ASYMPTOTICS FOR SYSTEMIC RISK WITH DEPENDENT HEAVY-TAILED LOSSES

BY

JIAJUN LIU AND YANG YANG

ABSTRACT

Systemic risk (SR) is considered as the risk of collapse of an entire system, which has played a significant role in explaining the recent financial turmoils from the insurance and financial industries. We consider the asymptotic behavior of the SR for portfolio losses in the model allowing for heavy-tailed primary losses, which are equipped with a wide type of dependence structure. This risk model provides an ideal framework for addressing both heavy-tailedness and dependence. As some extensions, several simulation experiments are conducted, where an insurance application of the asymptotic characterization to the determination and approximation of related SR capital has been proposed, based on the SR measure.

KEYWORDS

asymptotics; dependence; heavy-tailedness; systemic risk; systemic risk contagion

1. INTRODUCTION

In the wake of the financial crisis and the collapses of Lehman Brother and AIG, systemic risk (SR), considered as the risk of collapse of an entire financial system, has been widely used to explain the recent financial turmoils for insurance/actuarial and financial industries; some recent papers include Acharya et al. (2012, 2017), Adrian and Brunnermeier (2016), Brownlees and Engle (2017), and Asimit and Li (2018), among many others. This topic is of particular relevance to insurers who play a significant role in the economy as suppliers of protection against financial and economic risks. Moreover, insurers share some general characteristics with banks, such as the management of cash flows over different risk horizons to meet claims arising from providing financial services. After the global 2007–2009 financial crisis, many

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macro-prudential policies have been promoted by regulatory bodies in the insurance industry. A list of global systemically important insurers were published by the Financial Stability Board who intends to carry out special policy measures for these institutions by January 2019. In the same direction, the US regulatory reform, known as the Dodd-Frank Act, imposed a new form of regulation on non-bank holding companies (including insurance companies), who were considered as "Systemically Important Financial Institutions". SR and reform proposals have led academics to focus on evaluating the financial distress of a system as a result of the failure of an individual in the entire system. Nevertheless, measuring SR has been a challenging task in insurance and financial industries. One challenge behind SR is modeling the extreme risks, which generally arise from individuals' large losses. Larger losses exhibit higher systemic importance, see Gravelle and Li (2013). Fortunately, the study of extreme risks has recently attracted increasing attention in insurance and finance. The presence of heavy-tail phenomena in data resulting from a wide range of application spanning finance, insurance, and risk management is well-documented. Gabaix et al. (2006) and Gabaix (2009) offered theoretical results and empirical estimators supporting the heavy-tailedness for financial returns on many stocks and stock indices in different markets. Acharya (2009) emphasized that joint risks rather than individual risks should be taken into considerations of regulating the SRs, which reminds us that modeling dependency among individual institutions, or between the individual and the economy (or the financial system) poses a few challenges.

Generally, there are two major approaches employed for measuring SRs in the literature. One consists of taking a structural approach using network analysis and works directly on the structure and the nature of relationships between institutions in the market; see e.g. Ledford and Tawn (1996), and Gray et al. (2011). Another reduced-form approach is to investigate the contagion effect of one institution on the market and its contribution to the entire SR. De Jonghe (2010) applied extreme value analysis to estimate tail indices for European financial institutions as their SR measure. Balla et al. (2014) investigated the extreme loss tail dependence between stock returns and derived extremal dependence-based SR indicators. Adrian and Brunnermeier (2016) introduced the CoVaR measure to quantify an institution's contribution to SR or to the risk of another institution.

Acharya et al. (2017) bridged the gap between the structural and reduced-form approaches by introducing the systemic expected shortfall into a simple economic model, which revisited the widely popular measure Marginal Expected Shortfall (MES). A comprehensive discussion was presented by Idierb et al. (2014), where the practical advantages of MES in detecting extreme risk exposure of a financial institution to SR were empirically explained. Nonparametric inferences for MES were provided in Cai et al. (2015) by a statistic extreme value approach. Asymptotic evaluations of the MES were investigated in Asimit and Li (2016). A variate of variants on MES including

Conditional Tail Expectation, Marginal Mean Excess, and their asymptotic approximations have appeared in the insurance and actuarial science literature, see Asimit et al. (2011), Hua and Joe (2011), Das and Fasen (2018), Asimit and Li (2018). The determinants found in the literature provide us a guidance for dependent heavy-tailed losses in our model. Accordingly, we follow the precise methodology from Acharya et al. (2012), see also Acharya et al. (2017) and Brownlees and Engle (2017), where SR represents the expected capital shortfall of a system when one component is in financial distress.

The design of this paper is to offer a methodological framework for modeling SRs with extreme risks taken into account. Specifically, we model extreme risks by some regularly varying-tailed random variables possessing tail asymptotic independence, and study the asymptotic behavior of the conditional expectations of the system's under-capitalization when one component is under-capitalized. The expected capital shortfall of the system is regarded as a definition for insurance but not for corporate finance, although we do not purpose, it is sufficiently general to be applied to financial and insurance industries. The asymptotic tail independence indicates that the cooccurrence of very high values is extremely unlikely for two jointly heavy-tailed random variables. A classical example for asymptotic tail independence can be found in Das et al. (2013) in which Pareto-type risk factors are dependent according to a Gaussian copula. Recently, Hua and Joe (2014) studied the behavior of conditional tail expectation in an asymptotic tail independence case. Das and Fasen (2018) investigated the asymptotic behavior of certain risk contagion measures with multivariate regularly varying, hidden regularly varying, or tail asymptotically independent risk vectors. In this paper, we address asymptotically independent risks via the tail asymptotic independence when investigating SRs.

Our main results provide asymptotic results for the SR as well as the systemic risk contagion with heavy-tailed losses in extreme regions, which are relevant to a large array of financial institutions, particularly insurance companies. An immediate application of the obtained results is to determine and approximate the related SR capital for such institutions, since it aims at estimating the capital shortfall in a potential future failure. We observe that although in our model the tail asymptotic independence is considered, the aggregate SR converges to infinity under some mild conditions, which implies that the aggregate SR remains significant with the underlying weak-dependent risks. In addition, the empirical estimator is no longer as effective when data are scarce or even unavailable in the tail region of interest, which is usually a challenging task for both insurance/financial institutions and regulators. In that case, our results provide better estimators when data are hardly available in the tail regions.

The rest of this paper is organized as follows. Section 2 introduces our SR model, Section 3 prepares some preliminaries. Our main results and two special cases for refinements are given in Section 4. Section 5 presents our numerical results.

2. Modeling systemic risk

In this section, we use a transparent definition by Acharya et al. (2012) to model the SR, see also Acharya et al. (2017), Brownlees and Engle (2017), Asimit and Li (2018). Assume that there are n lines of businesses or legal entities with potential net loss variables $Z_1, ..., Z_n$, which may depend on each other since these n lines operate in similar macroeconomic environments. Each line i generates a net loss variable Z_i incurred by a primary loss variable X_i at the terminal time and a stochastic discount factor θ_i over the period. Then,

$$S_n^\theta = \sum_{i=1}^n Z_i = \sum_{i=1}^n \theta_i X_i$$

represents the total amount of discounted losses potentially incurred from risky investments. For more detailed discussions on S_n^θ, three examples in insurance, finance, and risk management can be found in Tang and Yuan (2014), where the system could be referred to the entire industry or a group of companies, whereas the individual components could be a single company within the industry, a legal entity, or even a line of business.

For some regulated business, such as banks or insurance companies, a risk capital C_i allocated to each entity is usually held as a buffer to protect them from large losses. Hence, the regulator sets a total capital $\sum_{i=1}^n C_i$. Note that C_i given above could be nonpositive, in other words, no capital or even a negative amount of capital is allocated to line i in this case, which means that such a line should be rewarded with risk capital. See Erel et al. (2013) for related discussions.

Without loss of generality, we assume that the first line of business or legal entity is in financial distress. Therefore, the aggregate SR becomes

$$\mathrm{SR} := \mathbb{E}\left[\left(S_n^\theta - \sum_{i=1}^n C_i\right)^+ \middle| \theta_1 X_1 > C_1\right],$$

where $x^+ = \max\{x, 0\}$. The individual systemic contribution to the kth component, known as the SR contagion, is defined as

$$\mathrm{SR}_k := \mathbb{E}\left[(\theta_k X_k - C_k)^+ \,|\theta_1 X_1 > C_1\right], \quad k = 1, \ldots, n.$$

The above definitions strongly rely on the way how the regulatory capital is defined. Nowadays, a commonly used and practical capital decomposition rule is the Euler one, in which C_i is replaced by

$$C_{i,\mathrm{CVaR}}(q) := \mathbb{E}\left[\theta_i X_i | S_n^\theta > \mathrm{VaR}_q(S_n^\theta)\right], \quad i = 1, \ldots, n,$$

with

$$\mathrm{VaR}_q(S_n^\theta) = \inf\{y \in \mathbb{R} : \mathbb{P}(S_n^\theta \le y) \ge q\}, \qquad 0 < q < 1.$$

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See Section 6.3.2 of McNeil et al. (2015). This is the so-called CVaR_q-based regulatory environment, which requires a total capital

$$\mathrm{CVaR}_q(S_n^\theta) = \sum_{i=1}^{n} C_{i,\mathrm{CVaR}}(q) = \mathbb{E}\left[S_n^\theta | S_n^\theta > \mathrm{VaR}_q(S_n^\theta)\right].$$

It is well known that the Swiss Solvency Test that defines the insurance regulatory environment in Switzerland is a CVaR_q-based regulation regime example. Euler's principle is considered to provide a right guidance for performance measures, which allocates risk capital accounting to the risk contribution of each line. For more details, we refer the reader to McNeil et al. (2015) and Dhaene et al. (2012), among others. In this way, the SR and SR_k's under the CVaR_q-based allocations can be expressed as

$$\mathrm{SR}_{\mathrm{CVaR}}(q) := \mathbb{E}\left[\left(S_n^\theta - \sum_{i=1}^{n} C_{i,\mathrm{CVaR}}(q)\right)^+ \middle| \theta_1 X_1 > C_{1,\mathrm{CVaR}}(q)\right], \qquad (2.1)$$

and

$$\mathrm{SR}_{k,\mathrm{CVaR}}(q) := \mathbb{E}\left[(\theta_k X_k - C_{k,\mathrm{CVaR}}(q))^+ \middle| \theta_1 X_1 > C_{1,\mathrm{CVaR}}(q)\right], \quad k = 1, \ldots, n. \qquad (2.2)$$

Nevertheless, some other regulatory environments in European Union, Japan, Brazil, and Bermuda are VaR_q-based. Alternatively, if the entire system is VaR_q regulated, then the total capital is $\mathrm{VaR}_q(S_n^\theta)$, and a more practical solution (see Kalkbrener (2005) or Bluhm et al. (2006)) is to use a surrogate $\mathrm{CVaR}_{\eta(q)}$-type allocation rule in the following fashion

$$C_{i,\mathrm{VaR}}(q) := \mathbb{E}\left[\theta_i X_i | S_n^\theta > \mathrm{VaR}_{\eta(q)}(S_n^\theta)\right], \quad i = 1, \ldots, n,$$

where

$$\eta(q) := \inf_{u \in (0,1]} \left\{\mathrm{VaR}_q(S_n^\theta) \le \mathrm{CVaR}_u(S_n^\theta)\right\}.$$

Then, the SR and SR_k's under the VaR_q-based allocations can be expressed as

$$\mathrm{SR}_{\mathrm{VaR}}(q) := \mathbb{E}\left[\left(S_n^\theta - \sum_{i=1}^{n} C_{i,\mathrm{VaR}}(q)\right)^+ \middle| \theta_1 X_1 > C_{1,\mathrm{VaR}}(q)\right], \qquad (2.3)$$

and

$$\mathrm{SR}_{k,\mathrm{VaR}}(q) := \mathbb{E}\left[(\theta_k X_k - C_{k,\mathrm{VaR}}(q))^+ \middle| \theta_1 X_1 > C_{1,\mathrm{VaR}}(q)\right], \quad k = 1, \ldots, n. \qquad (2.4)$$

Typically, the expressions given by (2.1)–(2.4) are difficult to evaluate. We aim at the asymptotic formulas as $q \uparrow 1$, when the excessive prudence of the current regulatory framework requires a confidence level close to 1.

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3. Preliminaries

3.1. Heavy-tailed distributions

By definition, for a distribution function F on $\mathbb{R}$, we write $F \in \mathcal{R}_{-\alpha}$ for some $\alpha > 0$ if its right tail $\overline{F}(x) = 1 - F(x) > 0$ for all $x > 0$ is regularly varying with index $-\alpha$, that is,

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha},$$

holds for any fixed $y > 0$. The definition immediately indicates that $\overline{F}(x)$ is a power-like function in the sense that it differs from the power function $x^{-\alpha}$ by up to a slowly varying function at ∞. The reader is referred to Bingham et al. (1987) and Embrechts et al. (1997), among others.

A smaller value of α means a heavier right tail of F. This class has been extensively used to describe heavy-tail phenomena in insurance and finance, which contains many popular distributions such as the Pareto, log-gamma, Burr, and student's t distributions. It has been further shown by a few empirical studies that most financial risks are moderately heavy-tailed with parameter $\alpha > 1$. Some recent studies even argue that the tail exponent α in heavy-tailed models typically lies in the interval (2, 5) for financial returns on many stocks and stock indices in different markets, see Gabaix et al. (2006) and Gabaix (2009).

