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ON QUASI AND WEAK STEINBERG CHARACTERS OF GENERAL LINEAR GROUPS

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Abstract Let G be a finite group and r be a prime divisor of the order of G. An irreducible character of G is said to be quasi r-Steinberg if it is non-zero on every r-regular element of G. A quasi r-Steinberg character of degree $|Syl_r(G)|$ is said to be weak r-Steinberg if it vanishes on the r-singular elements of G. In this article, we classify the quasi r-Steinberg cuspidal characters of the general linear group $GL(n,q)$. Then we characterize the quasi r-Steinberg characters of $GL(2,q)$ and $GL(3,q)$. Finally, we obtain a classification of the weak r-Steinberg characters of $GL(n,q)$.

Keywords: general linear group; cuspidal character; Steinberg character; parabolic induction

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1. Introduction

The significance of Steinberg characters to the study of finite groups of Lie type was well established by Curtis, Humphreys and Steinberg in [3, 8, 16] and [17]. Motivated by the intrinsic property of the Steinberg character, Feit introduced the notion of an r-Steinberg character for any finite group G and for any prime divisor r of the order of G (see [5]). Recall that an element of a group is called r-regular if its order is co-prime to r. An irreducible character of G is said to be r-Steinberg if each r-regular element, say g, of G takes the value $\pm|C_G(g)|_r$ on it. Here $|C_G(g)|_r$ is the highest power of r dividing the order of the centralizer $C_G(g)$ of g in G.

Feit conjectured [5] that if a finite simple group has an r-Steinberg character, then it is isomorphic to a simple group of Lie type in characteristic r. Darafsheh obtained an affirmative answer to this conjecture for the alternating and projective special linear groups (see [4]). Later, Tiep [18] extended the study to the rest of the finite simple groups and gave a positive answer.

In the last decade, several variants of r-Steinberg character have emerged as an important tool for studying the structure of finite groups through their characters (see [13]). One of the variants of r-Steinberg character is an r-vanishing character which is an irreducible character that vanishes on all the r-singular elements of the group. Note here that an element of a group is called r-singular if its order is divisible by r. In particular, if the degree of an r-vanishing character is $|Syl_r(G)|$, where $|Syl_r(G)|$ is the order of the Sylow r-subgroup of G, then the character is said to be Steinberg-like. In [9], Malle and Zaleski obtained a classification of the Steinberg-like characters of all finite simple groups.

Recently, Paul and Singla [12] introduced the notion of a quasi r-Steinberg character and a weak r-Steinberg character:

Definition 1.1. *Let G be a finite group and r be a prime dividing its order. An irreducible character of G is said to be* ***quasi r-Steinberg*** *if it takes non-zero value on every r-regular element of G. Further, a quasi r-Steinberg character which is Steinberg-like is said to be a* ***weak r-Steinberg*** *character.*

It follows that the r-Steinberg and the weak r-Steinberg characters are quasi r-Steinberg, but the converse need not be true. The above two variants of r-Steinberg characters were introduced to answer a question posed by Dipendra Prasad that asked whether the existence of a weak r-Steinberg character of a finite group G implies that G is a group of Lie type. It is well known that every finite group of Lie type has a r-Steinberg character for a prime r. Hence if a group does not have a nonlinear quasi r-Steinberg character, then it cannot be a finite group of Lie type of characteristic r. Therefore, one naturally requires the classification of quasi r-Steinberg characters of any finite group G.

In [12], the authors classified all the quasi r-Steinberg characters of symmetric and alternating groups and their double covers. A classification of the quasi r-Steinberg characters for the complex reflection groups has recently been done in [11].

In the present article, we study the quasi and weak r-Steinberg characters of the general linear group $GL(n,q)$ over a finite field F_q, where q is a power of prime p. Since every linear character of G is quasi r-Steinberg for any prime divisor r of $|G|$, one aims at the classification of the nonlinear quasi r-Steinberg characters of G. The paper is divided into six sections. In the second section, we introduce the notation which we follow throughout the article. In the third section, we classify the quasi r-Steinberg cuspidal characters of $GL(n,q)$ and the following is the first main result of the paper:

Theorem 1.2. *Let $n \geq 2$ be an integer and r be a prime divisor of the order of $GL(n,q)$. Then a cuspidal character of $GL(n,q)$ is quasi r-Steinberg if and only if one of the following holds:*

1. $n=2$ *and* $q = r^{\beta}+1$, *for some* $\beta \in \mathbb{N}$.
2. $n=3$ *and* $q = 3$, *when* $r=2$.
3. $n=3$ *and* $q = 2$, *when* $r=3$.

In the fourth and the penultimate section, we characterize the quasi r-Steinberg characters of $GL(2,q)$ and $GL(3,q)$, respectively.

Let χ_l be a character of F_q^* indexed by $l \in \{0,1,\ldots,(q-2)\}$. Further, let $\chi_k^{(1)}$ denote a linear and χ_t denote a cuspidal character of $GL(2,q)$ indexed by some $k \in$

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$\{0, 1, \ldots, (q-2)\}$ and $t \in \{1, 2, \ldots, (q^2-1)\}$. Assume that $L_{(2,1)}$ denotes the standard Levi complement of the standard parabolic subgroup (say $P_{(2,1)}$) of $GL(3, q)$. We denote the characters of $GL(3, q)$ obtained by parabolic induction of the irreducible characters $\chi_k^{(1)} \bigotimes \chi_l$ and $\chi_t \bigotimes \chi_l$ of $L_{(2,1)}$ by $\theta_{k,l}^{(4)}$ and $\theta_{t,l}^{(7)}$, respectively. The following is the second main result of the paper, which gives a classification of the weak r-Steinberg characters of $GL(n, q)$.

Theorem 1.3. *Let r be a prime divisor of $|GL(n, q)|$ different from p. Then an irreducible character χ of $GL(n, q)$ is weak r-Steinberg if and only if one of the following holds:*

1. *χ is a parabolically induced character of $GL(2, q)$ where $(q+1)$ is an r-power, for some odd prime r.*
2. *χ is a character of type $\theta_{k,l}^{(4)}$ of $GL(3, q)$, where*
   - *If q is odd, then $(1+q+q^2)$ is an r-power, $3 \nmid (q-1)$ and $(l-k)$ is not invertible in $\mathbb{Z}_{q-1}$.*
   - *If q is even, then $(1+q+q^2)$ is an r-power.*
3. *χ is the character $\theta_{t,l}^{(7)}$ of $GL(3, 2)$ and $r = 7$.*
4. *χ is a cuspidal character of $GL(3, 2)$ and $r = 3$.*

A proof of Theorem 1.3 is included in § 6.

2. Notation and preliminaries

For a finite group G, we denote the set of its irreducible characters by $Irr(G)$. We say that the conjugacy class $[g]$ of an element g of $GL(n, q)$ is primary if its characteristic polynomial has a unique irreducible factor. Further, g is said to be irreducible if its characteristic polynomial over F_q is irreducible of degree n. For any $n \geq 1$, assume that $F_{q^n}^* = \langle \epsilon_n \rangle$. Then we denote the irreducible element $diag(\epsilon_n, \epsilon_n^q, \ldots, \epsilon_n^{q^{n-1}})$ of $GL(n, q)$ by E_n. Let $\hat{}: F_{q^n}^* \to \mathbb{C}^*$ be the homomorphism defined by the rule $\hat{(\epsilon_n^s)} = \hat{\epsilon_n^s} = e^{s\frac{2\pi i}{q^n-1}}$, where $0 \leq s \leq (q^n - 2)$.

The character tables of $GL(2, q)$ and $GL(3, q)$ can be found in [15] and are also included in Appendix of this paper. We follow [1] for the notation of the conjugacy classes and the characters of these groups. For any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ of n, let P_λ denote the standard parabolic subgroup of $GL(n, q)$. The unipotent radical and the Levi complement of P_λ are denoted by U_λ and L_λ, respectively. Assume that for any $1 \leq i \leq m$, $\chi_{\lambda_i} \in Irr(GL(\lambda_i, q))$. An irreducible character of $GL(n, q)$ obtained by parabolic induction of the character of P_λ lifted from $\bigotimes_{i=1}^{m} \chi_{\lambda_i} \in Irr(L_\lambda)$ is called a parabolically induced character and is denoted by $\bigodot_{i=1}^{m} \chi_{\lambda_i}$.

