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UPPER BOUNDS ON $|L(1,\chi)|$ AND APPLICATIONS

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ABSTRACT. We give upper bounds on the modulus of the values at $s = 1$ of Artin L-functions of abelian extensions unramified at all the infinite places. We also explain how we can compute better upper bounds and explain how useful such computed bounds are when dealing with class number problems for CM-fields. For example, we will reduce the determination of all the non-abelian normal CM-fields of degree 24 with Galois group $\mathrm{SL}_2(F_3)$ (the special linear group over the finite field with three elements) which have class number one to the computation of the class numbers of 23 such CM-fields.

1. **Introduction.** It is well known that there exists $c > 0$ such that for any primitive Dirichlet character modulo $f > 1$ we have $|L(1,\chi)| \leq \frac{1}{2}\log f + c$. Letting $\zeta_{\mathbf{E}}$, $d_{\mathbf{E}}$ and $h_{\mathbf{E}}$ denote the Dedekind zeta function, the absolute value of the discriminant and the class number of a number field $\mathbf{E}$, in [Lou3] we generalized this result and proved:

THEOREM 1. *Let $\mathbf{k}$ be a given number field. There exists a constant $\mu_{\mathbf{k}} > 0$ (depending on $\mathbf{k}$ only) such that for any non-trivial character χ on the Galois group of any abelian extension $\mathbf{K}/\mathbf{k}$ which is assumed to be unramified at all the infinite places we have*

$$|L(1,\chi)| \leq \mathrm{Res}_{s=1}(\zeta_{\mathbf{k}})\Big(\frac{1}{2}\log f_\chi + 2\mu_{\mathbf{k}}\Big) \tag{1}$$

together with the following two improvements:

$$|L(1,\chi)| \leq \mathrm{Res}_{s=1}(\zeta_{\mathbf{k}})\Big(\frac{1}{2}\log f_\chi + \mu_{\mathbf{k}}\Big) \quad \text{if } f_\chi \geq e^{2\mu_{\mathbf{k}}} \tag{2}$$

and

$$|L(1,\chi)| \leq \mu_{\mathbf{k}}\,\mathrm{Res}_{s=1}(\zeta_{\mathbf{k}}) \quad \text{if } f_\chi = 1. \tag{3}$$

Here, we let F_χ denote the conductor of χ and set $f_\chi = N_{\mathbf{k}/\mathbf{Q}}(F_\chi)$.

COROLLARY 2. *Let $\mathbf{K}/\mathbf{k}$ be an unramified at all the infinite places abelian extension of degree m. Then*

$$\mathrm{Res}_{s=1}(\zeta_{\mathbf{K}}) \leq \big(\mathrm{Res}_{s=1}(\zeta_{\mathbf{k}})\big)^m\Big(\frac{1}{2(m-1)}\log(d_{\mathbf{K}}/d_{\mathbf{k}}^m) + 2\mu_{\mathbf{k}}\Big)^{m-1} \tag{4}$$

Moreover, if $\mathbf{K}/\mathbf{k}$ is unramified at all the places, then

$$\mathrm{Res}_{s=1}(\zeta_{\mathbf{K}}) \leq \mathrm{Res}_{s=1}(\zeta_{\mathbf{k}})B_{\mathbf{k}}^{m-1} \quad \text{where } B_{\mathbf{k}} \stackrel{\text{def}}{=} \mu_{\mathbf{k}}\,\mathrm{Res}_{s=1}(\zeta_{\mathbf{k}}). \tag{5}$$

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PROOF. Use (1),

$$\mathrm{Res}_{s=1}(\zeta_{\mathbf{K}}) = \mathrm{Res}_{s=1}(\zeta_{\mathbf{k}}) \prod_{\chi\neq 1} L(1,\chi) \leq \mathrm{Res}_{s=1}(\zeta_{\mathbf{k}}) \Big(\frac{1}{m-1}\sum_{\chi\neq 1}|L(1,\chi)|\Big)^{m-1}$$

and $\prod_{\chi\neq 1} f_\chi = d_{\mathbf{K}}/d_{\mathbf{k}}^m$ to get (4). Use (3) and

$$\mathrm{Res}_{s=1}(\zeta_{\mathbf{K}}) = \mathrm{Res}_{s=1}(\zeta_{\mathbf{k}}) \prod_{\chi\neq 1} L(1,\chi)$$

to get (5). ∎

According to (2), for any primitive even Dirichlet character χ of conductor $f_\chi > 1$ we have

(6) $$|L(1,\chi)| \leq \frac{1}{2}\log f_\chi + \mu_{\mathbf{Q}}.$$

(and we will prove that $\mu_{\mathbf{Q}} = \big(2+\gamma-\log(4\pi)\big)/2 = 0.023095708966\cdots$). Hence, for any real quadratic field $\mathbf{k}$ of discriminant $d_{\mathbf{k}}$ we get

(7) $$\mathrm{Res}_{s=1}(\zeta_{\mathbf{k}}) \leq \frac{1}{2}\log d_{\mathbf{k}} + \mu_{\mathbf{Q}},$$

and more generally, for any real abelian field $\mathbf{k}$ of degree n and conductor $f_{\mathbf{k}}$ we have

(8) $$\mathrm{Res}_{s=1}(\zeta_{\mathbf{k}}) \leq \Big(\frac{1}{2(n-1)}\log d_{\mathbf{k}} + \mu_{\mathbf{Q}}\Big)^{n-1} \leq \Big(\frac{1}{2}\log f_{\mathbf{k}} + \mu_{\mathbf{Q}}\Big)^{n-1}$$

(use (6), the conductor-discriminant formula and the arithmetic-geometric mean inequality). Moreover we proved in [Lou3] that if $\mathbf{k}$ is a real quadratic field then we have

(9) $$B_{\mathbf{k}} = \mu_{\mathbf{k}}\,\mathrm{Res}_{s=1}(\zeta_{\mathbf{k}}) \leq \frac{1}{8}\log^2 d_{\mathbf{k}}.$$

However, our proof of (9) in [Lou3] was not that enlightening and did not point to any easy to handle method which would enable us to get a result similar to (9) for totally real fields $\mathbf{k}$ of any degree $n \geq 2$. We then used (4), (5), (7) and (9) to get upper bounds on residues at $s = 1$ of Dedekind zeta functions of various totally real number fields which were abelian extensions of real quadratic fields $\mathbf{k}$. These bounds were in turn used to get lower bounds on relative class numbers of various CM-fields and, finally, these lower bounds were used to solved various class number problems for non-abelian CM-fields (see [Lef], [LLO], [LO] and [LOO]). We refer the reader to [Was] for all the prerequisites on CM-fields we will assume him to be familiar with. Let us only mention that the analytic relative class number formula

(10) $$h_{\mathbf{N}}^- = \frac{Q_{\mathbf{N}} w_{\mathbf{N}}}{(2\pi)^n}\sqrt{\frac{d_{\mathbf{N}}}{d_{\mathbf{N}^+}}\frac{\mathrm{Res}_{s=1}(\zeta_{\mathbf{N}})}{\mathrm{Res}_{s=1}(\zeta_{\mathbf{N}^+})}},$$

makes it reasonable to seek upper bounds on residues at $s = 1$ of Dedekind zeta functions of totally real number fields $\mathbf{N}^+$ to obtain lower bounds on relative class numbers $h_{\mathbf{N}}^-$ of

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CM-fields $\mathbf{N}$ (here, $\mathbf{N}$ is a CM-field of degree $2n$ and $w_{\mathbf{N}}$ and $Q_{\mathbf{N}} \in \{1,2\}$ denote its number of roots of unity and Hasse unit index, respectively). The general upper bound

$$\mathrm{Res}_{s=1}(\zeta_{\mathbf{N}^+}) \leq \left(\frac{e \log d_{\mathbf{N}^+}}{2(n-1)}\right)^{n-1} \tag{11}$$

(see [Lou3] and [Lou4]) would provide worse lower bounds on $h_{\mathbf{N}}^-$ than the ones we obtained above (for example, compare the two lower bounds (42) and (43)). Maybe the best illustration of the usefulness of our bound (4) is the solution of the class number one problem for the dihedral CM-fields (see [Lef]). For simplicity's sake we assume that $\mathbf{N}$ is a dihedral CM-field of degree $2n = 4m$ with $m \geq 3$ odd. We let $\mathbf{M}$ denote the imaginary biquadratic bicyclic subfield of $\mathbf{N}$ and $\mathbf{k}$ denote the real quadratic subfield of $\mathbf{M}$. Then $\mathbf{N}^+/\mathbf{k}$ is cyclic of degree m. We note that $h_{\mathbf{M}}^-$ divides $h_{\mathbf{N}}^-$ and that $(\zeta_{\mathbf{N}}/\zeta_{\mathbf{M}})(s) \geq 0$ for any $s \in]0,1[$. Now, assume that $h_{\mathbf{N}}^- = 1$. Then $h_{\mathbf{M}}^- = 1$. However, it is known that there are exactly 147 imaginary biquadratic bicyclic fields $\mathbf{M}$ such that $h_{\mathbf{M}}^- = 1$ and, moreover, one can easily check that for all these 147 fields $\mathbf{M}$ we have $\zeta_{\mathbf{M}}(s) < 0$ for $s \in]0,1[$. Therefore, if $h_{\mathbf{N}}^- = 1$ then $\zeta_{\mathbf{N}}\big(1 - (2/\log d_{\mathbf{N}})\big) \leq 0$. However, it is known that for any CM-field such that $\zeta_{\mathbf{N}}\big(1 - (2/\log d_{\mathbf{N}})\big) \leq 0$ we (roughly speaking) have $\mathrm{Res}_{s=1}(\zeta_{\mathbf{N}}) \geq 2/e \log d_{\mathbf{N}}$. Now, noticing that we have $d_{\mathbf{N}} \geq d_{\mathbf{N}^+}^2$, if we use (10) and (11) we get

$$h_{\mathbf{N}}^- \geq \frac{1}{n-1} \frac{\sqrt{d_{\mathbf{N}^+}}}{(\frac{\pi e}{n-1} \log d_{\mathbf{N}^+})^n}, \tag{12}$$

from which we can deduce that there are only finitely many dihedral CM-fields of degree $2n = 4m \equiv 4 \pmod 8$ with relative class number one and that all satisfy $d_{\mathbf{N}^+}^{1/n} \leq 30000$, too large a bound to enable us to solve the (relative) class number one problem for such dihedral CM-fields. But now, using (4), noticing that there are at most 147 occurrences for $\mathbf{k}$ (which all satisfy $d_{\mathbf{k}} \leq 65689$) and using the bounds (7) and (9) or, more efficiently, computing numerically $\mu_{\mathbf{k}}$ and $\mathrm{Res}_{s=1}(\zeta_{\mathbf{k}})$ for all of them, we end up with an explicit upper bound

$$\mathrm{Res}_{s=1}(\zeta_{\mathbf{N}^+}) = O(\log^{m-1} d_{\mathbf{N}^+}),$$

hence with an explicit lower bound

$$h_{\mathbf{N}}^- \gg c_m \frac{\sqrt{d_{\mathbf{N}^+}}}{\log^m d_{\mathbf{N}^+}},$$

whose exponent m is half as large as the one n in (12). This lower bound is now good enough to determine all the dihedral CM-fields with (relative) class numbers equal to one (see [Lef]).