If $F \in \mathcal{R}_{-\alpha}$ for some $\alpha > 0$, then it follows from Theorem 1.5.6 of Bingham et al. (1987) that, for every $\epsilon > 0$, there is some $x_0 > 0$ such that for all $x, y \geq x_0$,

$$(1-\epsilon)\left(\left(\frac{x}{y}\right)^{\alpha+\epsilon} \wedge \left(\frac{x}{y}\right)^{\alpha-\epsilon}\right) \leq \frac{\overline{F}(y)}{\overline{F}(x)} \leq (1+\epsilon)\left(\left(\frac{x}{y}\right)^{\alpha+\epsilon} \vee \left(\frac{x}{y}\right)^{\alpha-\epsilon}\right). \quad (3.1)$$

Moreover, Proposition 1.3.6 of Bingham et al. (1987) indicates that for every $\epsilon > 0$,

$$x^{-(\alpha+\epsilon)} = o(\overline{F}(x)). \quad (3.2)$$

3.2. Tail Asymptotic Independence

Studying the tail behavior of portfolio risk S_n^θ in SR modeling requires carefully addressing some dependence among the primary losses $X_1, \ldots, X_n$. Asymptotic independence represents that the probability of two or more components being simultaneously large can be negligible compared with one component being large. Generally speaking, for all $i \neq j \geq 1$, two random variables X_i and X_j, distributed by F_i and F_j, respectively, are said to be asymptotically independent if they have a zero coefficient of (upper) tail dependence

$$\lambda_{ij} = \lim_{u\uparrow 1} \Pr\left(X_i > F_i^{\leftarrow}(u) \,\middle|\, X_j > F_j^{\leftarrow}(u)\right) = 0,$$

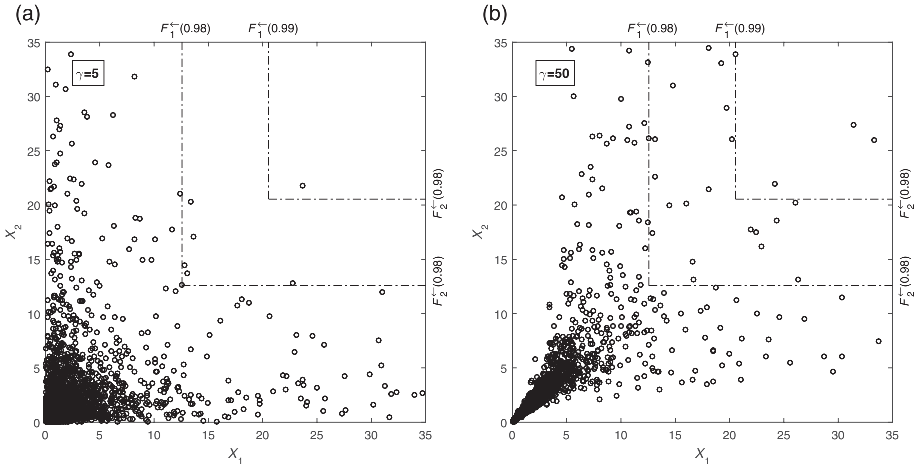

FIGURE 1: 5000 samples of (X_1, X_2), where X_1 and X_2 both follow Pareto (1.5,1) and are dependent via a Frank copula with parameter γ.

where $F^{\leftarrow}$ is the generalized inverse of F, see McNeil et al. (2015). However, $\lambda_{ij} = 0$ does not imply that the multivariate distribution can be automatically factorized asymptotically, as shown in Figure 1, where we visualize the tail risk scenarios. Choose 5000 samples of X_1 and X_2 representing two risks, both following the Pareto distribution (4.1) with parameters $\alpha = 1.5$ and $\vartheta = 1$, and they are associated through the Frank copula (4.2) with parameter $\gamma = 5$ or 50. The samples exceeding 98% (99%) quantile in the upper-right corner increase as parameter γ changes from 5 to 50. Although risks are less likely to be jointly extremely large in the upper-right corner, it is important to describe the tail risk given a high-risk scenario.

Indeed, the asymptotic independence contains a rich collection of dependence structures. Ledford and Tawn (1997) described the asymptotic independence in a more informative way, where for all $i \neq j \geq 1$, the asymptotic independence on two random variables with identical distributions X_i and X_j can be restated as

$$\lim_{x \to \infty} \Pr\left(X_i > x | X_j > x\right) = 0.$$

For nonindentically distributed random variables X_i and X_j, the upper tail asymptotic independence and the tail asymptotic independence (TAI) are defined as

$$\lim_{x_i \wedge x_j \to \infty} \Pr\left(X_i > x_i | X_j > x_j\right) = 0, \tag{3.3}$$

and

$$\lim_{x_i \wedge x_j \to \infty} \mathbb{P}(|X_i| > x_i | X_j > x_j) = 0, \tag{3.4}$$

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respectively by Geluk and Tang (2009), which allow the convergence to be independent from the path by which x_i and x_j go to infinity. Hua and Joe (2011) defined the tail order function in the form of copula: a bivariate copula $C(u, v)$ and its corresponding survival copula $\overline{C}$ satisfies $\overline{C}(u, v) = u + v - 1 + C(1 - u, 1 - v)$ for $u, v \in [0, 1]$. With a slowly varying function $l(\cdot)$ and an upper tail order κ, it can be described as $\lim_{u\downarrow 0} \frac{\overline{C}(u,u)}{u^\kappa l(u)} = 1$. The upper tailed order $\kappa > 1$ is called the extreme residual dependence by De Haan and Zhou (2011), which implies $\lambda_{ij} = 0$. In other words, there is much dependence left in the tails (see Figure 1), even within the framework of the asymptotic independence.

In this paper, we use the pairwise TAI structure to model the primary losses. Specifically, random variables $X_1, \ldots, X_n$ are said to be pairwise *tail asymptotically independent* (TAI), if for any $1 \leq i \neq j \leq n$ the relation (3.4) holds. The definition of TAI indicates that neither too positively nor too negatively can X_i and X_j be dependent. In spite of this, a wide range of dependence structures are included such as mutual independence, pairwise negative dependence (see, Lehmann (1966)), the Ali-Mikhail-Haq, Clayton, Frank, see Geluk and Tang (2009). Clearly, the TAI implies asymptotic independence of X_i and X_j.

Our study on asymptotically independent risks via the TAI structure becomes particularly relevant in the following circumstances. First, it is well documented that companies may be weakly dependent in good times while regulations would be drafted to prevent the strong dependence among them in a crisis. For empirical results of asymptotically independent stock returns of institutions, see Balla et al. (2014). Second, the underlying risks are asymptotically independent when the portfolio is well diversified. More detailed discussions on asymptotically independent risks can be found in Das et al. (2013) and Yuan (2017).

3.3. Modeling assumptions

Consider the model in which the primary losses $X_1, \ldots, X_n$ be n pairwise TAI real-valued random variables with marginal distributions $F_1, \ldots, F_n$, respectively, while the stochastic discount factors $\theta_1, \ldots, \theta_n$ are nonnegative, not degenerate at 0, and arbitrarily dependent on each other, but independent of $X_1, \ldots, X_n$. The right tails of $\theta_1, \ldots, \theta_n$ are lighter than $X_1, \ldots, X_n$.

Assume that all primary losses $X_1, \ldots, X_n$ are comparable in the right tail, that is, there exists a representative random variable X with distribution $F \in \mathcal{R}_{-\alpha}$ for some $\alpha > 1$, such that $\lim_{x\to\infty} \frac{\overline{F_i}(x)}{\overline{F}(x)} = b_i$ for some positive constants $b_i, \ldots, b_n$. We remark that, on the one hand, the tail index equivalence hypothesis has been adopted in other theoretical works on portfolio diversification with heavy-tails, see, e.g., Hyung and de Vries (2002, 2005, 2007), Ibragimov and Walden (2008) and Zhou (2010). Besides, empirical analyses performed by Moore et al. (2013) support the assumption of the tail index equivalence. On the other hand, although the insurance risks may have different tail indices α, the famous Breiman result (see, Breiman (1965)) ensures that the smallest

α's (the largest risks) dominate. In addition, if we want to include also the less severe losses, all marginal tails can be unified to the smallest α via a statistic approach. More related discussions can be found in (Resnick, 1987, Proposition 5.10), Kley and Klüppelberg (2016), and Kley et al. (2016).

Moreover, assume that the left tail of each primary loss is lighter than its right tail, that is,

$$F_i(-x) = o(\overline{F_i}(x)), \quad i = 1, \ldots, n. \tag{3.5}$$

Note that (3.5) is a mild condition from the practical viewpoint. In actuarial science, the individual loss X_i, considered as insurance risk, is interpreted as the total claim amount minus the total premium amount in period i. Then each individual loss X_i could be decomposed as $X_i = A_i - B_i$, where the nonnegative random variables A_i and B_i are independent. See Tang and Yuan (2015) for more details. If, further, $F_{A_i} \in \mathcal{R}_{-\alpha}$ and $\mathbb{E}[B_i^\beta] < \infty$ for some $\beta > \alpha$, then (3.5) is satisfied. In practice, it is reasonable to assume that the claim amount A_i tends to be heavy-tailed while the premium amount B_i is empirically light-tailed, see Zhou (2010). Moreover, X_i could also be represented as the loss given default of obligor i, or the financial loss. In such a situation, X_i is nonnegative, implying (3.5) is automatically satisfied. We refer the reader to Tang and Yuan (2013) and Tang et al. (2019).

4. Systemic risks under regularly varying losses

4.1. Exploratory numerical studies

Under the modeling assumptions in Section 3.3, we shall study the asymptotic behaviors of $\mathrm{SR}_{\mathrm{CVaR}}(q)$ and $\mathrm{SR}_{k,\mathrm{CVaR}}(q)$ to reflect our focus on the tail SR. By a heuristic analysis of $\mathrm{SR}_{\mathrm{CVaR}}(q)$ and $\mathrm{SR}_{k,\mathrm{CVaR}}(q)$, we expect that as $q \uparrow 1$ the value of $\mathrm{SR}_{\mathrm{CVaR}}(q)$ or $\mathrm{SR}_{k,\mathrm{CVaR}}(q)$ depends on the tail behavior of the primary losses and their tail dependence structure. To obtain a rough idea on how primary losses and the dependence structure affect the SR, we conduct the following numerical studies to analyze the SR in extreme regions.

Numerical study 4.1 Suppose that there are n lines of businesses or legal entities. Each line i generates a primary loss X_i with the Pareto distribution

$$F_i(x) = 1 - \left(\frac{\vartheta_i}{x + \vartheta_i}\right)^\alpha, \quad x > 0, \tag{4.1}$$

for some shape parameter $\alpha > 0$ and scale parameter $\vartheta_i \in \mathbb{R}$. $X_1, \ldots, X_n$, are dependent through a Frank copula

$$C(\mathbf{u}) = -\frac{1}{\gamma} \ln\left(1 + \frac{\prod_{i=1}^n (\mathrm{e}^{-\gamma u_i} - 1)}{(\mathrm{e}^{-\gamma} - 1)^{n-1}}\right), \quad \mathbf{u} \in (0, 1)^n, \tag{4.2}$$

for some parameter $\gamma \neq 0$. Let the stochastic discount factors θ_i's follow a common exponential distribution with rate $\lambda = 1.5$ and be dependent through a Gumbel copula

$$C(\mathbf{u}) = \exp\left\{ -\left(\sum_{i=1}^{n} (- \ln u_i)^{\eta} \right)^{\frac{1}{\eta}} \right\}, \quad \mathbf{u} \in (0, 1)^n, \tag{4.3}$$

for some parameter $\eta > 1$.

To obtain the empirical estimators for SR and SR_k in (2.1) and (2.2), we first need to construct the empirical estimator for $C_{k,\mathrm{CVaR}}(q)$. Generate $N = 10^8$ samples $(\theta_{1,i}, ..., \theta_{n,i}, X_{1,i}, ..., X_{n,i})$, $i = 1, ..., N$. Denote by $S^{\theta}_{n,(1)} \leq \cdots \leq S^{\theta}_{n,(N)}$ the order statistics of $S^{\theta}_{n,i} = \sum_{k=1}^{n} \theta_{k,i} X_{k,i}$, $i = 1, ..., N$, then the empirical estimator for $C_{k,\mathrm{CVaR}}(q)$ is given by

$$\widehat{C_k} = \frac{\sum_{i=1}^{N} \theta_{k,i} X_{k,i} \mathbf{1}_{\left(S^{\theta}_{n,i} > S^{\theta}_{n,(\lfloor Nq \rfloor)}\right)}}{\sum_{i=1}^{N} \mathbf{1}_{\left(S^{\theta}_{n,i} > S^{\theta}_{n,(\lfloor Nq \rfloor)}\right)}}, \qquad k = 1, \ldots, n,$$

where $\lfloor x \rfloor$ denotes the largest integer not exceeding $x \in \mathbb{R}$, and $\mathbf{1}_A$ is the indicator function of a set A. Hence, the empirical estimator for $\mathrm{SR}_{\mathrm{CVaR}}(q)$ and $\mathrm{SR}_{k,\mathrm{CVaR}}(q)$ is given by

$$\widehat{\mathrm{SR}} = \frac{\sum_{i=1}^{N} \left(S^{\theta}_{n,i} - \sum_{k=1}^{n} \widehat{C_k} \right)^{+} \mathbf{1}_{\left(\theta_{1,i} X_{1,i} > \widehat{C_1}\right)}}{\sum_{i=1}^{N} \mathbf{1}_{\left(\theta_{1,i} X_{1,i} > \widehat{C_1}\right)}}, \tag{4.4}$$

$$\widehat{\mathrm{SR}_k} = \frac{\sum_{i=1}^{N} \left(\theta_{k,i} X_{k,i} - \widehat{C_k} \right)^{+} \mathbf{1}_{\left(\theta_{1,i} X_{1,i} > \widehat{C_1}\right)}}{\sum_{i=1}^{N} \mathbf{1}_{\left(\theta_{1,i} X_{1,i} > \widehat{C_1}\right)}}. \tag{4.5}$$

For simplicity, we consider $n = 3$ lines of businesses or legal entities. Set the parameters to $\alpha = 1.5$ or 1.3, $\vartheta_i = i$, $\gamma = 2$ and $\eta = 6$, $i = 1, 2, 3$. The empirical estimators (4.4) and (4.5) are used to estimate the aggregate SR $\mathrm{SR}_{\mathrm{CVaR}}(q)$ and the systemic risk contagion $\mathrm{SR}_{k,\mathrm{CVaR}}(q)$. The plots of SRs against $\mathrm{VaR}_q[X_1]$ for q from 0.9459 to 0.9999 are demonstrated in Figure 2.

