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ON A PROBLEM OF R. G. D. RICHARDSON

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In 1913 Richardson published necessary and sufficient conditions for a system of three Sturm–Liouville equations, linked by three parameters, to possess eigenfunctions with arbitrarily many zeros. His work contains errors, but we give conditions of his type valid for k self-adjoint equations, with k parameters.

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1. Introduction

In 1912 R. G. D. Richardson published a landmark paper **[12]** on coupled Sturm–Liouville problems of the form

$$-(p_m y_m')' + q_m y_m = \sum_{n=1}^{k} \lambda_n r_{mn} y_m \quad 1 \leqq m \leqq k \tag{1.1}$$

for $k=2$. Richardson assumed Dirichlet end conditions on finite intervals, and analytic coefficient functions $p_n > 0$, q_m and r_{mn}, and he obtained necessary and sufficient conditions on the r_{mn} guaranteeing existence of solutions with specified numbers of zeros for the eigenfunctions y_m. A year later he attempted **[13]** the more difficult case $k=3$. His methods involved detailed examination of eigencurves and surfaces defined by single equations in two and three parameters, and led to a unified oscillation theory for indefinite, as well as the better known left ("polar") and right definite cases. In many ways these investigations were far ahead of their time: for example, oscillation theory for indefinite problems really restarted in the early 1970's (cf. **[6]**, **[7]** etc.) and a systematic study of the geometry of eigencurves and surfaces resurfaced with Turyn **[14]** in 1979, although specific properties of eigencurves can be found in, say, **[10]**, **[1]**, **[9]** and **[8]**.

Richardson's geometric reasoning can easily be put into a more modern (and rigorous) analytical framework when $k=2$, but there are difficulties when $k=3$, as observed by Turyn (loc. cit). In fact we shall show that Richardson's necessary and sufficient conditions in **[13]** are both in error, but we shall examine special cases for $k=3$ where his reasoning does lead to correct conclusions. Our general results will be couched in terms of self-adjoint operators acting on k Hulbert spaces H_m, $m=1,\ldots,k$, and the Sturm–Liouville setting (1.1) will be deduced as a special case. Necessary and sufficient conditions for the existence of eigenvalues can also be found in **[15]** and relevant references therein, but under special "definiteness" conditions. Our methods here are rather different, since, following Richardson, we presuppose no definiteness conditions.

Our plan is as follows. In Section 2 we shall discuss a general necessary condition and compare it with "left" and "right" definiteness, In Section 3 we specialise this to the Sturm–Liouville case and we compare it with Richardson's $k=3$ condition, which we show is not in fact necessary. In Section 4 we establish a sufficient condition by a degree theoretic argument. Finally Section 5 is devoted to comparisons between our sufficient conditions, our necessary conditions, left definiteness (which although related is not directly comparable with our conditions) and Richardson's sufficient conditions. We conclude with an example to show that his $k=3$ condition is not in fact sufficient.

2. A necessary condition

Let T_m, V_{mn}, $n=1,\dots,k$, be self-adjoint operators on separable Hilbert spaces H_m, T_m being bounded below with compact resolvents and V_{mn} being bounded, and let $W_m(\boldsymbol{\lambda})=T_m-V_m(\boldsymbol{\lambda})$, where $V_m(\boldsymbol{\lambda})=\sum_{n=1}^k V_{mn}\lambda_n$, $m=1,\dots,k$. Here and below we assume $\boldsymbol{\lambda}=[\lambda_1\dots\lambda_k]^T\in\mathbb{R}^k$. Let the eigenvalues of $W_m(\boldsymbol{\lambda})$ be listed as

$$\rho_m^1(\boldsymbol{\lambda})\leqq\rho_m^2(\boldsymbol{\lambda})\leqq\dots,$$

counted by multiplicity. With U_m as the unit sphere of H_m, the multiparameter eigenvalue problem

$$W_m(\boldsymbol{\lambda})u_m=0,\quad u_m\in U_m\cap D(T_m),\quad m=1,\dots,k, \tag{2.1}$$

can be studied via the equations

$$\rho_m^{i_m}(\boldsymbol{\lambda})=0,\quad m=1,\dots,k. \tag{2.2}$$

Indeed any solution of (2.2) yields nontrivial nullspaces for each $W_m(\boldsymbol{\lambda})$, and conversely any $\boldsymbol{\lambda}$ satisfying (2.1) also satisfies (2.2) for some *index* $\mathbf{i}=(i_1,\dots,i_k)$. For examples of this setting, we refer, for instance, to [**15**].

