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Rotation Algebras and the Exel Trace Formula

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Abstract. We show that if u and v are any two unitaries in a unital C^*-algebra such that $\|uv-vu\| < 2$ and uvu^*v^* commutes with u and v, then the C^*-subalgebra $A_{u,v}$ generated by u and v is isomorphic to a quotient of some rotation algebra A_θ, provided that $A_{u,v}$ has a unique tracial state. We also show that the Exel trace formula holds in any unital C^*-algebra. Let $\theta \in (-1/2, 1/2)$ be a real number. For any $\epsilon > 0$, we prove that there exists $\delta > 0$ satisfying the following: if u and v are two unitaries in any unital simple C^*-algebra A with tracial rank zero such that

$$\|uv - e^{2\pi i\theta}vu\| < \delta \quad \text{and} \quad \frac{1}{2\pi i}\tau(\log(uvu^*v^*)) = \theta,$$

for all tracial states τ of A, then there exists a pair of unitaries $\widetilde{u}$ and $\widetilde{v}$ in A such that

$$\widetilde{u}\widetilde{v} = e^{2\pi i\theta}\widetilde{v}\widetilde{u}, \quad \|u - \widetilde{u}\| < \epsilon \quad \text{and} \quad \|v - \widetilde{v}\| < \epsilon.$$

1 Introduction

Let θ be a real number in $(-1/2, 1/2)$ and let A_θ be the corresponding rotation algebra, defined as the universal C^*-algebra generated by a pair of unitaries u_θ and v_θ subject to the relation $u_\theta v_\theta = e^{2\pi i\theta}v_\theta u_\theta$. Let A be a unital C^*-algebra and let u and v be two unitaries with $\|uv - vu\| < 2$. Consider the C^*-subalgebra $A_{u,v}$ generated by u and v. One might ask when $A_{u,v}$ is isomorphic to a quotient of A_θ if uvu^*v^* commutes with u and v. This may seem like a rather unreasonable question; however, if $A_{u,v}$ has a unique tracial state, the answer is always "yes", and it has a simple proof.

This brings us to the following question.

(**Q**1): Let $\epsilon > 0$. Is there a $\delta > 0$ such that if u and v are two unitaries in a unital simple C^*-algebra A with tracial rank zero satisfying

$$\|uv - e^{2\pi i\theta}vu\| < \delta \quad \text{and} \quad \frac{1}{2\pi i}\tau(\log(uvu^*v^*)) = \theta \tag{1.1}$$

for all tracial state τ of A, then there exists a pair of unitaries $\widetilde{u}$ and $\widetilde{v}$ in A such that

$$\widetilde{u}\widetilde{v} = e^{2\pi i\theta}\widetilde{v}\widetilde{u}, \quad \|u - \widetilde{u}\| < \epsilon \quad \text{and} \quad \|v - \widetilde{v}\| < \epsilon\,?$$

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Note that δ is a universal constant independent of u, v, and C^*-algebra A.

A related old problem from the 1950s, popularized by Halmos, asks: if a pair of hermitian matrices almost commute, then are they necessarily close to a pair of commuting hermitian matrices [1, 5, 16, 32]? Voiculescu realized that the answer is negative if the word hermitian is replaced by unitary. In fact, Voiculescu showed that, when $\theta = 0$, something like (1.1) in question (**Q**1) is necessary.

However, despite Voiculescu's example, the related problem about almost commuting hermitians was solved affirmatively by the second author in [20] (see also [14] for a simplified exposition). The problem of whether a pair of almost commuting unitaries can be approximated by a pair of commuting unitaries was further studied by [3, 5, 12, 13, 27] and others. Exel and Loring, following Voiculescu's example, showed that the condition (1.1) is necessary for (**Q**1) in the case that $\theta = 0$, and they recognized that the obstacle in Voiculescu's example is the bott element ([13]). Things moved quickly in the mid 1990's resulting in the proof in ([20]). It has been proved that (**Q**1) has an affirmative answer when $\theta = 0$ (see [8, 15, 28]). The trace formula for the bott element provided by Exel ([11]) is a very convenient tool. In fact, the recent development in the connection to the Elliott program of classification of amenable C^*-algebras shows that the Exel trace formula has many further applications. The Exel trace formula brought together the bott element, a topological obstruction, and rotation number, a dynamical description. Originally, the Exel trace formula was proved in matrix algebras ([11]). We note that it in fact holds in general C^*-algebras. One might say that this paper provides further understanding of the Exel trace formula in the context of rotation algebras.

Shortly after the first version of these notes was posted, Terry Loring informed us about his joint work on (**Q**1). In fact, Eilers and Loring showed in [6] that, for rational values in $(-1/2, 1/2)$ (Eilers, Loring and Pedersen showed that in [7] , for rational value $\frac{1}{2}$), the answer to (**Q**1) is affirmative if the class of all unital simple C^*-algebras with real rank zero is replaced by the class of finite-dimensional simple C^*-algebras. It should be noted that, when A is a matrix algebra M_n, $\frac{1}{2\pi i}\tau(\log(uvu^*v^*))$ is always a rational number. Moreover, when θ is an irrational number, A_θ is always infinite dimensional. Therefore, there is no homomorphism from A_θ into M_n. It seems natural to study (**Q**1) in the class of unital simple AF-algebras, or even in the broader class of unital simple C^*-algebras of tracial rank zero (see Definition 4.1). We show that the answer to (**Q**1) is in the affirmative.

This paper is organized as follows. In Section 2, we list some notation and known results about certain universal C^*-algebras generated by two unitaries. In Section 3, we give a proof that the Exel trace formula holds for any unital C^*-algebra. In Section 4, we show that the answer to question (**Q**1) is affirmative for irrational numbers θ. In the last section, we show that, when θ is rational, we also have an affirmative answer to a version of (**Q**1). In fact, we allow a somewhat larger class of C^*-algebras, *i.e.,* the class of unital simple C^*-algebras of real rank zero and stable rank one.

2 Preliminaries

All statements in this section are known. We review them here for the convenience of the reader.

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Definition 2.1 Let A be a unital C^*-algebra. Denote by $T(A)$ the tracial state space of A. We will use τ for $\tau \otimes \mathrm{Tr}$ on $M_n(A)$, where Tr is the standard trace on M_n, $n = 1, 2, \ldots$. Denote by $\rho_A\colon K_0(A) \to \mathrm{Aff}(T(A))$ the order-preserving map induced by $\rho_A([p])(\tau) = \tau(p)$ for all projections $p \in A \otimes M_n$, $n = 1, 2, \ldots$.