The first purpose of this paper is to prove Theorem 1 (see Section 1.4).

The second purpose of this paper is to give bounds on $B_{\mathbf{k}} = \mu_{\mathbf{k}} \mathrm{Res}_{s=1}(\zeta_{\mathbf{k}})$ for totally real fields $\mathbf{k}$ of degree $n \geq 2$ (see Theorems 5 and 11). Contenting ourselves with totally real fields $\mathbf{k}$ is no serious restriction to us, for we aim at using our present results to get good upper for residues at $s = 1$ of the Dedekind zeta functions of various totally real

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number fields $\mathbf{K} = \mathbf{N}^+$ which are maximal totally real subfields of CM-fields $\mathbf{N}$. We will prove in Theorem 5 that we have

$$B_{\mathbf{k}} = \mu_{\mathbf{k}} \operatorname{Res}_{s=1}(\zeta_{\mathbf{k}}) \leq \frac{1}{2^n n!} \log^n d_{\mathbf{k}}, \tag{13}$$

provided that $d_{\mathbf{k}}$ is large enough. This will provide us with a less technical proof and a generalization of (9) to any totally real number field. Moreover, Theorem 11 will provide us with the better bound

$$B_{\mathbf{k}} = \mu_{\mathbf{k}} \operatorname{Res}_{s=1}(\zeta_{\mathbf{k}}) \leq \frac{n-1}{2}\left(\frac{1}{2(n-1)} \log d_{\mathbf{k}} + \mu_{\mathbf{Q}}\right)^n \leq \frac{n-1}{2}\left(\frac{1}{2} \log f_{\mathbf{k}} + \mu_{\mathbf{Q}}\right)^n, \tag{14}$$

provided that $\mathbf{k}$ is (real) abelian of conductor $f_{\mathbf{k}}$.

The third purpose of this paper is to explain how one can efficiently compute numerically the value of any $B_{\mathbf{k}} = \mu_{\mathbf{k}} \operatorname{Res}_{s=1}(\zeta_{\mathbf{k}})$ (see Sections 3.3 and 4.2). In fact, in the last section of this paper, we will firstly use (5), (8) and (14) to determine the reasonable bound $f_{\mathbf{k}} \leq 83000$ on the conductors $f_{\mathbf{k}}$ of the real cyclic cubic subfields $\mathbf{k}$ of the normal CM-fields $\mathbf{N}$ of degree 24 with Galois group $SL_2(F_3)$ which have class number one (and we will point out that $\mathbf{N}$ is well determined by $\mathbf{k}$), and we will secondly compute numerically all the $B_{\mathbf{k}}$ and $\operatorname{Res}_{s=1}(\zeta_{\mathbf{k}})$ for the 4784 possible occurrences of $\mathbf{k}$ with $f_{\mathbf{k}} \leq 10^5$ and we will then use (5) to prove that only 23 out of these 4784 cyclic cubic fields can be cyclic cubic subfields of normal CM-fields $\mathbf{N}$ of degree 24 with Galois group $SL_2(F_3)$ and class number one (see Proposition 16). This example clearly shows how useful (13), (14) and such computed bounds on residues can be, for it is much easier to compute $B_{\mathbf{k}}$ than to compute $h_{\mathbf{N}}^-$. We also refer the reader to [CK] and [Lef] for other examples.

1.1. *Definition of $\lambda_{\mathbf{k}}$ and $\mu_{\mathbf{k}}$.* Let $\mathbf{k}$ be a number field of degree $n = r_1 + 2r_2$, where r_1 and r_2 denote the number of real and complex places of $\mathbf{k}$, respectively. Let $\zeta_{\mathbf{k}}$ and $d_{\mathbf{k}}$ be the Dedekind zeta function and the absolute value of the discriminant of $\mathbf{k}$, respectively. We set

$$A_{\mathbf{k}} = 2^{-r_2}\pi^{-n/2}\sqrt{d_{\mathbf{k}}},$$
$$\Gamma_{\mathbf{k}}(s) = \Gamma^{r_1}(s/2)\Gamma^{r_2}(s)$$

and

$$F_{\mathbf{k}}(s) = A_{\mathbf{k}}^s \Gamma_{\mathbf{k}}(s)\zeta_{\mathbf{k}}(s).$$

It is well known that $F_{\mathbf{k}}$ satisfies the functional equation $F_{\mathbf{k}}(1-s) = F_{\mathbf{k}}(s)$, has only two poles, at $s = 1$ and $s = 0$, both simple, and we set

$$\lambda_{\mathbf{k}} = \operatorname{Res}_{s=1}(F_{\mathbf{k}}) = A_{\mathbf{k}}\Gamma_{\mathbf{k}}(1) \operatorname{Res}_{s=1}(\zeta_{\mathbf{k}}) = (2\pi)^{-r_2}\sqrt{d_{\mathbf{k}}} \operatorname{Res}_{s=1}(\zeta_{\mathbf{k}}),$$

which yields $\operatorname{Res}_{s=0}(F_{\mathbf{k}}) = -\lambda_{\mathbf{k}}$. Note that we have $\lambda_{\mathbf{k}} > 0$. We finally set

$$\mu_{\mathbf{k}} = \lim_{s \searrow 1}\left\{\frac{1}{\lambda_{\mathbf{k}}}F_{\mathbf{k}}(s) - \left(\frac{1}{s-1} - \frac{1}{s}\right)\right\}. \tag{15}$$

In particular, we have $\lambda_{\mathbf{Q}} = 1$, $\mu_{\mathbf{Q}} = \big(2 + \gamma - \log(4\pi)\big)/2 = 0.023\cdots$.

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1.2. *Definition of the functions $H_{\mathbf{k}}$, $S_{\mathbf{k}}$, Λ_χ and S_χ.* We set

$$\zeta_{\mathbf{k}}(s) = \sum_{m\geq 1} z_m m^{-s}$$

to define coefficients z_m (and note that we have $z_m \geq 0$) and define

$$H_{\mathbf{k}}(x) = \frac{1}{2\pi i}\int_{\Re(s)=\alpha} \Gamma_{\mathbf{k}}(s)x^{-s}\,ds \quad (x > 0 \text{ and } \alpha > 0)$$

(and note that we have $H_{\mathbf{k}}(x) \geq 0$ for $x > 0$) and

$$S_{\mathbf{k}}(x) = \frac{1}{2\pi i}\int_{\Re(s)=\alpha} F_{\mathbf{k}}(s)x^{-s}\,ds = \sum_{m\geq 1} z_m H_{\mathbf{k}}(mx/A_{\mathbf{k}}) \quad (x > 0 \text{ and } \alpha > 1). \tag{16}$$

Now, let χ denote a Dirichlet character associated to an abelian extension $\mathbf{K}/\mathbf{k}$ unramified at all the infinite places, let F_χ denote the conductor of χ (which is an integral ideal of $\mathbf{k}$) and set

$$f_\chi = N_{\mathbf{k}/\mathbf{Q}}(F_\chi),$$
$$L(s,\chi) = \sum_{m\geq 1} \phi_m m^{-s}$$

with

$$\phi_m = \sum_{N_{\mathbf{k}/\mathbf{Q}}(\mathbf{I})=m} \chi(\mathbf{I})$$

(where this sum ranges over all the integral ideals of $\mathbf{k}$ of norm m),

$$A_\chi = A_{\mathbf{k}}\sqrt{f_\chi},$$

and

$$\Lambda_\chi(s) = A_\chi^s \Gamma_{\mathbf{k}}(s)L(s,\chi),$$

which is entire and satisfies the functional equation

$$\Lambda_\chi(1-s) = W_\chi \Lambda_{\bar{\chi}}(s)$$

for some root number W_χ of absolute value equal to one. Notice that

$$L(1,\chi) = \frac{1}{\lambda_{\mathbf{k}}\sqrt{f_\chi}} \operatorname{Res}_{s=1}(\zeta_{\mathbf{k}})\Lambda_\chi(1) \tag{17}$$

We finally set

$$S_\chi(x) = \frac{1}{2\pi i}\int_{\Re(s)=\alpha} \Lambda_\chi(s)x^{-s}\,ds = \sum_{m\geq 1} \phi_m H_{\mathbf{k}}(mx/A_\chi) \quad (x > 0 \text{ and } \alpha > 1) \tag{18}$$

and notice that $|\phi_m| \leq z_m$ yields

$$|S_\chi(x)| \leq S_{\mathbf{k}}(xA_{\mathbf{k}}/A_\chi) = S_{\mathbf{k}}(x/\sqrt{f_\chi}).$$

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Observe that it is of paramount importance that $\mathbf{K}/\mathbf{k}$ be unramified at all the infinite places, for the proof of Theorem 1 stems from the fact that $\Gamma_{\mathbf{k}}$ is the Gamma factor which appears in the functional equations of both $F_{\mathbf{k}}$ and Λ_χ, which enables us first to express both $S_{\mathbf{k}}$ and S_χ in terms of $H_{\mathbf{k}}$, and second to obtain $|S_\chi(x)| \leq S_{\mathbf{k}}(x/\sqrt{f_\chi})$. We refer the reader to [Lou1] and [Lou4, Th. 6] to see how complicated and less satisfactory become generalizations of Theorem 1 when $\mathbf{K}/\mathbf{k}$ is not assumed to be unramified at all the infinite places.