We use lower case letters to denote quadratic forms, e.g., $t_m(u)=(u,T_mu)$, and $V(u)$ will be the $k\times k$ matrix with (m,n) entry $v_{mn}(u_m)$ and with mth row $\mathbf{v}_m(u_m)$. Our study of the equations (2.2) depends mainly on the following representation of the functions $\rho_m^{i_m}$ given by the minimax principle

$$\rho_m^{i_m}(\boldsymbol{\lambda})=\min\{\max\{\mathrm{t_m(u)}-\mathrm{v_m(u)}\boldsymbol{\lambda}: u\in U_m\cap E\}: E\subset D(T_m),\dim E=i_m\}. \tag{2.3}$$

For any subspace G of H_m, we write

$$\iota_m(\boldsymbol{\lambda},G)=\inf\{\mathrm{v_m(u)}\boldsymbol{\lambda}: u\in U_m\cap G\}.$$

It is convenient to make the assumption

(A) For each m and λ, the maximum of the spectrum of $V_m(\lambda)$ is not an eigenvalue of finite multiplicity.

Then we define

$$\sigma_m(\lambda)=\sup\{\iota_m(\lambda,G): G\subset D(T_m), \dim G=i\}$$

for any fixed finite i and by virtue of (A) we have

$$\sigma_m(\lambda)=\sup\{\mathbf{v}_m(u)\lambda: u\in U_m\}. \tag{2.4}$$

We shall need three lemmas involving single equations, i.e. with m fixed.

Lemma 2.1. *Let $i=i_m$ be fixed and let the sequence $\lambda^j\in\mathbb{R}^k$ satisfy $\rho_m^i(\lambda^j)\geqq 0$ for all j, with $\|\lambda^j\|\to\infty$ and $\lambda^j/\|\lambda^j\|\to\lambda^*$. Then $\sigma_m(\lambda^*)\leqq 0$.*

Proof. For any i dimensional subspace F of $D(T_m)$, (2.3) gives

$$t_m(u^j)-\mathbf{v}_m(u^j)\lambda^j\geqq 0 \tag{2.5}$$

for some $u^j\in U_m\cap F$. Assuming, without loss of generality, that $u^j\to u\in U_m\cap F$, we may divide (2.5) by $\|\lambda^j\|$ and proceed to the limit to give

$$\mathbf{v}_m(u)\lambda^*\leqq 0$$

for some $u\in U_m\cap F$. By definition of $\sigma_m(\lambda^*)$, this suffices for the result. □

Lemma 2.2. *As for Lemma 2.1, but with the inequalities reversed.*

Proof. By assumption, $\rho_m^1(\lambda)\leqq\rho_m^i(\lambda)\leqq 0$. Then (2.3) gives (2.5) (with the inequality reversed) for some sequence $u^j\in U_m\cap D(T_m)$. We again divide by $\|\lambda^j\|$ and obtain

$$\mathbf{v}_m(u^j)\lambda^*\geqq\frac{t_m(u^j)}{\|\lambda^j\|}-\mathbf{v}_m(u^j)\left(\frac{\lambda^j}{\|\lambda^j\|}-\lambda^*\right).$$

Since $t_m(u^j)$ is bounded below, we see that, for every given $\varepsilon>0$, $\mathbf{v}_m(u^j)\lambda^*>-\varepsilon$ whenever j is sufficiently large. The result now follows from (2.4). □

Lemma 2.3. *Let $i(j)$ be a sequence of positive integers converging to infinity and let $\rho_m^{i(j)}(\lambda^j)=0$ for all j. Then $\|\lambda^j\|\to\infty$.*

Proof. For each i, $\rho_m^i(\lambda^j)\leqq 0$ as $j\to\infty$. Thus if λ^j has an accumulation point λ^* in $\mathbb{R}^k$,

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then, by continuity of ρ_m^i, it follows that $\rho_m^i(\lambda^*) \leqq 0$ for all i. This contradicts the fact that $\rho_m^i(\lambda^*) \to \infty$ as $i \to \infty$. □

We are now ready for the basic necessary condition.