Definition 2.2 Let A be a unital C^*-algebra and let $u \in A$ be a unitary. Define $\mathrm{Ad}\, u(a) = u^* a u$ for all $a \in A$.

The C^*-algebra $C(\mathbb{T}^2)$ of all continuous complex valued functions on the two-torus is well known to be the universal C^*-algebra generated by two commuting unitary elements.

Definition 2.3 Let $\epsilon \in [0, 2)$. Recall that the *soft torus* $\mathcal{T}_\epsilon$ is the universal C^*-algebra generated by a pair of unitaries $\mathfrak{u}_\epsilon$ and $\mathfrak{v}_\epsilon$ subject to $\|\mathfrak{u}_\epsilon\mathfrak{v}_\epsilon - \mathfrak{v}_\epsilon\mathfrak{u}_\epsilon\| \le \epsilon$.

Given $\theta \in \mathbb{R}$, let A_θ be the universal C^*-algebra generated by a pair of unitaries u_θ and v_θ subject to $u_\theta v_\theta = e^{2\pi i\theta} v_\theta u_\theta$. If θ is irrational (resp., rational), A_θ is called an irrational (resp., rational) rotation algebra. The algebras A_θ are usually called noncommutative tori, since $C(\mathbb{T}^2) \cong A_0$, the C^*-algebra of continuous functions on the two-torus $\mathbb{T}^2$.

Let B_ϵ be the universal C^*-algebra generated by a set of unitaries $\{x_n : n \in \mathbb{Z}\}$ subject to $\|x_{n+1} - x_n\| \le \epsilon$ for all $n \in \mathbb{Z}$.

Let α_ϵ be the automorphism of B_ϵ specified by $\alpha_\epsilon(x_n) = x_{n+1}$. More details for the soft torus $\mathcal{T}_\epsilon$ and B_ϵ can be found in [11].

Theorem 2.4 ([11, Theorem 2.2]) *Let z denote the canonical generator of the C^*-algebra $C(\mathbb{T})$, and let $\psi_\epsilon\colon B_\epsilon \to C(\mathbb{T})$ be the unique homomorphism such that $\psi_\epsilon(x_n) = z$ for all n. If $\epsilon < 2$, then ψ_ϵ is a homotopy equivalence between B_ϵ and $C(\mathbb{T})$.*

Proposition 2.5 ([11, Proposition 2.3]) *For all $\epsilon \in [0, 2)$ one has an isomorphism $\varphi\colon \mathcal{T}_\epsilon \to B_\epsilon \rtimes_{\alpha_\epsilon} \mathbb{Z}$ such that $\varphi(\mathfrak{u}_\epsilon) = x_0$ and $\varphi(\mathfrak{v}_\epsilon\mathfrak{u}_\epsilon\mathfrak{v}_\epsilon^*) = x_1$.*

This is proved in [11, Proposition 2.3], but we would like to emphasize that $\varphi(\mathfrak{u}_\epsilon) = x_0$ and $\varphi(\mathfrak{v}_\epsilon\mathfrak{u}_\epsilon\mathfrak{v}_\epsilon^*) = x_1$.

In what follows we will identify x_0 with $\mathfrak{u}_\epsilon$ and $x_1 = \mathfrak{v}_\epsilon\mathfrak{u}_\epsilon\mathfrak{v}_\epsilon^*$.

Let z and w denote the coordinate functions on $\mathbb{T}^2$ so that z and w represent two unitaries in $C(\mathbb{T}^2)$. There is a unital homomorphism $\varphi_\epsilon\colon \mathcal{T}_\epsilon \to C(\mathbb{T}^2)$ such that $\varphi_\epsilon(\mathfrak{u}_\epsilon) = z$ and $\varphi_\epsilon(\mathfrak{v}_\epsilon) = w$.

By the proof of [11, Theorem 2.4], we have the following commutative diagram with exact rows, where ψ_{ϵ_*} and φ_{ϵ_*} are isomorphisms:

$$\begin{tikzcd}
0 \arrow[r] & K_0(B_\epsilon) \arrow[r] \arrow[d, "\psi_{\epsilon_*}"'] & K_0(\mathcal{T}_\epsilon) \arrow[r, "\partial"] \arrow[d, "\varphi_{\epsilon_*}"'] & K_1(B_\epsilon) \arrow[r] \arrow[d, "\psi_{\epsilon_*}"'] & 0 \\
0 \arrow[r] & K_0(C(\mathbb{T})) \arrow[r] & K_0(C(\mathbb{T}^2)) \arrow[r, "\partial"] & K_1(C(\mathbb{T})) \arrow[r] & 0
\end{tikzcd}$$

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Definition 2.6 By the above diagram, there is an element $b \in K_0(C(\mathbb{T}^2))$ such that $\partial(b) = [z]$ in $K_1(C(\mathbb{T}))$. Denote by b_ϵ the element in $K_0(\mathcal{T}_\epsilon)$ defined by $b_\epsilon = \varphi_{\epsilon *}^{-1}(b)$. Then $\partial(b_\epsilon) = [x_0] = [\mathfrak{u}_\epsilon]$ in $K_1(B_\epsilon)$.

We may assume that there are projections $p_\epsilon,\ q_\epsilon \in M_K(\mathcal{T}_\epsilon)$ such that $[p_\epsilon] - [q_\epsilon] = b_\epsilon$, where K is an integer. Note that

$$|\tau \circ \rho_{\mathcal{T}_\epsilon}(b_\epsilon)| \leq 2K \tag{2.1}$$

for all $\tau \in T(\mathcal{T}_\epsilon)$.

Definition 2.7 (see [13] with reversed roles for u and v) Define

$$f(e^{2\pi it}) = \begin{cases} 1-2t, & \text{if } 0 \leq t \leq 1/2, \\ -1+2t, & \text{if } 1/2 < t \leq 1, \end{cases}$$

$$g(e^{2\pi it}) = \begin{cases} (f(e^{2\pi it}) - f(e^{2\pi it})^2)^{\frac{1}{2}}, & \text{if } 0 \leq t \leq 1/2, \\ 0, & \text{if } 1/2 < t \leq 1, \end{cases}$$

$$h(e^{2\pi it}) = \begin{cases} 0, & \text{if } 0 \leq t \leq 1/2, \\ (f(e^{2\pi it}) - f(e^{2\pi it})^2)^{\frac{1}{2}}, & \text{if } 1/2 < t \leq 1. \end{cases}$$

These are non-negative continuous functions defined on the unit circle.

Let A be a unital C^*-algebra, and $u, v \in A$ be two unitaries, define

$$e(u,v) = \begin{pmatrix} f(u) & g(u) + h(u)v^* \\ g(u) + vh(u) & 1 - f(u) \end{pmatrix}.$$

This is a self-adjoint element. Suppose that $uv = vu$, then $e(u,v)$ is a projection.

