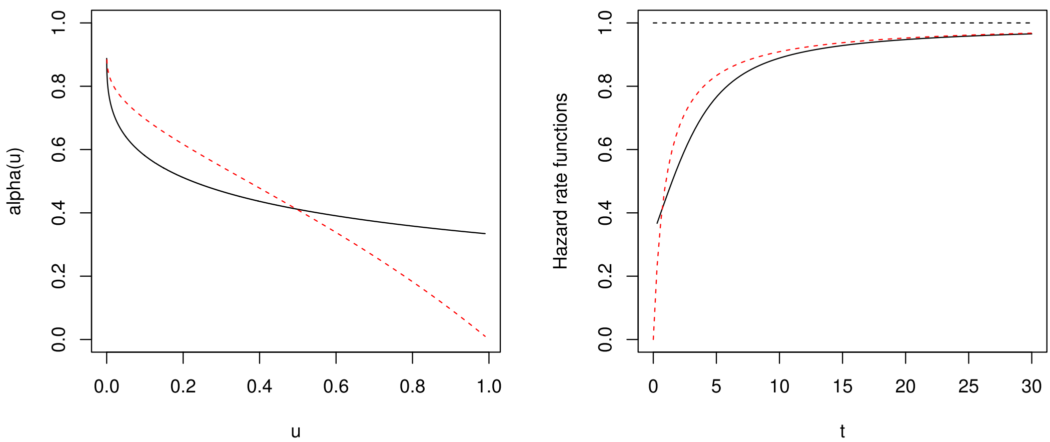

FIGURE **2.** Alpha (left) and hazard rate (right) functions for the sum of two standard exponential distributions with a Clayton copula (continuous black line) and a product copula (dashed red line); see Example 4.1. Note that the hazard rate functions are not ordered and that both are increasing to 1.

is decreasing (increasing). As α is strictly decreasing (see Figure 2, continuous black line, left), then S is IFR (and not DFR). Even more, from Proposition 4.1 in Burkschat and Navarro [10], the hazard rate r_S of S satisfies

$$\lim_{t\to+\infty} r_S(t) = \lim_{u\to 0^+} \alpha(u) = 1.$$

The plot of r_S can be seen in Figure 2 (continuous black line, right). From this plot, we see that $X \leq_{hr} X+Y$. This result can also be obtained from Proposition 3.6(ii), since $\bar{q}(u)/u$ is decreasing in $[0,1]$ (see Figure 3, continuous black line, left). Even more, as $\bar{q}'(u)$ is decreasing in $[0,1]$ (see Figure 3, continuous black line, right), from Proposition 3.6(iv), we get $X \leq_{lr} X+Y$.

A similar reasoning can be used for S^*, obtaining $\bar{q}^*(u) = u - u\ln(u)$ and

$$\alpha^*(u) = \frac{-\ln(u)}{1-\ln(u)}.$$

As α^* is strictly decreasing in $[0,1]$ (see Figure 2, dashed red line, left), S^* is IFR (and not DFR). It can also be seen that as $\lim_{u\to0^+}\alpha^*(u) = 1$, then the hazard rate r_{S^*} of S^* satisfies $\lim_{t\to+\infty} r_{S^*}(t) = 1$ as expected from Theorem 1 in Block et al. [8]. The plot can be seen in Figure 2 (dashed red line, right). Note that the hazard rate functions of S and S^* are not ordered. However, they are ordered with the hazard rate of X (the constant line at 1) since $\bar{q}^*(u)/u$ is decreasing (see Figure 3, dashed red line, left). Even more, as $(\bar{q}^*)'(u) = -\ln(u)$ is decreasing in $[0,1]$ (see Figure 3, dashed red line, right), from Proposition 3.6(iv), we get $X \leq_{lr} X^* + Y^*$.

In a similar way, for a general Clayton copula, we can obtain the following result from Theorem 3.3.