Figure 2 suggests the comonotonicity of the individual risk $\mathrm{VaR}_q(X)$, and the aggregate SR $\mathrm{SR}_{\mathrm{CVaR}}(q)$ (or the SR contagion $\mathrm{SR}_{1,\mathrm{CVaR}}(q)$) shows a rapid growth as q close to 1. Moreover, we observe in (a) and (b) of Figure 2 that the points all appear roughly on a straight line, which suggests that the $\mathrm{SR}_{\mathrm{CVaR}}(q)$ or $\mathrm{SR}_{1,\mathrm{CVaR}}(q)$ grows corresponding to $\mathrm{VaR}_q[X_1]$ as $q \uparrow 1$; that is, when $q \uparrow 1$,

$$\mathrm{SR}_{\mathrm{CVaR}}(q) \approx c_1(\alpha)\mathrm{VaR}_q[X_1], \text{ and } \mathrm{SR}_{1,\mathrm{CVaR}}(q) \approx c_2(\alpha)\mathrm{VaR}_q[X_1]$$

for some positive coefficients $c_1(\alpha)$, $c_2(\alpha)$. However, the values of $\mathrm{SR}_{2,\mathrm{CVaR}}(q)$ and $\mathrm{SR}_{3,\mathrm{CVaR}}(q)$ decay as the individual risk increases, which indicates that the

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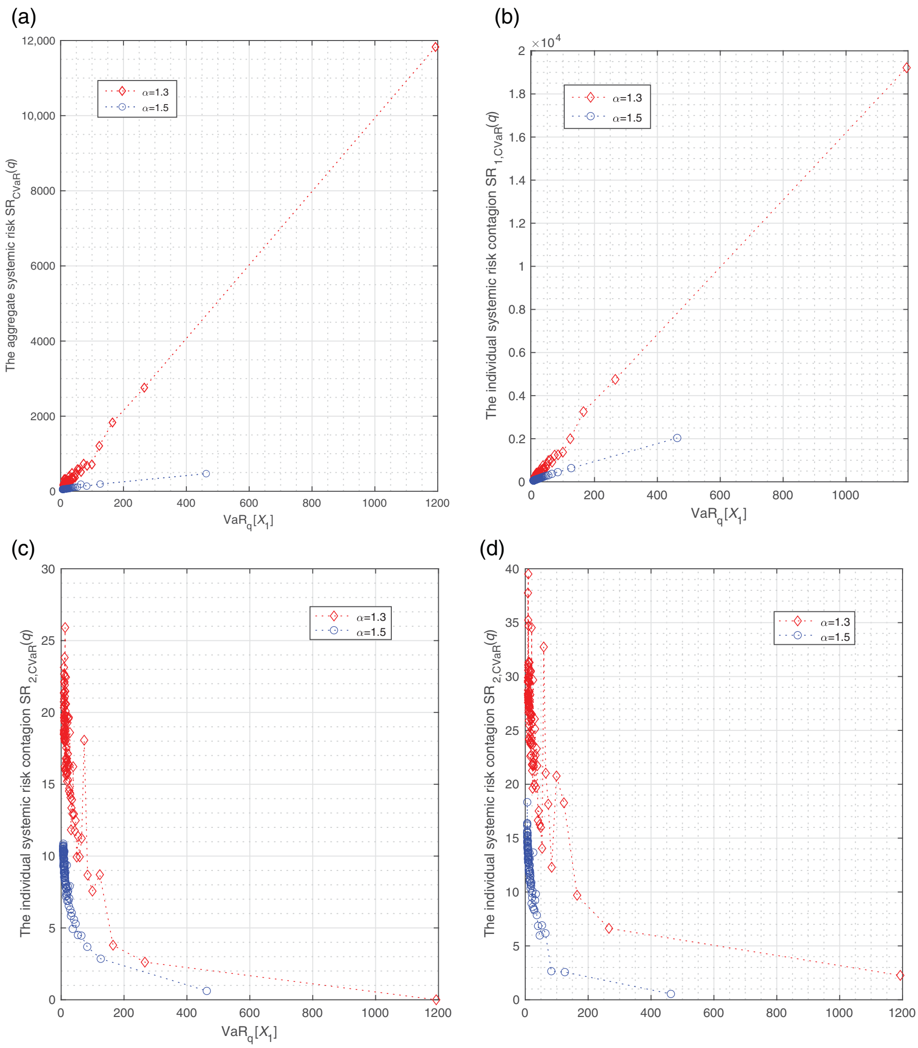

FIGURE 2: Comparison of the comovement of $\widehat{SR}$, $\widehat{SR}_k$, and $VaR_q(X_1)$ with $N = 10^8$.

values of $SR_{2,CVaR}(q)$ and $SR_{3,CVaR}(q)$ are negligible to $VaR_q[X_1]$ as $q \uparrow 1$. Such observations will be theoretically verified in refinements in Subsection 4.2.

Numerical study 4.2 Using a copula-based approach to describe the tail risk contagion requires an appropriate choice of the copula, and the possibility of choosing a wrong copula leads to significant model misspecification risks.

In this numerical study, we provide some numerical results to illustrate our characterizations of tail risk and the individual SR contagion $SR_{2,CVaR}$, where the impact of the misspecification on $SR_{2,CVaR}$ is investigated by examples that the tail dependence structure of the primary loss variables is misspecified.

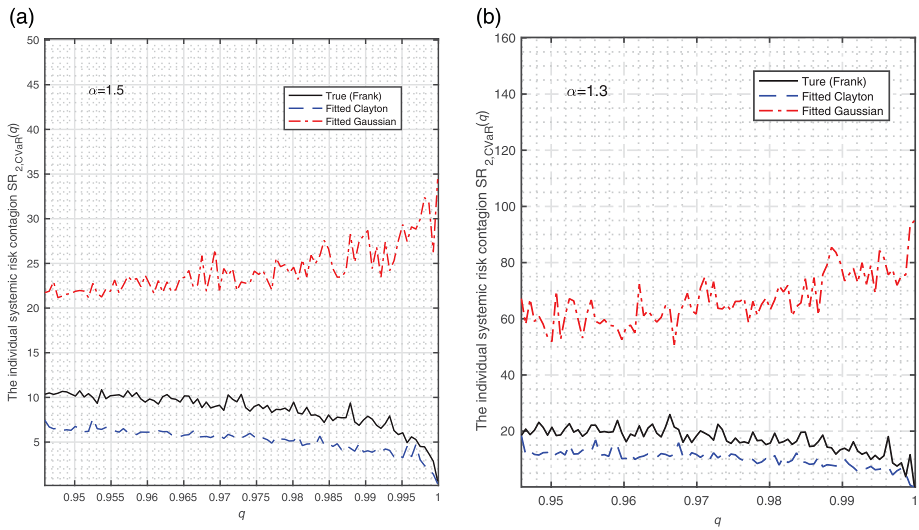

FIGURE 3: The empirically estimated $\mathrm{SR}_{2,\mathrm{CVaR}}$ based on the true copula (Frank), the fitted Clayton copula, and the fitted Gaussian copula, for q from 0.9459 to 0.9999, with $N = 10^8$.

Explicitly, we extract the primary loss variables (X_1, X_2) from Numerical study 4.1, where a Frank copula is regarded as the benchmark tail dependence between X_1 and X_2. Then we use a Clayton copula and a Gaussian copula to fit the generated data (a synthetic data set containing 10^8 samples of (X_1, X_2)) to see how the fitted cases differ from the benchmark. For both copulas, we applied the function *copulafit* in the *Matlab* to obtain maximum likelihood estimates for the parameters, where the parameter of the Clayton copula is estimated to be 0.3384, and that of the Gaussian copula is 0.3006. The obtained $\mathrm{SR}_{2,\mathrm{CVaR}}$'s are compared for the three cases to show the impact of misspecification in Figure 3. Clear discrepancies among the estimated $\mathrm{SR}_{2,\mathrm{CVaR}}$'s for the three cases can be observed in Figure 3, especially in the tail area, which implies that a misspecified dependence structure may lead to a significant over- or underestimation of the SR contagion on its extreme region. Moreover, one may note severe divergences on the tail behaviors of the estimated $\mathrm{SR}_{2,\mathrm{CVaR}}$'s as $q \uparrow 1$. For example, the estimated $\mathrm{SR}_{2,\mathrm{CVaR}}$'s decrease to 0 under the true Frank-copula specification and the Clayton-copula specification but not for the case under the Gaussian-copula specification. This will be further discussed in Subsection 4.3.

4.2. Main Results

As the main contribution of this paper, we derive the asymptotic behavior for the SR.

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Theorem 4.1. *Consider the aggregate SR in* (2.1) *under the modeling assumptions in Subsection 3.3. If* $\mathbb{E}[\theta_i^\beta] < \infty$ *for some* $\beta > \alpha > 1$, $i = 1, \ldots, n$, *then it holds that*

$$\lim_{q\uparrow 1} \frac{\mathrm{SR}_{\mathrm{CVaR}}(q)}{\mathrm{VaR}_q(X)} = \frac{\alpha}{(\alpha-1)^2} \frac{\left(b_1 \mathbb{E}[\theta_1^\alpha]\right)^\alpha}{\left(\sum_{i=1}^n b_i \mathbb{E}[\theta_i^\alpha]\right)^{\alpha - \frac{1}{\alpha}}}. \tag{4.6}$$

Our Theorem 4.1 quantitatively captures the aggregate SR with pairwise TAI losses, provided that one component of the system is undercapitalised. Relation (4.6) gives an infinite limit for the SR. On the one hand, the aggregate SR becomes much larger when financial distress is observed in one single component, which, thus, leads to a significant impact of the SR on the entire system. On the other hand, the whole system is vulnerable when the primary losses are heavy-tailed. Our second result aims to assess risk contagion among all other components in a system.

Theorem 4.2. *Consider the individual SR contagion in* (2.2) *under the conditions of Theorem 4.1, it holds that*

$$\lim_{q\uparrow 1} \frac{\mathrm{SR}_{k,\mathrm{CVaR}}(q)}{\mathrm{VaR}_q(X)} = \begin{cases} \frac{\alpha}{(\alpha-1)^2} \frac{b_1 \mathbb{E}[\theta_1^\alpha]}{\left(\sum_{i=1}^n b_i \mathbb{E}[\theta_i^\alpha]\right)^{1-\frac{1}{\alpha}}}, & k = 1, \\ 0, & k = 2, \ldots, n. \end{cases}$$

Theorem 4.2 shows that $\mathrm{SR}_{1,\mathrm{CVaR}}$ is proportional to $\mathrm{SR}_{\mathrm{CVaR}}$. Indeed, it holds from Theorems 4.1 and 4.2 that $\lim_{q\uparrow 1} \frac{\mathrm{SR}_{1,\mathrm{CVaR}}(q)}{\mathrm{SR}_{\mathrm{CVaR}}(q)} = \left(\frac{b_1 \mathbb{E}[\theta_1^\alpha]}{\sum_{i=1}^n b_i \mathbb{E}[\theta_i^\alpha]}\right)^{1-\alpha}$. This reminds us that all the other components spread the risks in the system. Nevertheless, the impact of the individual SR contagion on all the other components becomes negligible comparing with the value of the individual risk, when the tail asymptotic independence arises. This phenomenon might inspire regulatory bodies to promote regulations to weaken the dependence among components in one system, so that a chain reaction in the system is unlikely to happen. Then all other components remain solvent even through the entire system is overall under-capitalised. However, our Theorem 4.2 fails to capture the precise approximations for $\mathrm{SR}_{k,\mathrm{CVaR}}(q)$, $k = 2, \ldots, n$, under the TAI framework. We shall further consider the two special cases for refinements and look for the precise approximations for $\mathrm{SR}_{k,\mathrm{CVaR}}(q)$, $k = 2, \ldots, n$, in Subsections 4.3.

4.3. Refinements

In this section, we further investigate the asymptotic behavior of $\mathrm{SR}_{k,\mathrm{CVaR}}(q)$, $k = 2, \ldots, n$, for some refinements in the two special cases.

In the first case, we restrict the primary losses to the following pairwise strongly asymptotic independence structure. Random variables $X_1, \ldots, X_n$ are said to be pairwise *strongly tail asymptotically independent* (SAI), if for any

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$1 \leq i < j \leq n$, there exists some positive constant ρ such that

$$\lim_{x_i \wedge x_j \to \infty} \frac{\Pr(X_i > x_i, X_j > x_j)}{\Pr(X_i > x_i)\Pr(X_j > x_j)} = \rho.$$

Clearly, the above pairwise SAI structure can be rewritten in terms of copulas as following: if $C(\cdot, \cdot)$ is the copula of (X_i, X_j), $1 \leq i < j \leq n$, then,

$$\lim_{(u,v)\downarrow(0,0)} \frac{\overline{C}(u, v)}{uv} = \rho. \tag{4.7}$$

Related discussions on the pairwise SAI structure can be found in Li (2018). Although the pairwise SAI structure seems a bit strong, it includes many commonly-used copulas. For modeling purpose, we give some examples in terms of n-dimensional symmetric copulas.

Example 4.1. *If $X_1 \ldots, X_n$ are dependent through the Ali-Mikhail-Haq (AMH) copula*

$$C(\mathbf{u}) = (1-\gamma)\left(\prod_{i=1}^{n}\left(\frac{1-\gamma}{u_i}+\gamma\right)-\gamma\right)^{-1}, \quad \gamma \in (-1, 1],$$

then they are pairwise SAI with $\rho = 1 + \gamma$ in (4.7).

Example 4.2. *If $X_1 \ldots, X_n$ are dependent through the Clayton copula*

$$C(\mathbf{u}) = \left(\sum_{i=1}^{n} u_i^{-\gamma} - n + 1\right)^{-\frac{1}{\gamma}}, \quad \gamma \in (0, \infty),$$

then they are pairwise SAI with $\rho = 1 + \gamma$ in (4.7).