In $M_2(C(\mathbb{T}^2))$, $e(z,w)$ is a non-trivial rank one projection. Then

$$b = [e(z,w)] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] \tag{2.2}$$

(where b is from Definition 2.6) is often called the bott element for $C(\mathbb{T}^2)$.

There is $\delta_0 > 0$ (independent of unitaries u, v and A, see [29] for existence of δ_0) such that if $\|[u,v]\| < \delta_0$, then the spectrum of the element $e(u,v)$ has a gap at 1/2. The bott element of u and v is an element in $K_0(A)$ as defined by

$$\operatorname{bott}(u,v) = [\chi_{(1/2,\infty)}(e(u,v))] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right].$$

Note that (when $\|uv - vu\| < \delta_0$) there is a continuous function $\chi\colon [0,\infty] \to [0,1]$ such that

$$\chi(e(u,v)) = \chi_{(1/2,\infty)}(e(u,v)).$$

The reader is referred to [12, 13, 27] for more information about the bott element.

The following proposition is also known.

Proposition 2.8 *If* $\epsilon \in [0, \delta_0)$, *then* $b_\epsilon = \operatorname{bott}(\mathfrak{u}_\epsilon, \mathfrak{v}_\epsilon)$.

Proof When $\epsilon \in [0,\delta_0)$,

$$(\varphi_\epsilon \otimes \mathrm{id}_{M_2})(\chi(e(\mathfrak{u}_\epsilon, \mathfrak{v}_\epsilon))) = \chi(e(\varphi_\epsilon(\mathfrak{u}_\epsilon), \varphi_\epsilon(\mathfrak{v}_\epsilon))) = \chi(e(z,w)) = e(z,w).$$

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It follows that

$$\varphi_{\epsilon_*}\left(\left[\chi(e(\mathfrak{u}_\epsilon,\mathfrak{v}_\epsilon))\right]-\left[\begin{pmatrix}1&0\\0&0\end{pmatrix}\right]\right)=b.$$

Therefore, $b_\epsilon=\mathrm{bott}(\mathfrak{u}_\epsilon,\mathfrak{v}_\epsilon)$. ∎

3 Exel Trace Formula

Let A be a unital C^*-algebra and let $\alpha\colon A\to A$ be an automorphism. If τ is a trace on A that is invariant under the action α and if u is an implementing unitary of α, then $\tau\circ E$ gives a trace $\widetilde{\tau}$ on $A\rtimes_\alpha\mathbb{Z}$, where $E\colon A\rtimes_\alpha\mathbb{Z}\to A$ is the expectation defined by $E(\sum_{i=-n}^{n}a_iu^i)=a_0$.

Definition 3.1 ([10, Definition II.9]) If $\tau\in T(A)$ is a fixed tracial state on a unital C^*-algebra A, we say that the pair (A,τ) is an *integral C^*-algebra* if $\rho_A(x)(\tau)\subset\mathbb{Z}$ for all $x\in K_0(A)$.

Let A be a unital C^*-algebra and let $U_n(A)$ be the group of all unitary elements of $A\otimes M_n$, $n=1,2,\dots$. We denote by $U_\infty(A)$ the inductive limit of the sequence of groups

$$U_1(A)\xrightarrow{i_1}U_2(A)\xrightarrow{i_2}\cdots\xrightarrow{i_{n-1}}U_n(A)\xrightarrow{i_n}U_{n+1}(A)\xrightarrow{i_{n+1}}\cdots,$$

where i_n is defined by

$$i_n(u)=u\oplus 1_A\in U_{n+1}(A)\quad\text{for all } u\in U_n(A)\text{ and all } n\in\mathbb{N}.$$

We often use $U(A)$ for $U_1(A)$.

Definition 3.2 ([10, Definition II.2] and [17]) Let A be a unital C^*-algebra and let $\tau\in T(A)$. We say that a group homomorphism

$$\det_\tau\colon U_\infty(A)\longrightarrow\mathbb{T}$$

is a determinant associated with the tracial state τ if for all self-adjoint elements $h\in M_n(A)$, one has $\det_\tau(e^{ih})=e^{i\tau(h)}$.

It is proved by Exel ([10, Theorem II.10]) that such a determinant exists if and only if (A,τ) is an integral C^*-algebra.

Let α be an automorphism of a unital C^*-algebra A. Denote by $\partial\colon K_0(A\rtimes_\alpha\mathbb{Z})\to K_1(A)$ the connecting map of the Pimsner-Voiculescu sequence ([30]).

Let us recall the following two results.

Theorem 3.3 ([10, Theorem V.12]) *Let (A,τ) be an integral unital C^*-algebra and let α be a trace-preserving automorphism of A. Then for every a in $K_0(A\rtimes_\alpha\mathbb{Z})$ we have*

$$\exp\left(2\pi i\widetilde{\tau}\circ\rho_{A\rtimes_\alpha\mathbb{Z}}(a)\right)=\det_\tau\left(\alpha(u^{-1})u\right),$$

where u is any unitary element of $U_\infty(A)$ whose K_1-class is $\partial(a)$.

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Lemma 3.4 ([11, Lemma 3.3]) *Let α be an automorphism of a C^*-algebra A, and let τ_1 and τ_2 be traces on $A \rtimes_\alpha \mathbb{Z}$ such that $\tau_1 = \tau_2$ on A. Then $\tau_1 \circ \rho_{A\rtimes_\alpha\mathbb{Z}} = \tau_2 \circ \rho_{A\rtimes_\alpha\mathbb{Z}}$ on $K_0(A \rtimes_\alpha \mathbb{Z})$.*

Note that if $\epsilon \in [0,2)$, then $\|uvu^*v^* - 1\| = \epsilon < 2$. Then -1 is not in the spectrum of uvu^*v^*. Therefore, there is a continuous branch of logarithm defined on the compact subset $F_\epsilon = \{e^{it} : t \in [-\pi + 2\arccos(\epsilon/2), \pi - 2\arccos(\epsilon/2)]\}$. In what follows, unless otherwise stated, we use log defined on F_ϵ. Moreover, if $0 < \epsilon_1 < \epsilon$, we may assume that log is defined on F_ϵ.

