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CENTRAL SEQUENCES IN SUBHOMOGENEOUS UNITAL C*-ALGEBRAS

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Abstract

Suppose that $\mathcal{A}$ is a unital subhomogeneous C*-algebra. We show that every central sequence in $\mathcal{A}$ is hypercentral if and only if every pointwise limit of a sequence of irreducible representations is multiplicity free. We also show that every central sequence in $\mathcal{A}$ is trivial if and only if every pointwise limit of irreducible representations is irreducible. Finally, we give a nice representation of the latter algebras.

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1. Introduction

The notion of central sequences (with respect to the $\|\,\|_2$-norm defined by the trace) on II_1 factor von Neumann algebras has played an important role in the study of these algebras. For example, McDuff [4] proved that if there are two central sequences in a type II_1 factor von Neumann algebra $\mathcal{M}$ that do not asymptotically commute, then $\mathcal{M}$ is isomorphic to $\mathcal{M} \otimes \mathcal{R}$, where $\mathcal{R}$ is the hyperfinite II_1 factor. A II_1 factor has property Γ if and only if there is a central sequence that is bounded away from the centre.

The analogous notion for C*-algebras is more complicated. Suppose that $\mathcal{A}$ is a separable unital C*-algebra. A bounded sequence $\{x_n\}$ in $\mathcal{A}$ is called a *central sequence* if, for every $a \in \mathcal{A}$,

$$\lim_{n\to\infty} \|ax_n - x_n a\| = 0.$$

A central sequence $\{x_n\}$ for $\mathcal{A}$ is called *hypercentral* if and only if, for every central sequence $\{y_n\}$,

$$\lim_{n\to\infty} \|x_n y_n - y_n x_n\| = 0.$$

Let $\mathcal{Z}(\mathcal{A})$ denote the centre of the C*-algebra $\mathcal{A}$. A sequence $\{x_n\}$ is called a *trivial central sequence* if

$$\lim_{n\to\infty} \operatorname{dist}(x_n, \mathcal{Z}(\mathcal{A})) = 0.$$

The study of central sequences and hypercentral sequences for a C*-algebra was initiated by Akemann and Pedersen [1] and Phillips [6], who related central sequences to the automorphism groups of the algebras.

The question of whether there are central sequences that are bounded away from the centre was solved by Akemann and Pedersen [1], who proved that every central sequence in a separable C*-algebra $\mathcal{A}$ is trivial exactly when $\mathcal{A}$ is a continuous-trace C*-algebra. Thus, most C*-algebras have a large supply of nontrivial central sequences. The question of when every two central sequences are asymptotically commuting, that is, when every central sequence is hypercentral, has been partially solved. Ando and Kirchberg [2] proved that if $\mathcal{A}$ is a separable C*-algebra that is not type I (GCR), then there is a central sequence that is not hypercentral. However, if $\mathcal{A}$ is a unital type I separable C*-algebra with an infinite-dimensional irreducible representation, then Phillips [6, Theorem 3.6] proved that there is a central sequence that is not hypercentral. Thus, if a unital separable C*-algebra has the property that every two central sequences are asymptotically commuting, then every irreducible representation must be finite dimensional. In this paper we focus on the class of *subhomogeneous* C*-algebras, that is, ones for which there is a positive integer so that every irreducible representation is at most n dimensional. For subhomogeneous separable unital C*-algebras, we give characterisations, in terms of finite-dimensional irreducible representations, of the property that every central sequence is trivial (Theorem 2.6) and of the property that every central sequence is hypercentral (Theorem 2.3). Thus, for subhomogeneous C*-algebras our results give a nice comparison of these two properties. Most of the results in this paper appeared in the second author's PhD dissertation [5].

2. Theorems and proofs

The following lemma is contained in [1] and [6].