Notice that the choice $\mathbf{k} = \mathbf{Q}$ the field of rational numbers yields

$$S_{\mathbf{Q}}(x) = 2\sum_{n\geq 1} e^{-\pi n^2 x^2}. \tag{19}$$

1.3. *Integral representations of Λ_χ, $F_{\mathbf{k}}$ and $\mu_{\mathbf{k}}$.* By shifting the line of integration $\Re(s) = \alpha$ in (16) and (18) to the left to the line $\Re(s) = 1-\alpha$ we pick up residues at $s = 1$ and $s = 0$, and by using the functional equations satisfied by $F_{\mathbf{k}}$ and Λ_χ to come back to the line of integration $\Re(s) = \alpha$, we obtain the following functional equations:

$$S_{\mathbf{k}}(x) = \frac{1}{x}S_{\mathbf{k}}\Big(\frac{1}{x}\Big) - \lambda_{\mathbf{k}} + \frac{\lambda_{\mathbf{k}}}{x} \quad \text{and} \quad S_\chi(x) = \frac{W_\chi}{x}S_{\bar\chi}\Big(\frac{1}{x}\Big). \tag{20}$$

Therefore, we finally obtain:

$$\begin{aligned}\Lambda_\chi(s) &= \int_0^\infty S_\chi(x)x^s\frac{dx}{x}\\ &= \int_1^\infty S_\chi(x)x^s\frac{dx}{x} + \int_1^\infty S_\chi\Big(\frac{1}{x}\Big)x^{-s}\frac{dx}{x}\\ &= \int_1^\infty S_\chi(x)x^s\frac{dx}{x} + W_\chi\int_1^\infty S_{\bar\chi}(x)x^{1-s}\frac{dx}{x}\end{aligned}$$

and

$$\Lambda_\chi(1) = \int_1^\infty S_\chi(x)\,dx + W_\chi\int_1^\infty S_{\bar\chi}(x)\frac{dx}{x}. \tag{21}$$

In the same way, we get

$$\mu_{\mathbf{k}} \stackrel{\text{def}}{=} \lim_{s\searrow 1}\Big\{\frac{1}{\lambda_{\mathbf{k}}}F_{\mathbf{k}}(s) - \Big(\frac{1}{s-1}-\frac{1}{s}\Big)\Big\} = \frac{1}{\lambda_{\mathbf{k}}}\int_1^\infty S_{\mathbf{k}}(x)\,dx + \frac{1}{\lambda_{\mathbf{k}}}\int_1^\infty S_{\mathbf{k}}(x)\frac{dx}{x}. \tag{22}$$

Notice that we get $\mu_{\mathbf{k}} > 0$.

1.4. *Definition of $f \longmapsto I_{\mathbf{k}}(f)$ and proof of Theorem 1.* We set

$$f = A_\chi/A_{\mathbf{k}} = \sqrt{f_\chi} \geq 1,$$

which yields

$$|S_\chi(x)| \leq S_{\mathbf{k}}(x/f).$$

Setting

$$I_{\mathbf{k}}(f) = \int_1^\infty S_{\mathbf{k}}(x/f)\,dx + \int_1^\infty S_{\mathbf{k}}(x/f)\frac{dx}{x} \tag{23}$$

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and using (22) and (21), we obtain

$$I_{\mathbf{k}}(1) = \lambda_{\mathbf{k}}\mu_{\mathbf{k}} \quad \text{and} \quad |\Lambda_\chi(1)| \leq I_{\mathbf{k}}(f). \tag{24}$$

To begin with, if $f_\chi = 1$ then $f = 1$ and using (24) and (17) we get

$$|L(1,\chi)| = \frac{1}{\lambda_{\mathbf{k}}}\operatorname{Res}_{s=1}(\zeta_{\mathbf{k}})|\Lambda_\chi(1)| \leq \frac{1}{\lambda_{\mathbf{k}}}\operatorname{Res}_{s=1}(\zeta_{\mathbf{k}})I_{\mathbf{k}}(1) = \mu_{\mathbf{k}}\operatorname{Res}_{s=1}(\zeta_{\mathbf{k}})$$

and (3) is proved.

Now, for any χ, using (20), we have

$$\begin{aligned}
I_{\mathbf{k}}(f) &= f\int_{1/f}^{\infty} S_{\mathbf{k}}(x)\,dx + \int_{1/f}^{\infty} S_{\mathbf{k}}(x)\frac{dx}{x}\\
&= f\int_{1}^{\infty} S_{\mathbf{k}}(x)\,dx + \int_{1}^{\infty} S_{\mathbf{k}}(x)\frac{dx}{x} + f\int_{1}^{f} S_{\mathbf{k}}(1/x)\frac{dx}{x^2} + \int_{1}^{f} S_{\mathbf{k}}(1/x)\frac{dx}{x}\\
&= f\int_{1}^{\infty} S_{\mathbf{k}}(x)\,dx + \int_{1}^{\infty} S_{\mathbf{k}}(x)\frac{dx}{x} + f\int_{1}^{f} S_{\mathbf{k}}(x)\frac{dx}{x} + \int_{1}^{f} S_{\mathbf{k}}(x)\,dx + \lambda_{\mathbf{k}}(f-1)\log f\\
&\leq (f+1)\Big(\int_{1}^{\infty} S_{\mathbf{k}}(x)\,dx + \int_{1}^{\infty} S_{\mathbf{k}}(x)\frac{dx}{x}\Big) + \lambda_{\mathbf{k}}(f-1)\log f\\
&= (f+1)I_{\mathbf{k}}(1) + \lambda_{\mathbf{k}}(f-1)\log f\\
&= f\lambda_{\mathbf{k}}(\log f + \mu_{\mathbf{k}}) + \lambda_{\mathbf{k}}(\mu_{\mathbf{k}} - \log f),
\end{aligned}$$

and using (24) and (17) we get

$$\begin{aligned}
|L(1,\chi)| &= \frac{1}{f\lambda_{\mathbf{k}}}\operatorname{Res}_{s=1}(\zeta_{\mathbf{k}})|\Lambda_\chi(1)|\\
&\leq \frac{1}{f\lambda_{\mathbf{k}}}\operatorname{Res}_{s=1}(\zeta_{\mathbf{k}})I_{\mathbf{k}}(f) \leq \operatorname{Res}_{s=1}(\zeta_{\mathbf{k}})\Big(\Big(1-\frac{1}{f}\Big)\log f + \Big(1+\frac{1}{f}\Big)\mu_{\mathbf{k}}\Big)
\end{aligned}$$

from which we get (1) and (2) of Theorem 1.

Let us point out that Theorem 5 and Lemma 10 will be proved in much the same way.

2. **A bound on $\mu_{\mathbf{k}}\operatorname{Res}_{s=1}(\zeta_{\mathbf{k}})$ when k is totally real.** From now on, we assume that $\mathbf{k}$ is a totally real number field of degree n, which yields $\lambda_{\mathbf{k}} = \sqrt{d_{\mathbf{k}}}\operatorname{Res}_{s=1}(\zeta_{\mathbf{k}})$ and $I_{\mathbf{k}}(1) = \lambda_{\mathbf{k}}\mu_{\mathbf{k}} = \sqrt{d_{\mathbf{k}}}\mu_{\mathbf{k}}\operatorname{Res}_{s=1}(\zeta_{\mathbf{k}}) = \sqrt{d_{\mathbf{k}}}B_{\mathbf{k}}$. The aim of this section is to determine bounds on $I_{\mathbf{k}}(1)$. We first set some notation. For $n \geq 1$ we define

$$\begin{gathered}
F_{\mathbf{Q}}(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{s-1} + \frac{\gamma - \log(4\pi)}{2} + O(s-1),\\
F_n(s) = F_{\mathbf{Q}}^n(s) = A_n^s\Gamma^n(s/2)\zeta^n(s) \quad \text{with } A_n = \pi^{-n/2},\\
\zeta^n(s) = \sum_{m\geq 1} Z_m m^{-s},\\
f = A_{\mathbf{k}}/A_n = \sqrt{d_{\mathbf{k}}},\\
H_n(x) = \frac{1}{2\pi i}\int_{\Re(s)=\alpha}\Gamma^n(s/2)x^{-s}\,ds \quad (x>0 \text{ and } \alpha>0).
\end{gathered}$$

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(note that $H_{\mathbf{k}}(x) = H_n(x)$) and define

$$S_n(x) = \sum_{m\geq 1} Z_m H_n(mx/A_n) = \frac{1}{2\pi i}\int_{\Re(s)=\alpha} F_n(s)x^{-s}\,ds \quad (x > 0 \text{ and } \alpha > 1). \tag{25}$$

Since $0 \leq z_n \leq Z_n$, we get $S_{\mathbf{k}}(x) \leq S_n(x/f)$ and (23) yields

$$f\mu_{\mathbf{k}}\,\mathrm{Res}_{s=1}(\zeta_{\mathbf{k}}) = I_{\mathbf{k}}(1) \leq \int_1^\infty \Big(1+\frac{1}{x}\Big) S_n(x/f)\,dx. \tag{26}$$

The aim of this section is to compute bounds on the right hand side of this inequality. Shifting the line of integration $\Re(s) = \alpha$ in (25) to the left to the line $\Re(s) = 1-\alpha$ we pick up residues at $s = 1$ and $s = 0$, and using the functional equation $F_n(1-s) = F_n(s)$ to come back to the line of integration $\Re(s) = \alpha$, we obtain

$$S_n(x) = \mathrm{Res}_{s=1}\big(F_n(s)x^{-s}\big) + \mathrm{Res}_{s=0}\big(F_n(s)x^{-s}\big) + \frac{1}{x}S_n\Big(\frac{1}{x}\Big)$$

and note that both these residues depend on x. Since $F_n(s) = F_n(1-s)$, we get

$$\mathrm{Res}_{s=0}\big(F_n(s)x^{-s}\big) = -\,\mathrm{Res}_{s=1}\big(F_n(1-s)x^{s-1}\big) = -\,\mathrm{Res}_{s=1}\big(F_n(s)x^{s-1}\big),$$

setting

$$G_n(s) = F_n(s)(x^{-s} - x^{s-1}) \tag{27}$$

we get

$$S_n(x) = \mathrm{Res}_{s=1}(G_n) + \frac{1}{x}S_n\Big(\frac{1}{x}\Big), \tag{28}$$

(and note that this residue depends on x), which yields

$$\begin{aligned} F_n(S) &= \int_0^\infty S_n(x)x^S\frac{dx}{x} = \int_1^\infty S_n\Big(\frac{1}{x}\Big)x^{-S}\frac{dx}{x} + \int_1^\infty S_n(x)x^S\frac{dx}{x} \\ &= \int_1^\infty S_n(x)(x^{1-S}+x^S)\frac{dx}{x} - \int_1^\infty \mathrm{Res}_{s=1}(G_n)x^{-S}\,dx. \end{aligned}$$

LEMMA 3. *Set*

$$I_n(S) = \int_1^\infty S_n(x)(x^{1-S}+x^S)\frac{dx}{x}.$$

Then, $S > 1$ implies

$$I_n(S) = F_n(S) + \mathrm{Res}_{s=1}\Big(s \longmapsto \frac{F_n(s)}{S+s-1}\Big) - \mathrm{Res}_{s=1}\Big(s \longmapsto \frac{F_n(s)}{S-s}\Big). \tag{29}$$

PROOF. Using (27), we have

$$\begin{aligned} I_n(S) &= F_n(S) + \int_1^\infty \mathrm{Res}_{s=1}\big(s\longmapsto G_n(s)\big)x^{-S}\,dx \\ &= F_n(S) + \mathrm{Res}_{s=1}\Big(s\longmapsto \int_1^\infty G_n(s)x^{-S}\,dx\Big) \end{aligned}$$

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$$= F_n(S) + \mathrm{Res}_{s=1}\Big(s \longmapsto \int_1^\infty F_n(s)(x^{-s-S} - x^{s-S-1})\,dx\Big)$$
$$= F_n(S) + \mathrm{Res}_{s=1}\Big(s \longmapsto F_n(S)\Big(\frac{1}{S+s-1} - \frac{1}{S-s}\Big)\Big)$$
$$= F_n(S) - \mathrm{Res}_{s=1}\Big(s \longmapsto \frac{F_n(s)}{S-s}\Big) + \mathrm{Res}_{s=1}\Big(s \longmapsto \frac{F_n(s)}{S+s-1}\Big). \qquad \blacksquare$$