We first list the types of the conjugacy classes of $GL(2, q)$ and $GL(3, q)$. We denote by $C^{(1)}$ and $C^{(2)}$ the types of conjugacy classes of the elements of $GL(2, q)$ whose characteristic polynomial is $(x - \alpha)^2$, for some $\alpha \in F_q^*$, parameterized by the partitions $(1, 1)$

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and (2) of 2, respectively. We let $C^{(3)}$ denote the type of the conjugacy classes of elements having distinct eigen values in F_q^*. Further, note that for some $y \in F_{q^2} \setminus F_q$, the matrix $v = \begin{pmatrix} 0 & 1 \\ -y^{q+1} & y + y^q \end{pmatrix}$ is an element of $GL(2,q)$. We denote by $C^{(4)}$ the types of conjugacy classes constituted by such irreducible elements of $GL(2,q)$.

Further, let $T^{(1)}$, $T^{(2)}$ and $T^{(3)}$ denote the types of conjugacy classes of the elements of $GL(3,q)$ with characteristic polynomial $(x-\alpha)^3$, for some $\alpha \in F_q^*$; parameterized by the partitions $(1,1,1),(2,1)$ and (3) of 3, respectively. For any $\alpha \neq \beta \neq \gamma \neq \alpha \in F_q^*$, we denote by $T^{(4)}$ and $T^{(5)}$ the types of conjugacy classes of the elements whose characteristic polynomial is $(x-\alpha)^2(x-\beta)$, for the partitions $(1,1)$ and (2) of 2, respectively. Further, $T^{(6)}$ denotes the conjugacy class of elements of the form $diag(\alpha,\beta,\gamma)$ and $T^{(7)}$ denotes the conjugacy class of the elements with characteristic polynomial $(x^2+ax+b)(x-\alpha)$, where (x^2+ax+b) is an irreducible polynomial over F_q. Finally, $T^{(8)}$ corresponds to the conjugacy class of irreducible elements.

Now we determine the notation for the characters of $GL(2,q)$ and $GL(3,q)$. Let $0 \leq k,l \leq (q-2)$ be some integers. Then the linear character of $GL(2,q)$ indexed by k is denoted as $\chi_k^{(1)}$. The character of degree q obtained by tensoring the linear character $\chi_k^{(1)}$ with the Steinberg character $St^{(1,1)}$ is denoted by $\chi_k^{(2)}$. Further, for any $k \neq l$, the irreducible character parabolically induced from $\chi_k \bigotimes \chi_l \in Irr(L_{(1,1)})$ is denoted by $\chi_{k,l}$. Let T be the set of integers $1 \leq t \leq (q^2-1)$ such that $(q+1) \nmid t$ and tq is excluded whenever t is included. Clearly, $|T| = \dfrac{q^2-q}{2}$. It is known that the number of cuspidal characters of $GL(2,q)$ is same as the size of T. A cuspidal character indexed by some $t \in T$ is denoted by χ_t.

We denote the linear character of $GL(3,q)$ indexed by k as $\theta_k^{(1)}$. The types of characters obtained by tensoring the Steinberg characters $St^{(2,1)}$ and $St^{(1,1,1)}$ with the linear character $\theta_k^{(1)}$ are denoted by $\theta_k^{(2)}$ and $\theta_k^{(3)}$, respectively. For any integer $m \in \{0,1,\ldots,(q-2)\}$, let $\theta_{k,l}^{(4)}, \theta_{k,l}^{(5)}$ and $\theta_{k,l,m}^{(6)}$ denote the characters $\chi_k^{(1)} \bigodot \chi_l$, $\chi_k^{(2)} \bigodot \chi_l$ and $\chi_{k,m} \odot \chi_l (k \neq l \neq m \neq k)$, respectively. Further, for any $t \in T$ we let $\theta_{t,l}^{(7)}$ denote the character $\chi_t \odot \chi_l$. Finally, we consider the set V of integers $v \in \{0,1,\ldots,q^3-1\}$ such that $(q^2+q+1) \nmid v$ and vq, vq^2 are excluded whenever v is being chosen. Then the cuspidal characters of $GL(3,q)$, indexed by $v \in V$, are denoted by $\theta_v^{(8)}$ and are $|V| = \frac{q^3-q}{3}$ many.

We recall here that a cuspidal character of $GL(n,q)$ vanishes on all the non-primary conjugacy classes. For a proof of this fact, one can refer to [7, pp. 64, 67 (Theorem 3.2 and Proposition 4.1)].

3. Quasi Steinberg cuspidal characters of $GL(n,q)$

In this section, we classify the quasi r-Steinberg cuspidal characters of $GL(n,q)$. Observe that if $r = p$, then for any $n \geq 2$ and $q \geq 3$, the matrix $A = diag(\epsilon_1, 1, \ldots, 1)$ is an r-regular element of $GL(n,q)$. Since conjugacy class of A is not primary, a cuspidal character would vanish on it. Therefore, no cuspidal character is quasi p-Steinberg. Now assume that r is a prime different from p.

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We first obtain the quasi r-Steinberg cuspidal characters of $GL(2,q)$. When $q = 2$, $|GL(2,2)| = 6$ and so r can be either 2 or 3. As $r \neq p, r = 3$. Since it follows from the character table that the cuspidal character of $GL(2,2)$ does not vanish on any conjugacy class, it implies that it is quasi 3-Steinberg. Thus we now characterize the quasi r-Steinberg cuspidal characters of $GL(2,q)$ when $q \geq 3$:

Proposition 3.1. *Let r be a prime dividing the order of the group $GL(2,q)$, where $q \geq 3$ is a prime power. Then a cuspidal character of $GL(2,q)$ is quasi r-Steinberg if and only if $q = (r^\beta + 1)$, for some $\beta \in \mathbb{N}$.*

Proof. Let χ_t be a cuspidal character of $GL(2,q)$. Note that if $r \nmid (q-1)$, then $A = diag(\epsilon_1, 1)$ is an r-regular element. Since $\chi_t(A) = 0$, χ_t is not quasi r-Steinberg. Therefore, assume that $(q-1) = r^\beta\gamma$, where $\beta \geq 1$ and $(r,\gamma) = 1$. Now the following cases arise:

Case I: $r \mid (q+1)$. As r also divides $(q-1)$, $r = 2$. If $\gamma > 1$, then $A_1 = diag(\epsilon_1^{\frac{q-1}{\gamma}}, 1)$ is 2-regular and its conjugacy class is not primary. Since $\chi_t(A_1) = 0, \chi_t$ is not quasi 2-Steinberg.

On the other hand, if $(q-1) = 2^\beta$, then $(q+1) = 2^\beta + 2$. If $\beta = 1$, then $q = 3$. Note that the only 2-regular elements of $GL(2,3)$ are the identity matrix, say I_2, and the matrix $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Since no cuspidal character vanishes on both of these elements, it follows that all the cuspidal characters of $GL(2,3)$ are quasi 2-Steinberg. But if $\beta > 1$, then F_q^* is a 2-group and in this case the only 2-regular element of $GL(2,q)$ is A_2. The conjugacy class of A_2 is of type $C^{(2)}$ and no cuspidal character vanishes on $C^{(2)}$.

Now we examine whether χ_t vanishes on some irreducible 2-regular element of $GL(2,q)$ or not. Since $(q^2-1) = 2(q-1)(2^{\beta-1}+1)$, the 2-regular elements of $F_{q^2}^*$ are of the form $\epsilon_2^{2(q-1)l}$ for some divisor l of $(2^{\beta-1}+1) = \dfrac{(q+1)}{2}$. Thus an irreducible 2-regular element, say A_3, of $GL(2,q)$ is of form $diag\left(\epsilon_2^{2(q-1)l}, \epsilon_2^{2q(q-1)l}\right)$. Note that $\chi_t(A_3) = \left(\hat{\epsilon}_2^{2t(q-1)l} + \hat{\epsilon}_2^{2qt(q-1)l}\right) = 0$ if and only if $2t(q-1)l = \dfrac{(q^2-1)}{2} - 2t(q-1)l$. It further implies that $4tl = 2^{\beta-1}+1$, which is not possible. Therefore, every cuspidal character is quasi 2-Steinberg.