Theorem 3.5 *Let $\epsilon \in [0,2)$, $\mathfrak{u}_\epsilon, \mathfrak{v}_\epsilon \in U(\mathfrak{T}_\epsilon)$ be generators of $\mathfrak{T}_\epsilon$. Then*

$$\rho_{\mathfrak{T}_\epsilon}(b_\epsilon)(\tau) = \frac{1}{2\pi i}\tau(\log(\mathfrak{u}_\epsilon\mathfrak{v}_\epsilon\mathfrak{u}_\epsilon^*\mathfrak{v}_\epsilon^*)) \quad \text{for all} \quad \tau \in T(\mathfrak{T}_\epsilon).$$

In particular, when $\epsilon \in [0, \delta_0)$,

$$\rho_{\mathfrak{T}_\epsilon}\big(\mathrm{bott}(\mathfrak{u}_\epsilon, \mathfrak{v}_\epsilon)\big)(\tau) = \frac{1}{2\pi i}\tau\big(\log(\mathfrak{u}_\epsilon\mathfrak{v}_\epsilon\mathfrak{u}_\epsilon^*\mathfrak{v}_\epsilon^*)\big) \quad \text{for all} \quad \tau \in T(\mathfrak{T}_\epsilon).$$

Proof Identify B_ϵ as a subalgebra of $\mathfrak{T}_\epsilon$ under the isomorphism of Proposition 2.5. Let $\tau \in T(\mathfrak{T}_\epsilon)$. Then τ is given by restriction an α_ϵ-invariant trace on B_ϵ. Moreover, τ is an integral trace on B_ϵ since any tracial state is an integral trace on the homotopy class of $C(\mathbb{T})$ by Theorem 2.4. Let $\widetilde{\tau}$ be the canonical extension of $\tau|_{B_\epsilon}$. By Lemma 3.4 and Theorem 3.3 we obtain

$$\begin{aligned}
\exp\big(2\pi i\tau \circ \rho_{\mathfrak{T}_\epsilon}(b_\epsilon)\big) &= \exp\big(2\pi i\widetilde{\tau} \circ \rho_{\mathfrak{T}_\epsilon}(b_\epsilon)\big) = \det{}_\tau\big(\alpha_\epsilon(\mathfrak{u}_\epsilon^*)\mathfrak{u}_\epsilon\big) \\
&= \det{}_\tau\big(\alpha_\epsilon(x_0^*)x_0\big) = \det{}_\tau(x_1^*x_0) \\
&= \det{}_\tau\big(\exp\big(\log(x_1^*x_0)\big)\big) = \exp\big(\tau\big(\log(x_1^*x_0)\big)\big) \\
&= \exp\big(\tau\big(\log(\mathfrak{v}_\epsilon\mathfrak{u}_\epsilon^*\mathfrak{v}_\epsilon^*\mathfrak{u}_\epsilon)\big)\big) = \exp\big(\tau\big(\log(\mathfrak{u}_\epsilon\mathfrak{v}_\epsilon\mathfrak{u}_\epsilon^*\mathfrak{v}_\epsilon^*)\big)\big).
\end{aligned}$$

So there is an integer $k_\tau \in \mathbb{Z}$ such that

$$\rho_{\mathfrak{T}_\epsilon}(b_\epsilon)(\tau) - \frac{1}{2\pi i}\tau\big(\log(\mathfrak{u}_\epsilon\mathfrak{v}_\epsilon\mathfrak{u}_\epsilon^*\mathfrak{v}_\epsilon^*)\big) = k_\tau.$$

Note that by (2.1),

$$|k_\tau| \leq 2K + 1 \quad \text{for all} \quad \tau \in T(\mathfrak{T}_\epsilon).$$

Fix $\xi = (1,1) \in \mathbb{T} \times \mathbb{T} = \mathbb{T}^2$. Let $P_\xi \colon C(\mathbb{T}^2) \to \mathbb{C}1_{\mathfrak{T}_\epsilon} \subset \mathfrak{T}_\epsilon$ be the point-evaluation defined by $P_\xi(f) = f(\xi)1_{\mathfrak{T}_\epsilon}$ for all $f \in C(\mathbb{T}^2)$. Define $\pi_\xi \colon \mathfrak{T}_\epsilon \to \mathbb{C}$ by $\pi_\xi = P_\xi \circ \varphi_\epsilon$, where $\varphi_\epsilon \colon \mathfrak{T}_\epsilon \to C(\mathbb{T}^2)$ is defined just before Definition 2.6. Note that $(\pi_\xi)_{*0}(b_\epsilon) = (P_\xi)_*(b) = 0$, because $f(1) = 1$ and $g(1) = h(1) = 0$ using (2.2). Let $u = \mathfrak{u}_\epsilon \oplus I_m$ and $v = \mathfrak{v}_\epsilon \oplus I_m$, where I_m is the identity of $M_m(\mathfrak{T}_\epsilon)$, then

$$\Phi(a) = \begin{pmatrix} a & 0 & 0 & \cdots \\ 0 & \pi_\xi(a) & 0 & \cdots \\ \vdots & \vdots & \ddots & \\ & & & \pi_\xi(a) \end{pmatrix}_{(m+1)\times(m+1)}$$

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defines a homomorphism $\Phi\colon \mathcal{T}_\epsilon \to M_{m+1}(\mathcal{T}_\epsilon)$ such that $\Phi(\mathfrak{u}_\epsilon) = u$ and $\Phi(\mathfrak{v}_\epsilon) = v$.

Let $\tau_0 \in T(M_{m+1}(\mathcal{T}_\epsilon))$. Then $\tau_0 \circ \Phi$ is an an integral trace on B_ϵ, since any tracial state is an integral trace on the homotopy class of $C(\mathbb{T})$ by Theorem 2.4. It follows that

$$\tau_0 \circ \rho_{M_{m+1}(\mathcal{T}_\epsilon)}\big(\Phi_{*0}(b_\epsilon)\big) - \frac{1}{2\pi i}\tau_0 \circ \Phi\big(\log(\mathfrak{u}_\epsilon \mathfrak{v}_\epsilon \mathfrak{u}_\epsilon^* \mathfrak{v}_\epsilon^*)\big) = k_{\tau_0\circ\Phi} \in \mathbb{Z}.$$

On the other hand, one may write $\tau_0 = \frac{1}{m+1}(\tau \oplus \cdots \oplus \tau)$ for some $\tau \in T(\mathcal{T}_\epsilon)$. We compute that

$$\tau_0 \circ \Phi\big(\log(\mathfrak{u}_\epsilon \mathfrak{v}_\epsilon \mathfrak{u}_\epsilon^* \mathfrak{v}_\epsilon^*)\big) = \frac{1}{m+1}\tau\big(\log(\mathfrak{u}_\epsilon \mathfrak{v}_\epsilon \mathfrak{u}_\epsilon^* \mathfrak{v}_\epsilon^*)\big).$$