Theorem 2.4. *Fix m and positive integers i_n for $n=1,\ldots,k$, $n \neq m$. If, for an infinite set of i_m, (2.2) admits eigenvalues $\boldsymbol{\lambda}=\boldsymbol{\lambda}^{(m)}$, then there is $\boldsymbol{\mu} \neq 0$* such that

$$\sigma_n(\boldsymbol{\mu})=0 \leqq \sigma_m(\boldsymbol{\mu}) \tag{2.6}$$

if $n=1,\ldots,k$, $n \neq m$.

Proof. By Lemma 2.3, the eigenvalues $\boldsymbol{\lambda}^{(m)}$ satisfy $\|\boldsymbol{\lambda}^{(m)}\| \to \infty$ as $i_m \to \infty$. Let $\boldsymbol{\mu}$ be an accumulation point of tthe sequence $\boldsymbol{\lambda}^{(m)}/\|\boldsymbol{\lambda}^{(m)}\|$. Then $\sigma_n(\boldsymbol{\mu})=0$ follows readily from Lemmas 2.1 and 2.2. Moreover $\rho_m^1(\boldsymbol{\lambda}^{(m)}) \leqq \rho_m^{i_m}(\boldsymbol{\lambda}^{(m)})=0$ so Lemma 2.2 gives $0 \leqq \sigma_m(\boldsymbol{\mu})$. □

Corollary 2.5. *If there is a fixed index* **i** *so that (2.2) admits eigenvalues of any index* $\mathbf{j} \geqq \mathbf{i}$ *then for any nonempty subset Σ of $\{1,\ldots,k\}$ there is $\boldsymbol{\mu}^\Sigma \neq \mathbf{0}$ so that (2.6) holds for $\boldsymbol{\mu}=\boldsymbol{\mu}^\Sigma$ if $m \in \Sigma$ and $n \notin \Sigma$.*

Here and below we use the componentwise order on $\mathbb{R}^k$.

3. Differential equations

With reference to (1.1), suppose $1/p_m$ and q_m are L_1 with p_m positive, and define

$$D(T_m)=\{y \in C^1[a_m,b_m] \cap BC: p_m y' \in AC[a_m,b_m],\ -(p_m y')'+q_m y \in L_2[a_m,b_m]\}$$

where BC represents separated boundary conditions. Then define

$$T_m y=-(p_m y')'+q_m y,\ (V_{mn}y)(x_m)=r_{mn}(x_m)y(x_m)$$

so [**11**, §§ 18, 19] the hypotheses placed on T_m and V_{mn} in Section 2 are satisfied provided the r_{mn} are L_∞.

It is well known (cf. [**15**, Theorem 3.5.1]) that a solution of (2.1) has index **i** if and only if the corresponding solution y_m of (1.1) has i_m-1 zeros in $]a_m,b_m[$. This leads us to the following condition which is of Richardson's type, and which is a special case of Corollary 2.5. By virtue of (2.4), the functions σ_m now become

$$\sigma_m(\boldsymbol{\lambda})=\operatorname{ess\,sup}\left\{\sum_{n=1}^k \lambda_n r_{mn}(x_m): a_m \leqq x_m \leqq b_m\right\}. \tag{3.1}$$

Corollary 3.1. *If (1.1) admits solutions with abritrarily many zeros for each y_m, then there exist $\boldsymbol{\mu}^m \neq 0$, $m=1,\ldots,k$, such that, for all $m,n=1,\ldots,k$,*

$$\sigma_n(\boldsymbol{\mu}^m)=0 \quad \textit{if} \quad m \neq n, \tag{3.2}$$

$$\sigma_n(\mu^m) \geqq 0 \quad \textit{if} \quad m = n. \tag{3.3}$$

For comparison, we state Richardson's necessary condition (for $k=3$ and with analytic p_m, q_m, r_{mn}) in the following form. Let $R(x)$ be the 3×3 matrix with (m,n)th entry $r_{mn}(x_m)$ in (1.1).