PROPOSITION 4. *Set*

$$I_n \stackrel{\text{def}}{=} I_n(1) = \int_1^\infty S_n(x)\,dx + \int_1^\infty S_n(x)\frac{dx}{x}.$$

Then

$$I_n = \mathrm{Res}_{s=1}\Big(s \longmapsto F_n(s)\Big(\frac{1}{s} + \frac{1}{s-1}\Big)\Big). \tag{30}$$

PROOF. On the one hand we have

$$\lim_{S\searrow 1} \mathrm{Res}_{s=1}\Big(s \longmapsto \frac{F_n(s)}{S+s-1}\Big) = \mathrm{Res}_{s=1}\Big(s \longmapsto \lim_{S\searrow 1}\frac{F_n(s)}{S+s-1}\Big) = \mathrm{Res}_{s=1}\Big(s \longmapsto \frac{F_n(s)}{s}\Big).$$

On the other hand, using

$$\frac{1}{S-s} = \frac{1}{S-1}\sum_{j\geq 0}\Big(\frac{s-1}{S-1}\Big)^j$$

and writing $F_n(s) = \sum_{i\geq -n} a_i(n)(s-1)^i$, we get

$$F_n(S) - \mathrm{Res}_{s=1}\Big(s \longmapsto \frac{F_n(s)}{S-s}\Big) = F_n(S) - \sum_{i=-n}^{-1} a_i(n)(S-1)^i = \sum_{i\geq 0} a_i(n)(S-1)^i$$

and

$$\lim_{S\searrow 1}\Big(F_n(S) - \mathrm{Res}_{s=1}\Big(s \longmapsto \frac{F_n(s)}{S-s}\Big)\Big) = a_0(n) = \mathrm{Res}_{s=1}\Big(s \longmapsto \frac{F_n(s)}{s-1}\Big).$$

Therefore, using (29) we get the desired result. ∎

According to (26) and (28), we obtain:

$$\begin{aligned}
I_{\mathbf{k}}(1) &\leq \int_1^\infty \Big(1+\frac{1}{x}\Big)S_n(x/f)\,dx \\
&= \int_{1/f}^\infty \Big(f+\frac{1}{x}\Big)S_n(x)\,dx \\
&= \int_1^\infty \Big(f+\frac{1}{x}\Big)S_n(x)\,dx + \int_1^f (f+x)S_n\Big(\frac{1}{x}\Big)\frac{dx}{x^2} \\
&= \int_1^\infty \Big(f+\frac{1}{x}\Big)S_n(x)\,dx + \int_1^f \Big(\frac{f}{x}+1\Big)S_n(x)\,dx - \int_1^f \Big(\frac{f}{x}+1\Big)\,\mathrm{Res}_{s=1}(G_n)\,dx \\
&\leq \int_1^\infty \Big(f+\frac{1}{x}+\frac{f}{x}+1\Big)S_n(x)\,dx - \int_1^f \Big(\frac{f}{x}+1\Big)\,\mathrm{Res}_{s=1}(G_n)\,dx \\
&= (f+1)I_n - \mathrm{Res}_{s=1}\Big(\int_1^f \Big(\frac{f}{x}+1\Big)G_n(s)\,dx\Big) \\
&= (f+1)I_n - \mathrm{Res}_{s=1}\Big(F_n(s)\Big(\frac{1}{s}+\frac{1}{s-1}\Big)(f+1-f^s-f^{1-s})\Big).
\end{aligned}$$

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According to (30) and since $\sqrt{d_{\mathbf{k}}}B_{\mathbf{k}} = \sqrt{d_{\mathbf{k}}}\mu_{\mathbf{k}}\operatorname{Res}_{s=1}(\zeta_{\mathbf{k}}) = \lambda_{\mathbf{k}}\mu_{\mathbf{k}} = I_{\mathbf{k}}(1)$, we get

THEOREM 5. *Let* $\mathbf{k}$ *be a totally real number field of degree* n *and set* $f = \sqrt{d_{\mathbf{k}}}$. *We have*

$$\sqrt{d_{\mathbf{k}}}B_{\mathbf{k}} \leq R_n(f) \stackrel{\text{def}}{=} \operatorname{Res}_{s=1}\Big(F_n(s)\Big(\frac{1}{s}+\frac{1}{s-1}\Big)(f^s+f^{1-s})\Big).$$

Moreover, we have the following Table:

$R_1(f) = (f-1)\log f + b_1(f+1)$
with $b_1 = \big(2+\gamma-\log(4\pi)\big)/2 = 0.023095\cdots$,
$R_2(f) = \frac{f+1}{2}\log^2 f - c_1(f-1)\log f + c_2(f+1)$
with $c_1 = \log(4\pi)-1-\gamma = 0.9538\cdots$ *and* $c_2 = 0.001029\cdots$,
$R_3(f) = \frac{f-1}{6}\log^3 f - d_1(f+1)\log^2 f + d_2(f-1)\log f + d_3(f+1)$
with $d_1 = \Big(3\big(\log(4\pi)-\gamma\big)-2\Big)/4 = 0.965\cdots$, $d_2 = 1.933\cdots$ *and* $d_3 = 0.0000517\cdots$.

Let $n \geq 2$ *be given. There exists* f_n *such that* $f \geq f_n$ *implies* $R_n(f) \leq \frac{f}{n!}\log^n(f)$. *In other words, if* n *is given then there exists* d_n *such that*

$$B_{\mathbf{k}} = \mu_{\mathbf{k}}\operatorname{Res}_{s=1}(\zeta_{\mathbf{k}}) \leq \frac{1}{2^n n!}\log^n d_{\mathbf{k}} \tag{31}$$

holds for any totally real number field $\mathbf{k}$ *of degree* n *such that* $d_{\mathbf{k}} \geq d_n$. *In particular, (31) holds for any totally real number field* $\mathbf{k}$ *of degree* $n = 2$ *or* $n = 3$.

PROOF. One can easily check that $R_n(f) = fP_n(\log f) + P_n(-\log f)$ where

$$P_n(X) = \operatorname{Res}_{s=0}\Big(\Big(\frac{1}{s}+\frac{1}{s+1}\Big)e^{sX}F_n(s+1)\Big) = \sum_{k=0}^{n}\frac{p_k(n)}{k!}X^k$$

with

$$p_k(n) = \operatorname{Res}_{s=0}\Big(s^k\Big(\frac{1}{s}+\frac{1}{s+1}\Big)F_n(s+1)\Big).$$

Since

$$F_n(s+1) = \frac{1}{s^n} - n\frac{\log(4\pi)-\gamma}{2}\frac{1}{s^{n-1}} + \cdots$$

and

$$\frac{1}{s}+\frac{1}{s+1} = \frac{1}{s} + \sum_{k\geq 0}(-1)^k s^k,$$

we get $p_n(n) = 1$, $p_{n-1}(n) = -\Big(n\big(\log(4\pi)-\gamma\big)-2\Big)/2$ and

$$\begin{aligned} R_n(f) &= \frac{f+(-1)^n}{n!}\log^n f + p_{n-1}(n)\frac{f+(-1)^{n-1}}{(n-1)!}\log^{n-1} f + \cdots \\ &= \frac{f}{n!}\log^n f + p_{n-1}(n)\frac{f}{(n-1)!}\log^{n-1} f + O(f\log^{n-2} f), \end{aligned}$$

and the desired result follows from $p_{n-1}(n) \leq 1+\gamma-\log(4\pi) < 0$ for $n \geq 2$. ∎

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3. Numerical computation of $\mu_{\mathbf{k}}\,\mathrm{Res}_{s=1}(\zeta_{\mathbf{k}})$.

3.1. *The mean value of $\mu_{\mathbf{k}}\,\mathrm{Res}_{s=1}(\zeta_{\mathbf{k}})$ over real quadratic fields.* According to the results of the previous section, for any real quadratic field $\mathbf{k}$ we have $B_{\mathbf{k}} = \mu_{\mathbf{k}}\,\mathrm{Res}_{s=1}(\zeta_{\mathbf{k}}) \leq \frac{1}{8}\log^2 d_{\mathbf{k}}$. However, numerical computation of $B_{\mathbf{k}}$ for various real quadratic fields $\mathbf{k}$ suggests that in general this bound is poor. In fact, the following result says that, roughly speaking, we may expect $B_{\mathbf{k}}$ to be close to $c'\log d_{\mathbf{k}}$ where $c' = \frac{\pi^2}{3}c = 1.45\cdots$.

PROPOSITION 6. *When $\mathbf{k}$ ranges over the real quadratic fields*

$$f(x) = \sum_{d_{\mathbf{k}}\leq x} \mu_{\mathbf{k}}\,\mathrm{Res}_{s=1}(\zeta_{\mathbf{k}})$$

is asymptotic to $cx\log x$ with $c = \frac{1}{4}\prod_p\Big(1-(p^3+p^2)^{-1}\Big) = 0.22037\cdots$.

PROOF. According to Lemma 9 below, if $\mathbf{k}$ is quadratic then

$$\mu_{\mathbf{k}}\,\mathrm{Res}_{s=1}(\zeta_{\mathbf{k}}) = \mu_{\mathbf{k}}L(1,\chi_{\mathbf{k}}) = L'(1,\chi_{\mathbf{k}}) + \Big(1-\log(4\pi)+\frac{1}{2}\log d_{\mathbf{k}}\Big)L(1,\chi_{\mathbf{k}}).$$

We then argue as in [Jut1] and [Jut2] to prove that $g(x) = \sum_{d_{\mathbf{k}}\leq x} 1$ which equals the number of real quadratic fields of discriminants less than or equal to x is asymptotic to $3x/\pi^2$, that $\sum_{d_{\mathbf{k}}\leq x} L(1,\chi_{\mathbf{k}})$ is asymptotic to c_1x, that $\sum_{d_{\mathbf{k}}\leq x}(\log d_{\mathbf{k}})L(1,\chi_{\mathbf{k}})$ is asymptotic to $c_1x\log x$, and that $\sum_{d_{\mathbf{k}}\leq x} L'(1,\chi_{\mathbf{k}})$ is asymptotic to $-c_2x$, with

$$a_m = \prod_{p\mid m}(1+p^{-1})^{-1},$$

$$c_1 = \frac{3}{\pi^2}\sum_{m\geq 1} a_m\frac{1}{m^2} = \frac{3}{\pi^2}\prod_p(1-p^{-2})\Big(1-(p^3+p^2)^{-1}\Big) = \frac{1}{2}\prod_p\Big(1-(p^3+p^2)^{-1}\Big)$$

and

$$c_2 = \sum_{m\geq 1} a_m\frac{\log(m^2)}{m^2} = 1.32\cdots. \qquad \blacksquare$$

Since we may expect $B_{\mathbf{k}}$ to be smaller than the bound (31) given in section above, let us now explain on a particular example how useful it might be to compute numerically $B_{\mathbf{k}}$.