Case II: $r \nmid (q+1)$. First assume that $(q-1)$ has a prime divisor, say r', which is different from r. Then $A_4 = diag(\epsilon_1^{\frac{q-1}{r'}}, 1)$ is an r-regular element whose conjugacy class is not primary. It follows from the previous argument that no cuspidal character is quasi r-Steinberg in this case.

Now assume that $(q-1) = r^\beta$, for some $\beta \in \mathbb{N}$. Since $r \nmid (q+1), r$ is some odd prime. As $(q-1) = r^\beta$, no element in $C^{(1)}$ and $C^{(3)}$ is r-regular. Also, A_2 is the only r-regular element whose conjugacy class is of type $C^{(2)}$. One can check that no cuspidal character vanishes on it. Further, since $(q^2-1) = r^\beta(r^\beta+2)$, the only irreducible r-regular element of $GL(2,q)$ is of form $A_5 = diag\left(\epsilon_2^{r^\beta l}, \epsilon_2^{qr^\beta l}\right)$, where l is some divisor of $(r^\beta + 2)$.

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Note that $\chi_t(A_5) = 0$ if and only if $\dfrac{(r^\beta + 2)}{tl} = 4$, which is not possible since r is odd. Therefore, even in this case every cuspidal character of is quasi r-Steinberg. □

Note that it follows from Theorem 49.8 in [2] that the degree of a cuspidal character of $GL(n,q)$ is $\prod_{i=1}^{n-1}(q^i - 1)$. Thus the degree of a cuspidal character of $GL(2,q)$ is $(q-1)$. Now the above discussion leads to the following:

Remark 3.2. If a cuspidal character of $GL(2,q)$ is quasi r-Steinberg, then its degree is an r-power.

In the following, we classify the quasi r-Steinberg cuspidal characters of $GL(n,q)$:

Proof of Theorem 1.2. Let ϕ be a cuspidal character of $GL(n,q)$. Assume that $q \geq 3$. If $r \nmid (q-1)$, then $A = diag(\epsilon_1, 1, \ldots, 1)$ is an r-regular element of $GL(n,q)$. Since ϕ vanishes on A, it is not quasi r-Steinberg. On the other hand, when $r \mid (q-1)$, the following cases arise:

Case I: $r \nmid (q+1)$. If $n \geq 3$, then $B_1 = diag(\epsilon_2^{q-1}, \epsilon_2^{q(q-1)}, 1, \ldots, 1)$ is an r-regular element of $GL(n,q)$. Since the conjugacy class $[B_1]$ is not primary, it follows that ϕ is not quasi r-Steinberg.

Case II: $r \mid (q+1)$. Since $r \mid (q-1)$, $r = 2$. Assume that $(q-1) = 2^\beta\gamma$, where $\beta \geq 1$ and $(\gamma, 2) = 1$. If $\gamma > 1$, then $B_2 = diag\left(\epsilon_1^{\frac{q-1}{\gamma}}, 1, \ldots, 1\right)$ is 2-regular and its conjugacy class is not primary. Therefore, the previous argument implies that ϕ is not quasi 2-Steinberg.

But if $\gamma = 1$, then $(q-1) = 2^\beta$. If $\beta = 1$, then $q = 3$. Now for any $n > 3$, $(3^n - 3^{n-3}) = 3^{n-3}(13)(2)$ is a divisor of $|GL(n,3)|$. Now as $B_3 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 2 & 1 \end{pmatrix}$ is of order 13, the element $B_4 = diag(B_3, 1, \ldots, 1) \in GL(n,3)$ is 2-regular. Therefore, for any $n > 3$, no cuspidal character of $GL(n,3)$ is quasi 2-Steinberg. When $n = 3$, the only 2-regular elements of $GL(3,3)$ are the elements of order 13. Now note that the degree of any cuspidal character of $GL(3,3)$ is $(3-1)(3^2-1) = 16$ and it follows from the character table of $GL(3,3)$ that no character of degree 16 vanishes on any element of order 13. Therefore, every cuspidal character of $GL(3,3)$ is quasi 2-Steinberg.

Further, if $\beta > 1$, then $(q+1) = 2(2^{\beta-1}+1)$. Observe that for any $n \geq 3$, the conjugacy class of the 2-regular element $diag(\epsilon_2^{2(q-1)}, \epsilon_2^{2q(q-1)}, 1, \ldots, 1)$ is not primary. Therefore, even in this case no cuspidal character is quasi r-Steinberg.

Now consider the group $GL(n,2)$, where $n \geq 3$. Note that if $r \neq 3$, then $(q+1) = 3$. Therefore, the argument in Case I provides that no cuspidal character of $GL(n,2)$ is quasi r-Steinberg. If $r = 3$, then for any $n \geq 4, B_4 = diag(E_3, 1, \ldots, 1)$ is a 3-regular element of $GL(n,2)$. As $\phi(B_4) = 0, \phi$ is not quasi 3-Steinberg. Further, it follows from the character table that every cuspidal character $GL(3,2)$ is quasi 3-Steinberg. Now the result follows from Proposition 3.1. □

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4. Quasi Steinberg characters of $GL(2, q)$

In this section, we obtain a characterization of the nonlinear quasi r-Steinberg characters of $GL(2, q)$. In this direction, we start with the parabolically induced characters of $GL(2, q)$. We first make the following remark towards this end:

Remark 4.1. Let n be an odd prime and r be a prime divisor of $|GL(n, q)|$ different from p. Assume that $r \mid (q^i - 1)$, for some $1 < i < n$. If r is also a divisor of $(q^n - 1)$, then since $gcd(q^i - 1, q^n - 1) = (q^{gcd(i,n)} - 1) = (q - 1)$, $r \mid (q - 1)$. It further implies that $r \nmid \dfrac{(q^i - 1)}{q - 1}$. Indeed if r divides $\dfrac{(q^i - 1)}{q - 1}$, then since $gcd\Big(\dfrac{q^i - 1}{q - 1}, q^n - 1\Big) = 1$, $r \mid 1$, which is not possible.

In the following, we establish a necessary condition for a parabolically induced character to be quasi r-Steinberg:

Lemma 4.2. *Let n be a prime and $r(\neq p)$ be a prime divisor of $|GL(n, q)|$. If a parabolically induced character ψ of $GL(n, q)$ is quasi r-Steinberg, then r is the only prime divisor of* $\dfrac{(q^n - 1)}{(q - 1)}$.

Proof. Assume that n is an odd prime. If r is 2, then since $\dfrac{(q^n - 1)}{(q - 1)} = \displaystyle\sum_{j=0}^{n-1} q^j$ is odd, $r \nmid \dfrac{(q^n - 1)}{(q - 1)}$. Thus $B = diag(\epsilon_n^{q-1}, \epsilon_n^{q(q-1)}, \ldots, \epsilon_n^{q^{n-1}(q-1)})$ is an r-regular element of $GL(n, q)$. Note that the order $\displaystyle\sum_{j=0}^{n-1} q^j$ of ϵ_n^{q-1} does not divide $(q^i - 1)$, for any $1 \leq i < n$. Therefore, $\epsilon_n^{q-1} \in F_{q^n}^* \setminus F_{q^i}^*$ $\forall$ $1 \leq i < n$ and so B is an irreducible element of $GL(n, q)$. Since the conjugacy class of an irreducible element of $GL(n, q)$ intersects P_λ trivially, $\psi(B) = 0$. Therefore, ψ is not quasi r-Steinberg.