By the definition of Φ and the fact that $(\pi_\xi)_{*0}(b_\epsilon) = 0$, we also have

$$\tau_0 \circ \rho_{M_{m+1}(\mathcal{T}_\epsilon)}\big(\Phi_{*0}(b_\epsilon)\big) = \frac{1}{m+1}\tau \circ \rho_{\mathcal{T}_\epsilon}(b_\epsilon).$$

It follows, by combining this with (2.1), that (K only depends on ϵ)

$$|k_{\tau_0\circ\Phi}| = \Big|\frac{k_\tau}{m+1}\Big| \leq \frac{2K+1}{m+1}. \tag{3.1}$$

This holds for all integers m. It follows that $k_{\tau_0\circ\Phi} = 0$. Then, by (3.1), $k_\tau = 0$ for all $\tau \in T(\mathcal{T}_\epsilon)$. Therefore,

$$\rho_{\mathcal{T}_\epsilon}(b_\epsilon)(\tau) = \frac{1}{2\pi i}\tau\big(\log(\mathfrak{u}_\epsilon \mathfrak{v}_\epsilon \mathfrak{u}_\epsilon^* \mathfrak{v}_\epsilon^*)\big) \quad \text{for all} \quad \tau \in T(\mathcal{T}_\epsilon). \qquad \blacksquare$$

Definition 3.6 Let A be a unital C^*-algebra and let u and v be two unitaries in A such that $\|uv - vu\| \leq \epsilon < 2$. Denote by $A_{u,v}$ the C^*-subalgebra of A generated by u and v. There is a surjective homomorphism $\phi_{u,v}\colon \mathcal{T}_\epsilon \to A_{u,v}$ such that $\phi_{u,v}(\mathfrak{u}_\epsilon) = u$ and $\phi_{u,v}(\mathfrak{v}_\epsilon) = v$. Put $b_{u,v} = (\phi_{u,v})_{*0}(b_\epsilon)$. If $\|uv - vu\| < \delta_0$, then

$$b_{u,v} = \mathrm{bott}(u, v).$$

Theorem 3.7 (The Exel trace formula) *Let A be a unital C^*-algebra. Then for any $u, v \in U(A)$ and $\|uv - vu\| < 2$, we have*

$$\rho_A\big(\imath_{*0}(b_{u,v})\big)(\tau) = \frac{1}{2\pi i}\tau\big(\log(uvu^*v^*)\big) \quad \textit{for all} \quad \tau \in T(A),$$

where $\imath\colon A_{u,v} \to A$ is the unital embedding. If, in addition, $\|uv - vu\| < \delta_0$, then

$$\rho_A\big(\mathrm{bott}(u, v)\big)(\tau) = \frac{1}{2\pi i}\tau\big(\log(uvu^*v^*)\big) \quad \textit{for all} \quad \tau \in T(A).$$

Proof Since $\|uv - vu\| = \epsilon < 2$, there is a unique homomorphism $\phi\colon \mathcal{T}_\epsilon \to A$ such that $\phi(\mathfrak{u}_\epsilon) = u$ and $\phi(\mathfrak{v}_\epsilon) = v$. Then $\tau \circ \phi$ is a tracial state on $\mathcal{T}_\epsilon$, and we get

$$\rho_{\mathcal{T}_\epsilon}(b_\epsilon)(\tau \circ \phi) = \frac{1}{2\pi i}\tau \circ \phi\big(\log(\mathfrak{u}_\epsilon \mathfrak{v}_\epsilon \mathfrak{u}_\epsilon^* \mathfrak{v}_\epsilon^*)\big) \quad \text{for all} \quad \tau \in T(A).$$

Note that $\phi(\mathcal{T}_\epsilon) = A_{u,v}$. So

$$\rho_A(\imath_{*0}(b_{u,v}))(\tau) = \frac{1}{2\pi i}\tau(\log(uvu^*v^*)) \quad \text{for all} \quad \tau \in T(A). \qquad \blacksquare$$

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Remark 3.8 The Exel trace formula was first found for matrix algebras ([11]). It was later proved that the same formula also holds in unital simple C^*-algebras of tracial rank no more than one ([26, Theorem 3.5]).

Theorem 3.9 *Let A be a unital C^*-algebra, and assume $u, v \in U(A)$ satisfy the condition $\|uv - vu\| < 2$ and uvu^*v^* commutes with u and v. Let $A_{u,v}$ be the C^*-subalgebra generated by unitaries u and v. Suppose that $\theta \in (-1/2, 1/2)$. If $\frac{1}{2\pi i}\tau(\log(uvu^*v^*)) = \theta$ for all $\tau \in T(A_{u,v})$, then $A_{u,v}$ is isomorphic to a quotient of A_θ. Moreover, $uv = e^{2\pi i\theta}vu$.*

In particular, if θ is an irrational number, then $A_{u,v} \cong A_\theta$.

Proof Let $w = uvu^*v^*$. Suppose that the spectrum of w has more than one point, say $e^{2\pi i\theta_1}$ and $e^{2\pi i\theta_2}$, since $\|w - 1\| = \|uvu^*v^* - 1\| = \|uv - vu\| < 2$, we have $|1 - e^{2\pi i\theta_j}| < 2$ for $j = 1, 2$.

Note that w commutes with u and v. Working in the enveloping von Neumann algebra $A_{u,v}^{**}$, let $p_{\theta_j} \in A_{u,v}^{**}$ be the spectrum projection of w associated with the point $\{e^{2\pi i\theta_j}\}$, $j = 1, 2$. Since $e^{2\pi i\theta_j}$ is in the spectrum of w, $p_{\theta_j} \neq 0$ in $A_{u,v}^{**}$, $j = 1, 2$. Moreover, p_{θ_j} is a closed projection of $A_{u,v}$. Since w commutes with u and v, p_{θ_j} is central. Define $\varphi_j(a) = ap_{\theta_j}$ for all $a \in A_{u,v}$, $j = 1, 2$. Then $\varphi_j(w) = p_{\theta_j}w = e^{2\pi i\theta_j}p_{\theta_j}$, $j = 1, 2$. It follows that

$$\varphi_j(u)\varphi_j(v) = p_{\theta_j}uv = p_{\theta_j}wvu = e^{2\pi i\theta_j}\varphi_j(v)\varphi_j(u), \quad j = 1, 2.$$

Thus, $\varphi_j\colon A_{u,v} \to \varphi_j(A_{u,v})$ is a unital surjective homomorphism from $A_{u,v}$ onto a quotient of A_{θ_j}, $j = 1, 2$. We have $T(\varphi_j(A_{u,v})) \neq \varnothing$, because $\varphi_j(A_{u,v}) \cong A_{\theta_j}$ when θ_j is irrational and all irreducible representations of quotients of A_{θ_j} are finite dimensional when θ_j is rational.