(**RNC**) *There exists vectors* ξ^m $(=-[\lambda^{(m)}\mu^{(m)}v^{(m)}]^T$ *in his notation*) *such that, for all* $m, n = 1, 2, 3$,

$$(R(x)\xi^m)_n \leqq 0 \quad \textit{for all} \quad x_n \quad \textit{if} \quad m \neq n \tag{3.4}$$

and

$$(R(x)\xi^m)_m = 0 \quad \textit{for some} \quad x_m. \tag{3.5}$$

In our notation (2.4), (3.1) this means

$$\sigma_n(\xi^m) \leqq 0 \quad \text{if} \quad m \neq n \tag{3.6}$$

and

$$\iota_m(\xi^m, U_m) \leqq 0 \leqq \sigma_m(\xi^m) \quad \text{for all} \quad m. \tag{3.7}$$

It is evident that our conditions in Corollary 3.1 imply RNC except for the left-hand inequality in (3.7), and the following example shows that this infimal condition is *not* necessary.

Example 3.2. Let $p_m \equiv 1$, $q_m \equiv 0$ for $m = 1, 2, 3$, and let

$$R(x) = \begin{bmatrix} 1-x_1 & -x_1 & 0 \\ -x_2 & 1-x_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We solve (1.1) over the interval $[0,1]$ with Dirichlet end conditions; since the coefficients are analytic, this is a problem of Richardson's type. We claim that solutions of (1.1) exist for each index $\mathbf{i} \geqq (1,1,1)$. This is a consequence of Theorem 4.1 below, but may also be seen as follows.

For $m=3$, we have the (right definite) problem $-y_3'' = \lambda_3 y_3$, which evidently has solutions of any index $i_3 \geqq 1$. For $m = 1, 2$ we have a left definite system [**15**, p. 43], since the cofactor matrix $C(x)$ of

$$\begin{bmatrix} 1-x_1 & -x_1 \\ -x_2 & 1-x_2 \end{bmatrix}$$

satisfies

$$C(x)\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

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and T_m is positive definite for $m=1,2$. Moreover the determinant of

$$\begin{bmatrix} 1-x_1 & -x_1 \\ -x_2 & 1-x_2 \end{bmatrix}$$

is not identically zero. Thus solutions λ_1, λ_2 exist of any index $(i_1,i_2)\geqq(1,1)$—see, e.g., [**15**, Theorem 3.5.2].

It follows that (3.2), (3.3) must hold: indeed it is enough to choose the $\boldsymbol{\mu}^m$ as the unit coordinate vectors in $\mathbb{R}^3$. On the other hand, (3.4) requires

$$\begin{bmatrix} 1-x_1 & -x_1 & 0 \\ -x_2 & 1-x_2 & 0 \end{bmatrix}\xi^3 \leqq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{for all} \quad x_1, x_2 \in [0,1],$$

and this forces $\xi^3=[0 \quad 0 \quad \alpha]^T$. Further, (3.5) requires $[0 \quad 0 \quad 1]\ \xi^3$ to vanish for some x_3, so $\alpha=0$ and thus RNC fails if one excludes the triviality $\xi^3=\mathbf{0}$.

We conclude this section by comparing the above conditions with the standard ones of right definiteness (RD) and left definiteness (LD). RD means that $V(u)$ has determinant of one sign for all choices of the u_m. This implies that, for each subset Σ (including $\emptyset$) of $\{1,\dots,k\}$, there is $\mathbf{v}^\Sigma\neq\mathbf{0}$ such that

$$\sigma_n(\mathbf{v}^\Sigma)\leqq 0\leqq \iota_m(\mathbf{v}^\Sigma, U_m)(\leqq\sigma_m(\mathbf{v}^\Sigma)) \tag{3.8}$$

for all $m\in\Sigma$ and $n\notin\Sigma$. See (2.4), (3.1) for the notation and [**2**, Theorem 9.7.1] for a stronger result. Since [**15**, Theorem 3.5.2] RD implies that eigenvalues exist for all indices $\mathbf{i}$, Corollary 2.5 shows that (2.6) must hold for some $\boldsymbol{\mu}^\Sigma$ for all $\Sigma\neq\emptyset$. A direct proof that RD implies (2.6) is not obvious in general, but for $k\leqq 3$, simple geometric arguments based on perturbing $\mathbf{v}^\Sigma$ will suffice. In similar fashion one can show that RD implies (3.7) and hence RNC.