3.2. *Usefulness of the numerical computation of $\mu_{\mathbf{k}}\,\mathrm{Res}_{s=1}(\zeta_{\mathbf{k}})$.* Let $\mathbf{N}$ denote a dihedral CM-field $\mathbf{N}$ of 2-power degree $2n = 8m = 2^r \geq 8$ and let $\mathbf{k}$ denote the only quadratic subfield of $\mathbf{N}$ such that the extension $\mathbf{N}/\mathbf{k}$ is cyclic. Thus $\mathbf{k}$ is real. In [LO] we proved that $\mathbf{N}$ has odd relative class number if and only if $\mathbf{N}$ is the narrow Hilbert 2-class field of $\mathbf{k}$, the 2-Sylow subgroup of the narrow ideal class group of $\mathbf{k}$ is cyclic of order $4m$ and the norm of the fundamental unit of $\mathbf{k}$ is equal to $+1$ (which implies $\mathbf{k} = \mathbf{Q}(\sqrt{pq})$ for two primes $2 \leq p < q$ not equal to 3 modulo 4 and such the Legendre symbols $(\frac{p}{q})$ is equal to $+1$), and we have the following lower bound:

$$h_{\mathbf{N}}^- \geq \epsilon_{\mathbf{k}}\frac{64}{em}\frac{(d_{\mathbf{k}}/16\pi^4)^m}{B_{\mathbf{k}}^{2m-2}(\log d_{\mathbf{k}}+0.1)^4}, \tag{32}$$

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where

$$\epsilon_{\mathbf{k}} = \max\Big(1 - (2\pi n e^{1/n}/\sqrt{d_{\mathbf{k}}}), \frac{2}{5}\exp(2\pi n/\sqrt{d_{\mathbf{k}}})\Big)$$

is asymptotic to 1 when $d_{\mathbf{k}}$ goes to infinity.

Now, using the bound $B_{\mathbf{k}} \le \frac{1}{8}\log^2 d_{\mathbf{k}}$ (see (31)), we get

$$h_{\mathbf{N}}^- \ge \frac{\epsilon_{\mathbf{k}}}{em}\Big(\frac{4d_{\mathbf{k}}}{\pi^4(\log d_{\mathbf{k}} + 0.1)^4}\Big)^m.$$

Therefore, $8m \ge 16$ and $h_{\mathbf{N}}^- = 1$ imply $d_{\mathbf{k}} \le 3\cdot 10^6$, and in [LO], thanks to an efficient technique for computing relative class numbers of such narrow Hilbert 2-class fields, we were able to compute the 9542 relative class numbers for all the $\mathbf{k}$'s with $d_{\mathbf{k}} \le 3\cdot 10^6$, which enabled us to determine all the dihedral CM-fields of 2-power degrees with relative class number one. But now, we can alleviate this amount of required relative class number computation: we compute $B_{\mathbf{k}}$ for each possible $\mathbf{k}$ and get rid of the $\mathbf{k}$'s for which (32) yields $h_{\mathbf{N}}^- > 1$. Note that it is much easier to compute $B_{\mathbf{k}}$ than to compute $h_{\mathbf{N}}^-$. For example, there are 105 real quadratic fields $\mathbf{k}$ with $d_{\mathbf{k}} \le 3\cdot 10^6$ for which $[\mathbf{N}:\mathbf{Q}] \ge 128$ and all of them are such that (32) yields $h_{\mathbf{N}}^- > 1$. In particular, $h_{\mathbf{N}}^- = 1$ implies $[\mathbf{N}:\mathbf{Q}] = 2n \le 64$. Let us also mention that there are 462 real quadratic fields $\mathbf{k}$ with $d_{\mathbf{k}} \le 3\cdot 10^6$ for which $[\mathbf{N}:\mathbf{Q}] = 64$ and 443 of them are such that (32) yields $h_{\mathbf{N}}^- > 1$. This first example clearly shows that being able to compute numerically $B_{\mathbf{k}}$ might be quite useful. In the last section of this paper we will give a still more convincing example.

3.3. *Numerical computation of* $\mu_{\mathbf{k}}\operatorname{Res}_{s=1}(\zeta_{\mathbf{k}})$ *when* $\mathbf{k}$ *is totally real.* So, let us now explain how, for any totally real number field $\mathbf{k}$ of degree n, we can compute the numerical value of $B_{\mathbf{k}} = \mu_{\mathbf{k}}\operatorname{Res}_{s=1}(\zeta_{\mathbf{k}})$. Since

$$\sqrt{d_{\mathbf{k}}}B_{\mathbf{k}} = \sqrt{d_{\mathbf{k}}}\mu_{\mathbf{k}}\operatorname{Res}_{s=1}(\zeta_{\mathbf{k}}) = \lambda_{\mathbf{k}}\mu_{\mathbf{k}} = I_{\mathbf{k}}(1) = \int_1^\infty S_{\mathbf{k}}(x)\,dx + \int_1^\infty S_{\mathbf{k}}(x)\frac{dx}{x}$$

and $S_{\mathbf{k}}(x) = \sum_{m\ge 1} z_m H_n(mx/A_{\mathbf{k}})$ (with $A_{\mathbf{k}} = \sqrt{d_{\mathbf{k}}/\pi^n}$), setting

$$(33)\quad K_{n,1}(B) = \int_1^\infty BH_n(Bx)\,dx = \frac{1}{2\pi i}\int_{\Re(s)=\alpha}\Gamma^n(s/2)\frac{B^{1-s}}{s-1}\,ds \quad (B>0 \text{ and } \alpha>1)$$

and

$$(34)\quad K_{n,2}(B) = \int_1^\infty BH_n(Bx)\frac{dx}{x} = \frac{1}{2\pi i}\int_{\Re(s)=\alpha}\Gamma^n(s/2)\frac{B^{1-s}}{s}\,ds \quad (B>0 \text{ and } \alpha>1),$$

we get

LEMMA 7. *Let* $\mathbf{k}$ *be a totally real number field of degree* n. *We have*

$$(35)\qquad \mu_{\mathbf{k}}\operatorname{Res}_{s=1}(\zeta_{\mathbf{k}}) = \pi^{-n/2}\sum_{m\ge 1}\frac{z_m}{m}\Big(K_{n,1}(m/A_{\mathbf{k}}) + K_{n,2}(m/A_{\mathbf{k}})\Big).$$

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Now,

$$\frac{1}{2\pi i}\int_{\Re(s)=\alpha}\frac{(Y/B^2)^{s/2}}{s-1} = \begin{cases}\sqrt{Y/B^2} & \text{if } Y > B^2,\\ 0 & \text{if } 0 < Y < B^2,\end{cases}$$

and $\{(y_1,\dots,y_n),\ y_i \geq 0 \text{ and } \prod_{i=1}^n y_i \geq B^2\}$ is included in $\{(y_1,\dots,y_n),\ y_i \geq 0$ and $\exists i/y_i \geq B^{2/n}\}$. Therefore, we get

$$\begin{aligned} K_{n,1}(B) &= \iint_{y_1y_2\cdots y_n\geq B^2} e^{-(y_1+\cdots+y_n)}\frac{dy_1}{\sqrt{y_1}}\cdots\frac{dy_n}{\sqrt{y_n}}\\ &\leq n\pi^{n/2}\int_{B^{2/n}}^{\infty} e^{-y}\frac{dy}{\sqrt{y}}\\ &\leq n\pi^{n/2}B^{-1/n}e^{-B^{2/n}}\end{aligned}$$

and

$$K_{n,2}(B) = B\iint_{y_1y_2\cdots y_n\geq B^2} e^{-(y_1+\cdots+y_n)}\frac{dy_1}{y_1}\cdots\frac{dy_n}{y_n} \leq K_{n,1}(B).$$

In particular, (35) is a rapidly absolutely convergent series suitable for numerical computations, each terms of which we can compute thanks to power series expansions of the functions $K_{m,i}$. For example, we have:

PROPOSITION 8. *Let* $\gamma = 0.577215664901532\cdots$ *denote Euler's constant. Take* $B > 0$ *and set* $s_1(0) = -\gamma$, $s_2(0) = \pi^2/6$ *and for* $k \geq 1$, *set*

$$s_1(k) = -\gamma + \sum_{i=1}^{k}\frac{1}{i}.$$

We have the following power series expansions:

$$K_{2,1}(B) = \pi + 4\sum_{k\geq 0}\Big(-\frac{1}{2k+1} - s_1(k) + \log B\Big)\frac{B^{2k+1}}{(2k+1)(k!)^2}$$

and

$$K_{2,2}(B) = \Big(\frac{\pi^2}{6} + 2\gamma^2 + 4\gamma\log B + 2\log^2 B\Big)B + 4\sum_{k\geq 1}\Big(-\frac{1}{2k} - s_1(k) + \log B\Big)\frac{B^{2k+1}}{(2k)(k!)^2}.$$

Set also

$$s_2(k) = \frac{\pi^2}{6} + \sum_{i=1}^{k}\frac{1}{i^2}.$$

We have the following power series expansions:

$$K_{3,1}(B) = \pi^{3/2} - \sum_{k\geq 0} a_k\frac{(-1)^k B^{2k+1}}{(2k+1)(k!)^3}$$

and

$$K_{3,2}(B) = -b_0B - \sum_{k\geq 1} b_k\frac{(-1)^k B^{2k+1}}{(2k)(k!)^3},$$

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with

$$a_k = \frac{8}{(2k+1)^2} + \frac{12s_1(k)}{2k+1} + 9\big(s_1(k)\big)^2 + 3s_2(k) - \Big(\frac{8}{2k+1} + 12s_1(k)\Big)\log B + 4\log^2 B,$$
$$b_0 = \frac{3\pi^2\gamma}{4} + \frac{9\gamma^3}{2} + \zeta(3) + (9\gamma^2 + \frac{\pi^2}{2})\log B + 6\gamma\log^2 B + \frac{4}{3}\log^3 B$$
$$b_k = \frac{8}{(2k)^2} + \frac{12s_1(k)}{2k} + 9\big(s_1(k)\big)^2 + 3s_2(k) - \Big(\frac{8}{2k} + 12s_1(k)\Big)\log B + 4\log^2 B.$$

Note that $\zeta(3) = 1.202056903159594\cdots$.