LEMMA 2.1. *Let $\mathcal{A}$ be a separable C*-algebra.*

(1) *If $\pi : \mathcal{A} \to \mathcal{B}$ is a surjective unital $*$-homomorphism and $\{b_n\}$ is a central sequence in $\mathcal{B}$, then there is a central sequence $\{a_n\}$ in $\mathcal{A}$ such that $\pi(a_n) = b_n$ for every $n \in \mathbb{N}$.*
(2) *Every central sequence of $\mathcal{A}$ is trivial if and only if $\mathcal{A}$ is a continuous-trace C*-algebra.*
(3) *If $\pi : \mathcal{A} \to \mathcal{B}$ is a surjective unital $*$-homomorphism and every central sequence in $\mathcal{A}$ is hypercentral, then every central sequence of $\mathcal{B}$ is hypercentral.*

Suppose that $\mathcal{A}$ is a unital C*-algebra and n is a positive integer. Let $\mathrm{Rep}_n(\mathcal{A})$ be the set of unital $*$-homomorphisms $\pi : \mathcal{A} \to \mathbb{M}_n(\mathbb{C})$ with the topology of pointwise convergence. We define $\mathrm{Irr}_n(\mathcal{A})$ to be the set of irreducible representations in $\mathrm{Rep}_n(\mathcal{A})$. Every representation $\pi \in \mathrm{Rep}_n(\mathcal{A})$ can be written as a direct sum

$$\pi = \sigma_1 \oplus \cdots \oplus \sigma_k$$

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with each σ_j irreducible. We say that $\pi \in \mathrm{Rep}_n(\mathcal{A})$ is *multiplicity free* if σ_i and σ_j are not unitarily equivalent whenever $1 \le i < j \le k$. Equivalently, π is multiplicity free if and only if $\pi(\mathcal{A})'$ is abelian.

If $\mathcal{A}$ is separable and $\{x_1, x_2, \ldots\}$ is dense in the unit ball of $\mathcal{A}$, then

$$d(\pi, \rho) = \sum_{k=1}^{\infty} \frac{1}{2^k} \|\pi(x_k) - \rho(x_k)\|$$

is a metric on $\mathrm{Rep}_n(\mathcal{A})$ that makes a compact metric space.

A C*-algebra $\mathcal{A}$ is *subhomogeneous* if there is a smallest positive integer N such that every irreducible representation of $\mathcal{A}$ is unitarily equivalent to a representation in

$$\cup_{1 \le k \le N} \mathrm{Irr}_k(\mathcal{A}).$$

We call N the *degree* of $\mathcal{A}$. It was shown by Robert [7] that a unital continuous-trace C*-algebra is subhomogeneous.

LEMMA 2.2. *Suppose that $\mathcal{A}$ is a unital separable C*-algebra, $s \in \mathbb{N}$ and $\{\pi_n\}$ is a sequence in $Irr_s(\mathcal{A})$ such that π_m and π_n are not unitarily equivalent whenever $1 \le m < n < \infty$. Suppose that $\pi_0 \in Rep_s(\mathcal{A})$, $\pi_n \to \pi_0$ pointwise on $\mathcal{A}$ and, for every $n \in \mathbb{N}$, π_0 is not unitarily equivalent to π_n. Let $\pi = \pi_0 \oplus \pi_1 \oplus \pi_2 \oplus \cdots$. Then:*

(1) $T = T_0 \oplus T_1 \oplus T_2 \oplus \cdots \in \pi(\mathcal{A})$ *if and only if* $T_0 \in \pi_0(\mathcal{A})$ *and*

$$\lim_{n\to\infty} \|T_n - T_0\| = 0;$$

(2) $A \in \pi_0(\mathcal{A})'$ *is in* $\mathbb{M}_s(\mathcal{A})$ *if and only if there is a central sequence* $\{a_n\}$ *in* $\mathcal{A}$ *and* $\pi_n(a_n) \to A \in \mathbb{M}_s(\mathbb{C})$.

PROOF. (1) Since $\pi(a) = \pi_0(a) \oplus \pi_1(a) \oplus \pi_2(a) \oplus \cdots$ and $\pi_n(a) \to \pi_0(a)$, we see that the condition is necessary. Since π_m and π_n are finite-dimensional representations for $0 \le m < n < \infty$, with no unitarily equivalent irreducible direct summands, there are elements $a_{mn} \in \mathcal{A}$ with $0 \le a_{mn} \le 1$ such that $\pi_m(a_{mn}) = 0$ and $\pi_n(a_{mn}) = 1$. Thus, $\lim_{n\to\infty} \|\pi_n(a_{01})\| = \|\pi_0(a_{01})\| = 0$. It follows that there is an $n_0 \in \mathbb{N}$ such that $0 \le \pi_n(a_{01}) \le 1/2$ whenever $n \ge n_0$. If $f : [0,1] \to [0,1]$ is continuous with $f(t) = 0$ when $t \in [0, 1/2]$ and $f(1) = 1$, then

$$\pi_1(f(a_{01})) = 1 \quad \text{and} \quad \pi_n(f(a_{01})) = 0 \quad \text{when } n \ge n_0.$$