Analogous reasoning holds for LD, which consists of a definiteness condition on the T_m, and an "ellipticity" condition which requires existence of a unit $\omega\in\mathbb{R}^k$ such that $C(u)\omega>\mathbf{0}$, where $C(u)$ is the cofactor matrix of $V(u)$. Rotating axes so that $\omega=[0\dots0 \quad 1]^T$, we may use [**3**, Theorem 6.3] to show that ellipticity implies the existence for each $\Sigma\neq\emptyset$ of $\mathbf{v}^\Sigma$, with $v_k^\Sigma=0$, such that (3.8) holds. Thus the above implications of RD also hold for LD.

4. A sufficient condition

The purpose of this section is to prove the following result for the operators T_m and V_{mn} of Section 2. We recall assumption (A) and the subsequent definition of σ_m.

Theorem 4.1. *Suppose that*

(i) *the i_mth eigenvalue of T_m, i.e. $\rho_m^{i_m}(\mathbf{0})$, os positive for $m=1,\dots,k$.*

(ii) *$V_{mn}\leqq 0$ if $m,n=1,\dots,k$, $m\neq n$.*

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(iii) *for some* $\boldsymbol{\mu}>\mathbf{0}$, $\sigma_m(\boldsymbol{\mu})>0$ *for all* $m=1,\dots,k$.

Then (2.1) has a solution $\boldsymbol{\lambda}>\mathbf{0}$ *of any given index* $\mathbf{j}\geqq\mathbf{i}$.

Remarks. (a) Theorem 4.1 applies to Example 3.2 with $\boldsymbol{\mu}=[1\quad 1\quad 1]^T$.

(b) By (iii), there are i_m-dimensional subspaces E_m of $D(T_m)$ such that, for some $\boldsymbol{\mu}>0$,

$$V(u)\boldsymbol{\mu}>\mathbf{0}\quad\text{for all}\quad u\in E:=\mathop{\mathrm{X}}_{m=1}^{k}(E_m\cap U_m).\tag{4.1}$$

We also note by (i) and the minimax principle (2.3) that there is $e\in E$ such that

$$t_m(e_m)\geqq\rho_m^{i_m}(\mathbf{0})>0,\quad m=1,\dots,k.\tag{4.2}$$

(c) If we drop our assumption (A) on the spectra of the $V_m(\lambda)$ then Theorem 4.1 remains true if we replace assumption (iii) by (4.1).

We shall need two lemmas, the first involving the set

$$D=\{\boldsymbol{\lambda}\in\mathbb{R}^k:\boldsymbol{\lambda}>\mathbf{0}\quad\text{and}\quad\mathbf{t}(u)>V(u)(\boldsymbol{\lambda}-\boldsymbol{\mu})\text{ for some }u\in E\}$$

where $\mathbf{t}=[t_1\dots t_k]^T$.

Lemma 4.2. *D is a nonempty open bounded subset of* $\mathbb{R}^k$.

Proof. Openness of D is elementary. For any $u\in E$, (ii) and (4.1) give $v_{mn}(u_m)\leqq 0$ whenever $m\neq n$ and $V(u)\boldsymbol{\mu}>0$, for some $\boldsymbol{\mu}>0$. By a well known result, cf. [**15**, Lemma 5.5.1], $V(u)^{-1}$ exists with all elements nonnegative. In particular

$$\boldsymbol{\lambda}(e):=V(e)^{-1}\mathbf{t}(e)\in D.\tag{4.3}$$

Moreover, for any $\boldsymbol{\lambda}\in D$ there is $u\in E$ such that

$$\mathbf{0}<\boldsymbol{\lambda}<\boldsymbol{\mu}+V(u)^{-1}\mathbf{t}(u).$$

These bounds are uniform because $V(u)^{-1}\mathbf{t}(u)$ is continuous on the compact set E. □

Our second lemma concerns the function $\mathbf{g}=[0,1]\times\mathbb{R}^k\to\mathbb{R}^k$ given by

$$g_m(\alpha,\boldsymbol{\lambda})=\inf\{\max\{t_m(u_m^\alpha)-\mathbf{v}_m(u_m^\alpha)\boldsymbol{\lambda}: u_m^\alpha=\alpha^{1/2}e_m+(1-\alpha)^{1/2}u_m,$$

$$u_m\in U_m\cap F_m,\operatorname{Re}(e_m,u_m)=0\}: F_m\subset D(T_m),\dim F_m=i_m\}.\tag{4.4}$$

Standard arguments using the semiboundedness of $t_m(u_m)$ show that g is continuous.