Let $\tau_j \in T(\varphi_j(A_{u,v}))$. Then $\tau_j \circ \varphi_j \in T(A_{u,v})$. We have

$$\begin{aligned}\frac{1}{2\pi i}\big(\tau_j \circ \varphi_j\big(\log(uvu^*v^*)\big)\big) &= \frac{1}{2\pi i}\Big(\tau_j\big(\log\big(\varphi_j(u)\varphi_j(v)\varphi_j(u)^*\varphi_j(v)^*\big)\big)\Big)\\ &= \theta_j.\end{aligned}$$

By assumption, $\theta_j = \theta$, $j = 1, 2$. So the spectrum of w has only one point, which is equal to $e^{2\pi i\theta}$. In other words, $w = e^{2\pi i\theta}$. It follows that $uv = e^{2\pi i\theta}vu$. Therefore, $A_{u,v}$ is isomorphic to a quotient of A_θ.

If θ is an irrational number, it is well known that irrational rotation algebra A_θ is simple, so $A_{u,v} \cong A_\theta$. ■

Remark 3.10 One might ask what happens when $\theta = \frac{1}{2}$ in above theorem. In fact, for any pair of unitaries u and v in a unital C^*-algebra A such that $\|uv + vu\| < 2$ and uvu^*v^* commutes with u and v, let $A_{u,v}$ be the C^*-subalgebra generated by u and v. If $\frac{1}{2\pi i}\tau(\log_0(uvu^*v^*)) = 1/2$ for all $\tau \in T(A_{u,v})$, where $\log_0$ is a continuous logarithm defined on a compact subset F of $\{e^{it} : t \in (0, 2\pi)\}$ with values in $\{ri : r \in (0, 2\pi)\}$, then $uv = -vu$.

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Corollary 3.11 *Let A be a unital C^*-algebra and let $u, v \in U(A)$ satisfy the condition $\|uv - vu\| < 2$, and assume uvu^*v^* commutes with u and v. Let $A_{u,v}$ be the C^*-subalgebra generated by unitaries u and v. If $A_{u,v}$ has a unique tracial state, then $A_{u,v}$ is isomorphic to some irrational rotation algebra A_θ or some matrix algebra M_n.*

Proof Let τ be the unique tracial state on $A_{u,v}$. If $\frac{1}{2\pi i}\tau(\log(uvu^*v^*)) = \theta$ is an irrational number, then by Theorem 3.9, $A_{u,v} \cong A_\theta$.

If $\frac{1}{2\pi i}\tau(\log(uvu^*v^*)) = \theta$ is a rational number, then $A_{u,v}$ is isomorphic to a quotient of rational rotation algebra A_θ. It follows from [31] that A_θ is strongly Morita equivalent to $C(\mathbb{T}^2)$. Therefore $A_\theta \otimes \mathcal{K} \cong C(\mathbb{T}^2) \otimes \mathcal{K}$, where $\mathcal{K}$ is the C^*-algebra of compact operator on an infinite dimensional separable Hilbert space. Let $\phi\colon A_\theta \otimes \mathcal{K} \to C(\mathbb{T}^2) \otimes \mathcal{K}$ denote the isomorphism. Then

$$A_\theta = (1_{A_\theta} \otimes e_{11})(A_\theta \otimes \mathcal{K})(1_{A_\theta} \otimes e_{11}) \cong \phi(1_{A_\theta} \otimes e_{11})(C(\mathbb{T}^2) \otimes \mathcal{K})\phi(1_{A_\theta} \otimes e_{11}).$$

Thus, we can find a projection $P_1 \in M_N(C(\mathbb{T}^2))$ that is equivalent to $\phi(1_{A_\theta} \otimes e_{11})$ for some $N \in \mathbb{N}$. So $A_\theta \cong P_1 M_N(C(\mathbb{T}^2))P_1$, where N is an integer and $P_1 \in M_N(C(\mathbb{T}^2))$ is a projection. Since each quotient of $P_1 M_N(C(\mathbb{T}^2))P_1$ is isomorphic to $P_1 M_N(C(X))P_1$ for some closed subset $X \subset \mathbb{T}^2$, we have $A_{u,v} \cong P_1 M_N(C(X))P_1$ for some closed subset $X \subset \mathbb{T}^2$. The assumption that $A_{u,v}$ has a unique tracial state implies that X is only one point. It follows that $A_{u,v} \cong M_n$ for some $n \in \mathbb{N}$. ∎

4 Stability of Irrational Rotation in Infinite Simple C^*-algebras

Eilers and Loring ([6, Corollary 7.6]) showed that the answer to (**Q**1) is affirmative for all rational numbers in $(-1/2, 1/2]$ if the class of unital simple C^*-algebras of tracial rank zero is replaced by the class of all matrix algebras. As mentioned in the introduction, to include irrational numbers, one may replace M_n, a finite dimensional simple C^*-algebra, by a unital infinite dimensional simple AF-algebra. To make it even more general, we will replace finite dimensional simple C^*-algebras (matrix algebras) by unital simple C^*-algebras with tracial rank zero.

We would also remark that an affirmative answer to (**Q**1) does not follow from Theorem 3.9, even with the additional assumption that uvu^*v^* commutes with u and v. Note that in Theorem 4.5 and Theorem 5.3, the τ are tracial states on A, while the τ in Theorem 3.9 are all tracial states on $A_{u,v}$.

We recall the definition of tracial (topological) rank of C^*-algebras.

Definition 4.1 ([22]) Let A be a unital simple C^*-algebra. Then A is said to have tracial (topological) rank zero if for any $\varepsilon > 0$, any finite set $\mathcal{F} \subset A$ and any nonzero positive element $c \in A$, there exists a finite dimensional C^*-subalgebra $B \subset A$ with $1_B = p$ such that

(i) $\|pa - ap\| < \varepsilon$ for all $a \in \mathcal{F}$;
(ii) $\operatorname{dist}(pap, B) < \epsilon$ for all $a \in \mathcal{F}$;
(iii) $1_A - p$ is Murray-von Neumann equivalent to a projection in $\overline{cAc}$.

If A has tracial rank zero, we write $\mathrm{TR}(A) = 0$.

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Definition 4.2 Let $L\colon A \to B$ be a linear map. Let $\delta > 0$ and $\mathcal{G} \subset A$ be a (finite) subset. We say L is $\mathcal{G}$-δ-multiplicative if

$$\|L(ab) - L(a)L(b)\| < \delta \quad \text{for all} \quad a, b \in \mathcal{G}.$$

We begin with the following lemma, which is known.