PROPOSITION 4.2: *If $X_1,\ldots,X_n$ have distribution functions $F_1,\ldots,F_n$ and the Clayton copula defined in (4.1) and $X_1^*,\ldots,X_n^*$ are independent and satisfy $X_i \leq_{st} X_i^*$ for $i = 1,\ldots,n-1$ and $Y_0 \leq_{st} X_n^*$ where the distribution function of Y_0 is $G_0(x) = F_n^{1+(n-1)/\alpha}(x)$*

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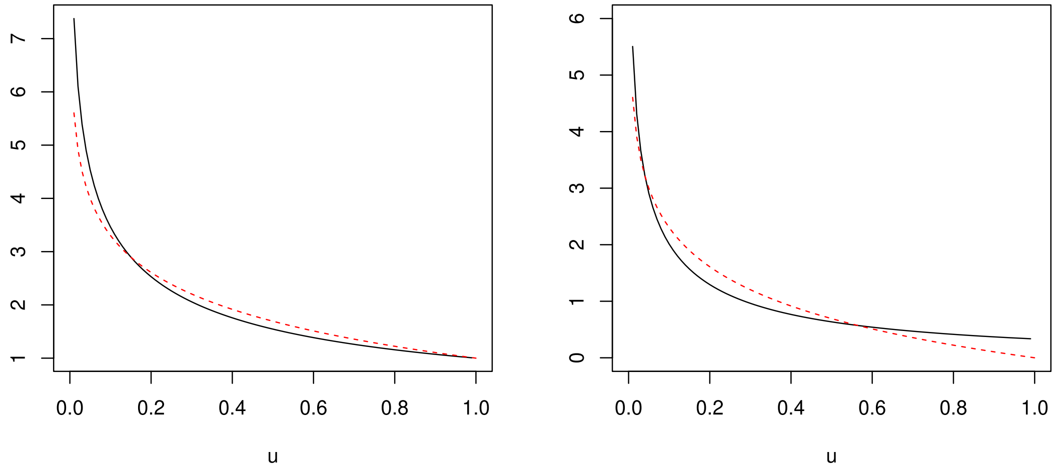

FIGURE **3.** Plots of functions $\bar{q}(u)/u$ (left) and $\bar{q}'(u)$ (right) for the sums of two standard exponential distributions with a Clayton copula (continuous black line) and a product copula (dashed red line); see Example 4.1. Note that all of them are decreasing.

for all x, then $X_1+\cdots+X_n \le_{st} X_1^*+\cdots+X_n^*$. The bound is attained when $F_n^*=G_0$ and $F_i^*=F_i \to 1$ for $i=1,\ldots,n-1$.

4.2. Ali-Mikhail-Haq Copula

Let us consider now the Ali-Mikhail-Haq (AMH) copula defined by,

$$C(u,v)=\frac{uv}{1-\theta(1-u)(1-v)}, \quad u,v\in[0,1] \tag{4.3}$$

for $\theta\in(-1,1)$. Since

$$\partial_1 C(u,v)=\frac{(1-\theta)v+\theta v^2}{[1-\theta(1-u)(1-v)]^2},$$

the distribution function of the sum is

$$H(t)=\int_{-\infty}^{\infty} f(x)\frac{G(t-x)[1-\theta\bar{G}(t-x)]}{[1-\theta\bar{F}(x)\bar{G}(t-x)]^2}\,dx. \tag{4.4}$$

Then, we consider two relevant models:

- AMH copula with standard exponential distributions:

$$H(t;\theta)=\frac{1-e^{-t}+\theta\, e^{-t}-\theta\, e^{-2t}-(\theta+1)t\, e^{-t}}{(1-\theta\, e^{-t})^2}, \quad t\ge 0. \tag{4.5}$$

  If we set $\theta=0$ in (4.5), we obtain the usual exponential convolution.
- AMH copula with Pareto marginals $P(1,1)$ with pdf (2.2),

$$H(t;\theta)=\frac{-t(2+t)}{4\theta-(2+t)^2}-\frac{4[2+\theta(t-2)+t]}{[4\theta-(2+t)^2]^{3/2}}\arctan\left(\frac{t}{\sqrt{4\theta-(2+t)^2}}\right), \quad t\ge 0.$$

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For general distributions, we can obtain the following result.