Example 4.3. *If $X_1 \ldots, X_n$ are dependent through the Frank copula in form of (4.2), then they are pairwise SAI with $\rho = \frac{\gamma e^{\gamma}}{e^{\gamma}-1}$ in (4.7).*

Example 4.4. *If $X_1 \ldots, X_n$ are dependent through the Farlie-Gumbel-Morgenstern (FGM) copula*

$$C(\mathbf{u}) = \left(1 + \gamma\prod_{i=1}^{n}(1-u_i)\right)\prod_{i=1}^{n} u_i, \quad \gamma \in (-1, 1],$$

then they are pairwise SAI with $\rho = 1 + \gamma$ in (4.7).

We remark that (3.4) implies (3.3) and

$$\lim_{x_i \wedge x_j \to \infty} \mathbb{P}(X_i < -x_i | X_j > x_j) = 0. \tag{4.8}$$

It is easy to check that relation (3.3) holds if $X_1 \ldots, X_n$ are pairwise SAI, while (4.8) is still needed. Recall that there exists a representative random variable X with distribution $F \in \mathcal{R}_{-\alpha}$ for some $\alpha > 1$, such that $\lim_{x\to\infty} \frac{\overline{F_i}(x)}{\overline{F}(x)} = b_i$ for some positive constants $b_1, ..., b_n$. Now we establish the precise approximations for $\mathrm{SR}_{k,\mathrm{CVaR}}(q)$, $k = 2, \ldots, n$, under the pairwise SAI structure.

Theorem 4.3. *Consider the individual SR contagion in* (2.2) *under the conditions of Theorem 4.1. If $X_1 \ldots, X_n$ are pairwise SAI and $\mathbb{E}[(\theta_1\theta_k)^\beta] < \infty$ for some $\beta > \alpha > 1$, then it holds that for $k = 2, \ldots, n$,*

$$\lim_{q\uparrow 1} \frac{\mathrm{SR}_{k,\mathrm{CVaR}}(q)}{\mathrm{VaR}_q(X)(1-q)} = \frac{\alpha^{1-\alpha}\rho b_k^{2-\alpha}\mathbb{E}[(\theta_1\theta_k)^\alpha](\mathbb{E}[\theta_k^\alpha])^{1-\alpha}}{(\alpha-1)^{2-\alpha}\mathbb{E}[\theta_1^\alpha]\left(\sum_{i=1}^n b_i\mathbb{E}[\theta_i^\alpha]\right)^{2-\alpha-\frac{1}{\alpha}}}.$$

Since $F \in \mathcal{R}_{-\alpha}$, we have the individual SR contagion $\mathrm{SR}_{k,\mathrm{CVaR}}(q)$ is asymptotically of the order $(1-q)^{1-1/\alpha}$. Thus, the individual SR contagion is asymptotically zero when $X_1 \ldots, X_n$ are pairwise SAI.

In the second case, we consider the most famous Gaussian copula

$$C(u_1, ..., u_n) = \phi_n(\phi^{\leftarrow}(u_1), ..., \phi^{\leftarrow}(u_n); \Sigma_n),$$

where ϕ is the standard-normal distribution function, $\phi_n(\,\cdot\,; \Sigma_n)$ is a n-variate normal distribution function with correlation matrix Σ_n. Then, there is a bivariate copula associated with pair (X_i, X_j) for all $1 \le i < j \le n$,

$$C(u, v) = \phi_2(\phi^{\leftarrow}(u), \phi^{\leftarrow}(v); \Sigma_2), \quad (u, v) \in [0, 1]^2, \tag{4.9}$$

with the correlation $\sigma_{12} = \sigma_{21} = \sigma \in (-1, 1)$. Clearly, its survival copula satisfies

$$\lim_{u\downarrow 0} \frac{\overline{C}(u, u)}{u^{\frac{2}{\sigma+1}} l(u)} = \lim_{u\downarrow 0} \frac{C(u, u)}{u^{\frac{2}{\sigma+1}} l(u)} = 1 \tag{4.10}$$

where $l(\cdot)$ is a slowly varying function at 0, see Ledford and Tawn (1996) and Reiss (1989). For a slowly variation function $l(\cdot)$, it follows from Theorem 1.5.6 of Bingham et al. (1987) that, for every ϵ, there is some $x_0 > 0$ such that for $x, y \ge x_0$

$$\frac{l(y)}{l(x)} \le (1+\epsilon)\left(\left(\frac{y}{x}\right)^{-\epsilon} \vee \left(\frac{y}{x}\right)^{\epsilon}\right). \tag{4.11}$$

Furthermore, it holds that

$$\lim_{u\downarrow 0} \frac{\overline{C}(ux, uy)}{u^{\frac{2}{\sigma+1}} l(u)} = x^{\frac{1}{\sigma+1}} y^{\frac{1}{\sigma+1}}, \tag{4.12}$$

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see Asimit and Li (2016). In (4.9) and (4.10), a common slowly varying function is

$$l(u) = C_\sigma(-\log u)^{-\frac{\sigma}{\sigma+1}},$$

where

$$C_\sigma = (1+\sigma)^{\frac{3}{2}}(1-\sigma)^{-\frac{1}{2}}(4\pi)^{-\frac{\sigma}{\sigma+1}}. \tag{4.13}$$

Our last result presents the asymptotic formulas for $\mathrm{SR}_{k,\mathrm{CVaR}}(q)$, $k=2,\ldots,n$, in the case that each pair of (X_i, X_j) follows a bivariate Gaussian copula (4.9), $1 \le i < j \le n$.

Theorem 4.4. *Consider the individual SR contagion in* (2.2) *under the conditions of Theorem 4.1. If each pair of (X_i, X_j) follows a bivariate Gaussian copula (4.9), $1 \le i < j \le n$, and $\mathbb{E}[\theta_1^{(\sigma+3)\zeta}] < \infty$, $\mathbb{E}[\theta_k^{(\sigma+3)\zeta}] < \infty$, $\mathbb{E}[(\theta_1\theta_k)^\zeta] < \infty$ for some $\zeta > \frac{\alpha}{\sigma+1}$, then it holds that for $k=2,\ldots,n$,*

$$\lim_{q\uparrow 1} \frac{\mathrm{SR}_{k,\mathrm{CVaR}}(q)}{\mathrm{VaR}_q(X)\,(1-q)^{\frac{1-\sigma}{1+\sigma}}\,(-\log(1-q))^{-\frac{\sigma}{\sigma+1}}} = \xi,$$

where the limit

$$\xi = \frac{(1+\sigma)\alpha^{1-\frac{\alpha(1-\sigma)}{1+\sigma}} b_1^{\frac{\sigma(1-\alpha)}{1+\alpha}} b_k^{\frac{2+\sigma-\alpha}{1+\sigma}} C_\sigma \mathbb{E}[(\theta_1\theta_k)^{\frac{\alpha}{1+\sigma}}](\mathbb{E}[\theta_k^\alpha])^{1-\frac{\alpha}{1+\sigma}}}{|\alpha-\sigma-1|(\alpha-1)^{1-\frac{\alpha(1-\sigma)}{1+\sigma}}\mathbb{E}[\theta_1^\alpha]^{\frac{1+\sigma-\alpha\sigma}{1+\sigma}}\left(\sum_{i=1}^n b_i\mathbb{E}[\theta_i^\alpha]\right)^{(1-\frac{1}{\alpha})(1-\alpha\frac{1-\sigma}{1+\sigma})}},$$

with C_σ defined in (4.13).

Since $F \in \mathcal{R}_{-\alpha}$, we have the individual SR contagion $\mathrm{SR}_{k,\mathrm{CVaR}}(q)$ is asymptotically of the order $(1-q)^{\frac{1-\sigma}{1+\sigma}-\frac{1}{\alpha}}(-\log(1-q))^{-\frac{\sigma}{1+\sigma}}$ when each pair of (X_i, X_j) follows a bivariate Gaussian copula. Thus, the $\mathrm{SR}_{k,\mathrm{CVaR}}(q)$ being asymptotically zero, or infinity is completely determined by the interplay between the values of σ and α. If $1 < \alpha < \frac{1+\sigma}{1-\sigma}$ (implying $\sigma > 0$) the $\mathrm{SR}_{k,\mathrm{CVaR}}(q)$ is asymptotically infinity; If either $\alpha \ge \frac{1+\sigma}{1-\sigma}$ or $\sigma \le 0$, the individual SR contagion is asymptotically zero.

Remark 4.1. *Suppose that $X_1,\ldots,X_n$ are mutually independent, then it holds that*

$$\lim_{q\uparrow 1}\frac{\mathrm{SR}_{k,\mathrm{CVaR}}(q)}{\mathrm{VaR}_q(X)(1-q)} = \frac{\alpha^{1-\alpha} b_k^{2-\alpha}\mathbb{E}[(\theta_1\theta_k)^\alpha](\mathbb{E}[\theta_k^\alpha])^{1-\alpha}}{(\alpha-1)^{2-\alpha}\mathbb{E}[\theta_1^\alpha]\left(\sum_{i=1}^n b_i\mathbb{E}[\theta_i^\alpha]\right)^{2-\alpha-\frac{1}{\alpha}}},$$

which coincides with Theorem 4.3 with $\rho = 1$ or Theorem 4.4 with $\sigma = 0$.

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4.4. An application

The characterization of the tail SR in Theorem 4.1 has immediate applications in the calculation of the required risk capital on the SR. According to the Swiss Solvency Test (SST) guidelines, the insurance company operating in Switzerland is given by the CVaR_q-based risk capital corresponding to a 99% level of confidence over a one-year horizon. In practice, many insurance companies select an even more conservative confidence level, e.g., 99.5%, 99.9%, or even 99.95%. To protect an insurance company from the SR under an extreme risk, the SR-based economic capital under the extreme risk condition can be defined similarly as the aggregate SR (2.1) given by the line with the largest loss in financial distress, that is

$$\mathrm{SR}_{M_n^\theta,\mathrm{CVaR}}(q) := \mathbb{E}\left[\left(S_n^\theta - \sum_{i=1}^n C_{i,\mathrm{CVaR}}(q)\right)^+ \middle| M_n^\theta > C_{M_n^\theta,\mathrm{CVaR}}(q)\right], \quad (4.14)$$

where $M_n^\theta = \bigvee_{i=1}^n \theta_i X_i$, and $C_{M_n^\theta,\mathrm{CVaR}}(q) := \mathrm{E}\left[M_n^\theta \middle| S_n^\theta > \mathrm{VaR}_q(S_n^\theta)\right]$. Here we provide an asymptotic formula as an alternative approximation.

Proposition 4.1. *Consider the SR under the extreme risk condition in (4.14) and under the conditions of Theorem 4.1, it holds that*

$$\lim_{q\uparrow 1} \frac{\mathrm{SR}_{M_n^\theta,\mathrm{CVaR}}(q)}{\mathrm{VaR}_q(X)} = \frac{\alpha}{(\alpha-1)^2}\left(\sum_{i=1}^n b_i \mathbb{E}[\theta_i^\alpha]\right)^{1/\alpha}. \quad (4.15)$$

5. Numerical Studies

In this section, we use some numerical examples to show the discrepant tail behaviors of the SR due to the heavy-tailedness of the primary losses and do a comparison of the empirical estimates and its asymptotic approximations for the SRs $\mathrm{SR}_{\mathrm{CVaR}}$ and $\mathrm{SR}_{1,\mathrm{CVaR}}$ (or $\mathrm{SR}_{\mathrm{VaR}}$ and $\mathrm{SR}_{1,\mathrm{VaR}}$), where it could be observed via the numerical studies that the asymptotic results may lead to a better estimate for the SRs of portfolio losses, when there is uncertainty in the tail part. For simplicity, we consider the case of $n = 3$ lines of businesses or legal entities with q from 0.9459 to 0.9999.

Similarly to Subsection 4.1, we assume that the primary losses X_i, $i = 1, 2, 3$, are distributed by a Pareto distribution (4.1) with α and $\vartheta_i = i$, respectively, and they are dependent via a Frank copula (4.2) with $\gamma = 2$. Again, let the stochastic discount factors θ_i follow an exponential distribution with rate $\lambda = 1.5$ and depend on each other through a Gumbel copula (4.3) with $\eta = 6$. Inspired by our asymptotic results in Theorems 4.1 and 4.2, we start by showing the discrepant tail behaviors of the SRs $\mathrm{SR}_{\mathrm{CVaR}}$ and $\mathrm{SR}_{1,\mathrm{CVaR}}$ when the heavy-tailedness of X_i's are changed according to α, such that $\alpha = 1.3, 1.4, 1.5$.