Lemma 4.3 ([24, Lemma 4.1]) *Let A be a separable unital C^*-algebra. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset A_{s.a.}$, there exists $\delta > 0$ and a finite subset $\mathcal{G} \subset A_{s.a}$ satisfying the following. For any $\mathcal{G}$-δ-multiplicative, contractive, completely positive, linear map $L\colon A \to B$, any unital C^*-algebra B with $T(B) \neq \varnothing$, and any tracial state $t \in T(B)$, there exists a $\tau \in T(A)$ such that*

$$||t \circ L(a) - \tau(a)\| < \epsilon \quad \textit{for all} \quad a \in \mathcal{F}.$$

Let $\theta \in (-1/2, 1/2)$ be an irrational number and $\epsilon = |1 - e^{2\pi i\theta}|$. Recalling Definition 2.6, we write

$$p_\epsilon = (a_{i,j})_{K\times K}, \quad q_\epsilon = (c_{i,j})_{K\times K} \quad \text{and} \quad b_\epsilon = [p_\epsilon] - [q_\epsilon]$$

where $a_{i,j}, b_{i,j} \in \mathcal{T}_\epsilon$.

Let $\phi_\theta\colon \mathcal{T}_\epsilon \to A_\theta$ be the homomorphism such that $\phi_\theta(\mathfrak{u}_\epsilon) = u_\theta$ and $\phi_\theta(\mathfrak{v}_\epsilon) = v_\theta$. Let A be a unital C^*-algebra and let $u, v \in A$ be two unitaries. Let $A_{u,v}$ be the C^*-subalgebra of A generated by u and v. If $\|uv - vu\| < \epsilon$, then there is a surjective homomorphism $\phi_{u,v}\colon \mathcal{T}_\epsilon \to A_{u,v}$ such that $\phi_{u,v}(\mathfrak{u}_\epsilon) = u$ and $\phi_{u,v}(\mathfrak{v}_\epsilon) = v$.

Lemma 4.4 *Let $\theta \in (-1/2, 1/2)$. For any $\epsilon_0 > 0$, any $\eta_1 > 0$, and any finite subset $\mathcal{G} \subset A_\theta$, there exists $\delta_{00} > 0$ satisfying the following. For any unital C^*-algebra A and any pair of unitaries u, $v \in A$, if $\|uv - e^{2\pi i\theta}vu\| < \delta_{00}$, then there exists a unital $\mathcal{G}$-η_1-multiplicative, contractive, completely positive, linear map $L\colon A_\theta \to A$ such that $\|L(u_\theta) - u\| < \epsilon_0$, $\|L(v_\theta) - v\| < \epsilon_0$ and*

$$(\imath \circ \phi_{u,v})_{*0}([p_\epsilon]) = [L \circ \phi_\theta]([p_\epsilon]),$$
$$(\imath \circ \phi_{u,v})_{*0}([q_\epsilon]) = [L \circ \phi_\theta]([q_\epsilon]) \quad \textit{in} \quad K_0(A),$$

where $\imath\colon A_{u,v} \to A$ is the unital embedding map. Moreover, if $\theta = p/q \in (-1/2, 1/2]$, where p and q are non-zero integers with $(p, q) = 1$ and $q > 0$, we may also assume that

$$[L]\big(\mathrm{bott}(u_\theta^q, v_\theta^q)\big) = \mathrm{bott}(u^q, v^q). \tag{4.1}$$

Proof Let $\eta > 0$ be any positive number with $\eta < \epsilon_0/2$ and let $N \geq 1$ be an integer. There is $\delta_{00} > 0$ such that if $\|uv - e^{2\pi i\theta}vu\| < \delta_{00}$, then there exist a surjective homomorphism $\phi_{u,v}\colon \mathcal{T}_\epsilon \to A_{u,v}$ such that $\phi_{u,v}(\mathfrak{u}_\epsilon) = u$ and $\phi_{u,v}(\mathfrak{v}_\epsilon) = v$, and a $\mathcal{G}$-η_1-multiplicative, contractive, completely positive, linear map $L\colon A_\theta \to A$ such that

$$\|\imath \circ \phi_{u,v}(a_{i,j}) - L(\phi_\theta(a_{i,j}))\| < 1/(4K^2),$$
$$\|\imath \circ \phi_{u,v}(c_{i,j}) - L(\phi_\theta(c_{i,j}))\| < 1/(4K^2)$$

for all $i, j \in \{1, 2, \dots, K\}$.

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Moreover, we may also assume that

$$\begin{aligned} &\|L(u_\theta) - u\| < \eta/4N < \epsilon_0 \quad \text{and} \quad \|L(v_\theta) - v\| < \eta/4N < \epsilon_0, \\ &\|L(u_\theta^N) - L(u_\theta)^N\| < \eta/4 \quad \text{and} \quad \|L(v_\theta^N) - L(v_\theta)^N\| < \eta/4. \end{aligned} \tag{4.2}$$

We then obtain

$$\Big\| [(\imath \circ \phi_{u,v}) \otimes \mathrm{id}_{M_K}](p_\epsilon) - [(L \circ \phi_\theta) \otimes \mathrm{id}_{M_K}](p_\epsilon) \Big\| < 1/4,$$
$$\Big\| [(\imath \circ \phi_{u,v}) \otimes \mathrm{id}_{M_K}](q_\epsilon) - [(L \circ \phi_\theta) \otimes \mathrm{id}_{M_K}](q_\epsilon) \Big\| < 1/4.$$

It follows that

$$(\imath \circ \phi_{u,v})_{*0}([p_\epsilon]) = [L \circ \phi_\theta]([p_\epsilon]) \quad \text{and} \quad (\imath \circ \phi_{u,v})_{*0}([q_\epsilon]) = [L \circ \phi_\theta]([q_\epsilon]).$$

In the case where $\theta = p/q$, as described in the lemma, we choose $N = q$. By (4.2), we have

$$\|L(u_\theta^q) - u^q\| < \eta/2 \quad \text{and} \quad \|L(v_\theta^q) - v^q\| < \eta/2.$$

Therefore, with sufficiently small η, by the definition of the bott element in Definition 2.7, (4.1) also holds. ∎