PROOF. Let us only prove the first expansion. We shift the line of integration $\Re(s) = \alpha$ in (33) to the left to $-\infty$. We pick residues at $s = 1$ and at each non-positive even integer $s = -2k$. Noticing that

$$\mathrm{Res}_{s=1}\Big(\Gamma^2(s/2)\frac{B^{1-s}}{s-1}\Big) = \pi$$

and using the functional equation satisfied by the Gamma function we get

$$\mathrm{Res}_{s=-2k}\Big(\Gamma^2(s/2)\frac{B^{1-s}}{s-1}\Big) = -4\Big(\frac{1}{2k+1} + \frac{\Gamma'}{\Gamma}(k+1) - \log B\Big)\frac{B^{2k+1}}{(2k+1)(k!)^2}$$

from which we easily get the desired result. ∎

4. **The case where k is abelian.** We improve our bounds on $B_{\mathbf{k}}$ and give a different and more efficient technique for computing numerically $B_{\mathbf{k}}$. Whenever χ is an even primitive Dirichlet character of conductor $f_\chi > 1$ we set

$$(36) \qquad \Lambda_\chi(s) = (f_\chi/\pi)^{s/2}\Gamma(s/2)L(s,\chi) = \int_1^\infty S_\chi(x)x^s\frac{dx}{x} + W_\chi\int_1^\infty S_{\bar\chi}(x)x^{1-s}\frac{dx}{x}$$

(see (21)). Let x_k be the group of primitive Dirichlet characters associated to k. Then

$$F_{\mathbf{k}}(s) = F_{\mathbf{Q}}(s)\prod_{\substack{\chi\in X_{\mathbf{k}}\\ \chi\neq 1}}\Lambda_\chi(s) \quad \text{and} \quad \lambda_{\mathbf{k}} = \prod_{\substack{\chi\in X_{\mathbf{k}}\\ \chi\neq 1}}\Lambda_\chi(1).$$

Since

$$F_{\mathbf{Q}}(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{s-1} - c + O(s-1)$$

with $c = 1 - \mu_{\mathbf{Q}} = \big(\log(4\pi) - \gamma\big)/2 = 0.976904291\cdots$, using (15) we get

$$\begin{aligned}\mu_{\mathbf{k}} &= \lim_{s\searrow 1}\Big\{\frac{1}{\lambda_{\mathbf{k}}}F_{\mathbf{k}}(s) - \Big(\frac{1}{s-1} - \frac{1}{s}\Big)\Big\}\\ &= 1 + \lim_{s\searrow 1}\Big\{F_{\mathbf{Q}}(s)\Big(\prod_{\substack{\chi\in X_{\mathbf{k}}\\ \chi\neq 1}}\frac{\Lambda_\chi(s)}{\Lambda_\chi(1)}\Big) - \frac{1}{s-1}\Big\}\\ &= 1 + \lim_{s\searrow 1}\Big\{\Big(\frac{1}{s-1} - c + O(s-1)\Big)\end{aligned}$$

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$$\Big(1+(s-1)\Big(\sum_{\substack{\chi\in X_{\mathbf{k}}\\ \chi\neq 1}}\frac{\Lambda'_\chi}{\Lambda_\chi}(1)\Big)+O\big((s-1)^2\big)\Big)-\frac{1}{s-1}\Big\}$$

$$=1-c+\sum_{\substack{\chi\in X_{\mathbf{k}}\\ \chi\neq 1}}\frac{\Lambda'_\chi}{\Lambda_\chi}(1)=\mu_{\mathbf{Q}}+\sum_{\substack{\chi\in X_{\mathbf{k}}\\ \chi\neq 1}}\frac{\Lambda'_\chi}{\Lambda_\chi}(1).$$

Using $(\Gamma'/\Gamma)(1/2)=-\gamma-\log 4$ and $d_{\mathbf{k}}=\prod_{\substack{\chi\in X_{\mathbf{k}}\\ \chi\neq 1}} f_\chi$, we obtain

LEMMA 9. *Let $\gamma=0.577215\cdots$ denote Euler's constant. Let $\mathbf{k}$ be a real abelian field of degree $n\geq 2$ and let $X_{\mathbf{k}}$ be the group of primitive Dirichlet characters associated to $\mathbf{k}$. Then,*

$$\mu_{\mathbf{k}}=1-\frac{n-2}{2}\gamma-\frac{n}{2}\log(4\pi)+\frac{1}{2}\log d_{\mathbf{k}}+\sum_{\substack{\chi\in X_{\mathbf{k}}\\ \chi\neq 1}}\frac{L'(1,\chi)}{L(1,\chi)}$$

and

$$B_{\mathbf{k}}=\mu_{\mathbf{k}}\operatorname{Res}_{s=1}(\zeta_{\mathbf{k}})=\mu_{\mathbf{Q}}\prod_{\substack{\chi\in X_{\mathbf{k}}\\ \chi\neq 1}}L(1,\chi)+\sum_{\substack{\chi\in X_{\mathbf{k}}\\ \chi\neq 1}}\frac{\Lambda'_\chi(1)}{\sqrt{f_\chi}}\prod_{\substack{\psi\in X_{\mathbf{k}}\\ \psi\neq 1,\chi}}L(1,\psi). \tag{37}$$

In particular, if $\mathbf{k}$ is a real quadratic field associated to a primitive quadratic Dirichlet character $\chi_{\mathbf{k}}$ of conductor $f_{\mathbf{k}}=d_{\mathbf{k}}$ we get

$$\mu_{\mathbf{k}}\operatorname{Res}_{s=1}(\zeta_{\mathbf{k}})=\mu_{\mathbf{Q}}L(1,\chi_{\mathbf{k}})+\frac{\Lambda'_{\chi_{\mathbf{k}}}(1)}{\sqrt{f_{\mathbf{k}}}}, \tag{38}$$

and if $\mathbf{k}$ is a cyclic cubic field associated to a primitive cubic Dirichlet character $\chi_{\mathbf{k}}$ of conductor $f_{\mathbf{k}}$ we get

$$\mu_{\mathbf{k}}\operatorname{Res}_{s=1}(\zeta_{\mathbf{k}})=\mu_{\mathbf{Q}}|L(1,\chi_{\mathbf{k}})|^2+2\Re\Big(\frac{\Lambda'_{\chi_{\mathbf{k}}}(1)}{\sqrt{f_{\mathbf{k}}}}L(1,\bar{\chi}_{\mathbf{k}})\Big). \tag{39}$$

4.1. *A better bound on $\mu_{\mathbf{k}}\operatorname{Res}_{s=1}(\zeta_{\mathbf{k}})$ when $\mathbf{k}$ is abelian.*

LEMMA 10. *Let χ be an even primitive Dirichlet character of conductor $f_\chi>1$. We have*

$$|\Lambda'_\chi(1)|/\sqrt{f_\chi}\leq\Big(\frac{1}{8}\log^2 f_\chi-\frac{1}{2}\mu_{\mathbf{Q}}^2\Big)=\Big(\frac{1}{4}\log f_\chi-\frac{1}{2}\mu_{\mathbf{Q}}\Big)\Big(\frac{1}{2}\log f_\chi+\mu_{\mathbf{Q}}\Big).$$

PROOF. Noticing that $|S_\chi(x)|\leq S_{\mathbf{Q}}(x/\sqrt{f_\chi})$ and using (36) we obtain

$$|\Lambda'_\chi(1)|\leq\int_1^\infty S_{\mathbf{Q}}(x/\sqrt{f_\chi})(\log x)\Big(1+\frac{1}{x}\Big)\,dx.$$

We set $f=\sqrt{f_\chi}$ and must prove $|\Lambda'_\chi(1)|\leq\frac{f}{2}\log^2 f-\frac{f}{2}\mu_{\mathbf{Q}}^2$. Using the functional equation $S_{\mathbf{Q}}(1/x)=xS_{\mathbf{Q}}(x)+x-1$ (see (20)), we obtain

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$$\begin{aligned}
|\Lambda_\chi'(1)| &\le \int_1^\infty S_{\mathbf{Q}}(x/f)(\log x)\Big(1+\frac{1}{x}\Big)\,dx \\
&= \int_{1/f}^\infty S_{\mathbf{Q}}(x)\big(\log(fx)\big)\Big(f+\frac{1}{x}\Big)\,dx \\
&= \int_1^\infty S_{\mathbf{Q}}(x)\big(\log(fx)\big)\Big(f+\frac{1}{x}\Big)\,dx + \int_1^f S_{\mathbf{Q}}(1/x)\big(\log(f/x)\big)\Big(\frac{f}{x^2}+\frac{1}{x}\Big)\,dx \\
&= \int_1^\infty S_{\mathbf{Q}}(x)\big(\log(fx)\big)\Big(f+\frac{1}{x}\Big)\,dx + \int_1^f S_{\mathbf{Q}}(x)\big(\log(f/x)\big)\Big(\frac{f}{x}+1\Big)\,dx \\
&\quad + \int_1^f \big(\log(f/x)\big)(x-1)\Big(\frac{f}{x^2}+\frac{1}{x}\Big)\,dx \\
&= \int_1^\infty S_{\mathbf{Q}}(x)\big(\log(fx)\big)\Big(f+\frac{1}{x}\Big)\,dx + \int_1^f S_{\mathbf{Q}}(x)\big(\log(f/x)\big)\Big(\frac{f}{x}+1\Big)\,dx \\
&\quad + \frac{f-1}{2}\log^2 f-(f+1)\log f+2(f-1) \\
&= (f+1)\log f\int_1^\infty S_{\mathbf{Q}}(x)\Big(1+\frac{1}{x}\Big)\,dx+(f-1)\int_1^\infty S_{\mathbf{Q}}(x)(\log x)\Big(1-\frac{1}{x}\Big)\,dx \\
&\quad + \int_f^\infty S_{\mathbf{Q}}(x)\big(\log(x/f)\big)\Big(\frac{f}{x}+1\Big)\,dx+\frac{f-1}{2}\log^2 f-(f+1)\log f+2(f-1) \\
&= \frac{f-1}{2}\log^2 f-(1-a)(f+1)\log f+(2+b)(f-1)+R(f)
\end{aligned}$$

where we have set

$$a=\int_1^\infty S_{\mathbf{Q}}(x)\Big(1+\frac{1}{x}\Big)\,dx,\quad b=\int_1^\infty S_{\mathbf{Q}}(x)(\log x)\Big(1-\frac{1}{x}\Big)\,dx$$

and

$$R(f)=\int_f^\infty S_{\mathbf{Q}}(x)\big(\log(x/f)\big)\Big(\frac{f}{x}+1\Big)\,dx=f\int_1^\infty xS_{\mathbf{Q}}(fx)\frac{x+1}{x^2}(\log x)\,dx.$$

Noticing that

$$F_{\mathbf{Q}}(s)-\Big(\frac{1}{s-1}-\frac{1}{s}\Big)=\int_1^\infty S_{\mathbf{Q}}(x)(x^s+x^{1-s})\frac{dx}{x}=\mu_{\mathbf{Q}}+\nu_{\mathbf{Q}}(s-1)+O\big((s-1)^2\big)$$

we obtain $a=\mu_{\mathbf{Q}}=\big(2+\gamma-\log(4\pi)\big)/2=0.023095\cdots$ and $b=\nu_{\mathbf{Q}}=0.000248155\cdots$. Finally, since $x\ge 1$ implies $\big((x+1)\log x\big)/x^2\le 1$, using (19) we have

$$R(f)\le 2f\sum_{n\ge1}\int_1^\infty xe^{-\pi n^2f^2x^2}\,dx=\sum_{n\ge1}\frac{1}{\pi n^2 f}e^{-\pi n^2 f^2}\le e^{-\pi f^2}\sum_{n\ge1}\frac{1}{\pi n^2 f}=\frac{\pi}{6f}e^{-\pi f^2}$$

and

$$|\Lambda_\chi'(1)|\le\frac{f-1}{2}\log^2 f-(1-a)(f+1)\log f+(2+b)(f-1)+\frac{\pi}{6f}e^{-\pi f^2}.$$