Thus, $p_1 = f(a_{01}) \prod_{k=2}^{n_0-1} a_{k1}$ has the property that $\pi_1(p_1) = 1$ and $\pi_n(p_1) = 0$ when $n \ne 1$. Similarly, for $n = 2, 3, \ldots$, there is a $p_n \in \mathcal{A}$ such that $\pi_n(p_n) = 1$ and $\pi_k(p_n) = 0$ when $k \ne n$. Since $\pi_n : \mathcal{A} \to \mathbb{M}_n(\mathbb{C})$ is surjective for each $n \in \mathbb{N}$, we see that the set $\mathcal{S}$ of all $0 \oplus T_1 \oplus T_2 \oplus \cdots$ with only finitely many nonzero summands is contained in $\pi(\mathcal{A})$. The closure $\mathcal{S}^-$ of $\mathcal{S}$, which is the set of all $0 \oplus T_1 \oplus T_2 \oplus \cdots$ with $\lim_{n\to\infty} \|T_n\| = 0$, is also contained in $\pi(\mathcal{A})$. If $T_0 = \pi_0(a)$ for some $a \in \mathcal{A}$ and $\lim_{n\to\infty} T_n = T_0$, then

$$T_0 \oplus T_1 \oplus \cdots - \pi(a) \in \mathcal{S}^- \subset \pi(\mathcal{A}),$$

so

$$T_0 \oplus T_1 \oplus \cdots \in \pi(\mathcal{A}).$$

(2) Suppose that $\{a_n\}$ is a central sequence and $\pi_n(a_n) \to A \in \mathbb{M}_s(\mathbb{C})$. Suppose that $T \in \pi_0(\mathcal{A})$. Then, by (1), $\hat{T} = T \oplus T \oplus \cdots \in \pi(\mathcal{A})$. Thus,

$$\|AT - TA\| = \lim_{n\to\infty} \|\pi_n(a_n)T - T\pi_n(a_n)\| \leq \lim_n \|\pi(a_n)\hat{T} - \hat{T}\pi(a_n)\| = 0.$$

It follows that $A \in \pi_0(\mathcal{A})'$. Now suppose that $A \in \pi_0(\mathcal{A})'$ and, for each $n \in \mathbb{N}$, define $A_n = A_n(0) \oplus A_n(1) \oplus A_n(2) \oplus \cdots$, where $A_n(n) = A$ and $A_n(k) = 0$ when $k \neq n$. From (1), $A_n \in \pi(T)$ and, for every $T \in \pi_0(\mathcal{A})$,

$$A_n\hat{T} - \hat{T}A_n = 0.$$

Thus, $\{A_n\}$ is a central sequence in $\pi(\mathcal{A})$ and there is a central sequence $\{a_n\}$ in $\mathcal{A}$ such that, for each $n \in \mathbb{N}$, $\pi(a_n) = A_n$ and $\pi_n(a_n) = A$. □

THEOREM 2.3. *Suppose that $\mathcal{A}$ is a separable unital C*-algebra.*

(1) *If every central sequence of $\mathcal{A}$ is hypercentral, then, for every $s \in \mathbb{N}$, every representation in $Irr_s(\mathcal{A})^-$ is multiplicity free.*
(2) *If $\mathcal{A}$ is subhomogeneous and, for every $s \in \mathbb{N}$, every representation in $Irr_s(\mathcal{A})^-$ is multiplicity free, then every central sequence in $\mathcal{A}$ is hypercentral.*