Now consider the case when r is an odd prime. Note that if $r \mid (q+1)$, then $r \nmid \displaystyle\sum_{j=0}^{n-1} q^j$. Thus the previous argument implies that ψ is not quasi r-Steinberg. Now let $r \nmid (q+1)$. Suppose $\displaystyle\sum_{j=0}^{n-1} q^j = r^\delta k$, where $(r, k) = 1$ and $\delta \geq 0$ is an integer. If $k = 1$, then $\dfrac{(q^n - 1)}{(q - 1)}$ is an r-power and our claim is established. On the other hand, if $k > 1$, then the order of $\epsilon = \epsilon_n^{(q-1)r^\delta}$ is k and so $B' = diag(\epsilon, \epsilon^q, \ldots, \epsilon^{q^{n-1}})$ is an r-regular element of $GL(n, q)$. Further, note that B' is irreducible if and only if $o(\epsilon) = k \nmid (q^i - 1)$, for any $1 \leq i \leq (n-1)$. Since $\Big(q^i - 1, \dfrac{(q^n - 1)}{(q - 1)}\Big) = 1$, it follows that $\dfrac{(q^i - 1)}{k}$ is not an integer. Thus B' is an irreducible r-regular element and ψ vanishes on it. It implies that no parabolically induced character of $GL(n, q)$ is quasi r-Steinberg in this case.

Finally, we consider the case of $n = 2$. If $r \nmid (q + 1)$, then the element $diag(\epsilon_2^{(q-1)}, \epsilon_2^{q(q-1)})$ is an irreducible r-regular element of $GL(2, q)$ and the previous

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argument again implies that no parabolically induced character is quasi r-Steinberg. Further, assume that $(q+1) = r^{\beta}\gamma$, where $\beta \geq 1$ and $(r, \gamma) = 1$. If $\gamma > 1$, then in the following we prove that there always exists an r-regular element on which ψ vanishes:

- $r \nmid (q-1)$: If $\gamma \mid (q-1)$, then $\gamma = 2$. Let $(q-1) = 2^{\alpha}\kappa$, where $(2, \kappa) = 1$. Then $diag(\epsilon_2^{\frac{(q^2-1)}{2^{\alpha+1}}}, \epsilon_2^{\frac{q(q^2-1)}{2^{\alpha+1}}})$ is an irreducible r-regular element. If $\gamma \nmid (q-1)$, then $diag(\epsilon_2^{\frac{(q^2-1)}{\gamma}}, \epsilon_2^{\frac{q(q^2-1)}{\gamma}})$ is an irreducible r-regular element.
- $r \mid (q-1)$: If $\gamma \mid (q-1)$, then $\gamma = 2$, which is not possible as $r = 2$. Therefore, $\gamma \nmid (q-1)$; and hence s_3 is an irreducible r-regular element.

Therefore, ψ is not quasi r-Steinberg. Further, for $\gamma = 1$, $(q+1) = r^{\beta}$. Now the result follows. □

We now classify the quasi r-Steinberg characters of $GL(2, q)$:

Theorem 4.3. *Let $q \geq 3$ be a prime power, r be a prime dividing $|GL(2, q)|$ and $\chi \in \left\{\chi_k^{(2)}, \chi_{k,l} \mid 0 \leq k < l \leq (q-2)\right\}$.*

1. *If $r = p$, then χ is quasi p-Steinberg if and only if χ is of type $\chi_k^{(2)}$.*
2. *If $r \neq p$, then χ is quasi r-Steinberg if and only if χ is of type $\chi_{k,l}$ and $q = (r^{\delta} - 1)$.*

Proof. First assume that $r = p$. Note that any character of degree q vanishes only on the elements of the form $t_{\alpha} = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} \in C^{(2)}$, for some $\alpha \in F_q^*$. Since $o(t_{\alpha}) = lcm(o(\alpha), p)$, no element of $C^{(2)}$ is p-regular. Therefore, for any $0 \leq k \leq (q-2)$, $\chi_k^{(2)}$ is a quasi p-Steinberg character. Also, a parabolically induced character vanishes on the irreducible r-regular element E_2 and hence is not quasi p-Steinberg.

Further, let $r \neq p$. In this case, t_1 is an r-regular element of $GL(2, q)$. Since $\chi_k^{(2)}$ vanishes on t_1, it is not quasi r-Steinberg. Now we study when a parabolically induced character of $GL(2, q)$ is quasi r-Steinberg. In this direction, a necessary condition provided by Lemma 4.2 is that $(q+1) = r^{\beta}$, for some $\beta \in \mathbb{N}$.

Let $\chi_{k,l}$ be a parabolically induced character, where $0 \leq k < l \leq (q-2)$. It follows from the character table that it takes 0 value only on the elements whose conjugacy class is of type $C^{(3)}$ or $C^{(4)}$. Observe that if r is some odd prime, then $r \nmid (q-1)$. Indeed, if $r \mid (q-1)$, then $r = 2$, which is not the case. Now since $(q^2-1) = (q-1)r^{\beta}$, it follows that no irreducible element of $GL(2, q)$ is r-regular. Therefore, the elements whose conjugacy class is of type $C^{(4)}$ are not r-regular.

Further, let $V = diag(\alpha, \beta)$ has conjugacy class of type $C^{(3)}$. Then $\chi_{k,l}(V) = \hat{\alpha}^k\hat{\beta}^l + \hat{\alpha}^l\hat{\beta}^k = 0$ if and only if F_q^* contains the primitive second root of unity. Since r being odd implies q is even, this is also not the case. Therefore, $\chi_{k,l}$ is quasi r-Steinberg.

Now if $(q+1) = 2^{\delta}$, then $(q-1) = 2(2^{\delta-1} - 1)$. Note that $\delta = 2$ implies $q = 3$. Since there is no 2-regular element in $C^{(3)}$ and $C^{(4)}$, the parabolically induced characters of

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$GL(2,3)$ are quasi 2-Steinberg. Further, if $\delta \geq 3$, then there is no 2-regular element in $C^{(4)}$. On the other hand, assume that the conjugacy class of $W = diag(\epsilon_1^{s_1}, \epsilon_1^{s_2})$, for some $0 \leq s_1 < s_2 \leq (q-2)$, is of type $C^{(3)}$. Then $\chi_{k,l}(W) = \hat{\epsilon_1}^{s_1 k + s_2 l} + \hat{\epsilon_1}^{s_1 l + s_2 k} = 0$ if and only if $(s_2 - s_1)(l-k) = \dfrac{(q-1)}{2}$. Observe that a pair of such distinct elements s_1 and s_2 exists if $(l-k)$ is invertible in $\mathbb{Z}_{q-1}$. Further, note that for any $1 \leq i \leq 2$, $o(\epsilon_i^{s_i}) = \dfrac{(q-1)}{s_i}$ and so $o(W) = lcm\Big(\dfrac{(q-1)}{s_1}, \dfrac{(q-1)}{s_2}\Big)$. Since s_1 and s_2 have different parity, 2 divides $o(W)$. Therefore, W is not a 2-regular element. Thus an element of $C^{(3)}$ on which $\chi_{k,l}$ vanishes, is not 2-regular. It implies that if $(l-k)$ is invertible in $\mathbb{Z}_{q-1}$, then $\chi_{k,l}$ is quasi 2-Steinberg. On the other hand, if $(l-k)$ is not invertible in $\mathbb{Z}_{q-1}$, then there does not exist any element in $C^{(3)}$ on which $\chi_{k,l}$ vanishes. Now the result follows. □

5. Quasi Steinberg characters of $GL(3,q)$

In this section, we classify the quasi r-Steinberg characters of $GL(3,q)$. Since the discussion on the cuspidal characters has been done in § 3, we now continue the study for the remaining nonlinear characters of $GL(3,q)$:

Theorem 5.1. *Let us denote by S the set constituted by all the irreducible characters of $GL(3,q)$ except the linear and cuspidal ones. If $r = p$, then the only characters in S that are quasi p-Steinberg are of type $\theta_k^{(3)}$. If $r \neq p$, then we have the following:*

1. *If q is even, then the characters of type $\theta_{k,l}^{(4)}$ are quasi r-Steinberg if and only if $(1+q+q^2)$ is an r-power.*
2. *If q is odd, then the characters of type $\theta_{k,l}^{(4)}$ are quasi r-Steinberg if and only if $(l-k)$ is not invertible in $\mathbb{Z}_{q-1}$, $(1+q+q^2)$ is an r-power and $3 \nmid (q-1)$.*
3. *The characters of type $\theta_{t,l}^{(7)}$ are quasi r-Steinberg if and only if $r = 7$ and $q = 2$.*

Proof. Case I: $r = p$. Note that $B = diag(E_2, 1)$ is an r-regular element and its conjugacy class is of type $T^{(7)}$. It follows from the character table of $GL(3,q)$ that any character of type $\theta_k^{(2)}$ vanishes on B. Therefore, it is not quasi p-Steinberg. Now note that a character of type $\theta_k^{(3)}$ takes 0 value only on the conjugacy classes of type $T^{(2)}, T^{(3)}$ and $T^{(5)}$. Since there are no p-regular elements in these classes, $\theta_k^{(3)}$ is quasi p-Steinberg. Further, any parabolically induced character is not quasi p-Steinberg as it vanishes on the irreducible r-regular element E_n.