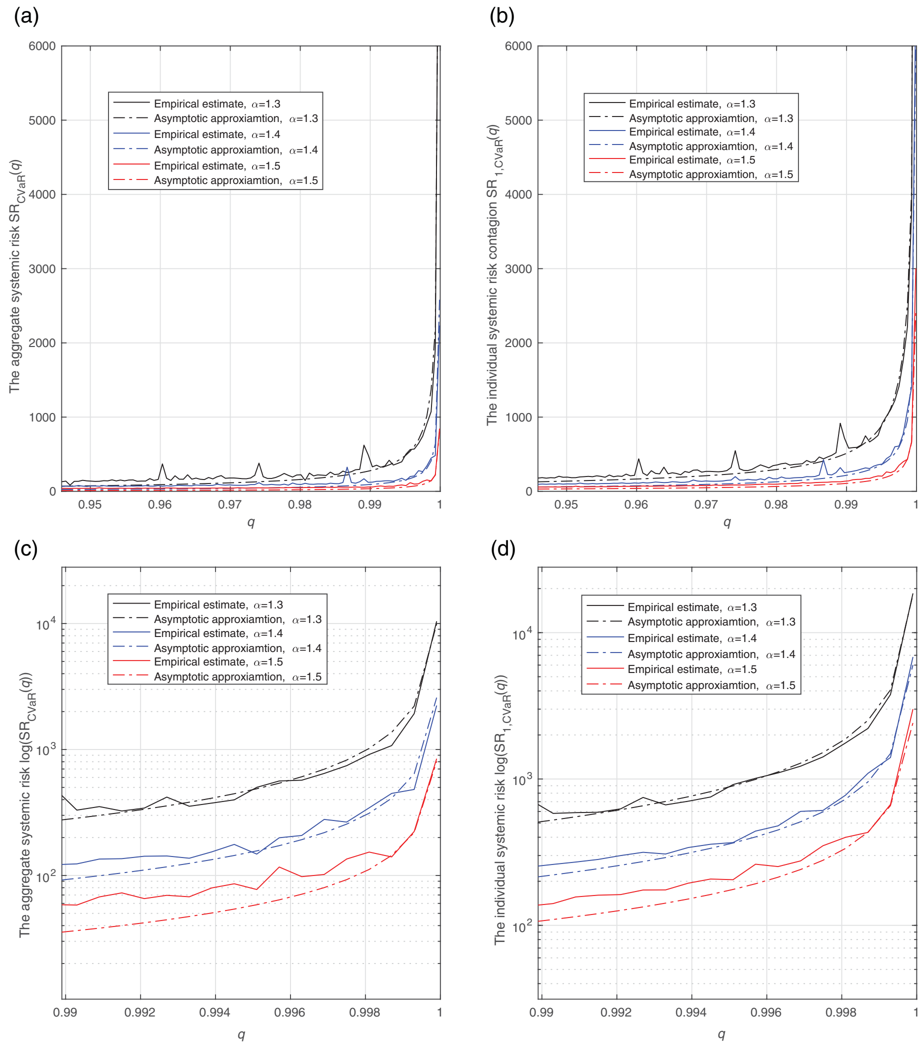

FIGURE 4: Comparisons between the asymptotic approximations of $\mathrm{SR}_{\mathrm{CVaR}}$ and $\mathrm{SR}_{1,\mathrm{CVaR}}$ ($\log(\mathrm{SR}_{\mathrm{CVaR}})$ and $\log(\mathrm{SR}_{1,\mathrm{CVaR}})$) and their empirical estimates (obtained based on $\alpha = 1.3, 1.4, 1.5$, with a sample size of $N = 10^8$).

We would like to point out that although we do not provide graphs here, our experiments show that values of both $\mathrm{SR}_{\mathrm{CVaR}}$ and $\mathrm{SR}_{1,\mathrm{CVaR}}$ are not sensitive to different TAI structures (i.e., the Frank, Clayton, FGM, Gaussian copulas, or with different γ) between the primary losses X_1, X_2, X_3, and the dependence structures between the stochastic discount factors θ_1, θ_2, θ_3. However, an increase on the value of λ will increase the level of $\mathrm{SR}_{\mathrm{CVaR}}$ and $\mathrm{SR}_{1,\mathrm{CVaR}}$. These are obvious according to our asymptotic results in Theorem 4.1 and Theorem 4.2.

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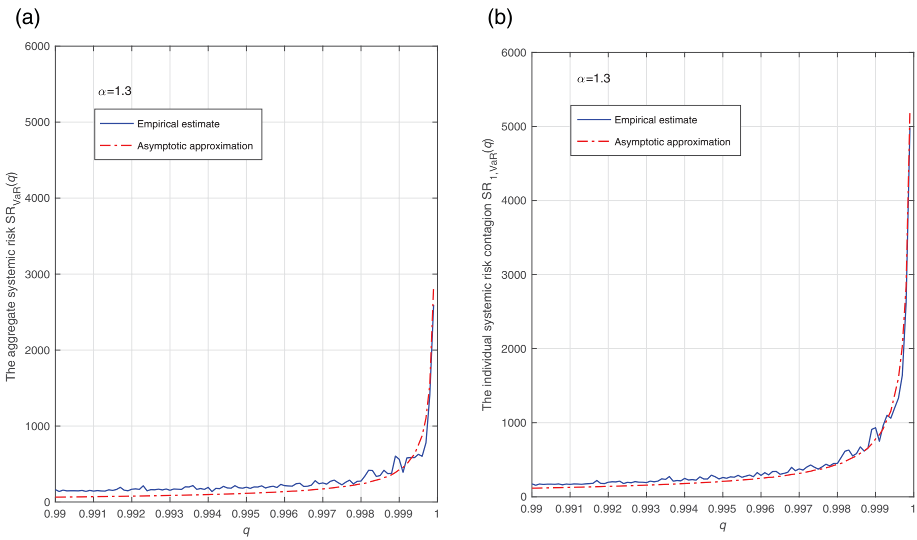

FIGURE 5: Comparisons between the asymptotic approximations of SR_{VaR} and $SR_{1,VaR}$ and their empirical estimates(obtained based on $\alpha = 1.3$, with a sample size of $N = 10^8$).

In addition to the movement of SR_{CVaR} and $SR_{1,CVaR}$ due to changes of the heavy-tailedness on the primary risk variables, one may observe from Figure 4 that the empirical estimations get more fluctuations in the tail part, that is, when the heavy-tailedness of primary losses arises. The graphs in Figure 4 show that the absolute differences between the empirical estimates and approximations are always small for q reasonably close to 1. The fluctuation for large q close to 1 should be due to the variation of empirical estimations, which become less stable when the event $\theta_1 X_1$ exceeding $C_{1,CVaR}(q)$ becomes rarer as q gets closer to 1.

Although we only provide the results for the SR and SR_k's under the $CVaR_q$-based allocations in Theorem 4.1 and Theorem 4.2, all the results can also be easily derived under the VaR_q-based allocations (i.e., $SR_{VaR}(q)$, $SR_{k,VaR}(q)$) by replacing the $VaR_q(X)$ with $VaR_{\eta(q)}(X)$. Hence, the asymptotic approximations in Theorem 4.1 and Theorem 4.2 can also be applied to approximate the $SR_{VaR}(q)$ and $SR_{k,VaR}(q)$, under the VaR_q- based allocations. With $\alpha = 1.3$, we provide comparisons between the asymptotic approximations of the $SR_{VaR}(q)$ and $SR_{k,VaR}(q)$ and their empirical estimates in Figure 5. We would like to highlight that the simulation costs on traditional empirical estimates for SR_{VaR} and $SR_{1,VaR}$ are much higher than SR_{CVaR} and $SR_{1,CVaR}$ due to the complex structures related to $\eta(q)$, where our asymptotic approximations simplify the computations and provide a better estimation on its tail part. In view of the fact that the traditional empirical estimates for SR need a large sample size when the confidence level q is closed to 1, our asymptotic estimates have apparent advantages in this situation.

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TABLE 1

EMPIRICAL ESTIMATE AND ASYMPTOTIC APPROXIMATION ON $\mathrm{SR}_{M_n^\theta,\mathrm{CVaR}}(q)$ AND ITS RATIO (EMPIRICAL ESTIMATE/ASYMPTOTIC APPROXIMATION) FOR $\alpha = 1.5$, WITH A SAMPLE SIZE OF $N = 10^8$.

Confidence level	Empirical estimate	Asymptotic approximation	Ratio
99%	1.709×10^7	1.625×10^7	1.0520
99.5%	2.674×10^7	2.626×10^7	1.0183
99.9%	7.787×10^7	7.831×10^7	0.9945
99.95%	1.201×10^8	1.248×10^8	0.9628

TABLE 2

ASYMPTOTIC APPROXIMATION ON $\mathrm{SR}_{\mathrm{CVaR}}(q)$ WITH A FINANCIAL DISTRESS IN THE iTH LINE FOR $\alpha = 1.5$ AND A SAMPLE SIZE OF $N = 10^8$.

Confidence level	Systemic risk with a financial distress in the first line	Systemic risk with a financial distress in the second line	Systemic risk with a financial distress in the third line
99%	7.523×10^4	1.702×10^6	1.055×10^7
99.5%	1.216×10^5	2.751×10^6	1.706×10^7
99.9%	3.625×10^5	8.203×10^6	5.086×10^7
99.95%	5.777×10^5	1.307×10^7	8.1044×10^7

In the last part of this section, we apply our approximations to evaluate the SR capital for a Swiss-based insurance company. Let us assume that a Swiss-based insurance company holds a portfolio of three business lines, where the primary loss random variable X_i, $i = 1, 2, 3$ for ith business line is Pareto distributed with $\alpha = 1.5$ and $\vartheta_i = 10^4 \cdot i$. The dependence structure of primary losses is described by a Frank copula in (4.2) with $\gamma = 2$. Since an insurer makes risky investments, who suffers the financial risk θ_i, e.g., $\theta_i = \frac{1}{1+R_i}$ with an overall return rate $R_i \in [-1, \infty)$. Without loss of generality, we assume that the financial risk follows the experiential distribution with rate $\lambda_i = i$ and be dependent through a Gumbel copula with $\eta = 6$. By the SST guidelines, the amount of economic capital against the SR on the extreme risk prepared by the insurer, is given by $\mathrm{SR}_{M_n^\theta,\mathrm{CVaR}}(q)$.

Table 1 elucidates relation (4.15) numerically, which herein considers four different confidence levels, i.e., 99%, 99.5%, 99.9%, and 99.95%. Further, Table 2 applies Theorem 4.1, which are to this end calculated for $\mathrm{SR}_{\mathrm{CVaR}}(q)$ with a financial distress in each business line for various confidence level q. From Table 1 and Table 2, we can see that $\mathrm{SR}_{M_n^\theta,\mathrm{CVaR}}(q)$ has a higher level than $\mathrm{SR}_{\mathrm{CVaR}}(q)$ with a financial distress in each line of business. In addition, the change in the values of $\mathrm{SR}_{M_n^\theta,\mathrm{CVaR}}(q)$ and $\mathrm{SR}_{\mathrm{CVaR}}(q)$ is more pronounced as the confidence level becomes less severe. Among the results of $\mathrm{SR}_{\mathrm{CVaR}}(q)$, the third line business always has a highest $\mathrm{SR}_{\mathrm{CVaR}}(q)$ as expected, according to a

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highest scale parameter ϑ_3. This application of the characterization shows that it is helpful in the determination of the extra risk capital needed for the whole system to mitigate impact of SRs for the insurer.

6. Concluding Remarks

In this paper, we study the tail behavior of the SRs for portfolio losses in the presence of extreme risks. We consider a portfolio loss model in which the primary losses are heavy-tailed and equipped with a wide type of dependence structure (TAI), and the stochastic discount factors can be arbitrarily dependent.

Under the TAI structure, we obtain an asymptotic characterization for the SRs based on the popular measure MES. Unlike the pioneering works by Cai et al. (2015), Asimit and Li (2016), Das and Fasen (2018), we consider the marginal excess of a systemic portfolio risks provided one line of business or legal entity is in a financial distress, where the risk capital allocation by Euler's principle is also taken into consideration. Distinguished from Asimit et al. (2011), a nonpositive amount of capital is also allowed to be allocated to any line of business. In many practical cases, our formulas can serve as some accurate approximations for the reasonably large confidence level q close to 1. Applications of the characterization show that it is helpful in the determination of the extra risk capital needed for the whole system to mitigate the impact of SRs, to insurance and financial institutions.

We further study the risk contagion $\mathrm{SR}_{k,\mathrm{CVaR}(q)}, k = 2, \ldots, n$, for some refinements in two special cases. In particular, we notice that the Gaussian copula can still produce significant tail risk contagion, and, therefore, could still be a good candidate for modeling tail risk. This may provide us some ideas of the potential extensions in the future, including capturing the tail risk contagion under some more general asymptotic dependence structures, which may be helpful to quantitatively analyze and distinguish the different impacts of the dependence structures between primary losses and has important implications in the SR management.

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JIAJUN LIU (Corresponding author)
Department of Statistics and Actuarial Science
Xi'an Jiaotong-Liverpool University
E-Mail: jiajun.liu@xjtlu.edu.cn

YANG YANG
Department of Statistics
Nanjing Audit University
E-Mail: yangyangmath@163.com

APPENDIX A

Throughout the paper, all limit relationships are according to $x \to \infty$ unless otherwise stated. For two positive functions $g_1(\cdot)$ and $g_2(\cdot)$, we write $g_1(x) = o(g_2(x))$ if $\lim g_1(x)/g_2(x) = 0$, write $g_1(x) = O(g_2(x))$ if $\limsup g_1(x)/g_2(x) < \infty$, and we write $g_1(x) \sim bg_2(x)$ to mean strong equivalence, i.e., $\lim g_1(x)/g_2(x) = b$ for some positive constant b, and we write $g_1(x) \asymp g_2(x)$ to mean weak equivalence, i.e., $0 < \liminf g_1(x)/g_2(x) \le \limsup g_1(x)/g_2(x) < \infty$. We also denote $\liminf g_1(x)/g_2(x) \ge 1$ and $\limsup g_1(x)/g_2(x) \le 1$ by $g_1(x) \gtrsim g_2(x)$ and $g_1(x) \lesssim g_2(x)$, respectively. For any $x, y \in \mathbb{R}$, write $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$, and $x^+ = x \vee 0$, $x^- = -(x \wedge 0)$.

A.1. Lemmas

Mimicking the proofs of Theorem 3.1 (a) of Zhang et al. (2009) and Lemma 3.1 of Chen and Yuen (2009), we can obtain the following Lemmas A.1–A.3.