PROOF. (1) Suppose that every central sequence of $\mathcal{A}$ is hypercentral. Assume, by way of a contradiction, that there are an s in $\mathbb{N}$ and a sequence $\{\pi_k\}$ in $\mathrm{Irr}_s(\mathcal{A})$ such that $\pi_k(a) \to \pi(a)$ for every $a \in A$, for some $\pi \in \mathrm{Rep}_s(\mathcal{A})$, and π is not multiplicity free. Thus, $\pi(\mathcal{A})'$ contains projections P and Q such that $PQ - QP \neq 0$. Let

$$\pi_\infty = \pi \oplus \pi_1 \oplus \pi_2 \oplus \cdots .$$

If infinitely many of the π_n are unitarily equivalent, we can replace the sequence with a new sequence such that, for every $n \in \mathbb{N}$, there is a unitary matrix U_n such that $\pi_n(a) = U_n\pi_1(a)U_n^*$ for every $a \in \mathcal{A}$. In this case, if we find a subsequence $\{U_{n_k}\}$ that converges to a unitary U, then π_0 is unitarily equivalent to π_1, which contradicts the assumption that π_0 is multiplicity free. Thus, we can assume that π_m and π_n are not unitarily equivalent when $0 \leq m < n < \infty$. Suppose that $A, B \in \pi_0(\mathcal{A})'$. By Lemma 2.2, there are central sequences $\{a_n\}, \{b_n\}$ in $\mathcal{A}$ such that

$$\lim_{n\to\infty} \pi_n(a_n) = A \quad \text{and} \quad \lim_{n\to\infty} \pi_n(b_n) = B.$$

Since $\{a_n\}$ is hypercentral,

$$\begin{aligned}\|AB - BA\| &= \lim_n \|\pi_n(a_n)\pi_n(b_n) - \pi_n(b_n)\pi_n(a_n)\| \\ &\leq \lim_n \|\pi(a_n)\pi(b_n) - \pi(b_n)\pi(a_n)\| = 0.\end{aligned}$$

Thus, $\pi_0(\mathcal{A})'$ is abelian, which means that π_0 is multiplicity free.

(2) Suppose that $\mathcal{A}$ is subhomogeneous. Then there is an N such that every irreducible representation of $\mathcal{A}$ is unitarily equivalent to a representation in $\cup_{s=1}^N \mathrm{Irr}_s(\mathcal{A})$.

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Now suppose that there is a central sequence $\{a_n\}$ in $\mathcal{A}$ that is not hypercentral. Then there are a subsequence $\{a_{n_k}\}$, a central sequence $\{b_n\}$ in $\mathcal{A}$ and an $\varepsilon > 0$ such that, for every $k \in \mathbb{N}$,

$$\|a_{n_k} b_{n_k} - b_{n_k} a_{n_k}\| \geq \varepsilon.$$

By relabelling the subsequences, we can assume that, for every $n \in \mathbb{N}$,

$$\|a_n b_n - b_n a_n\| \geq \varepsilon.$$

For each $n \in \mathbb{N}$, there is a representation $\pi_n \in \cup_{s=1}^{N} \mathrm{Irr}_s(\mathcal{A})$ such that, for every $n \in \mathbb{N}$,

$$\|\pi_n(a_n b_n - b_n a_n)\| = \|a_n b_n - b_n a_n\|.$$

By again restricting to a subsequence, we can assume that there is an s, with $1 \leq s \leq N$, such that $\pi_n \in \mathrm{Irr}_s(\mathcal{A})$. Since $\mathrm{Rep}_s(\mathcal{A})$ is compact, we can also assume that there is a $\pi_0 \in \mathrm{Rep}_s(\mathcal{A})$ such that $\pi_n(a) \to \pi_0(a)$ for every $a \in \mathcal{A}$. If π_0 is unitarily equivalent to infinitely many of $\pi_1, \pi_2, \ldots$, then $\|\pi_0(a_{n_k} b_{n_k} - b_{n_k} a_{n_k})\| \geq \varepsilon$ for infinitely many $k \in \mathbb{N}$ and $\{\pi_0(a_{n_k})\}$ and $\{\pi_0(b_{n_k})\}$ are central sequences for $\pi_0(\mathcal{A}) = \mathcal{M}_s(\mathbb{C})$, which is impossible. Hence, we can assume that, for every $n \in \mathbb{N}$, π_0 is not unitarily equivalent to π_n. Finally, we can assume that

$$\pi_n(a_n) \to A \quad \text{and} \quad \pi_n(b_n) \to B$$

in $\mathbb{M}_n(\mathbb{C})$. By Lemma 2.2, $A, B \in \pi_0(\mathcal{A})'$. Hence, $\pi_0(\mathcal{A})'$ is not abelian and

$$\|AB - BA\| = \lim_n \|\pi_n(a_n)\pi_n(b_n) - \pi_n(b_n)\pi_n(a_n)\| \geq \varepsilon.$$

Thus, $\pi_0(\mathcal{A})'$ is not abelian, so π_0 is not multiplicity free. □

To compare the property that every central sequence in a subhomogeneous C*-algebra is hypercentral with the property that every central sequence is trivial, we will show that the latter is equivalent to the property that $\mathrm{Irr}_s(\mathcal{A})$ is closed for every $s \in \mathbb{N}$.