PROPOSITION 4.3: *If X and Y have distribution functions F and G and the AMH copula (4.3), then $X + Y \leq_{st} X^* + Y^*$ when X^* and Y^* are independent and satisfy $X \leq_{st} X^*$ and $Y_0 \leq_{st} Y^*$, where the distribution function of Y_0 is $G_0(x) = (1-\theta)G(x) + \theta G^2(x)$ when $\theta \geq 0$ and $G_0(x) = G(x)/(1-\theta+\theta G(x))$ when $\theta < 0$, for all x. The bounds are sharp.*

PROOF: The first partial derivative $\partial_1 C$ of C obtained above is decreasing in u when $\theta > 0$ and increasing in u when $\theta < 0$. Hence,

$$\inf_{u\in[0,1]} \partial_1 C(u,v) = \partial_1 C(1,v) = (1-\theta)v + \theta v^2$$

when $\theta \geq 0$ and

$$\inf_{u\in[0,1]} \partial_1 C(u,v) = \partial_1 C(0,v) = \frac{v}{1-\theta+\theta v}$$

when $\theta < 0$. Then, the result holds from Theorem 3.3. ■

4.3. Farlie-Gumbel-Morgenstern Copula

In this section, we obtain the distribution of the sum of two random variables under the Farlie-Gumbel-Morgenstern copula with arbitrary marginals. For the case of mixed Erlang marginals, we have the work by Cossette *et al.* [16] and the other references quoted in the Introduction.

We will obtain a new simple formula in terms of the pdf of convolutions of order statistics. This formulation permits to get explicit expressions for the pdf of the sum in the case of working with two very "different" risks with Pareto and Erlang distributions. The bivariate FGM copula is defined as

$$C(u,v) = uv[1+\alpha(1-u)(1-v)], \quad u,v \in [0,1] \tag{4.6}$$

for $-1 \leq \alpha \leq 1$. The independent case is represented by $\alpha = 0$.

We consider two nonnegative and absolutely continuous random variables X and Y with pdf f and g, respectively. If X_1 and X_2 are two i.i.d. (independent and identically distributed) copies from X, the minimum and maximum are denoted by $X_{1:2} = \min(X_1, X_2)$ and $X_{2:2} = \max(X_1, X_2)$, respectively. The pdf of the minimum $X_{1:2}$ and $Y_{1:2}$ will be denoted by $f_{1:2}$ and $g_{1:2}$, respectively.

THEOREM 4.4: *Let (X,Y) be a random vector having the FGM copula (4.6) and marginal distributions with pdf f and g. Then, the pdf of the aggregated risk $S = X + Y$ is*

$$f_S(x;\alpha) = (1+\alpha)\, f * g(x) - \alpha\, f_{1:2} * g(x) - \alpha\, f * g_{1:2}(x) + \alpha\, f_{1:2} * g_{1:2}(x), \tag{4.7}$$

for all $-1 \leq \alpha \leq 1$, where $*$ denotes the (usual) convolution operator.

PROOF: If $\alpha \in [-1,1]$, the joint pdf $c = \partial_{1,2} C$ of the bivariate FGM copula is

$$c(u,v) = 1 + \alpha - 2\alpha(1-u) - 2\alpha(1-v) + 4\alpha(1-u)(1-v)$$

for $u, v \in [0,1]$. Hence, by using (2.11), we obtain (4.7) directly. ■

Taking the n-dimensional version of the FGM copula, it is also possible to get a formula similar to (4.7) in higher dimensions.

Remark 4.5: *From (4.7), the distribution of the sum under the FGM copula is a linear combination of usual convolutions. Note that this linear combination contains negative coefficients when* $\alpha \neq 0$. *In this case, it is also called "negative mixture" (see, e.g., [37]). Hence, we can use here the results obtained for generalized mixtures (which include both positive and negative mixtures) to determine the asymptotic behaviour of the hazard rate of* $S = X + Y$ *(see [37] and the references therein) extending the results for the usual convolutions given in Block et al. [7,8] . Thus, Lemma 3.3 in [37] can be used jointly with (4.7) to prove that the limit behavior of the hazard of the FGM-convolution coincides with that of the usual convolution (the leading term in (4.7)) when* $\alpha \neq -1$. *For completeness, we include this lemma after this remark. If* $\alpha = -1$, *then the leading term in (4.7) is either* $f_{1:2} * g$ *or* $f * g_{1:2}$. *Formula (4.7) can also be written in terms of pdf of the maximum using that* $f_{2:2} = 2f - f_{1:2}$. *The cross moments* $\alpha_{r,s} = E(X^r Y^s)$ *can be obtained easily from (4.7). Moreover, the Tail Value-at-Risk (TVaR) of a linear combination of pdf is a linear combination of the TVaR of the components.*