Lemma 4.3. *For each* $\alpha\in[0,1]$, *the map* $\mathbf{g}(\alpha,\cdot)$ *does not vanish on the boundary* ∂D *of D.*

Proof. Elementary considerations show that if $\boldsymbol{\lambda}\in\partial D$ then for some m either

(a) $\lambda_m = 0$

or

$$\text{(b)} \quad \max\{t_m(u_m) - \mathbf{v}_m(u_m)(\boldsymbol{\lambda} - \boldsymbol{\mu}) : u_m \in E_m \cap U_m\} = 0. \tag{4.5}$$

(a) For any i_m-dimensional subspace F_m of $D(T_m)$, choose $u_m \in F_m \cap U_m$ so that $t_m(u_m) \geqq \rho_m^{i_m}(\mathbf{0})$, $\mathrm{Re}(e_m, u_m) = 0$ and $\mathrm{Re}(T_m e_m, u_m) \geqq 0$ — see (i). In the notation of (4.4), then, $t_m(u_m^\alpha) \geqq \rho_m^{i_m}(\mathbf{0})$ by (4.2). Thus from (ii) and $\lambda_m = 0$ we have

$$t_m(u_m^\alpha) - \mathbf{v}_m(u_m^\alpha)\boldsymbol{\lambda} \geqq \rho_m^{i_m}(\mathbf{0}).$$

It follows that $g_m(\alpha, \boldsymbol{\lambda}) \geqq \rho_m^{i_m}(\mathbf{0}) > 0$.

(b) For any $u_m \in U_m \cap E_m$ such that $\mathrm{Re}(e_m, u_m) = 0$, we have by (4.1) and (4.5)

$$t_m(u_m^\alpha) - \mathbf{v}_m(u_m^\alpha)\boldsymbol{\lambda} = t_m(u_m^\alpha) - \mathbf{v}_m(u_m^\alpha)(\boldsymbol{\lambda} - \boldsymbol{\mu}) - \mathbf{v}_m(u_m^\alpha)\boldsymbol{\mu} < 0$$

and so $g_m(\alpha, \boldsymbol{\lambda}) < 0$. □

We are now ready to prove Theorem 4.1. Since $V(e)$ is invertible (as above) and

$$\mathbf{g}(1, \boldsymbol{\lambda}) = \mathbf{t}(e) - V(e)\boldsymbol{\lambda}$$

vanishes only at $\boldsymbol{\lambda}(e) \in D$ by (4.3) we have by Lemma 4.2 that the degree $\deg(\mathbf{g}(\alpha, \cdot), D, \mathbf{0})$ is well defined and nonzero for $\alpha = 1$. By Lemma 4.3, this remains true for $\alpha = 0$. Now (2.3) shows that

$$g_m(0, \boldsymbol{\lambda}) = \rho_m^{i_m}(\boldsymbol{\lambda}),$$

so there must be a solution to (2.2) in D. For $\mathbf{j} \geqq \mathbf{i}$ we note that (i)–(iii) hold with $\mathbf{i}$ replaced by $\mathbf{j}$, so solutions also exist for such indices $\mathbf{j}$. □

5. Discussion

We shall now compare the conditions of Theorem 4.1 with various others, specialising for convenience to (1.1) with continuous r_{mn}. Let us start with uniform left definiteness (ULD) where it is required that the operators T_m should be positive definite and

$$C(x)\boldsymbol{\omega} > \mathbf{0} \quad \text{for all} \quad x \in \mathop{\mathsf{X}}_{m=1}^{k} [a_m, b_m]$$

for some $\boldsymbol{\omega} \in \mathbb{R}^k$, where $C(x)$ denotes the cofactor matrix of $R(x)$. Continuity forces the components of $C(x)\boldsymbol{\omega}$ to have positive lower bounds, hence the "uniformity". It can be shown [**4**, Lemma 2.1] that, after a nonsingular change of $\boldsymbol{\lambda}$ coordinates, ULD implies

(i)$_u$ $\quad \rho_m^1(0) > 0 \quad \text{for all} \quad m = 1, \ldots, k$

$(\mathrm{ii})_u$ $r_{mn}(x_m)<0$ for all $x_m \in [a_m, b_m]$ if $m \neq n$

$(\mathrm{iii})_u$ $r_{mm}(x_m)>0$ for all $x_m \in [a_m, b_m]$ if $m=1,\dots,k$

$(\mathrm{iv})_u$ $c_{mn}(x)>0$ for all $x_m \in [a_m, b_m]$, $m, n=1,\dots,k$.