Suppose that s is a positive integer. We define an *s-equivalence system* to be a tuple (X, α, β), where:

(1) X is a compact metric space;
(2) $\alpha \subset X \times X$ is a closed equivalence relation;
(3) $\beta : \alpha \to \mathcal{U}_s$ (the set of unitary operators in $\mathbb{M}_s(\mathbb{C})$) is such that

$$\beta(x, y)^* \beta(x, z) = \beta(y, z) \quad \text{whenever } (x, y), (y, z) \in X.$$

If (X, α, β) is an s-equivalence system, we define $C(X, \alpha, \beta, \mathbb{M}_s(\mathbb{C}))$ to be the set of all functions $f \in C(X, \mathbb{M}_s(\mathbb{C}))$ such that, whenever $x, y \in \alpha$,

$$\beta(x, y) f(x) \beta(x, y)^* = f(y).$$

It is clear that $C(X, \alpha, \beta, \mathbb{M}_s(\mathbb{C}))$ is a unital C*-algebra that contains every function of the form

$$f(x) = h(x) I_s$$

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with $h \in C(X)$ such that $h(x) = h(y)$ whenever $(x, y) \in \alpha$. We say that the system (X, α, β) is *regular* if, for every $x \in X$,

$$\{f(x) : f \in C(X, \alpha, \beta, \mathbb{M}_s(\mathbb{C}))\} = \mathbb{M}_s(\mathbb{C}).$$

LEMMA 2.4. *If (X, α, β) is a regular s-equivalence system, then every central sequence in $C(X, \alpha, \beta, \mathbb{M}_s(\mathbb{C}))$ is trivial.*

PROOF. Suppose that $\{a_n\}$ is a central sequence in $C(X, \alpha, \beta, \mathbb{M}_s(\mathbb{C}))$.
Claim:

$$\lim_{n\to\infty} \sup_{x\in X} \operatorname{dist}(a_n(x), \mathbb{C}I_s) = 0.$$

Assume, via contradiction, that the claim is false. By considering a subsequence, we can assume that there are an $\varepsilon > 0$ and a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} x_n = x_0 \in X \text{ and } a(x_n) \to A \in \mathbb{M}_s(\mathbb{C})$$

and, for every $n \in \mathbb{N}$,

$$\operatorname{dist}(a_n(x_n), \mathbb{C}I_s) \geq \varepsilon.$$

It follows that dist $(A, \mathbb{C}I_s) \geq \varepsilon$. Hence, there is a $B \in \mathbb{M}_s(\mathbb{C})$ such that

$$\|AB - BA\| = \delta > 0.$$

We can choose $f \in C(X, \alpha, \beta, \mathbb{M}_s(\mathbb{C}))$ so that $f(x_0) = B$. Thus,

$$0 = \lim_{n\to\infty} \|fa_n - a_n f\| \geq \lim_{n\to\infty} \|f(x_n)a_n(x_n) - a_n(x_n)f(x_n)\| = \|f(x_0)A - Af(x_0)\| \geq \delta.$$

This contradiction proves the claim.

If $T \in \mathbb{M}_s(\mathbb{C})$, $\lambda \in \mathbb{C}$ and $\|T - \lambda I_s\| = \operatorname{dist}(T, \mathbb{C}I_s)$, then

$$|\tau_s(T) - \lambda| = \|\tau_s(T - \lambda I_s)\| \leq \operatorname{dist}(T, \mathbb{C}I_s).$$

Thus, $\|T - \tau_s(T)\| \leq 2\operatorname{dist}(T, \mathbb{C}I_s)$. For each $n \in \mathbb{N}$, define $z_n = \tau_s \circ a_n$. It is clear that $z_n \in \mathcal{Z}(C(X, \alpha, \beta, \mathbb{M}_s(\mathbb{C})))$ and

$$\lim_{n\to\infty} \|a_n - z_n\| = \lim_{n\to\infty} \sup_{x\in X} \|a_n(x) - \tau_s(a_n(x))I_s\| \leq 2 \lim_{n\to\infty} \sup_{x\in X} \operatorname{dist}(a_n(x), \mathbb{C}I_s) = 0.$$