Lemma 4.6 (Navarro and Shaked [37]): *Let* $\bar{G} = \sum_{i=1}^{k} w_i \bar{G}_i$ *be a survival function obtained as a generalized mixture of the absolutely continuous survival functions* $\bar{G}_1, \ldots, \bar{G}_k$ *for some weights* $w_1, \ldots, w_k \in \mathbb{R}$. *Let* $r, r_1, \ldots, r_k$ *be the respective hazard rate functions. If*

$$1 < \liminf_{t\to\infty} \frac{r_i(t)}{r_1(t)} \quad \text{and} \quad \limsup_{t\to\infty} \frac{r_i(t)}{r_1(t)} < \infty$$

for $i = 2, \ldots, k$, *then* $w_1 > 0$ *and* $\lim_{t\to\infty} r(t)/r_1(t) = 1$.

As we have mentioned before, the exponential distribution is very important in reliability theory where the hazard rate is used to describe the aging process. It is well known that the exponential model has a constant hazard rate. Ross [43] p. 299 (see also [7]) proved that if X and Y are independent and have exponential distributions with hazard rates λ and μ, then the survival function of $S = X + Y$ is

$$\bar{F}_S(t) = \bar{F} * \bar{G}(t) = \frac{\mu}{\mu - \lambda}\, e^{-\lambda t} - \frac{\lambda}{\mu - \lambda}\, e^{-\mu t} \tag{4.8}$$

for $t \geq 0$, when $\mu \neq \lambda$. Note that it is a negative mixture of exponential distributions. Hence, its pdf is

$$f_S(t) = f * g(t) = \frac{\lambda\mu}{\mu - \lambda}\, e^{-\lambda t} - \frac{\lambda\mu}{\mu - \lambda}\, e^{-\mu t}$$

for $t \geq 0$. However, if $\lambda = \mu$, then

$$\bar{F}_S(t) = \bar{F} * \bar{G}(t) = (1 + \lambda t)\, e^{-\lambda t} \tag{4.9}$$

and $f_S(t) = f * g(t) = \lambda^2 t\, e^{-\lambda t}$ for $t \geq 0$, that is, it has a Gamma (or Erlang) distribution.

Let us consider now that X and Y are dependent with an FGM copula and let $S = X + Y$. If X and Y have exponential distributions with hazard rates λ and μ, then $X_{1:2}$ and $Y_{1:2}$ also have exponential distributions with hazard rates 2λ and 2μ, respectively. Then, the different options for the survival function of S can be reduced to the following three cases:

Case I: $\mu > \lambda$ and $\mu \neq 2\lambda$. Here, the distributions in (4.7) can be obtained from (4.8). Then,

$$\bar{F}_S(t) = (1+\alpha)\frac{\mu\, e^{-\lambda t} - \lambda\, e^{-\mu t}}{\mu - \lambda} - \alpha\frac{\mu\, e^{-2\lambda t} - 2\lambda\, e^{-\mu t}}{\mu - 2\lambda} - \alpha\frac{2\mu\, e^{-\lambda t} - \lambda\, e^{-2\mu t}}{2\mu - \lambda} + \alpha\frac{\mu\, e^{-2\lambda t} - \lambda\, e^{-2\mu t}}{\mu - \lambda} \tag{4.10}$$

for $t \geq 0$. Note that it is a generalized mixture of exponential distributions (since some weights are negative).

Case II: $\mu > \lambda$ and $\mu = 2\lambda$. In this case, the second distribution in (4.7) is computed from (4.9) and the others from (4.8). Then,

$$\bar{F}_S(t) = (1+\alpha)\frac{\mu\, e^{-\lambda t} - \lambda\, e^{-\mu t}}{\mu - \lambda} - \alpha(1+\mu t)\, e^{-\mu t} - \alpha\frac{2\mu\, e^{-\lambda t} - \lambda\, e^{-2\mu t}}{2\mu - \lambda} + \alpha\frac{\mu\, e^{-2\lambda t} - \lambda\, e^{-2\mu t}}{\mu - \lambda}$$

for $t \geq 0$. Note that it is a generalized mixture of exponential and Gamma (Erlang) distributions.