Case II: $r \neq p$. Consider the element $C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Since its order is p, it is r-regular and its conjugacy class is of type $T^{(3)}$. Since the characters $\theta_k^{(2)}, \theta_k^{(3)}$ and $\theta_k^{(5)}$ vanish on $T^{(3)}$, they are not quasi r-Steinberg. Now we study the parabolically induced characters $\theta_{k,l}^{(4)}$, $\theta_{k,l,m}^{(6)}$ and $\theta_{t,l}^{(7)}$.

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Observe that if $r \mid (q+1)$, then $r \nmid (q^2+q+1)$. Thus it follows from Lemma 4.2 that no parabolically induced character is quasi r-Steinberg. □

Remark 5.2. For an odd prime n, if a parabolically induced character of $GL(n,q)$ is quasi r-Steinberg, then $r \nmid (q+1)$.

Therefore, in the following, we assume that r does not divide $(q+1)$:

The characters of type $\theta^{(6)}_{k,l,m}$:

Note that the order of the element $D_1 = diag(\epsilon_2^{(q-1)}, \epsilon_2^{q(q-1)}, 1)$ is $(q+1)$. Since $r \nmid (q+1)$, D_1 is an r-regular element. It follows from the character table that any character of type $\theta^{(6)}_{k,l,m}$ vanishes on D_1 and hence is not quasi r-Steinberg.

The following lemma gives a necessary condition for a parabolically induced character of $GL(3,q)$ to be quasi r-Steinberg:

Lemma 5.3. *Let $q \geq 3$. If a parabolically induced character of $GL(3,q)$ is quasi r-Steinberg, then $r \nmid (q-1)$.*

Proof. On contrary, assume that $q = (1+rm)$, for some $m \in \mathbb{N}$. Then $(1+q+q^2) = (3+3rm+r^2m^2)$. Clearly, if r is different from 3, then $r \nmid (1+q+q^2)$. Further, if $r=3$, then $(1+q+q^2) = 3(1+3(m+m^2))$, which is not a 3-power. In either case, we get a contradiction to Lemma 4.2. □

In other words, if $(1+q+q^2)$ is an r-power, then $r \nmid (q-1)$. Also if $r \nmid (q-1)$, then since we have already observed that $r \nmid (q+1)$, it implies that $r \nmid (q^2-1)$ as well. With this note, we now continue our investigation:

The characters of type $\theta^{(4)}_{k,l}$:

It follows from the character table that the classes on which a character of type $\theta^{(4)}_{k,l}$ can vanish are of type $T^{(5)}$ or $T^{(6)}$. Therefore, if the value of $\theta^{(4)}_{k,l}$ on every r-regular element from these conjugacy classes is non-zero, then it follows from Lemma 4.2 that it is quasi r-Steinberg if and only if $(1+q+q^2)$ is an r-power.

Let $A_{\alpha,\beta}$ and $B_{\alpha,\beta,\gamma}$ be some elements of $GL(3,q)$ whose conjugacy class is of type $T^{(5)}$ and $T^{(6)}$, respectively. Note that $\theta^{(4)}_{k,l}(A_{\alpha,\beta}) = (\hat{\alpha^{k+l}}\hat{\beta^l} + \hat{\beta^{2l}}\hat{\alpha^k})$ and $\theta^{(4)}_{k,l}(B_{\alpha,\beta,\gamma}) = \hat{\alpha^k}\hat{\beta^l}\hat{\gamma^l} + \hat{\alpha^l}\hat{\beta^k}\hat{\gamma^l} + \hat{\alpha^l}\hat{\beta^l}\hat{\gamma^k}$. Therefore, if $\theta^{(4)}_{k,l}$ vanishes on some conjugacy class of type $T^{(5)}$ or $T^{(6)}$, then F_q^* contains a primitive second or third root of unity, respectively.

Now the following corollary to Lemma 4.2 classifies the quasi r-Steinberg characters of type $\theta^{(4)}_{k,l}$ of $GL(3,q)$ for even q:

Corollary 5.4. *Let q be even. Then $\theta^{(4)}_{k,l}$ is quasi r-Steinberg character of $GL(n,q)$ if and only if $(1+q+q^2)$ is a power of r.*

Proof. Since q is even, F_q^* does not contain the primitive second root of unity and hence $\theta^{(4)}_{k,l}$ does not vanish on $T^{(5)}$. Now we check whether F_q^* contains a primitive third

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root of unity. If q is of the form $(3m+2)$, then $3 \nmid (q-1)$. As F_q^* does not have a primitive third root of unity, it follows that $\theta_{k,l}^{(4)}$ does not vanish on $T^{(6)}$. Thus it follows from Lemma 4.2 that $\theta_{k,l}^{(4)}$ is quasi r-Steinberg if and only if $(1+q+q^2)$ is a power of r. On the other hand, if $q=(3m+1)$, then $(1+q+q^2)=3(1+3m^2+3m)$ is not a prime power. So $\theta_{k,l}^{(4)}$ is not a quasi r-Steinberg character of $GL(3,q)$ for such q. □

Example. Since $q=8$ is of the form $(3m+2)$ and $(1+q+q^2)=73$ is a prime, any character of type $\theta_{k,l}^{(4)}$ is a quasi 73-Steinberg character of $GL(3,8)$.

Let us now consider the case when q is odd. It follows from Lemma 5.3 that if $\theta_{k,l}^{(4)}$ is quasi r-Steinberg, then $r \nmid (q-1)$. Therefore, the elements in the conjugacy classes of type $T^{(4)}, T^{(5)}, T^{(6)}$ and $T^{(7)}$ are r-regular.

If $\alpha = \epsilon_1^{s_1}$ and $\beta = \epsilon_1^{s_2}$ are some elements of F_{q^*} with $1 \leq s_1 < s_2 \leq (q-1)$, then $\theta_{k,l}^{(4)}(A_{\alpha,\beta}) = \hat{\epsilon_1}^{s_1(k+l)+s_2 l} + \hat{\epsilon_1}^{(2s_1 l+s_2 k)}$. Therefore, $\theta_{k,l}^{(4)}(A_{\alpha,\beta}) = 0$ if and only if $(s_1-s_2)(k-l) = \dfrac{(q-1)}{2}$. It further implies that if k and l are such that $(k-l)$ is not invertible in $\mathbb{Z}_{q-1}$, then $\theta_{k,l}^{(4)}$ does not vanish on any element whose conjugacy class is of type $T^{(5)}$. On the other hand, if $(k-l, q-1)=1$, then for the choice of $s_1 = \left(\dfrac{q-1}{2(k-l)} + s_2\right), \theta_{k,l}^{(4)}(A_{\alpha,\beta}) = 0$.

Here note that if $\theta_{k,l}^{(4)}$ is quasi r-Steinberg, then $3 \nmid (q-1)$. Indeed if $3 \mid (q-1)$, then it follows from a similar discussion as in Corollary 5.4 that $(1+q+q^2)$ is not an r-power. Thus we obtain the following necessary conditions for $\theta_{k,l}^{(4)}$ to be quasi r-Steinberg:

Corollary 5.5. *Let q be odd. If a character of type $\theta_{k,l}^{(4)}$ is quasi r-Steinberg character of $GL(3,q)$, then r is the only prime divisor of $(1+q+q^2)$, $(l-k)$ is not invertible in $\mathbb{Z}_{q-1}$ and $3 \nmid (q-1)$.*

Now we obtain the sufficient conditions for a character of type $\theta_{k,l}^{(4)}$ to be quasi r-Steinberg. Clearly, if $3 \nmid (q-1)$, then $\theta_{k,l}^{(4)}(B_{\alpha,\beta,\gamma}) \neq 0 \ \forall \ \alpha,\beta,\gamma \in F_q^*$. Therefore, the sufficient conditions for a character $\theta_{k,l}^{(4)}$ to be quasi r-Steinberg are that r is the only prime divisor of $(1+q+q^2)$, $(l-k)$ is not invertible in $\mathbb{Z}_{q-1}$ and $3 \nmid (q-1)$. It now establishes the parts (i) and (ii) of the Theorem.