Hence, $\{a_n\}$ is trivial. □

LEMMA 2.5. *Suppose that $\mathcal{A}$ is a separable unital C^*-algebra, $s \in \mathbb{N}$, $Irr_s(\mathcal{A})$ is closed and every irreducible representation of $\mathcal{A}$ is unitarily equivalent to a representation in $Irr_s(\mathcal{A})$. Then there is a regular s-equivalence system (X, α, β) such that $\mathcal{A}$ is isomorphic to $C(X, \alpha, \beta, \mathbb{M}_s(\mathbb{C}))$.*

PROOF. Let $X = \operatorname{Irr}_s(\mathcal{A})$ and let α denote unitary equivalence. For each $\pi_0 \in X$ and each π such that $\pi \, \alpha \, \pi_0$, we can choose a unitary $\beta(\pi_0, \pi) = U_{(\pi_0,\pi)}$ such that

$$U_{(\pi_0,\pi)}\pi_0(\cdot)U^*_{(\pi_0,\pi)} = \pi(\cdot).$$

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If $\rho \in X$ and $\rho\,\alpha\,\pi$, we define $U_{(\pi,\rho)} = U_{(\pi_0,\rho)}U^*_{(\pi_0,\pi)}$. We define

$$\Gamma : \mathcal{A} \to C(X, \alpha, \beta, \mathbb{M}_s(\mathbb{C}))$$

by

$$\Gamma(a)(\pi) = \pi(a).$$

It follows that β is regular. A pure state on $C(X, \alpha, \beta, \mathbb{M}_s(\mathbb{C}))$ can be extended to a pure state φ on $C(X, \mathbb{M}_s(\mathbb{C}))$, which has the form

$$\varphi(f) = \langle f(x_0)e, e\rangle$$

for some $x_0 \in X$, which, restricted to $C(X, \alpha, \beta, \mathbb{M}_s(\mathbb{C}))$, is a pure state. It is clear that the set of pure states on $C(X, \alpha, \beta, \mathbb{M}_s(\mathbb{C}))$ is closed and that $\Gamma(\mathcal{A})$ separates the pure states on $\mathcal{A}$. It follows from Glimm's Stone–Weierstrass theorem [3] that $\Gamma(\mathcal{A}) = C(X, \alpha, \beta, \mathbb{M}_s(\mathbb{C}))$. □

THEOREM 2.6. *Suppose that $\mathcal{A}$ is a separable unital subhomogeneous C*-algebra. The following are equivalent:*

(1) *$\mathcal{A}$ is a continuous-trace C*-algebra;*
(2) *every central sequence in $\mathcal{A}$ is trivial;*
(3) *for every $s \in \mathbb{N}$, $Irr_s(\mathcal{A})$ is closed in $Rep_s(\mathcal{A})$;*
(4) *$\mathcal{A}$ is isomorphic to a finite direct sum of C*-algebras of the form $C(X, \alpha, \beta, \mathbb{M}_s(\mathbb{C}))$ with (X, α, β) a regular s-equivalence system.*

PROOF. (1) ⇔ (2). This is proved in [1].

(2) ⇒ (3). Suppose that $\mathrm{Irr}_s(\mathcal{A})$ is not closed. Then there are a $\pi_0 \in \mathrm{Rep}_s(\mathcal{A})$ that is not irreducible and a sequence $\{\pi_n\}$ in $\mathrm{Irr}_s(\mathcal{A})$ such that $\pi_n(a) \to \pi_0(a)$. Arguing as in the proof of Theorem 2.3(1), we can assume that π_m and π_n are not unitarily equivalent for $1 \le m < n < \infty$. Let $\pi = \pi_0 \oplus \pi_1 \oplus \cdots$. Since π_0 is not irreducible, there is a projection P with $0 \ne P \ne 1$ such that $P \in \pi_0(\mathcal{A})'$. Thus, by Lemma 2.2, there is a central sequence $\{a_n\}$ in $\mathcal{A}$ such that

$$\pi_n(a_n) \to P.$$

Thus,

$$\tfrac{1}{2} = \mathrm{dist}(P, \mathbb{C}I_s) = \lim_{n\to\infty} \mathrm{dist}(\pi_n(a_n), \mathbb{C}I_r) \le \lim_{n\to\infty} \mathrm{dist}(a_n, \mathcal{Z}(\mathcal{A})) = 0.$$

This contradiction proves that (2) ⇒ (3).