Case III: $\mu = \lambda$. In this case, the second and the fourth distributions in (4.7) are computed from (4.9) and the others from (4.8). Then,

$$\bar{F}_S(t) = (1+\alpha)(1+\lambda t)\, e^{-\lambda t} - 2\alpha\left(2e^{-\lambda t} - e^{-2\lambda t}\right) + \alpha(1+2\lambda t)\, e^{-2\lambda t}$$

for $t \geq 0$. Again it is a generalized mixture of exponential and Gamma (Erlang) distributions.

In a similar way, by using formula (4.7), we can obtain the distribution of the sum of two random variables with Pareto and Erlang distributions assuming an FGM copula. As a previous step, we need the convolution of Pareto and Erlang distributions and the distribution of the minimum of Pareto and Erlang distributions. The following theorem provides the convolution of Pareto and Gamma distributions (see [36]).

THEOREM 4.7 (Nadarajah and Kotz [36]): *The pdf of $S = X + Y$ for two-independent random variables with $X \sim P(a, \sigma)$ and $Y \sim G(b, \lambda)$ is*

$$f_S(t; a, \sigma, b, \lambda) = \frac{a t^b \exp(-t/\lambda)}{\sigma \lambda^b \Gamma(1+b)} \Phi_1\left(1, a+1, b+1; -\frac{t}{\sigma}, -\frac{t}{\lambda}\right), \quad t > 0, \tag{4.11}$$

where Φ_1 denotes the Humbert series function (degenerate Appell hypergeometric function) defined as,

$$\Phi_1(\alpha, \beta, \gamma; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_{m+n}} \frac{x^m y^n}{m!n!}, \tag{4.12}$$

where $(\alpha)_n = \alpha \ldots (\alpha + n - 1)$ is the ascending factorial.

The distribution obtained in the preceding theorem will be denoted as $\mathrm{PGC}(a, \sigma, b, \lambda)$ (Pareto-Gamma convolution) and its pdf by $f_{\mathrm{PGC}}(x; a, \sigma, b, \lambda)$.

In order to use formula (4.7), we need the distributions of the minimum of Pareto and Erlang distributions in samples of size two. The proofs of these lemmas are straightforward and so they will be omitted.

LEMMA 4.8: *Let $X_1, \ldots, X_n$ be i.i.d. with $X_1 \sim P(a, \sigma)$. Then, $X_{1:n} \sim P(an, \sigma)$.*

LEMMA 4.9: *Let X_1 and X_2 be i.i.d. random variables with $X_1 \sim E(k, \beta)$. Then,*

$$f_{1:2}(t) = \sum_{j=0}^{k-1} \frac{1}{2^{j+k-1}} \binom{j+k-1}{k-1} f_E(t; j+k, 2\beta), \tag{4.13}$$

that is, it is a finite mixture of Erlang distributions.

Now, we are ready to state the following result.

THEOREM 4.10: *Let (X, Y) having the FGM copula in (4.6) and marginal distributions $X \sim P(a, \sigma)$ and $Y \sim E(k, \beta)$, with $a, \sigma, \beta > 0$ and $k \in \mathbb{N}$. Then, the pdf of $S = X + Y$ is a linear combination of $2k + 2$ Pareto-Erlang convolutions with pdf*

$$\begin{aligned} f(t; \alpha, a, \sigma, k, \beta) &= (1+\alpha) f_{PGC}(t; a, \sigma, k, 1/\beta) - \alpha f_{PGC}(t; 2a, \sigma, k, 1/\beta) \\ &\quad - \alpha \sum_{j=0}^{k-1} w_{j,k} f_{PGC}(t; a, \sigma, j+k, 2/\beta) + \alpha \sum_{j=0}^{k-1} w_{j,k} f_{PGC}(t; 2a, \sigma, j+k, 2/\beta), \end{aligned} \tag{4.14}$$

where $w_{j,k} = \binom{j+k-1}{k-1} 2^{-j-k+1}$, $-1 \le \alpha \le 1$, and f_{PGC} denotes the pdf of the Pareto-Erlang convolution given in (4.11).