We now make a remark about the existence of infinitely many primes q such that many characters of type $\theta_{k,l}^{(4)}$ are quasi r-Steinberg characters of $GL(3,q)$, for some r. For instance, in addition to the aforementioned sufficient conditions, let us assume that q is such that $4 \nmid (q-1)$. Since q is odd, it implies that 2 is not invertible in $\mathbb{Z}_{q-1}$. Therefore, as a consequence of the above discussion, we have that whenever $k=(l+2)$, $\theta_{k,l}^{(4)}(A_{\alpha,\beta}) \neq 0 \ \forall \ \alpha,\beta \in F_q^*$. This now leads to the following corollary which establishes our remark:

Corollary 5.6. *As per Bunyakovsky conjecture there exist infinitely many q for which $(1+q+q^2)$ is a prime. Then any such q is of the form $q \not\equiv 1 \mod 12$ and if $(1+q+q^2) =$*

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r (a prime), then the character $\theta^{(4)}_{k,l}$, *where* $k = (l+2)$, *is a quasi r-Steinberg character of* $GL(3, q)$.

Proof. Bunyakovsky conjecture predicts that there exist infinitely many q for which $(1+q+q^2)$ is a prime. We first note that such a q cannot be of the form $(12m+1)$. Indeed, then $(1+q+q^2) = 3(1+12m(1+4m))$, which cannot be a prime. It implies that for any q for which $(1+q+q^2)$ is a prime, say r, $q \not\equiv 1 \mod (12)$. Now it follows from Corollary 5.5 and the discussion following it, that for any such q, $\theta^{(4)}_{l+2,l}$ is a quasi r-Steinberg character of $GL(3, q)$. □

The characters of type $\theta^{(7)}_{t,l}$:

It follows from character table that $\theta^{(7)}_{t,l}$ vanishes on all the conjugacy classes of type $T^{(6)}$ and $T^{(8)}$. Now we study the values of $\theta^{(7)}_{t,l}$ on the conjugacy classes of type $T^{(7)}$. Note that for $E_{i,\alpha} = diag(F_i, \alpha)$, where $F_i = diag(\epsilon_2^i, \epsilon_2^{qi})$ is an irreducible in $GL(2, q)$, $[E_{i,\alpha}]$ is of type $T^{(7)}$. Here $\theta^{(7)}_{t,l}(E_{i,\alpha}) = \hat{\alpha}^t(\hat{\epsilon_2}^{li} + \hat{\epsilon_2}^{lqi})$. If $\theta^{(7)}_{t,l}$ vanishes on $E_{i,\alpha}$, then $F^*_{q^2}$ contains the primitive second root of unity. Note that if $\theta^{(7)}_{t,l}$ is quasi r-Steinberg, then $r \nmid (q-1)$ (see Lemma 5.3). Now the further discussion is as follows:

- **q ≥ 4:** Let α, β, γ be distinct elements in F^*_q. Now since $\theta^{(7)}_{k,l}$ vanishes on the r-regular element $B_2 = diag(\alpha, \beta, \gamma)$, it is not quasi r-Steinberg.
- **q = 2:** Note that $(1+q+q^2) = 7$. Since $(q-1) = 1$, there is no element in $T^{(6)}$; and as $F^*_{2^2}$ does not contain the primitive second root of unity, $\theta^{(7)}_{t,l}$ does not vanish on $T^{(7)}$ either. Since there does not exist any irreducible 7-regular element in $GL(3, 2)$, it follows that $\theta^{(7)}_{t,l}$ is a quasi 7-Steinberg character of $GL(3, 2)$. Further, if r is some prime different from 7, then it follows from Lemma 4.2 that $\theta^{(7)}_{t,l}$ is not quasi r-Steinberg.
- **q = 3:** In this case, $(1+q+q^2) = 13$. Note that the element $E_{2,1} = diag(F_1, 1)$, where $F_1 = diag(\epsilon_2^2, \epsilon_2^6)$, is a 13-regular element whose class is of type $T^{(7)}$. Since $\theta^{(7)}_{t,l}$ vanishes on $E_{2,1}$, it is not quasi 13-Steinberg. Now one can conclude from Lemma 4.2 that $\theta^{(7)}_{t,l}$ is not a quasi r-Steinberg character of $GL(3, 3)$, for any r.

6. Weak Steinberg characters of general linear groups

In this section, we classify the weak r-Steinberg characters of $GL(n, q)$. Towards this end, we first make the following remark:

Remark 6.1. If χ is a weak r-Steinberg character of $GL(n, q)$, then $r \nmid (q-1)$. Indeed if $r \mid (q-1)$, then the central elements αI_n, where $1 \neq \alpha \in F^*_q$, are r-singular. It implies that $\chi(\alpha I_n) = 0$, which is a contradiction.

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Recall that by definition, a weak r-Steinberg character is quasi r-Steinberg. In the following, we first determine which quasi r-Steinberg characters of $GL(2,q)$ and $GL(3,q)$ are weak r-Steinberg:

Lemma 6.2. *Let $q \geq 3$ be a prime power. Then a nonlinear quasi r-Steinberg character χ of $GL(2,q)$ is weak r-Steinberg if and only if one of the following holds:*

1. *χ is of type $\chi_k^{(2)}$ and $r = p$.*
2. *χ is a parabolically induced character and $(q+1)$ is an r-power, for some odd prime r.*

Proof. Note that $|Syl_p(GL(2,q))| = q$. If $r = p$, then the characters of type $\chi_k^{(2)}$ are weak p-Steinberg as their degree is q and they vanish on all p-singular elements of $GL(2,q)$. Now we consider the case when r is some prime other than p. In this direction, first recall that a cuspidal character of $GL(2,q)$ is quasi r-Steinberg if and only if $(q-1) = r^{\beta}$. Since $|GL(2,q)| = q(q-1)^2(q+1), |Syl_r(GL(2,q))| \geq (q-1)^2$. Since the degree of a cuspidal character is $(q-1)$, it is not weak r-Steinberg.

Further, note that it follows from Theorem 4.3 that a parabolically induced character $\chi_{k,l}$ is quasi r-Steinberg if and only if its degree, $(q+1)$, is an r-power. If $r = 2$, then $(q-1)$ is also divisible by 2 and hence $\chi_{k,l}(1) \neq |Syl_r(GL(2,q))|$. On the other hand, if $r \neq 2$, then q is even and $\chi_{k,l}(1) = |Syl_r(GL(2,q))|$. Since there does not exist any r-singular element in $C^{(1)}, C^{(2)}$ and $C^{(3)}$, $\chi_{k,l}$ is r-vanishing too. Hence the result follows. □

Lemma 6.3. *A cuspidal character of $GL(3,q)$ is weak r-Steinberg if and only if $q=2$ and $r=3$. Otherwise, an irreducible nonlinear character of $GL(3,q)$ is weak r-Steinberg if and only if it is quasi r-Steinberg.*

Proof. Since the degree of a cuspidal character is $(q-1)(q^2-1)$, it follows from Theorem 1.2 that a quasi r-Steinberg cuspidal character of $GL(3,q)$ has a prime power degree if and only if $(q,r) \in \{(3,2),(2,3)\}$. If $q = 3$, then since $2 \mid (3-1)$, it follows from Remark 6.1 that no cuspidal character of $GL(3,3)$ is weak 2-Steinberg. Further, if $q = 2$, then the degree of a cuspidal character of $GL(3,2)$ is 4 which is $|Syl_3(GL(3,2))|$. Now since any cuspidal character of $GL(3,2)$ is 3-vanishing, it is weak 3-Steinberg. Hence the first part of result is established.

Observe that the degree of a character of type $\theta_k^{(2)}$ is not a prime power and that any character of type $\theta_k^{(3)}$ is weak r-Steinberg if and only if $r = p$. Recall that if a parabolically induced character of $GL(3,q)$ is quasi r-Steinberg, then $(1+q+q^2)$ is an r-power. It follows that $r \neq p$.