It should be noted that these four conditions are not quite sufficient for the existence of eigenvalues. If

$(\mathrm{v})_u$ $$\det R(x) \neq 0 \quad \text{for some } x$$

then solutions of any index are guaranteed [**15**, Theorem 3.5.2] but otherwise there may be no eigenvalues. As a trivial example, one could take:

Example 5.1.

$$k=2,\ p_m=1,\ q_m=0,\ R(x)=\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

The two equations (1.1) then force $\lambda_1-\lambda_2$ to take opposite signs for any index. It follows that there are no eigenvalues under Dirichlet boundary conditions.

We now show that if we strengthen $(\mathrm{v})_u$ to

$(\mathrm{v})'_u$ $$\det R(x)>0 \quad \text{for some } x$$

then $(\mathrm{i})_u$, $(\mathrm{ii})_u$, $(\mathrm{iv})_u$ and $(\mathrm{v})'_u$ together imply our sufficiency conditions (i), (ii), (iii) of Theorem 4.1. First, $(\mathrm{i})_u$ strengthens (i) by requiring $\mathbf{i}=\mathbf{1}$ and $(\mathrm{ii})_u$ strengthens (ii) by requiring strict inequality (and hence a negative upper bound). Moreover, a well known result for real matrices [15, Lemma 5.5.1] shows that $(\mathrm{iv})_u$, $(\mathrm{v})'_u$ imply (iii) which in the present context becomes

$(\mathrm{iii})_s$ $$R(x)\tilde{\boldsymbol{\mu}}>0 \quad \text{for some } x \text{ and some } \tilde{\boldsymbol{\mu}}>0.$$

Hence we can state that, apart from $(\mathrm{v})'_u$, which must be compatible with the initial change of λ coordinates, our sufficient condition of Theorem 4.1 is weaker than ULD. On the other hand the following example satisfies the conditions of Theorem 4.1 but drastically fails ULD.

Example 5.2. As for Example 5.1 but with

$$R(x)=\begin{bmatrix} \cos x_1 & -\sin x_1 \\ -1 & 0 \end{bmatrix}$$

on $[a_m, b_m]=[0,\pi]$ and $\boldsymbol{\mu}=[-1 \quad 1]^T$. It is impossible to satisfy *any* of $(\mathrm{ii})_u$, $(\mathrm{iii})_u$ or $(\mathrm{iv})_u$ even after a change of coordinates.

Let us also point out that $(\mathrm{i})_u$, $(\mathrm{ii})_u$, $(\mathrm{iii})_u$ and $(\mathrm{v})_u$ are sufficient for the existence of eigenvalues of arbitrary index if $k \leqq 3$. This is trivial if $k=1$. If $k=2$ then $(\mathrm{ii})_u$ and $(\mathrm{iii})_u$ imply $(\mathrm{iv})_u$ so we have a uniform left definite problem (1.1). If $k=3$ then $(\mathrm{i})_u$, $(\mathrm{ii})_u$ and $(\mathrm{iii})_u$ imply local definiteness of (1.1) in the sense of [**15**, Theorem 3.3.2] which together with $(\mathrm{v})_u$ gives the existence of eigenvalues of any index. If $k>3$, however, then $(\mathrm{i})_u$, $(\mathrm{ii})_u$, $(\mathrm{iii})_u$ and $(\mathrm{v})_u$ do not guarantee existence of eigenvalues.

Weaker forms of $(\mathrm{ii})_u$, $(\mathrm{iii})_u$ are

$(\mathrm{ii})_N$ $\quad r_{mn}(x_m) \leqq 0$ for all x_m if $m \neq n$,

$(\mathrm{iii})_N$ $\quad r_{mm}(x_m) \geqq 0$ for some x_m, $m=1,\dots,k$,

which constitute Richardson's "normal form" for (1.1) (cf. [**13**, p. 301]) if the *necessary* conditions of Corollary 3.1 are satisfied with linearly indpendent $\boldsymbol{\mu}^m$. In this case the $\boldsymbol{\mu}^m$ can be taken as $\boldsymbol{\lambda}$ coordinate vectors after a nonsingular transformation. We should point out that this normal form cannot be guaranteed, since it may be impossible to choose linearly independent $\boldsymbol{\mu}^m$: cf. Example 5.2.