The desired result follows. ∎

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THEOREM 11. *Let* $\mathbf{k}$ *be a real abelian field of degree* $n \geq 2$ *and conductor* $f_{\mathbf{k}}$. *We have*

$$(40)\quad B_{\mathbf{k}} = \mu_{\mathbf{k}} \operatorname{Res}_{s=1}(\zeta_{\mathbf{k}}) \leq \frac{n-1}{2}\Big(\frac{1}{2(n-1)}\log d_{\mathbf{k}} + \mu_{\mathbf{Q}}\Big)^n \leq \frac{n-1}{2^{n+1}}(\log f_{\mathbf{k}} + 2\mu_{\mathbf{Q}})^n.$$

PROOF. Using (37), (6) and previous lemma, we get

$$\begin{aligned} B_{\mathbf{k}} &\leq \mu_{\mathbf{Q}} \prod_{\substack{\chi\in X_{\mathbf{k}}\\ \chi\neq 1}} \Big(\frac{1}{2}\log f_\chi + \mu_{\mathbf{Q}}\Big) + \sum_{\substack{\chi\in X_{\mathbf{k}}\\ \chi\neq 1}} \Big(\frac{1}{4}\log f_\chi - \frac{1}{2}\mu_{\mathbf{Q}}\Big) \prod_{\substack{\psi\in X_{\mathbf{k}}\\ \psi\neq 1}} \Big(\frac{1}{2}\log f_\psi + \mu_{\mathbf{Q}}\Big)\\ &= \Big(\frac{1}{4}\log d_{\mathbf{k}} + \frac{3-n}{2}\mu_{\mathbf{Q}}\Big) \prod_{\substack{\chi\in X_{\mathbf{k}}\\ \chi\neq 1}} \Big(\frac{1}{2}\log f_\chi + \mu_{\mathbf{Q}}\Big)\\ &\leq \Big(\frac{1}{4}\log d_{\mathbf{k}} + \frac{3-n}{2}\mu_{\mathbf{Q}}\Big)\Big(\frac{1}{2(n-1)}\log d_{\mathbf{k}} + \mu_{\mathbf{Q}}\Big)^{n-1}\\ &\leq \frac{n-1}{2}\Big(\frac{1}{2(n-1)}\log d_{\mathbf{k}} + \mu_{\mathbf{Q}}\Big)^n. \end{aligned}$$

■

4.2. *Numerical computation of* $\mu_{\mathbf{k}} \operatorname{Res}_{s=1}(\zeta_{\mathbf{k}})$ *when* $\mathbf{k}$ *is abelian.* According to (36), we have

$$\Lambda'_\chi(1) = \int_1^\infty S_\chi(x)(\log x)\,dx - W_\chi \int_1^\infty S_{\bar\chi}(x)(\log x)\frac{dx}{x}.$$

Since

$$S_\chi(x) = \frac{1}{2\pi i}\int_{\Re(s)=\alpha} \Lambda_\chi(s)x^{-s}\,ds,$$

setting

$$\begin{aligned} K_1(B) &= \frac{1}{2\pi i}\int_{\Re(s)=\alpha} \Gamma(s/2)\frac{B^{1-s}}{(s-1)^2}\,ds\\ &= 2B\int_1^\infty e^{-B^2t^2}\log t\,dt \leq \frac{2B}{e}\int_1^\infty te^{-B^2t^2}\,dt = \frac{e^{-B^2}}{eB} \end{aligned}$$

and

$$K_2(B) = \frac{1}{2\pi i}\int_{\Re(s)=\alpha} \Gamma(s/2)\frac{B^{1-s}}{s^2}\,ds = 2B\int_1^\infty e^{-B^2t^2}\log t\frac{dt}{t} \leq K_1(B),$$

we obtain:

LEMMA 12. *Let* χ *be an even primitive Dirichlet character of conductor* $f_\chi > 1$. *We have*

$$\Lambda'_\chi(1)/\sqrt{f_\chi} = \frac{1}{\sqrt{\pi}}\Big(\sum_{m\geq 1}\frac{\chi(m)}{m}K_1(B_m) - W_\chi \sum_{m\geq 1}\frac{\bar\chi(m)}{m}K_2(B_m)\Big)$$

where $B_m = \sqrt{\pi m^2/f_\chi}$.

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Note that if **k** is quadratic then this formula boils down to

$$\Lambda'_{\chi_{\mathbf{k}}}(1)/\sqrt{d_{\mathbf{k}}} = \frac{1}{\sqrt{\pi}}\sum_{m\geq 1}\frac{\chi_{\mathbf{k}}(m)}{m}\big(K_1(B_m) - K_2(B_m)\big).$$

Setting

$$R_M = \sum_{m\leq M}\frac{\chi(m)}{m}K_1(B_m) - W_\chi\sum_{m\geq 1}\frac{\bar{\chi}(m)}{m}K_2(B_m),$$

we note that if M is any integer greater than or equal to $\sqrt{\lambda f/\pi}$, then we have

$$\begin{aligned}|\Lambda'_\chi(1)/\sqrt{f_\chi} - R_M| &\leq \sum_{m>M}\frac{2e^{-B_m^2}}{emB_m} \leq \frac{2\sqrt{f/\pi}}{eM^3}\int_M^\infty me^{-\pi m^2/f}\,dm\\ &= \frac{(f/\pi)^{3/2}}{eM^3}e^{-\pi M^2/f} \leq \frac{e^{-\lambda}}{e\lambda^{3/2}}.\end{aligned}$$

Finally, as there is no known general formulas for Gauss sums we need compute

$$W_\chi = \frac{1}{\sqrt{f_\chi}}\sum_{x=1}^{f_\chi-1}\chi(x)e^{2\pi ix/f_\chi} = \frac{2}{\sqrt{f_\chi}}\sum_{1\leq x<f_\chi/2}\chi(x)\cos(2\pi x/f_\chi),$$

and it is not much more time consuming to also compute

$$L(1,\chi) = -\frac{W_\chi}{\sqrt{f_\chi}}\sum_{x=1}^{f_\chi-1}\bar{\chi}(x)\log\sin(\pi x/f_\chi) = -\frac{2W_\chi}{\sqrt{f_\chi}}\sum_{1\leq x<f_\chi/2}\bar{\chi}(x)\log\sin(\pi x/f_\chi).$$

We note that if **k** is real quadratic then $W_\chi = 1$ need not be computed, and it is more efficient to use [WB] to compute the regulator and class number of **k**, from which we deduce the exact value of $L(1,\chi_{\mathbf{k}})$.

Moreover, in the same way we proved Proposition 8 we would prove:

PROPOSITION 13. *We have*

$$K_1(B) = -\sqrt{\pi}\Big(\frac{\gamma}{2} + \log 2 + \log B\Big) + 2B + 2\sum_{k\geq 1}\frac{(-1)^k B^{2k+1}}{(2k+1)^2(k!)}$$

and

$$K_2(B) = \Big(\frac{\pi^2}{24} + \frac{\gamma^2}{4} + \gamma\log B + \log^2 B\Big)B + 2\sum_{k\geq 1}\frac{(-1)^k B^{2k+1}}{(2k)^2(k!)}.$$

PROPOSITION 14. *If* **k** *is a real quadratic field, then* $d_{\mathbf{k}} \leq 10^5$ *implies* $\mu_{\mathbf{k}} \leq 7$, $\mathrm{Res}_{s=1}(\zeta_{\mathbf{k}}) \leq 5$ *and* $\mu_{\mathbf{k}}\,\mathrm{Res}_{s=1}(\zeta_{\mathbf{k}}) \leq 11$ *(note that* $\frac{1}{8}\log^2(10^5) = 16.56\cdots$*). If* **k** *is cyclic cubic field of prime conductor* $p \equiv 1 \pmod 6$*, then* $p \leq 10^5$ *implies* $\mu_{\mathbf{k}} \leq 12$, $\mathrm{Res}_{s=1}(\zeta_{\mathbf{k}}) \leq 21$ *and* $\mu_{\mathbf{k}}\,\mathrm{Res}_{s=1}(\zeta_{\mathbf{k}}) \leq 91$ *(note that* $\frac{1}{6}\log^3(10^5) = 254.33\cdots$*).*

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5**. On the class number one problem for some non-abelian normal** CM-**fields of degree** 24**.** From now on, we let $\mathbf{N}$ be a non-abelian normal CM-field of degree 24 with Galois group $\mathrm{SL}_2(F_3)$, the special linear group over the finite field with three elements, and we let $\mathbf{N}^+$ be the maximal totally real subfield of $\mathbf{N}$. Therefore, $\mathbf{N}^+$ is a non-abelian normal field with Galois group A_4, the alternating group of degree 4 and order 12. Since A_4 has a unique (normal) subgroup of index three, we let $\mathbf{k}$ denote the unique (cyclic) cubic subfield of $\mathbf{N}^+$ and let $f_\mathbf{k}$ denote the conductor of $\mathbf{k}$. We note that the extension $\mathbf{N}^+/\mathbf{k}$ is abelian with Galois group isomorphic to the four group $(\mathbf{Z}/2\mathbf{Z})^2$.

To begin with, we give lower bounds on the relative class numbers $h_\mathbf{N}^-$ of such $\mathbf{N}$'s.

First, one proves that the Dedekind zeta function of $\mathbf{N}$ satisfies

$$\zeta_\mathbf{N}\big(1-(2/\log d_\mathbf{N})\big)\leq 0.$$

Indeed, $\zeta_{\mathbf{N}^+}/\zeta_\mathbf{k}$ is the cube of the entire Artin's L-function associated to the character of degree 3 of the alternating group $\mathrm{Gal}(\mathbf{N}^+/\mathbf{Q})$ of degree 4 and order 12, and $\zeta_\mathbf{k}(s)=\zeta(s)|L(s,\chi_\mathbf{k})|^2\leq 0$ for any $s\in\,]0,1[$. Therefore, if $\zeta_{\mathbf{N}^+}(s_0)>0$ for some $s_0\in\,]0,1[$, then $\zeta_{\mathbf{N}^+}$ has at least a triple zero on $]s_0,1[$. Now, one proves that the Dedekind zeta function of any number field $\mathbf{M}$ has at most two real zeros in the range $1-(1/\log d_\mathbf{M})\leq s<1$. Putting everything together, we deduce that $\zeta_{\mathbf{N}^+}$ does not have any real zero in the range $1-(1/\log d_{\mathbf{N}^+})\leq s<1$, hence in the range $1-(2/\log d_\mathbf{N})\leq s<1$, which implies $\zeta_{\mathbf{N}^+}\big(1-(2/\log d_\mathbf{N})\big)\leq 0$. Since $\zeta_\mathbf{N}/\zeta_{\mathbf{N}^+}$ is the square of the entire Artin's L-function associated to the character of degree 2 of the quaternion group $\mathrm{Gal}(\mathbf{N}/\mathbf{k})$ then $\zeta_\mathbf{N}/\zeta_{\mathbf{N}^+}$ is entire and $(\zeta_\mathbf{N}/\zeta_{\mathbf{N}^+})(s_0)\leq 0$ for any $s_0\in\,]0,1[$. Hence, we do have $\zeta_\mathbf{N}\big(1-(2/\log d_\mathbf{N})\big)\leq 0$.