PROOF: The proof is direct using Theorem 4.4, Lemmas 4.9 and 4.8 and taking into account that the convolution operator is closed under linear combinations of densities. ■

REMARK 4.11: *The moments $E(S^r)$ can be obtained easily since the moments of Pareto and Erlang distributions are available (see [29]). Moreover, the TVaR of S is a linear combination of the TVaR of the PGC components in (4.14).*

4.4. More General Copulas

In this section, we work with more general copulas included in the following polynomial family.

DEFINITION 4.12: *A copula $C(u_1, \ldots, u_p)$ is said to be a polynomial copula if it can be written as*

$$C(u_1, \ldots, u_p) = \sum_{i=1}^{n} \alpha_i u_1^{r_{1,i}} \cdots u_p^{r_{p,i}}, \quad u_1, \ldots, u_p \in [0, 1] \tag{4.15}$$

where $n \in \mathbb{N}$ and $\{(\alpha_i, r_{1,i}, \ldots, r_{p,i}),\ i = 1, \ldots, n\}$ are real numbers such that (4.15) is a genuine copula.

A lot of copulas fit to the previous definition which includes the following well-known copulas: Drouet-Mari and Kotz [22], FGM and its extensions, Ibragimov [28], cubic (see [42]), Mai and Scherer [33] and Nadarajah [35]. For the bivariate polynomial copula, (4.15) becomes in:

$$C(u, v) = \sum_{j=1}^{n} a_i u^{r_i} v^{s_i}, \quad u, v \in [0, 1]. \tag{4.16}$$

For this copula, we have the following result. The proof is straightforward from (2.11) and will be omitted.

Theorem 4.13: *Let (X, Y) be a random vector having the polynomial copula (4.16) with parameters $\{(a_i, r_i, s_i),\ i = 1, \ldots, n\}$ and marginal distributions with pdf f and g. Then, the pdf of $S = X + Y$ is*

$$f_S(t; \mathbf{a}, \mathbf{r}, \mathbf{s}) = \sum_{i=1}^{n} a_i\, f_{r_i:r_i} * g_{s_i:s_i}(t), \tag{4.17}$$

where $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{r} = (r_1, \ldots, r_n)$, $\mathbf{s} = (s_1, \ldots, s_n)$, $*$ is the convolution operator and $f_{r_i:r_i}$ and $g_{s_i:s_i}$, $i = 1, \ldots, n$, represent the pdf of the maximum of $\{X_1, \ldots, X_{r_i}\}$ and $\{Y_1, \ldots, Y_{s_i}\}$, respectively, where X_i are Y_j are i.i.d. random variables with pdf f and g, respectively.

We consider the following example. Note that we obtain again a negative mixture of usual convolutions.

Example 4.14: *For the Drouet-Mari and Kotz [22] copula with pdf*

$$c(u, v) = 1 + \theta \left(u^r - \frac{1}{r+1} \right) \left(u^s - \frac{1}{s+1} \right), \quad u, v \in [0, 1], \tag{4.18}$$

for $r, s \in \mathbb{N}$ and $0 < \theta \leq \min\{\frac{(r+1)(s+1)}{s}, \frac{(r+1)(s+1)}{r}\}$, the pdf of $S = X + Y$ is

$$\begin{aligned} f_S(t; \theta, r, s) = (1 + w)\, f * g(t) - w\, f_{r+1:r+1} * g(t) - w\, f * g_{s+1:s+1}(t) \\ + w\, f_{r+1:r+1} * g_{s+1:s+1}(t), \end{aligned}$$

being $w = \theta/[(r+1)(s+1)]$.

5. APPLICATION

Applications of the distribution of the sum of dependent risks can be found in actuarial sciences, risk management and in many other scientific disciplines. In this section, we present an application in actuarial science.

We consider the set of bivariate data of loss and alae (allocated loss adjustment expenses), which can be found in Klugman *et al.* [31] Chap. 12. We are interested in modeling the distribution of the total expense, that is, the sum of loss and alae assuming two different structures of dependence. We begin by considering the distribution of the sum of two exponentials with the FGM copula in Eq. (4.10). The data are a set of 24 bivariate observations and present a limited degree of dependence, in concordance with the FGM copula (see [46]), which presents a linear relation with the Pearson correlation coefficient ρ with $\rho \in [-1/3, 1/3]$. In particular, for exponential distributions $\rho = \alpha/4$. The data are quite concentrated, except for some extreme values, which is a usual situation in this kind of data (see [11] Chap. 2).