Lemma A.1. *Under the conditions of Theorem 4.1, it holds that*

$$\mathbb{P}\left(\max_{1\le i\le n} S_i^\theta > x\right) \sim \mathbb{P}(S_n^\theta > x) \sim \mathbb{P}\left(\sum_{i=1}^n \theta_i X_i^+ > x\right) \sim \mathbb{P}\left(\bigvee_{i=1}^n \theta_i X_i > x\right) \sim \sum_{i=1}^n \mathbb{P}(\theta_i X_i > x).$$

Lemma A.2. *Let X be a real-valued random variable with distribution $F \in \mathcal{R}_{-\alpha}$ for some $\alpha > 0$ sasisfying $F(-x) = o(\overline{F}(x))$, and θ be another nonnegative random variable satisfying $\mathbb{E}[\theta^\beta] < \infty$ for some $\beta > \alpha$. Then, it holds that for any $y > 0$,*

$$\lim_{x\to\infty} \frac{\mathbb{P}(\theta X < -xy)}{\mathbb{P}(\theta X > x)} = 0.$$

Lemma A.3. *Under the conditions of Theorem 4.1, it holds that for any $y > 0$ and $k = 2, \ldots, n$,*

$$\lim_{x\to\infty} \frac{\mathbb{P}(\theta_1 X_1 > x, \theta_k X_k > xy)}{\mathbb{P}(\theta_1 X_1 > x)} = 0.$$

The next lemma plays an important role in the proofs of our main results.

Lemma A.4. *Under the conditions of Theorem 4.1, it holds that for $k=1,\ldots,n$,*

$$\mathbb{E}\left[\theta_k X_k \mathbf{1}_{(S_n^\theta > x)}\right] \sim \mathbb{E}\left[\theta_k X_k \mathbf{1}_{(\theta_k X_k > x)}\right]. \tag{A.1}$$

Proof. Without loss of generality, we only consider the case $k=1$. Motivated by Theorem 4.1 of Tang and Yuan (2014), we define a new probability measure $\mathbb{Q}$ by

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \frac{\theta_1 X_1^+}{\mathbb{E}[\theta_1]\mathbb{E}[X_1^+]},$$

where the symbol $\mathbb{E}$ without superscript means the expectation still under probability measure $\mathbb{P}$. Under $\mathbb{Q}$, the tail probabilities of X_1 and $\theta_1 X_1$ are given by

$$\mathbb{Q}(X_1 > x) = \frac{x\overline{F_1}(x) + \int_x^\infty \overline{F_1}(y)\mathrm{d}y}{\mathbb{E}[X_1^+]}, \quad x > 0,$$

and

$$\mathbb{Q}(\theta_1 X_1 > x) = \frac{x\mathbb{P}(\theta_1 X_1 > x) + \int_x^\infty \mathbb{P}(\theta_1 X_1 > y)\mathrm{d}y}{\mathbb{E}[\theta_1]\mathbb{E}[X_1^+]}, \quad x > 0.$$

By $F_1 \in \mathcal{R}_{-\alpha}$, Breiman's theorem (see, Breiman (1965)) leads to $F_{\theta_1 X_1} \in \mathcal{R}_{-\alpha}$, and Karamata's theorem (see Theorem 1.5.11(ii) of Bingham et al., 1987) further gives that the distribution functions of X_1 and $\theta_1 X_1$ under $\mathbb{Q}$ are both regularly varying tailed with the same index $-\alpha+1$.

For the desired (A.1), it suffices to prove

$$\mathbb{E}\left[\theta_1 X_1 \mathbf{1}_{(\theta_1 X_1 > x)}\right] \lesssim \mathbb{E}\left[\theta_1 X_1 \mathbf{1}_{(S_n^\theta > x)}\right], \tag{A.2}$$

and

$$\mathbb{E}\left[\theta_1 X_1^+ \mathbf{1}_{\left(\sum_{i=1}^n \theta_i X_i^+ > x\right)}\right] \sim \mathbb{E}\left[\theta_1 X_1 \mathbf{1}_{(\theta_1 X_1 > x)}\right]. \tag{A.3}$$

We first deal with (A.2). Clearly,

$$\begin{aligned}
&\mathbb{E}\left[\theta_1 X_1 \mathbf{1}_{(S_n^\theta > x)}\right] \\
&= \mathbb{E}\left[\theta_1 X_1^+ \mathbf{1}_{(S_n^\theta > x)}\right] - \mathbb{E}\left[\theta_1 X_1^- \mathbf{1}_{(S_n^\theta > x)}\right] \\
&\geq \mathbb{E}\left[\theta_1 X_1^+ \mathbf{1}_{\left(\theta_1 X_1 - \sum_{i=2}^n \theta_i X_i^- > x\right)}\right] - \mathbb{E}\left[\theta_1 X_1^- \mathbf{1}_{\left(\sum_{i=2}^n \theta_i X_i > x\right)}\right] \\
&=: I_1 - I_2.
\end{aligned} \tag{A.4}$$

Note that, under $\mathbb{Q}$,

$$I_1 = \mathbb{E}[\theta_1]\mathbb{E}[X_1^+]\mathbb{Q}\left(\theta_1 X_1 - \sum_{i=2}^n \theta_i X_i^- > x\right). \tag{A.5}$$

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For any $v > 1$, we have

$$\begin{aligned}
&\mathbb{Q}\left(\theta_1 X_1 - \sum_{i=2}^{n} \theta_i X_i^- > x\right) \\
&\geq \mathbb{Q}\left(\theta_1 X_1 > vx, \sum_{i=2}^{n} \theta_i X_i^- \leq (v-1)x\right) \\
&= \mathbb{Q}\left(\theta_1 X_1 > vx\right) - \mathbb{Q}\left(\theta_1 X_1 > vx, \sum_{i=2}^{n} \theta_i X_i^- > (v-1)x\right). \qquad \text{(A.6)}
\end{aligned}$$

By Lemma A.2, we have that for any $i = 2, \ldots, n$, under $\mathbb{P}$,

$$\begin{aligned}
\mathbb{P}\left(\theta_1 X_1 > vx, \theta_i X_i^- > \frac{(v-1)x}{n-1}\right) &\leq \mathbb{P}\left(\theta_i X_i < -\frac{(v-1)x}{n-1}\right) \\
&= o\left(1\right) \mathbb{P}(\theta_1 X_1 > x), \qquad \text{(A.7)}
\end{aligned}$$

and, moreover, for any $c < 1$ and all $y \geq vx \to \infty$,

$$\begin{aligned}
&\mathbb{P}\left(\theta_1 X_1 > y, \theta_i X_i^- > \frac{(v-1)x}{n-1}\right) \\
&\leq \mathbb{P}\left(\theta_1 X_1 > y, \theta_i X_i < -\frac{(v-1)x}{n-1}, \theta_1 \leq x^c, \theta_i \leq x^c\right) \\
&\quad + \mathbb{P}\left(\theta_1 X_1 > y, \theta_1 > x^c\right) + \mathbb{P}\left(\theta_1 X_1 > y, \theta_i > x^c\right) \\
&= o(\mathbb{P}(\theta_1 X_1 > y)), \qquad \text{(A.8)}
\end{aligned}$$

where in the last step we applied (3.4) to the first term and Lemma 7 of Tang and Yuan (2014) to the last two terms. By (A.7) and (A.8), we have

$$\begin{aligned}
&\mathbb{E}[\theta_1]\mathbb{E}[X_1^+]\mathbb{Q}\left(\theta_1 X_1 > vx, \theta_i X_i < -\frac{(v-1)x}{n-1}\right) \\
&= vx\mathbb{P}\left(\theta_1 X_1 > vx, \theta_i X_i < -\frac{(v-1)x}{n-1}\right) + \int_{vx}^{\infty} \mathbb{P}\left(\theta_1 X_1 > y, \theta_i X_i < -\frac{(v-1)x}{n-1}\right) \mathrm{d}y \\
&= o(1)\mathbb{Q}(\theta_1 X_1 > x),
\end{aligned}$$

which implies

$$\begin{aligned}
&\mathbb{Q}\left(\theta_1 X_1 > vx, \sum_{i=2}^{n} \theta_i X_i^- > (v-1)x\right) \\
&\leq \sum_{i=2}^{n} \mathbb{Q}\left(\theta_1 X_1 > vx, \theta_i X_i^- > \frac{(v-1)x}{n-1}\right). \\
&= o(1)\mathbb{Q}(\theta_1 X_1 > x). \qquad \text{(A.9)}
\end{aligned}$$

Combining (A.5), (A.6), (A.9), and letting $v \downarrow 1$, yields

$$I_1 \gtrsim \mathbb{E}\left[\theta_1 X_1 \mathbf{1}_{(\theta_1 X_1 > x)}\right], \qquad \text{(A.10)}$$

because, under $\mathbb{Q}$, the distribution of $\theta_1 X_1$ belongs to $\mathcal{R}_{-\alpha+1}$.

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As for I_2, by Lemma A.2, for any $0 < \epsilon < 1$,

$$\begin{aligned}\mathbb{E}\left[\theta_1 X_1^- \mathbf{1}_{\left(\theta_1 X_1^- > \epsilon x\right)}\right] &= \epsilon x \mathbb{P}(\theta_1 X_1^- > \epsilon x) + \int_{\epsilon x}^{\infty} \mathbb{P}(\theta_1 X_1^- > y)\mathrm{d}y \\ &= \epsilon \left(x\mathbb{P}(\theta_1 X_1 < -\epsilon x) + \int_{x}^{\infty} \mathbb{P}(\theta_1 X_1 < -\epsilon z)\mathrm{d}z \right) \\ &= o(1) \left(x\mathbb{P}(\theta_1 X_1 > x) + \int_{x}^{\infty} \mathbb{P}(\theta_1 X_1 > z)\mathrm{d}z \right) \\ &= o(1)\mathbb{E}\left[\theta_1 X_1 \mathbf{1}_{(\theta_1 X_1 > x)}\right]. \end{aligned} \tag{A.11}$$

By using Lemma A.1, (A.11) and Breiman's theorem, we have

$$\begin{aligned} I_2 &= \mathbb{E}\left[\theta_1 X_1^- \mathbf{1}_{\left(\theta_1 X_1^- \le \epsilon x, \sum_{i=2}^n \theta_i X_i > x\right)}\right] + \mathbb{E}\left[\theta_1 X_1^- \mathbf{1}_{\left(\theta_1 X_1^- > \epsilon x, \sum_{i=2}^n \theta_i X_i > x\right)}\right] \\ &\le \epsilon x \mathbb{P}\left(\sum_{i=2}^n \theta_i X_i > x\right) + \mathbb{E}\left[\theta_1 X_1^- \mathbf{1}_{\left(\theta_1 X_1^- > \epsilon x\right)}\right] \\ &= \epsilon x \sum_{i=2}^n \mathbb{P}(\theta_i X_i > x) + o(1)\mathbb{E}\left[\theta_1 X_1 \mathbf{1}_{(\theta_1 X_1 > x)}\right] \\ &= \epsilon \cdot O(1) x \mathbb{P}(\theta_1 X_1 > x) + o(1)\mathbb{E}\left[\theta_1 X_1 \mathbf{1}_{(\theta_1 X_1 > x)}\right] \\ &\le \left(\epsilon \cdot O(1) + o(1)\right) \mathbb{E}\left[\theta_1 X_1 \mathbf{1}_{(\theta_1 X_1 > x)}\right]. \end{aligned} \tag{A.12}$$

Therefore, relation (A.2) follows from (A.4), (A.10), (A.12) and by the arbitrariness of $\epsilon > 0$.

Now we consider (A.3). Clearly,

$$\begin{aligned}\mathbb{E}\left[\theta_1 X_1^+ \mathbf{1}_{\left(\sum_{i=1}^n \theta_i X_i^+ > x\right)}\right] &= \mathbb{E}\left[\theta_1 X_1 \mathbf{1}_{(\theta_1 X_1 > x)}\right] + \mathbb{E}\left[\theta_1 X_1^+ \mathbf{1}_{\left(\theta_1 X_1^+ \le x, \sum_{i=1}^n \theta_i X_i^+ > x\right)}\right] \\ &=: I_3 + I_4. \end{aligned} \tag{A.13}$$

For any $0 < \epsilon < 1$,

$$I_4 \le \epsilon x \mathbb{P}\left(\sum_{i=1}^n \theta_i X_i^+ > x\right) + x\mathbb{P}\left(\epsilon x < \theta_1 X_1 \le x, \sum_{i=1}^n \theta_i X_i^+ > x\right). \tag{A.14}$$

By Lemma A.1 and Breiman's theorem, we have

$$\mathbb{P}\left(\sum_{i=1}^n \theta_i X_i^+ > x\right) \sim \sum_{i=1}^n \mathbb{P}(\theta_i X_i > x) = O(1)\mathbb{P}(\theta_1 X_1 > x). \tag{A.15}$$

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Again by Lemma A.1 and Breiman's theorem,

$$
\begin{aligned}
&\mathbb{P}\left(\epsilon x < \theta_1 X_1 \le x, \sum_{i=1}^{n} \theta_i X_i^+ > x\right) \\
&\le \mathbb{P}\left(\sum_{i=1}^{n} \theta_i X_i^+ > x\right) - \mathbb{P}(\theta_1 X_1 > x) - \mathbb{P}\left(\theta_1 X_1 \le \epsilon x, \sum_{i=2}^{n} \theta_i X_i^+ > x\right) \\
&= \sum_{i=2}^{n} \mathbb{P}(\theta_i X_i > x) + o(1)\mathbb{P}(\theta_1 X_1 > x) - \mathbb{P}\left(\theta_1 X_1 \le \epsilon x, \sum_{i=2}^{n} \theta_i X_i^+ > x\right) \\
&= \mathbb{P}\left(\theta_1 X_1 > \epsilon x, \sum_{i=2}^{n} \theta_i X_i^+ > x\right) + o(1)\mathbb{P}(\theta_1 X_1 > x) \\
&\le \sum_{i=2}^{n} \mathbb{P}\left(\theta_1 X_1 > \epsilon x, \theta_i X_i > \frac{x}{n-1}\right) + o(1)\mathbb{P}(\theta_1 X_1 > x) \\
&= o(1)\mathbb{P}(\theta_1 X_1 > x), \qquad \text{(A.16)}
\end{aligned}
$$

where in the last step we used Lemma A.3. Plugging (A.15) and (A.16) into (A.14) yields

$$I_4 = o(1)x\mathbb{P}(\theta_1 X_1 > x) = o(1)\mathbb{E}[\theta_1 X_1 \mathbf{1}_{(\theta_1 X_1 > x)}],$$

by the arbitrariness of $\epsilon > 0$. Thus, the desired relation (A.3) holds from (A.13). □