Assume that the integers k, l and t are such that the characters $\theta_{k,l}^{(4)}$ and $\theta_{t,l}^{(7)}$ are quasi r-Steinberg (c.f. Theorem 5.1). Since they satisfy the degree condition, we check whether they are r-vanishing or not. It follows from Lemma 5.3 that $r \nmid (q-1)$ and hence no element in the classes of type $\{T^{(i)} \mid 1 \leq i \leq 6\}$ is r-singular. Further, if $r \mid (q+1)$, then no parabolically induced character of $GL(3,q)$ is quasi r-Steinberg. Therefore, $r \nmid (q^2-1)$. It implies that no element in any class of type $T^{(7)}$ is r-singular. Moreover, as r is a divisor of $GL(3,q)$, it follows that $r \mid (q^3-1)$. Hence $GL(3,q)$ has irreducible r-singular elements. Since $\theta_{k,l}^{(4)}$ and $\theta_{t,l}^{(7)}$ vanish on them, they are r-vanishing. Therefore, the quasi r-Steinberg parabolically induced characters of $GL(3, q)$ are weak r-Steinberg. □

Now we characterize the weak r-Steinberg characters of $GL(n, q)$. Recall that an irreducible character χ of $GL(n, q)$ is of the form $(g_1^{n_1} g_2^{n_2} \dots g_k^{n_k})$, where for any $1 \leq i \leq k$, the degree of the simplex g_i is d_i and $\sum_{i=1}^{k} d_i n_i = n$ (see [6]). Further, it follows from [4] that if the degree of χ is some prime power, then

$$\chi(1) = \frac{\psi_n(q)}{\psi_{n_1}(q^{d_1})\psi_{n_2}(q^{d_2}) \dots \psi_{n_k}(q^{d_k})}, \tag{6.1}$$

where $\psi_x(q^y) = \prod_{i=1}^{x}(q^{iy} - 1)$. Now we state the Zsigmondy's theorem (see Theorem 3 in [14]), which is crucial to the upcoming discussion.

Let a and l be integers greater than 1. Then there exists a prime divisor s of $(a^l - 1)$ such that $s \nmid (a^k - 1)\ \forall\ 0 < k < l$, except exactly in the following cases:

- $l = 2, a = (2^t - 1)$, where $t \geq 2$.
- $l = 6, a = 2$.

We are now in a position to classify the weak r-Steinberg characters of $GL(n, q)$:

Proof of Theorem 1.2: Let χ be a weak r-Steinberg character of $GL(n, q)$. Since $r \neq p,\ \chi(1) = |(q^n - 1)(q^{n-1} - 1) \dots (q - 1)|_r$. The following cases arise:

Case I: $r \nmid (q^n - 1)$. By Zsigmondy's theorem there exists a prime divisor s of $(q^n - 1)$ such that $s \nmid (q^m - 1)\ \forall\ m < n$; except for $(n, q) = (2, 2^t - 1)$ and $(6, 2)$. Clearly, s is different from r. Since $\chi(1)$ is an r-power and $s \mid (q^n - 1)$, it follows from (6.1) that there exists some $1 \leq j \leq k$ such that $s \mid \psi_{n_j}(q^{d_j})$. Without loss of generality, assume that $j = 1$. Since s is a prime, s divides $(q^{id_1} - 1)$ for some $1 \leq i \leq n_1$. As $s \nmid (q^m - 1)\ \forall\ m < n$, it implies that $id_1 > (n - 1)$. Thus $n_1 d_1 = n$. Also note that $d_1 = 1$ implies that $\chi(1) = 1$. So $d_1 \geq 2$.

If $n = 2$, then $r \mid (q^2 - 1)$. But since $r \nmid (q^n - 1)$, it follows that $n \geq 3$. First assume that $n = 3$. If $r \nmid (q^3 - 1)$, then it follows from Lemma 4.2 that χ is not a parabolically induced character. Also, Lemma 6.3 implies that the cuspidal characters of $GL(3, 2)$ are weak r-Steinberg if and only if $r = 3$. Further, note that it follows from Equation (6.2) that the only choice of (n, q) for which $\chi(1)$ is a prime power is $(4, 2)$. The only character in the character table of $GL(4, 2)$ whose degree is a prime power is the character of degree 7. But this character is not quasi 7-Steinberg.

If $(n, q) = (2, 2^t - 1)$, it follows from Lemma 6.2 that χ is not weak r-Steinberg. Now assume that $(n, q) = (6, 2)$. Since $GL(6, 2)$ is a quasi-simple group, it follows from Theorem 1.2 in [10] that $GL(6, 2)$ does not have any character whose degree is a prime power.

Case II: $r \mid (q^n - 1)$. Since $r \nmid (q - 1)$, we have that $r \nmid (q^{n-1} - 1)$. In this case, the previous argument for $a = q$ and $l = (n - 1)$ yields that $k = 2$ and g_1 is a simplex of degree

d_1 with $n_1 d_1 = (n-1)$, where q and n are such that $(q, n-1) \notin \{(2^t-1, 2), (2, 6)\}$. Clearly, the degree of g_2 is 1 and hence the degree of χ is

$$\frac{(q^n-1)(q^{n-1}-1)\ldots(q-1)}{(q^{d_1}-1)(q^{2d_1}-1)\ldots(q^{n_1 d_1}-1)(q-1)} = (q^n-1)(q^{n-2}-1)\ldots(q^{n-d_1}-1). \quad (6.2)$$

If $n = 2$, then the degree of χ, $\chi(1) = (q+1)$. It follows from Lemma 6.2 that χ is weak r-Steinberg if and only if $(q+1) = r^\beta$, for an odd prime r. Further, if $n = 3$, then we have the following:

1. If $d_1 = 1$, then $\chi(1) = (q^2+q+1)$;
2. If $d_1 = 2$, then $\chi(1) = (q^3-1)$.

Now Lemma 6.3 gives the conditions under which the characters of these degrees are weak r-Steinberg. Finally, assume that $n \geq 4$. If $d_1 \geq 2$, then both (q^n-1) and $(q^{n-2}-1)$ occur as divisors of $\chi(1)$ and hence are r-powers. This contradicts the fact that $r \nmid (q-1)$. Therefore, $d_1 = 1$ and hence $\chi = (g_1^{n-1} g_2)$ has degree $\chi(1) = \dfrac{q^n-1}{q-1}$. Since

$$\chi(1) = |(q^n-1)(q^{n-1}-1)\cdots(q-1)|_r,$$

it further implies that $r \nmid (q^i-1)$, for any $1 \leq i \leq (n-1)$.

Note that if $n \geq 4$ is an even integer, then $diag(E_2, E_2, \ldots, E_2) \notin P_{n-1,1}$. Further, if $n \geq 5$ is an odd integer, then $diag(E_3, E_2, \ldots, E_2)$ intersects $P_{n-1,1}$ trivially. Since in either case χ vanishes on some r-regular element, it is not weak r-Steinberg.

Now if $(n-1, q) = (2, 2^t-1)$, then $n = 3$. The weak Steinberg characters of $GL(3, q)$ have been classified in Lemma 6.3. If $(n-1, q) = (6, 2)$, then since r is a divisor of (2^7-1), $r = 127$. Now note that it follows from the character table that $GL(7, 2)$ does not have a character of degree 127. Now the result follows. □

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Appendix

In the following, we give the character tables of $GL(2,q)$ and $GL(3,q)$. Note that in these tables, Class and Rep denote the type and the representative of conjugacy class.

Table A1. Character table of $GL(2,q)$.