Second, using $\zeta_\mathbf{N}\big(1-(2/\log d_\mathbf{N})\big)\leq 0$ and setting $\epsilon_\mathbf{N}=1-(24\pi e^{1/12}/d_\mathbf{N}^{1/24})$, we have:

$$\mathrm{Res}_{s=1}(\zeta_\mathbf{N})\geq 2\epsilon_\mathbf{N}/\log d_\mathbf{N}.$$

Using (10), we get:

PROPOSITION 15 (SEE [LLO]). *Let* $\mathbf{N}$ *be a normal* CM*-field of degree* 24 *with Galois group isomorphic to* $\mathrm{SL}_2(F_3)$*. If the relative class number* $h_\mathbf{N}^-$ *of* $\mathbf{N}$ *is odd then the quaternion octic extension* $\mathbf{N}/\mathbf{k}$ *is unramified at all the finite places, which yields* $d_\mathbf{N}=d_{\mathbf{N}^+}^2=d_\mathbf{k}^8=f_\mathbf{k}^{16}$*, and* $w_\mathbf{N}=Q_\mathbf{N}=2$*, which yields*

$$h_\mathbf{N}^-\geq\epsilon_\mathbf{k}\frac{f_\mathbf{k}^4/\log f_\mathbf{k}}{2e(2\pi)^{12}\,\mathrm{Res}_{s=1}(\zeta_{\mathbf{N}^+})}\tag{41}$$

where $\epsilon_\mathbf{k}=1-(24\pi e^{1/12}/f_\mathbf{k}^{2/3})$ *is asymptotic to* 1 *when* $f_\mathbf{k}$ *goes to infinity.*

Now, using (11) and (41), we get

$$h_\mathbf{N}^-\geq\epsilon_\mathbf{k}\frac{2f_\mathbf{k}^4}{11\big((8\pi e/11)\log f_\mathbf{k}\big)^{12}}\tag{42}$$

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and obtain $h_{\mathbf{N}}^- > 1$ for $f_{\mathbf{k}} \geq 970000$, quite a large bound.

But, using (5), (8) and (14), we get

$$\mathrm{Res}_{s=1}(\zeta_{\mathbf{N}^+}) \leq \mathrm{Res}_{s=1}(\zeta_{\mathbf{k}})\Big(\mu_{\mathbf{k}}\,\mathrm{Res}_{s=1}(\zeta_{\mathbf{k}})\Big)^3 \leq \frac{1}{2^{11}}(\log f_{\mathbf{k}} + 0.05)^{11}$$

which together with (41) imply

$$h_{\mathbf{N}}^- \geq \epsilon_{\mathbf{k}} \frac{f_{\mathbf{k}}^4}{4e\Big(\pi(\log f_{\mathbf{k}} + 0.05)\Big)^{12}} \tag{43}$$

which yields $h_{\mathbf{N}}^- > 1$ for $f_{\mathbf{k}} \geq 83000$, a much more reasonable bound. Nevertheless, this bound is still too large to solve easily the (relative) class number one problem for these $\mathbf{N}$'s. Indeed, according to [Lou2] we would have to do at least $\gg \sqrt{d_{\mathbf{N}}/d_{\mathbf{N}^+}}\log^6 d_{\mathbf{N}}/d_{\mathbf{N}^+} \gg f_{\mathbf{k}}^4 \log^6 f_{\mathbf{k}}$ elementary operations to compute each $h_{\mathbf{N}}^-$ and, moreover, it is not that easy to explicitly construct $\mathbf{N}$ from $\mathbf{k}$. However, according to Section 4.2, the computation of each $B_{\mathbf{k}}$ can be done in $\ll f_{\mathbf{k}}$ elementary operations and we might expect that the lower bound

$$h_{\mathbf{N}}^- \geq \epsilon_{\mathbf{k}} \frac{f_{\mathbf{k}}^4/\log f_{\mathbf{k}}}{2e(2\pi)^{12}\,\mathrm{Res}_{s=1}(\zeta_{\mathbf{k}})B_{\mathbf{k}}^3} \tag{44}$$

(use (41) and (5)) will imply $h_{\mathbf{N}}^- > 1$ for most of the fields $\mathbf{k}$ with $f_{\mathbf{k}} \leq 83000$. To simplify, we shall now focus on the class number one problem for these $\mathbf{N}$'s (and refer the reader to [LLO] for the solution of the relative class number one problem for these $\mathbf{N}$'s). To start with, we notice that thanks to class field theory and Proposition 15, if $h_{\mathbf{N}} = 1$ then $h_{\mathbf{k}} = 4$, hence $f_{\mathbf{k}}$ is a prime equal to 1 modulo 6. We computed the numerical values of $\mathrm{Res}_{s=1}(\zeta_{\mathbf{k}})$ and $B_{\mathbf{k}} = \mu_{\mathbf{k}}\,\mathrm{Res}_{s=1}(\zeta_{\mathbf{k}})$ for the 4784 possible $\mathbf{k}$ of prime conductors $f_{\mathbf{k}} \equiv 1 \pmod 6$ such that $f_{\mathbf{k}} \leq 10^5$ and found that (44) implies $h_{\mathbf{N}}^- > 1$ except for 250 cyclic cubic fields $\mathbf{k}$, the 56 of them with conductors greater than 5000 being given in the following Table. Note that only 10 out of them are such that their class numbers are equal to 4.

Case	$f_{\mathbf{k}}$	$h_{\mathbf{k}}$	Case	$f_{\mathbf{k}}$	$h_{\mathbf{k}}$	Case	$f_{\mathbf{k}}$	$h_{\mathbf{k}}$	Case	$f_{\mathbf{k}}$	$h_{\mathbf{k}}$
250	21787		236	12007		222	8893		208	6967	
249	19843		235	11971		221	8779		207	6301	
248	18307	4	234	11923		220	8707		206	6271	
247	15973		233	11551		219	8629		205	6091	
246	15679		232	11149	4	218	8317		204	6079	4
245	14407	4	231	11113		217	8191	4	203	5953	
244	14197		230	10957	4	216	8167		202	5821	
243	13063		229	10243		215	8011	4	201	5737	
242	12973		228	9973		214	7963		200	5569	
241	12799		227	9931		213	7723		199	5347	
240	12583		226	9817		212	7639	4	198	5323	
239	12391		225	9439		211	7369		197	5197	4
238	12343		224	9109	4	210	7333		196	5113	
237	12163		223	8929		209	7213		195	5101	

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Now, according to [Gra], there are 32 cyclic cubic fields of prime conductors $f_{\mathbf{k}} \leq 5000$ and class number 4, the ones given in the following Table and for 14 out of them (44) implies $h_{\mathbf{N}}^{-} > 1$

Case	$f_{\mathbf{k}}$	$h_{\mathbf{N}}^{-}$	Case	$f_{\mathbf{k}}$	$h_{\mathbf{N}}^{-}$	Case	$f_{\mathbf{k}}$	$h_{\mathbf{N}}^{-}$	Case	$f_{\mathbf{k}}$	$h_{\mathbf{N}}^{-}$
1	163		9	937		17	2311	>1	25	4099	>1
2	277		10	1009		18	2689		26	4261	>1
3	349		11	1399		19	2797	>1	27	4357	>1
4	397		12	1699		20	2803		28	4561	>1
5	547		13	1789		21	3037	>1	29	4567	
6	607		14	1879	>1	22	3271		30	4639	>1
7	709		15	1951	>1	23	3517	>1	31	4789	>1
8	853		16	2131		24	3727	>1	32	4801	>1

Moreover, according to the following Table, only 23 out of these 28 = 10 + 18 remaining cubic fields $\mathbf{k}$ are such that their narrow class numbers are equal to 4:

Case	$f_{\mathbf{k}}$	$h_{\mathbf{k}}^{+}$	Case	$f_{\mathbf{k}}$	$h_{\mathbf{k}}^{+}$	Case	$f_{\mathbf{k}}$	$h_{\mathbf{k}}^{+}$	Case	$f_{\mathbf{k}}$	$h_{\mathbf{k}}^{+}$
1	163	4	8	853	4	15	2689	4	22	8011	4
2	277	4	9	937	4	16	2803	4	23	8191	16
3	349	4	10	1009	16	17	3271	4	24	9109	16
4	397	4	11	1399	4	18	4567	4	25	10957	4
5	547	4	12	1699	16	19	5197	4	26	11149	4
6	607	4	13	1789	4	20	6079	4	27	14407	4
7	709	4	14	2131	4	21	7639	16	28	18307	4

Hence, we finally get the following results which clearly show how useful our bounds on $B_{\mathbf{k}}$ and our techniques for computing numerically $B_{\mathbf{k}}$ are:

PROPOSITION 16. *Let* $\mathbf{N}$ *be a normal* CM-*field of degree* 24 *with Galois group isomorphic to* $\mathrm{SL}_2(F_3)$, *the special linear group over the finite field with three elements. Assume that the class number of* $\mathbf{N}$ *is equal to* 1. *Then,*

1. The class number $h_{\mathbf{k}}$ *and narrow class number* $h_{\mathbf{k}}^{+}$ *of* $\mathbf{k}$ *are equal to* 4, *which implies that the conductor* $f_{\mathbf{k}}$ *of* $\mathbf{k}$ *is a prime equal to* 1 *modulo* 6.

2. $\mathbf{N}^{+}$ *is the narrow Hilbert* 2-*class field of* $\mathbf{k}$, *the narrow class number of* $\mathbf{N}^{+}$ *is equal to* 2 *and* $\mathbf{N}$ *is the second narrow Hilbert* 2-*class field of* $\mathbf{k}$.

3. Finally, $f_{\mathbf{k}}$ *is equal to one of the following* 23 *prime values:* $f_{\mathbf{k}}$ = 163, 277, 349, 397, 547, 607, 709, 853, 937, 1399, 1789, 2131, 2689, 2803, 3271, 4567, 5197, 6079, 8011*r*, 10957, 11149, 14407 *or* 18307.

PROOF. Use Proposition 15. ■

Finally, we refer the reader to [CK] and [Lef] for other examples of the use of the techniques developed in this paper.

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