Let us turn now to Richardson's Theorem [**13**, p. 299] which is for analytic coefficients and Dirichlet end conditions. Richardson assumes that nonzero solutions of (1.1) with $\boldsymbol{\lambda}=0$ and $y_m(a_m)=0$ have i_m-1 internal zeros: this is easily seen to imply (i) of Theorem 4.1. He also assumes (3.4) and the following strengthening of (3.5):

$$(R(x)\xi^m)_m \text{ takes both signs for } x_m \in [a_m, b_m]. \tag{5.1}$$

He then claims the existence of a solution to (1.1) so that each y_m has i_m-1 internal zeros, i.e., a solution of index **i**.

The following example shows that Richardson's conditions are not sufficient as stated.

Example 5.3. *Let* $k=3$, $p_m=-q_2=-q_3\equiv 1$, $q_1\equiv 0$,

$$R(x)=\begin{bmatrix} 0 & x_1 & -1 \\ \pi+x_2 & 0 & -1 \\ -\pi-x_3 & 0 & -1 \end{bmatrix} \text{with Dirichlet conditions on } [a_m,b_m]=\left[-\frac{\pi}{2},\frac{\pi}{2}\right],$$

$m=1,2,3$. We choose $\mathbf{i}=\mathbf{1}$ since each T_m is positive definite, and with $[0\ \ 1\ \ 1]^T$, $[1\ \ 0\ \ \pi]^T$ and $[-1\ \ 0\ \ \pi]^T$ as ξ^m we may check (3.4) and (5.1), so Richardson's conditions are satisfied. We shall prove by contradiction that no solution of (1.1) exists with index **1**.

First we claim that $m=1$ in (1.1) forces

$$\lambda_3 \geqq -1. \tag{5.2}$$

This follows because the eigenvalue $\lambda_3=\lambda_3(\lambda_2)$ of

$$-y_1''=(\lambda_2 x_1-\lambda_3)y_1, \quad y_1\left(\pm\frac{\pi}{2}\right)=0,$$

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corresponding to $i_1 = 1$, is a convex function having asymptotic slopes

$$\lambda_3'(\lambda_2) \to \begin{Bmatrix} \min \\ \max \end{Bmatrix} (x_1) = \mp \frac{\pi}{2}$$

as $\lambda_2 \to \mp\infty$: see, e.g. [5, Section 2]. Also the transformation $x_1 \to -x_1$ shows that λ_3 is even. Hence $\lambda_3(\lambda_2) \geqq \lambda_3(0) = -1$ for all λ_2.

On the other hand, $m=2$ and 3 in (1.1) give

$$y_2 y_3'' - y_2'' y_3 = \lambda_1 (2\pi + x_2 + x_3) y_2 y_3$$

so integration by parts yields

$$0 = \lambda_1 \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} (2\pi + x_2 + x_3) y_2 y_3 \, dx_2 \, dx_3.$$

Since the integrand does not vanish, we see that $\lambda_1 = 0$. Hence

$$-y_3'' + y_3 = -\lambda_3 y_3, \; y_3\left(\pm\frac{\pi}{2}\right) = 0$$

and this contradicts (5.2).

We conclude by returning to Richardson's "normal form", which exists when the ξ^m are linearly independent, as is the case in Example 5.3. In "normal form", Richardson's necessary conditions of the r_{mn} become $(\text{ii})_N$ (i.e. (ii) of Theorem 4.1) and

$$(\text{iii})_{\text{RN}} \quad r_{mm}(x_m) = 0 \text{ for some } x_m,$$

while his sufficiency conditions are $(\text{ii})_{\text{N}}$ and

$$(\text{iii})_{\text{RS}} \quad r_{mm} \text{ takes both signs on } [a_m, b_n].$$

In "normal form", Example 5.3 then satisfies (i) and (ii) of Theorem 4.1, and also $(\text{iii})_{\text{RS}}$. Thus $(\text{iii})_{\text{RS}}$ cannot be substituted for (iii) in Theorem 4.1.

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