Class	$C^{(1)}$	$C^{(2)}$	$C^{(3)}$	$C^{(4)}$
Rep	$\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}$	$\begin{pmatrix}\alpha & 1\\ 0 & \alpha\end{pmatrix}$	$\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}$	$\begin{pmatrix}0 & 1\\ -y^{q+1} & y+y^q\end{pmatrix}$
$\chi_k^{(1)}$	$\hat{\alpha}^{2k}$	$\hat{\alpha}^{2k}$	$\hat{\alpha}^k\hat{\beta}^k$	$\hat{y}^{k(q+1)}$
$\chi_k^{(2)}$	$q\hat{\alpha}^{2k}$	0	$\hat{\alpha}^k\hat{\beta}^k$	$-\hat{y}^{k(q+1)}$
$\chi_{k,l}$	$(q+1)\ \hat{\alpha}^{k+l}$	$\hat{\alpha}^{k+l}$	$\hat{\alpha}^k\hat{\beta}^l+\hat{\alpha}^l\hat{\beta}^k$	0
χ_t	$(q-1)\hat{\alpha}^k$	$-\hat{\alpha}^k$	0	$-(\hat{y}^k+\hat{y}^{kq})$

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where in this table:

- $\alpha, \beta \in F_q^*$ such that $\alpha \neq \beta$.
- $y \in F_{q^2} \setminus F_q; y^q$ is excluded whenever y is included.

Table A2. Character table of $GL(3, q)$.

Class	$T^{(1)}$	$T^{(2)}$	$T^{(3)}$
Rep	$\begin{pmatrix}\alpha & 0 & 0\\ 0 & \alpha & 0\\ 0 & 0 & \alpha\end{pmatrix}$	$\begin{pmatrix}\alpha & 1 & 0\\ 0 & \alpha & 0\\ 0 & 0 & \alpha\end{pmatrix}$	$\begin{pmatrix}\alpha & 1 & 0\\ 0 & \alpha & 1\\ 0 & 0 & \alpha\end{pmatrix}$
$\theta_k^{(1)}$	$\hat{\alpha}^{3k}$	$\hat{\alpha}^{3k}$	$\hat{\alpha}^{3k}$
$\theta_k^{(2)}$	$(q^2+q)\hat{\alpha}^{3k}$	$q\hat{\alpha}^{3k}$	0
$\theta_k^{(3)}$	$q^3\hat{\alpha}^{3k}$	0	0
$\theta_{k,l}^{(4)}$	$(q^2+q+1)\hat{\alpha}^{k+2l}$	$(q+1)\hat{\alpha}^{k+2l}$	$\hat{\alpha}^{k+2l}$
$\theta_{k,l}^{(5)}$	$q(q^2+q+1)\hat{\alpha}^{k+2l}$	$q\hat{\alpha}^{k+2l}$	0
$\theta_{k,l,m}^{(6)}$	$(q+1)(q^2+q+1)\hat{\alpha}^{k+l+m}$	$(2q+1)\hat{\alpha}^{k+l+m}$	$\hat{\alpha}^{k+l+m}$
$\theta_{t,l}^{(7)}$	$(q^3-1)\hat{\alpha}^{k+l}$	$-\hat{\alpha}^{k+l}$	$-\hat{\alpha}^{k+l}$
$\theta_v^{(8)}$	$(q-1)^2(q+1)\hat{\alpha}^k$	$-(q-1)\hat{\alpha}^k$	$\hat{\alpha}^k$

Class	$T^{(4)}$	$T^{(5)}$
Rep	$\begin{pmatrix}\alpha & 0 & 0\\ 0 & \alpha & 0\\ 0 & 0 & \beta\end{pmatrix}$	$\begin{pmatrix}\alpha & 1 & 0\\ 0 & \alpha & 0\\ 0 & 0 & \beta\end{pmatrix}$
$\theta_k^{(1)}$	$\hat{\alpha}^{2k}\hat{\beta}^k$	$\hat{\alpha}^{2k}\hat{\beta}^k$
$\theta_k^{(2)}$	$(q+1)\hat{\alpha}^{2k}\hat{\beta}^k$	$\hat{\alpha}^{2k}\hat{\beta}^k$
$\theta_k^{(3)}$	$q\hat{\alpha}^{2k}\hat{\beta}^k$	0
$\theta_{k,l}^{(4)}$	$(q+1)\hat{\alpha}^{k+l}\hat{\beta}^l + \hat{\alpha}^{2l}\hat{\beta}^k$	$\hat{\alpha}^{k+l}\hat{\beta}^l + \hat{\alpha}^{2l}\hat{\beta}^k$
$\theta_{k,l}^{(5)}$	$(q+1)\hat{\alpha}^{k+l}\hat{\beta}^l + q\hat{\alpha}^{2l}\hat{\beta}^k$	$\hat{\alpha}^{k+l}\beta^l$
$\theta_{k,l,m}^{(6)}$	$(q+1)(\hat{\alpha}^{k+l}\hat{\beta}^m + \hat{\alpha}^{k+m}\hat{\beta}^l + \hat{\alpha}^{m+l}\hat{\beta}^k)$	$\hat{\alpha}^{k+l}\hat{\beta}^m + \hat{\alpha}^{k+m}\hat{\beta}^l + \hat{\alpha}^{m+l}\hat{\beta}^k$
$\theta_{t,l}^{(7)}$	$(q-1)\hat{\alpha}^l\hat{\beta}^k$	$-\hat{\alpha}^l\hat{\beta}^k$
$\theta_v^{(8)}$	0	0

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Type	$T^{(6)}$
Rep	$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}$
$\theta_k^{(1)}$	$\hat{\alpha}^k \hat{\beta}^k \hat{\gamma}^k$
$\theta_k^{(2)}$	$2\hat{\alpha}^k \hat{\beta}^k \hat{\gamma}^k$
$\theta_k^{(3)}$	$\hat{\alpha}^k \hat{\beta}^k \hat{\gamma}^k$
$\theta_{k,l}^{(4)}$	$\hat{\alpha}^k \hat{\beta}^l \hat{\gamma}^l + \hat{\alpha}^l \hat{\beta}^k \hat{\gamma}^l + \hat{\alpha}^l \hat{\beta}^l \hat{\gamma}^k$
$\theta_{k,l}^{(5)}$	$\hat{\alpha}^k \hat{\beta}^l \hat{\gamma}^l + \hat{\alpha}^l \hat{\beta}^k \hat{\gamma}^l + \hat{\alpha}^l \hat{\beta}^l \hat{\gamma}^k$
$\theta_{k,l,m}^{(6)}$	$\hat{\alpha}^k(\hat{\beta}^l \hat{\gamma}^m + \hat{\beta}^m \hat{\gamma}^l) + \hat{\alpha}^m(\hat{\beta}^l \hat{\gamma}^k + \hat{\beta}^k \hat{\gamma}^l) + \hat{\alpha}^l(\hat{\beta}^k \hat{\gamma}^m + \hat{\beta}^m \hat{\gamma}^k)$
$\theta_{t,l}^{(7)}$	0
$\theta_v^{(8)}$	0

Class	$T^{(7)}$	$T^{(8)}$
Rep	$\begin{pmatrix} 0 & 1 & 0 \\ -x^{q+1} & x(x+x^q) & 0 \\ 0 & 0 & \alpha \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ s^{1+q+q^2} & -s(s^q + s^{q^2}) - s^{q+q^2} & s + s^q + s^{q^2} \end{pmatrix}$
$\theta_k^{(1)}$	$\hat{\alpha}^k \hat{x}^{k(q+1)}$	$\hat{s}^{k(1+q+q^2)}$
$\theta_k^{(2)}$	0	$-\hat{s}^{k(1+q+q^2)}$
$\theta_k^{(3)}$	$-\hat{\alpha}^k \hat{x}^{k(q+1)}$	$\hat{s}^{k(1+q+q^2)}$
$\theta_{k,l}^{(4)}$	$\hat{\alpha}^k \hat{x}^{l(q+1)}$	0
$\theta_{k,l}^{(5)}$	$-\hat{\alpha}^k \hat{x}^{l(q+1)}$	0
$\theta_{k,l,m}^{(6)}$	0	0
$\theta_{t,l}^{(7)}$	$-\hat{\alpha}^k(\hat{x}^l + \hat{x}^{ql})$	0
$\theta_v^{(8)}$	0	$\hat{s}^k + \hat{s}^{kq} + \hat{s}^{kq^2}$

where in this table:

- $\alpha, \beta, \gamma \in F_q^*$ such that $\alpha \neq \beta \neq \gamma \neq \alpha$.
- $x \in F_{q^2} \setminus F_q; x^q$ is excluded whenever x is included.
- $s \in F_{q^3} \setminus F_q; s^q$ and s^{q^2} are excluded whenever s is included.

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