For estimating the parameters, we proceed in two steps (see [1]). In a first step, we estimate the dependence parameter α, using the estimator $\hat{\alpha} := \min(4\hat{\rho}, 1)$ for $\hat{\rho} \geq 0$ (see [20]), obtaining $\hat{\alpha} = 0.33$. Then, we estimate the marginal exponential distributions by maximum-likelihood obtaining $\hat{\lambda} = 0.0436$ for the loss variable and $\hat{\mu} = 0.1884$ for the alae variable (both in thousand dollars).

On the other hand, we have considered the Clayton copula (4.1) with Pareto marginals (see also [44]). If all the marginal distributions are identically distributed Pareto, the

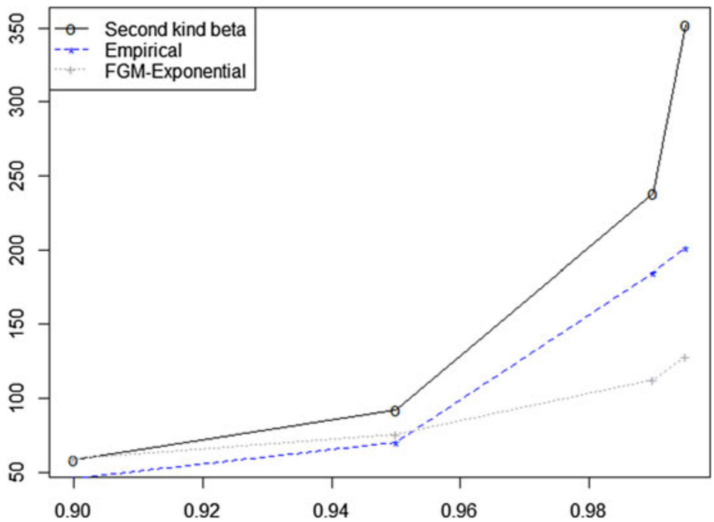

FIGURE 4. VaR quantities for $q = 0.9$, 0.95, 0.99 and 0.995 for the empirical distribution of the sum, for the distribution (4.10) and for second kind beta distribution defined in (5.1).

distribution of the sum is a second kind beta distribution with pdf,

$$h(x; p, \alpha, \beta) = \frac{x^{p-1}}{\beta^p B(p, \alpha)(1 + x/\beta)^{p+\alpha}}, \quad x \geq 0, \tag{5.1}$$

where $p = n$ is the number of marginal distributions and α, β are the shape and scale parameters in the Pareto distribution. In this case, since the marginals are not identically distributed, we approximated the distribution of the sum by Eq. (5.1), where we fit the three parameters p, α and β. For this task, we have considered maximum-likelihood estimation and we have used the R libraries actuar and fitdistrplus. In the actuar library, the second kind beta distribution is called generalized Pareto distribution. We have obtained the estimations $\hat{p} = 2.5978$, $\hat{\alpha} = 1.8816$ and $\hat{\beta} = 1.0000$.

For comparing the two models, we have obtained the VaR quantities (Value-at-Risk, i.e., $\mathrm{VaR}_X(q) = F_X^{-1}(q)$), for $q = 0.9$, 0.95, 0.99 and 0.995 for the distribution of the sum (4.10) with FGM copula and exponential marginals, for the empirical distribution and for the second kind beta distribution. The results are presented in Figure 4. Model (4.10) underestimates the empirical values for $q = 0.99$ and 0.995. However, model (5.1) presents more realistic estimates for extreme values of VaR, which is a more credible situation for loss data with heavy tails.

6. CONCLUSIONS

We have obtained several representations for the distributions of the sums of dependent random variables with a given copula. These representations allow us to obtain explicit expressions for them in specific models (copulas and/or marginals). They are also used to compare stochastically two sums with different models. In particular, we obtain sharp bounds for the distribution of the sum of dependent random variables based on that of independent random variables (Theorem 3.3). Particular results are obtained for sums of an exponential distribution with other distributions. We also determine, in some models, the asymptotic behavior of the hazard rate function of the sum extending the results given in preceding papers for independent random variables (convolutions).

The results given here can be used to solve other models by using a similar procedure. They can also be used to establish more comparison results, to obtain bounds and to determine the behavior of the hazard rate of the sum of dependent random variables.

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