Lemma A.5. *Under the conditions of Theorem 4.1, it holds that for $k = 1, \ldots, n$,*

$$\lim_{t\to\infty} \mathbb{P}(S_n^\theta > tx | \theta_k X_k > t) = \mathbf{1}_{(0<x\le 1)} + x^{-\alpha}\mathbf{1}_{(x>1)}. \qquad \text{(A.17)}$$

Proof. When $x > 1$, on the one hand,

$$
\begin{aligned}
&\mathbb{P}(S_n^\theta > tx, \theta_k X_k > t) \le \mathbb{P}\left(\sum_{i=1}^{n} \theta_i X_i^+ > tx, \theta_k X_k > t\right) \\
&\le \mathbb{P}\left(\sum_{i=1}^{n} \theta_i X_i^+ > tx\right) - \mathbb{P}\left(\bigcup_{i=1}^{n} \{\theta_i X_i^+ > tx\}, \theta_k X_k \le t\right) \\
&\le \mathbb{P}\left(\sum_{i=1}^{n} \theta_i X_i^+ > tx\right) - \mathbb{P}\left(\bigcup_{\substack{i=1 \\ i\ne k}}^{n} \{\theta_i X_i > tx\}, \theta_k X_k \le t\right) \\
&\le \mathbb{P}\left(\sum_{i=1}^{n} \theta_i X_i^+ > tx\right) - \sum_{\substack{i=1 \\ i\ne k}}^{n} \mathbb{P}\left(\theta_i X_i > tx\right) + \sum_{\substack{1\le i<j\le n \\ i\ne k\ne j}} \mathbb{P}(\theta_i X_i > tx, \theta_j X_j > tx) \\
&\quad + \sum_{\substack{i=1 \\ i\ne k}}^{n} \mathbb{P}(\theta_k X_k > t, \theta_i X_i > tx).
\end{aligned}
$$

Applying Lemma A.1 and Breiman's theorem (implying $F_{\theta_k X_k} \in \mathcal{R}_{-\alpha}$) leads to

$$\limsup_{t\to\infty} \mathbb{P}\left(S_n^\theta > tx|\theta_k X_k > t\right) \le x^{-\alpha}.$$

On the other hand, again by Lemma A.1,

$$\begin{aligned}
\mathbb{P}\left(S_n^\theta > tx, \theta_k X_k > t\right) &\ge \mathbb{P}(S_n^\theta > tx) - \mathbb{P}\left(\sum_{i=1}^n \theta_i X_i^+ > tx, \theta_k X_k \le t\right) \\
&= \mathbb{P}\left(S_n^\theta > tx\right) - \mathbb{P}\left(\sum_{i=1}^n \theta_i X_i^+ > tx\right) + \mathbb{P}\left(\sum_{i=1}^n \theta_i X_i^+ > tx, \theta_k X_k > t\right) \\
&\ge \mathbb{P}(S_n^\theta > tx) - \mathbb{P}\left(\sum_{i=1}^n \theta_i X_i^+ > tx\right) + \mathbb{P}(\theta_k X_k > tx, \theta_k X_k > t) \\
&= \mathbb{P}(S_n^\theta > tx) - \mathbb{P}\left(\sum_{i=1}^n \theta_i X_i^+ > tx\right) + \mathbb{P}\left(\theta_k X_k > tx\right) \\
&\sim \mathbb{P}\left(\theta_k X_k > tx\right),
\end{aligned}$$

which implies

$$\liminf_{t\to\infty} \mathbb{P}(S_n^\theta > tx|\theta_k X_k > t) \ge x^{-\alpha}.$$

When $0 < x \le 1$, on the one hand, we have $\mathbb{P}(S_n^\theta > tx, \theta_k X_k > t) \le \mathbb{P}(\theta_k X_k > t)$. On the other hand, applying a similar argument above and noting that $\mathbb{P}(\theta_k X_k > tx, \theta_k X_k > t) = \Pr\left(\theta_k X_k > t\right)$, we have $\mathbb{P}(S_n^\theta > tx, \theta_k X_k > t) \sim \mathbb{P}(\theta_k X_k > t)$. It ends the proof of the lemma. □

Lemma A.6. *Under the conditions of Theorem 4.1, it holds that*

$$\lim_{q\uparrow 1} \frac{C_{k,\mathrm{CVaR}}(q)}{\mathrm{VaR}_q(X)} = \frac{\alpha}{\alpha-1} \frac{b_k \mathbb{E}[\theta_k^\alpha]}{\left(\sum_{i=1}^n b_i \mathbb{E}[\theta_i^\alpha]\right)^{1-\frac{1}{\alpha}}}. \tag{A.18}$$

Proof. By Lemma A.1 and Breiman's theorem, we have

$$\mathbb{P}(S_n^\theta > x) \sim \overline{F}(x) \sum_{i=1}^n b_i \mathbb{E}[\theta_i^\alpha]. \tag{A.19}$$

By applying Lemma A.4, Karamata's theorem, Breiman's theorem and (A.19), we have

$$\begin{aligned}
\mathbb{E}\left[\theta_k X_k \middle| S_n^\theta > x\right] &\sim \frac{\mathbb{E}[\theta_k X_k \mathbf{1}_{(\theta_k X_k > x)}]}{\overline{F}(x)\sum_{i=1}^n b_i \mathbb{E}[\theta_i^\alpha]} \\
&= \frac{x\mathbb{P}(\theta_k X_k > x) + \int_x^\infty \mathbb{P}(\theta_k X_k > y)\mathrm{d}y}{\overline{F}(x)\sum_{i=1}^n b_i \mathbb{E}[\theta_i^\alpha]} \\
&\sim \frac{\alpha}{\alpha-1} \frac{x b_k \mathbb{E}[\theta_k^\alpha]}{\sum_{i=1}^n b_i \mathbb{E}[\theta_i^\alpha]}.
\end{aligned}$$

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By (A.19) and Proposition 0.8(v) of Resnick (1987), as $q \uparrow 1$,

$$\mathrm{VaR}_q(S_n^\theta) \sim F^{\leftarrow}\left(1 - \frac{1-q}{\sum_{i=1}^n b_i \mathbb{E}[\theta_i^\alpha]}\right) \sim \left(\sum_{i=1}^n b_i \mathbb{E}[\theta_i^\alpha]\right)^{\frac{1}{\alpha}} \mathrm{VaR}_q(X).$$

Thus, the desired (A.18) follows from the above two relations. □

The following lemma is a restatement of Lemma 3 (iii) of Li (2018).

Lemma A.7. *Under the conditions of Theorem 4.3, it holds that for $k = 2, ..., n$,*

$$\mathbb{P}(\theta_1 X_1 > x, \theta_k X_k > y) \sim \rho \mathbb{E}[(\theta_1\theta_k)^\alpha] \overline{F_1}(x)\overline{F_k}(y), \quad \text{as } (x, y) \to (\infty, \infty).$$

Next lemma will be used for the proof of Theorem 4.4

Lemma A.8. *Under the conditions of Theorem 4.4, it holds for that $k = 2, ..., n$,*

$$\mathbb{P}(\theta_1 X_1 > tx, \theta_k X_k > t) \sim (b_1 b_k)^{\frac{1}{1+\sigma}} x^{-\frac{\alpha}{1+\sigma}} \mathbb{E}[(\theta_1\theta_k)^{\frac{\alpha}{1+\sigma}}] \left(\overline{F}(t)\right)^{\frac{2}{1+\sigma}} l\left(\overline{F}(t)\right), \quad \text{as } t \to \infty.$$

Proof. Noting $\zeta > \frac{\alpha}{1+\sigma}$, Choose some $\frac{\alpha}{\zeta(1+\sigma)} < c < 1$, and some real number $M > 1$. Then split the tail probability $\mathbb{P}(\theta_1 X_1 > tx, \theta_k X_k > t)$ according to the set $\Delta_1 = [0, M]^2$, $\Delta_2 = \left(M, t^c\right]^2$, and $\Delta_3 = \left(t^c, \infty\right)^2$ as

$$\mathbb{P}(\theta_1 X_1 > tx, \theta_k X_k > t) = \sum_{i=1}^3 \mathbb{P}\left(\theta_1 X_1 > tx, \theta_k X_k > t, (\theta_1, \theta_k) \in \Delta_i\right) =: \sum_{i=1}^3 J_i.$$

In view of $F \in \mathcal{R}_{-\alpha}$ and $\overline{F_i}(x) \sim b_i \overline{F}(x), i = 1, \ldots, n$, according to the tail asymptotic behavior (4.12) for the Gaussian copula in Subsection 4.3 and $\mathbb{E}[(\theta_1\theta_k)^{\frac{\alpha}{1+\sigma}}] < \infty$, we have

$$\begin{aligned} J_1 &= \iint_{\Delta_1} \mathbb{P}\left(X_1 > \frac{tx}{u}, X_k > \frac{t}{v}\right) \mathbb{P}(\theta_1 \in \mathrm{d}u, \theta_k \in \mathrm{d}v) \\ &= \iint_{\Delta_1} \overline{C}\left(\overline{F_1}\left(\frac{tx}{u}\right), \overline{F_k}\left(\frac{t}{v}\right)\right) \mathbb{P}(\theta_1 \in \mathrm{d}u, \theta_k \in \mathrm{d}v) \\ &\sim (b_1 b_k)^{\frac{1}{1+\sigma}} x^{-\frac{\alpha}{1+\sigma}} \mathbb{E}\left[(\theta_1\theta_k)^{\frac{\alpha}{1+\sigma}} \mathbf{1}_{((\theta 1, \theta_k) \in \Delta_1)}\right] \left(\overline{F}(t)\right)^{\frac{2}{1+\sigma}} l\left(\overline{F}(t)\right) \\ &\sim (b_1 b_k)^{\frac{1}{1+\sigma}} x^{-\frac{\alpha}{1+\sigma}} \mathbb{E}\left[(\theta_1\theta_k)^{\frac{\alpha}{1+\sigma}}\right] \left(\overline{F}(t)\right)^{\frac{2}{1+\sigma}} l\left(\overline{F}(t)\right) \end{aligned}$$

where the last step we let $M \to \infty$.

For J_2, we have,

$$\begin{aligned} J_2 &= \iint_{\Delta_2 \cap \{u \le v\}} \overline{C}\left(\overline{F_1}\left(\frac{tx}{u}\right), \overline{F_k}\left(\frac{t}{v}\right)\right) \mathbb{P}(\theta_1 \in \mathrm{d}u, \theta_k \in \mathrm{d}v) \\ &\quad + \iint_{\Delta_2 \cap \{u > v\}} \overline{C}\left(\overline{F_1}\left(\frac{tx}{u}\right), \overline{F_k}\left(\frac{t}{v}\right)\right) \mathbb{P}(\theta_1 \in \mathrm{d}u, \theta_k \in \mathrm{d}v) \\ &=: J_{21} + J_{22}. \end{aligned}$$

Thus, by $\mathbb{E}\left[\theta_k^{(\sigma+3)\zeta}\right] < \infty$,

$$J_{21} \le \iint_{\Delta_2 \cap \{u \le v\}} \overline{C}\left(\overline{F_1}\left(\frac{tx}{v}\right), \overline{F_k}\left(\frac{t}{v}\right)\right) \mathbb{P}(\theta_1 \in \mathrm{d}u, \theta_k \in \mathrm{d}v)$$
$$\le C\left(\overline{F}(t)\right)^{\frac{2}{\sigma+1}} l\left(\overline{F}(t)\right) \mathbb{E}\left[\theta_k^{(\sigma+3)\zeta} \mathbf{1}_{(\theta_k \in \Delta_2)}\right],$$

where the last step we applied (4.12) and inequalities (3.1) and (4.11). This indicates $\lim_{M\to\infty} \limsup_{t\to\infty} \frac{J_{21}}{J_1} = 0$. In a similar argument, we also have $\lim_{M\to\infty} \limsup_{t\to\infty} \frac{J_{22}}{J_1} = 0$. Hence, $J_2 = o(1)J_1$ as $M \to \infty$.

By Markov's inequality and $\mathbb{E}\left[\theta_1^{(\sigma+3)\zeta}\right] < \infty$, $\mathbb{E}\left[\theta_k^{(\sigma+3)\zeta}\right] < \infty$, we have

$$J_3 \le \mathbb{P}(\theta_1 > t^c) + \mathbb{P}(\theta_k > t^c)$$
$$\le \left(\mathbb{E}\left[\theta_1^{(3+\sigma)\zeta}\right] + \mathbb{E}\left[\theta_k^{(3+\sigma)\zeta}\right]\right) t^{-(\sigma+3)c\zeta}$$
$$= o\left(\left(\overline{F}(t)\right)^{\frac{\sigma+3}{\sigma+1}}\right) = o(1)J_1,$$

where in the last step we used (3.2) by noting $c\zeta(1+\sigma) > \alpha$. Then, we conclude the lemma from the above three relations. □

The following lemmas will be used for the proof of Proposition 4.1.