(3) ⇒ (4). Suppose that $\mathrm{Irr}_s(\mathcal{A})$ is closed in $\mathrm{Rep}_s(\mathcal{A})$ for every $s \in \mathbb{N}$. Since $\mathcal{A}$ is subhomogeneous, there is a minimal positive integer N such that every irreducible representation of $\mathcal{A}$ is unitarily equivalent to a representation in $\cup_{s=1}^N \mathrm{Irr}_s(\mathcal{A})$. Suppose that $1 \le m < n \le N$. For each $\pi \in \mathrm{Irr}_m(\mathcal{A})$ and each $\rho \in \mathrm{Irr}_n(\mathcal{A})$, there is an $a \in \mathcal{A}$ with $0 \le a \le 1$ such that $\pi(a) = 0$ and $\rho(a) = 1$. Thus, there is an open subset $U_{\pi,\rho}$ containing π such that $0 \le \sigma(a) \le 1/2$ for every $\sigma \in U$. Choose a continuous $h : [0, 1] \to [0, 1]$ so that $h(1)$ and $h|_{[0,1/2]} = 0$. Then $b_{U,\pi,\rho} = h(a)$ satisfies $\sigma(b_{U,\pi,\rho}) = 0$

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for every $\sigma \in U_{\pi,\rho}$ and $\rho(b_{U,\rho}) = 1$. Since $\{U_{\pi,\rho} : \pi \in \mathrm{Irr}_m(\mathcal{A})\}$, there is a finite subcover $\{U_{\pi_k,\rho} : 1 \le k \le t\}$ such that, if

$$b_{m,\rho} = \Big(\prod_{1\le i\le t} b_{U_{\pi_i,\rho}}\Big)^* \Big(\prod_{1\le i\le t} b_{U_{\pi_i,\rho}}\Big),$$

then $0 \le b_{m,\rho} \le 1$ and $\pi(b_{m,\rho}) = 0$ for every $\pi \in \mathrm{Irr}_m(\mathcal{A})$ and $\rho(b_{m,\rho}) = 1$. We now let $c_{m,r} = 1 - b_{m,\rho}$. Then $\pi(c_{m,r}) = 1$ for every $\pi \in \mathrm{Irr}_m(\mathcal{A})$ and $0 \le \sigma(c_{m,\rho}) \le 1/2$. Proceeding as before, there is a $d_{m,n} \in \mathcal{A}$ such that $0 \le d_{m,n} \le 1$ and $\pi(d_{m,n}) = 1$ for $\pi \in \mathrm{Irr}_m(\mathcal{A})$ and $\sigma(d_{m,n}) = 1$ for $\sigma \in \mathrm{Irr}_n(\mathcal{A})$. It follows that, for $1 \le m \le N$, there is a $p_m \in \mathcal{A}$ with $0 \le p_m \le 1$ such that $\pi(p_m) = 1$ when $\pi \in \mathrm{Irr}_m(\mathcal{A})$ and $\pi(p_n) = 0$ when $\pi \in \mathrm{Irr}_k(\mathcal{A})$ with $k \ne m$. Thus, p_n is a central projection and $p_1 + \cdots + p_N = 1$. Thus, $\mathcal{A}$ is a direct sum

$$\mathcal{A} = \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_N,$$

where, for $1 \le m \le N$, every irreducible representation of $\mathcal{A}_m$ is unitarily equivalent to a representation in $\mathrm{Irr}_m(\mathcal{A}_m)$ and $\mathrm{Irr}_m(\mathcal{A}_m)$ is closed. Thus, by Lemma 2.5, $\mathcal{A}_m$ is isomorphic to $C(X_m, \alpha_m, \beta_m, \mathbb{M}_m(\mathbb{C}))$ for some regular m-equivalence system (X_m, α_m, β_m).

(4) $\Rightarrow$ (1). This follows from Lemma 2.4. □

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