Lemma A.9. *Under the conditions of Theorem 4.1, it holds that for $k = 1, \ldots, n$,*

$$\lim_{t\to\infty} \mathbb{P}(S_n^\theta > tx | M_n^\theta > t) = \mathbf{1}_{(0<x\le 1)} + x^{-\alpha}\mathbf{1}_{(x>1)}.$$

Proof. *When $0 < x \le 1$, on the one hand, it trivially holds that*

$$\Pr\left(S_n^\theta > tx, M_n^\theta > t\right) \le \Pr\left(M_n^\theta > t\right).$$

On the other hand, by BonferroniâĂŹs inequality, we have

$$\Pr\left(S_n^\theta > tx, \bigvee_{i=1}^n \theta_i X_i > t\right)$$
$$\ge \sum_{i=1}^n \Pr\left(S_n^\theta > tx, \theta_i X_i > t\right) - \sum_{1\le j<k\le n} \Pr\left(\theta_j X_j > t, \theta_k X_k > t\right)$$
$$\ge n\Pr\left(S_n^\theta > tx\right) - n\Pr\left(\sum_{i=1}^n \theta_i X_i^+ > tx\right) + \sum_{i=1}^n \Pr\left(\theta_i X_i > tx, \theta_i X_i > t\right)$$
$$- \sum_{1\le j<k\le n} \Pr\left(\theta_j X_j > t, \theta_k X_k > t\right)$$
$$\ge n\Pr\left(S_n^\theta > tx\right) - n\Pr\left(\sum_{i=1}^n \theta_i X_i^+ > tx\right) + (1+o(1))\sum_{i=1}^n \Pr\left(\theta_i X_i > t\right)$$
$$\sim \Pr\left(M_n^\theta > t\right),$$

where the last second step we applied a similar argument as we did in the proof of Lemma A.5 and the last step we applied Lemma A.1 and Breiman's theorem. Similarly, when $x \geq 1$, *we have,*

$$\Pr\left(S_n^\theta > tx, \bigvee_{i=1}^n \theta_i X_i > t\right) \sim \sum_{i=1}^n \Pr\left(\theta_i X_i > tx\right).$$

Applying Lemma A.1 and Breiman's theorem (implying $F_{S_n^\theta} \in \mathcal{R}_{-\alpha}$*) leads to*

$$\lim_{t\to\infty} \mathbb{P}\left(S_n^\theta > tx \,\middle|\, M_n^\theta > t\right) = x^{-\alpha}\mathbf{1}_{(x>1)}.$$

It ends the proof of the lemma. □

Lemma A.10. *Under the conditions of Theorem 4.1, it holds that*

$$\mathbb{E}\left[M_n^\theta \mathbf{1}_{\left(S_n^\theta > x\right)}\right] \sim \mathbb{E}\left[M_n^\theta \mathbf{1}_{\left(M_n^\theta > x\right)}\right]. \tag{A.20}$$

Proof. *For the desired* (A.20), *it suffices to prove*

$$\mathbb{E}\left[M_n^\theta \mathbf{1}_{\left(M_n^\theta > x\right)}\right] \lesssim \mathbb{E}\left[M_n^\theta \mathbf{1}_{\left(S_n^\theta > x\right)}\right]. \tag{A.21}$$

and

$$\mathbb{E}\left[M_n^{\theta+} \mathbf{1}_{\left(\sum_{i=1}^n \theta_i X_i^+ > x\right)}\right] \sim \mathbb{E}\left[M_n^\theta \mathbf{1}_{\left(M_n^\theta > x\right)}\right]. \tag{A.22}$$

For (A.21), *we have*

$$\begin{aligned}\mathbb{E}\left[M_n^\theta \mathbf{1}_{\left(S_n^\theta > x\right)}\right] &\geq \mathbb{E}\left[M_n^\theta \mathbf{1}_{\left(S_n^\theta > x, M_n^\theta > x\right)}\right] - \mathbb{E}\left[M_n^{\theta-} \mathbf{1}_{\left(S_n^\theta > x\right)}\right] \\ &=: K_1 - K_2. \end{aligned} \tag{A.23}$$

Note that, for K_1,

$$\begin{aligned} K_1 &= x\left(\Pr\left(S_n^\theta > x, M_n^\theta > x\right) + \int_1^\infty \Pr\left(S_n^\theta > x, M_n^\theta > tx\right) \mathrm{d}t\right) \\ &\sim \mathbb{E}\left[M_n^\theta \mathbf{1}_{\left(M_n^\theta > x\right)}\right], \end{aligned} \tag{A.24}$$

where the last step we applied Lemma A.9. Moreover, as for K_2, *by Lemma A.2 for any* $0 < \epsilon < 1$,

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$$
\begin{aligned}
\mathbb{E}\left[M_n^{\theta-}1_{\left(M_n^{\theta-}>\epsilon x\right)}\right] &= \epsilon\left(x\mathbb{P}(M_n^{\theta}<-\epsilon x)+\int_x^{\infty}\mathbb{P}(M_n^{\theta}<-\epsilon z)\mathrm{d}z\right)\\
&\le \epsilon\left(x\mathbb{P}(\theta_1X_1<-\epsilon x)+\int_x^{\infty}\mathbb{P}(\theta_1X_1<-\epsilon z)\mathrm{d}z\right)\\
&= o(1)\left(x\mathbb{P}(\theta_1X_1>x)+\int_x^{\infty}\mathbb{P}(\theta_1X_1>z)\mathrm{d}z\right)\\
&\le o(1)\mathbb{E}\left[M_n^{\theta}\mathbf{1}_{\left(M_n^{\theta}>x\right)}\right].
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
K_2 &\le \mathbb{E}\left[M_n^{\theta-}\mathbf{1}_{\left(M_n^{\theta-}\le\epsilon x, S_n^{\theta}>x\right)}\right]+\mathbb{E}\left[M_n^{\theta-}1_{\left(M_n^{\theta-}>\epsilon x\right)}\right]\\
&\le \epsilon x\mathbb{P}\left(S_n^{\theta}>x\right)+o(1)\mathbb{E}\left[M_n^{\theta}\mathbf{1}_{\left(M_n^{\theta}>x\right)}\right]\\
&\le \left(\epsilon O(1)+o(1)\right)\mathbb{E}\left[M_n^{\theta}\mathbf{1}_{\left(M_n^{\theta}>x\right)}\right],
\end{aligned} \tag{A.25}
$$

where the last step we applied Lemma A.1. Therefore, relation (A.21) *follows from* (A.23)*,* (A.24)*,* (A.25)*, and by the arbitrariness of* $\epsilon>0$.

Now we consider (A.22)*, obviously,*

$$
\begin{aligned}
\mathbb{E}\left[M_n^{\theta+}\mathbf{1}_{\left(\sum_{i=1}^n\theta_iX_i^+>x\right)}\right] &= \mathbb{E}\left[M_n^{\theta}\mathbf{1}_{\left(M_n^{\theta}>x\right)}\right]+\mathbb{E}\left[M_n^{\theta}\mathbf{1}_{\left(M_n^{\theta+}\le x,\sum_{i=1}^n\theta_iX_i^+>x\right)}\right]\\
&=: K_3+K_4.
\end{aligned} \tag{A.26}
$$

For any $0<\epsilon<1$*,*

$$
K_4\le\epsilon x\mathbb{P}\left(\sum_{i=1}^n\theta_iX_i^+>x\right)+x\mathbb{P}\left(\epsilon x<M_n^{\theta}\le x,\sum_{i=1}^n\theta_iX_i^+>x\right).
$$

Again by Lemma A.1,

$$
\begin{aligned}
&\mathbb{P}\left(\epsilon x<M_n^{\theta}\le x,\sum_{i=1}^n\theta_iX_i^+>x\right)\\
&\le \mathbb{P}\left(\sum_{i=1}^n\theta_iX_i^+>x\right)-\mathbb{P}(M_n^{\theta}>x)-\mathbb{P}\left(M_n^{\theta}\le\epsilon x,\sum_{i=1}^n\theta_iX_i^+>x\right)\\
&\le (1+o(1))\,\mathrm{Pr}\left(S_n^{\theta}>x\right)-\mathbb{P}(M_n^{\theta}>x)-\mathbb{P}\left(M_n^{\theta}\le\epsilon x, S_n^{\theta}>x\right)\\
&= o(1)\,\mathrm{Pr}\left(S_n^{\theta}>x\right)-\mathbb{P}(M_n^{\theta}>x)+\mathbb{P}\left(M_n^{\theta}>\epsilon x, S_n^{\theta}>x\right)\\
&= o(1)\mathbb{P}(M_n^{\theta}>x),
\end{aligned}
$$

where in the last step we used Lemma A.1 and Lemma A.9 for any $0<\epsilon<1$*. Hence,*

$$
K_4=o(1)\mathbb{E}\left[M_n^{\theta}\mathbf{1}_{\left(M_n^{\theta}>x\right)}\right]
$$

holds by the arbitrariness of $\epsilon>0$*. Thus, the desired relation* (A.22) *holds from* (A.26). □

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A.2. Proofs of main results

Proof of Theorem 4.1. Starting from (2.1), we rewrite $\mathrm{SR}_{\mathrm{CVaR}}(q)$ as

$$\begin{aligned}\mathrm{SR}_{\mathrm{CVaR}}(q) &= \int_{\sum_{i=1}^{n} C_{i,\mathrm{CVaR}}(q)}^{\infty} \mathbb{P}\left(S_n^{\theta} > x \,\middle|\, \theta_1 X_1 > C_{1,\mathrm{CVaR}}(q)\right) \mathrm{d}x \\ &= C_{1,\mathrm{CVaR}}(q) \int_{\frac{\sum_{i=1}^{n} C_{i,\mathrm{CVaR}}(q)}{C_{1,\mathrm{CVaR}}(q)}}^{\infty} \mathbb{P}\left(S_n^{\theta} > C_{1,\mathrm{CVaR}}(q)x \,\middle|\, \theta_1 X_1 > C_{1,\mathrm{CVaR}}(q)\right) \mathrm{d}x.\end{aligned}$$

Noticing $C_{1,\mathrm{CVaR}}(q) \to \infty$ as $q \uparrow 1$, and by Lemma A.1, Breiman's theorem and (3.1), there exist some $1 < \gamma < \alpha$ and some large M such that

$$\mathbb{P}\left(S_n^{\theta} > C_{1,\mathrm{CVaR}}(q)x \,\middle|\, \theta_1 X_1 > C_{1,\mathrm{CVaR}(q)}\right) \le M x^{-\gamma}$$

holds for q in some left neighborhood of 1 and $x > 1$, which is integrable on $(1, \infty)$. Hence, by Lemmas A.6 and A.5, and applying the Dominated Convergence Theorem, we derive

$$\begin{aligned}\lim_{q\uparrow 1} \frac{\mathrm{SR}_{\mathrm{CVaR}}(q)}{\mathrm{VaR}_q(X)} &= \frac{\alpha}{\alpha-1} \frac{b_1 \mathbb{E}[\theta_1^{\alpha}]}{\left(\sum_{i=1}^{n} b_i \mathbb{E}[\theta_i^{\alpha}]\right)^{1-\frac{1}{\alpha}}} \int_{\frac{\sum_{i=1}^{n} b_i \mathbb{E}[\theta_i^{\alpha}]}{b_1 \mathbb{E}[\theta_1^{\alpha}]}}^{\infty} x^{-\alpha} \mathrm{d}x \\ &= \frac{\alpha}{(\alpha-1)^2} \frac{\left(b_1 \mathbb{E}[\theta_1^{\alpha}]\right)^{\alpha}}{\left(\sum_{i=1}^{n} b_i \mathbb{E}[\theta_i^{\alpha}]\right)^{\alpha-\frac{1}{\alpha}}},\end{aligned}$$

which concludes relation (4.6).

Proof of Theorem 4.2. Similarly to the proof of Theorem 4.1,

$$\begin{aligned}\mathrm{SR}_{k,\mathrm{CVaR}}(q) &= \int_{C_{k,\mathrm{CVaR}}(q)}^{\infty} \mathbb{P}(\theta_k X_k > y \,|\, \theta_1 X_1 > C_{1,\mathrm{CVaR}}(q)\,) \mathrm{d}y \\ &= C_{1,\mathrm{CVaR}}(q) \int_{\frac{C_{k,\mathrm{CVaR}}(q)}{C_{1,\mathrm{CVaR}}(q)}}^{\infty} \mathbb{P}(\theta_k X_k > C_{1,\mathrm{CVaR}}(q)x \,|\, \theta_1 X_1 > C_{1,\mathrm{CVaR}}(q)\,) \mathrm{d}x.\end{aligned}$$

When $k = 1$, Breiman's theorem implies $F_{\theta_1 X_1} \in \mathcal{R}_{-\alpha}$, and keeping in mind $C_{1,\mathrm{CVaR}}(q) \to \infty$, $q \uparrow 1$, thus, by using (3.1), we have that for some $1 < \gamma < \alpha$, large $M > 0$ and q close to 1,

$$\mathbb{P}(\theta_k X_k > C_{1,\mathrm{CVaR}}(q)x \,|\, \theta_1 X_1 > C_{1,\mathrm{CVaR}}(q)\,) \le \mathbf{1}_{(0<x\le 1)} + M x^{-\gamma} \mathbf{1}_{(x>1)}, \tag{A.27}$$

which is integrable on $(0, \infty)$. Applying the Dominated Convergence Theorem and Lemma A.6, and by $F_{\theta_1 X_1} \in \mathcal{R}_{-\alpha}$ we have

$$\lim_{q\uparrow 1} \frac{\mathrm{SR}_{1,\mathrm{CVaR}}(q)}{\mathrm{VaR}_q(X)} = \frac{\alpha}{(\alpha-1)^2} \frac{b_1 \mathbb{E}[\theta_1^{\alpha}]}{\left(\sum_{i=1}^{n} b_i \mathbb{E}[\theta_i^{\alpha}]\right)^{1-\frac{1}{\alpha}}}.$$

When $k = 2, \ldots, n$, again by Breiman's theorem and (3.1), we can obtain that inequality (A.27) still holds for all q close to 1. By Lemma A.3, we have that for any fixed $x > 0$,

$$\lim_{t\to\infty} \mathbb{P}(\theta_k X_k > tx | \theta_1 X_1 > t) = 0,$$

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which, by using the Dominated Convergence Theorem and by Lemma A.6, leads to

$$\lim_{q\uparrow 1} \frac{\mathrm{SR}_{k,\mathrm{CVaR}}(q)}{\mathrm{VaR}_q(X)} = 0, \quad k = 2, \ldots, n.$$

Proofs of Theorems 4.3 and 4.4. The proofs are the same as that of Theorem 4.2 by addressing Lemma A.7 and Lemma A.8, respectively.

Proofs of Proposition 4.1 The proof is the same as that of Theorem 4.1 by addressing Lemma A.9 and Lemma A.10, respectively.

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