Theorem 4.5 *Let $\theta \in (-1/2, 1/2)$ be an irrational number. For any $\epsilon > 0$, there exists $\delta > 0$ satisfying the following. For any unital simple infinite dimensional C^*-algebra A with tracial rank zero and any pair of unitaries $u, v \in A$ such that*

$$\|uv - e^{2\pi i\theta} vu\| < \delta \quad \textit{and} \quad \frac{1}{2\pi i}\tau\big(\log(uvu^*v^*)\big) = \theta$$

for all $\tau \in T(A)$, there exists a pair of unitaries $\widetilde{u}, \widetilde{v} \in A$ such that

$$\widetilde{u}\widetilde{v} = e^{2\pi i\theta}\widetilde{v}\widetilde{u}, \quad \|\widetilde{u} - u\| < \epsilon, \quad \textit{and} \quad \|\widetilde{v} - v\| < \epsilon.$$

Proof Let $\epsilon_0 = |1 - e^{2\pi i\theta}| < 2$. We will apply [25, Theorem 3.2]. Let A_θ be the irrational rotation algebra generated by a pair of unitaries u_θ and v_θ such that $u_\theta v_\theta = e^{2\pi i\theta} v_\theta u_\theta$. By [9], A_θ is a unital simple $A\mathbb{T}$-algebra of real rank zero with

$$\big(K_0(A_\theta), K_0(A_\theta)_+, [1_{A_\theta}]\big) = \big(\mathbb{Z} + \mathbb{Z}\theta, (\mathbb{Z} + \mathbb{Z}\theta)_+, 1\big) \quad \text{and} \quad K_1(A_\theta) = \mathbb{Z} \oplus \mathbb{Z}.$$

To apply [25, Theorem 3.2], put $C = A_\theta$. Let τ be the unique tracial state on C. For each $t \in T(A)$, define $\gamma\colon C_{s.a.} \to \mathrm{Aff}(T(A))$ by $\gamma(c)(t) = \tau(c)$ for all $c \in C_{s.a.}$ and all $t \in T(A)$, where $C_{s.a.}$ is the set of all self-adjoint elements of C.

Fix $1 > \epsilon > 0$ and let $\mathcal{F} = \{1_{A_\theta}, u_\theta, v_\theta\}$. Let $\eta > 0$, $\delta_0 > 0$ (in place of δ), $\mathcal{G}_1 \subset C$ (in place of $\mathcal{G}$) be a finite subset, let $\mathcal{H} \subset C_{s.a.}$ be a finite subset, and let $\mathcal{P} \subset \underline{K}(C)$ be a finite subset required by [25, Theorem 3.2] for $\epsilon/2$ (in place of ϵ) and $\mathcal{F}$ given.

Note that $\tau \circ \rho_C(b_{u_\theta,v_\theta}) = \theta$. Therefore, $K_0(C)$ is generated by $[1_C]$ and b_{u_θ,v_θ}. Thus, we may assume, without loss of generality, that $\mathcal{P} = \{[1_C], b_{u_\theta,v_\theta}, [u_\theta], [v_\theta]\}$.

It follows from Lemma 4.3 that there exists a finite subset $\mathcal{H}_1 \subset C_{s.a.}$ and $\delta_2 > 0$ satisfying the following. For any $\mathcal{H}_1$-δ_2-multiplicative, contractive, completely positive linear map $L\colon C \to A$, for any unital C^*-algebra A with $T(A) \neq \varnothing$, and any

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tracial state $t \in T(A)$, we have

$$|t \circ L(c) - \tau(c)| < \eta \quad \text{for all} \quad c \in \mathcal{H}.$$

Let $\mathcal{G}_2 = \mathcal{H}_1 \cup \mathcal{G}_1$ and let $\delta_3 = \min\{\delta_1, \delta_2\}$. Choose $1 > \delta > 0$ such that there is a $\mathcal{G}_2$-δ_3-multiplicative, contractive, completely positive, linear map $L\colon C \to A$ (for any unital C^*-algebra A) such that

$$\|L(u_\theta) - u\| < \epsilon/2 \quad \text{and} \quad \|L(v_\theta) - v\| < \epsilon/2 \tag{4.3}$$

for any pair of unitaries u and v in A with $\|uv - e^{2\pi i\theta}vu\| < \delta$. Furthermore, by Lemma 4.4, we may also assume, by choosing even smaller δ, that

$$[L \circ \phi_\theta]([p_{\epsilon_0}]) = (\imath \circ \phi_{u,v})_{*0}([p_{\epsilon_0}]), \tag{4.4}$$

$$[L \circ \phi_\theta]([q_{\epsilon_0}]) = (\imath \circ \phi_{u,v})_{*0}([q_{\epsilon_0}]). \tag{4.5}$$

Now suppose that A is a unital simple C^*-algebra with tracial rank zero and let $u, v \in A$ be two unitaries such that

$$\|uv - e^{2\pi i\theta}vu\| < \delta \quad \text{and} \quad \frac{1}{2\pi i}t(\log(uvu^*v^*)) = \theta$$

for all $t \in T(A)$. Therefore, there exists a $\mathcal{G}_2$-δ_3-multiplicative, contractive, completely positive, linear map $L\colon C \to A$ such that (4.3)–(4.5) hold. Moreover, by the choices of δ and $\mathcal{G}$,

$$|t \circ L(c) - \tau(c)| < \eta \quad \text{for all} \quad c \in \mathcal{H} \text{ and } t \in T(A).$$

It follows from (4.4) and (4.5) that

$$[L](b_{u_\theta,v_\theta}) = (\imath \circ \phi_{u,v})_{*0}(b_{\epsilon_0}).$$

Thus, by the Exel trace formula of Theorem 3.5,

$$\begin{aligned} \rho_A\big([L](b_{u_\theta,v_\theta})\big)(t) &= \rho_A\big((\imath \circ \phi_{u,v})_{*0}(b_{\epsilon_0})\big)(t) \\ &= \rho_A\big(\imath_{*0}(b_{u,v})\big)(t) = \frac{1}{2\pi i}t\big(\log(uvu^*v^*)\big) = \theta \end{aligned} \tag{4.6}$$

for all $t \in T(A)$. Define $\kappa\colon \mathbb{Z} + \mathbb{Z}\theta \to K_0(A)$ by $\kappa([1]) = [1_A]$ and $\kappa(\theta) = \imath_{*0}(b_{u,v})$. Since $t(\imath_{*0}(b_{u,v})) = \theta = \tau(b_{u_\theta,v_\theta})$ for all $t \in T(A)$, κ is an order preserving homomorphism. Now, by [25, Theorem 5.2], we have a unital homomorphism $h\colon A_\theta \to A$ such that

$$h_{*0} = \kappa \tag{4.7}$$

$$h_{*1}([u_\theta]) = [u] \quad \text{and} \quad h_{*1}([v_\theta]) = [v]. \tag{4.8}$$

It follows from (4.7) and (4.8) that $[h]|_{\mathcal{P}} = [L]|_{\mathcal{P}}$. Moreover, by (4.6),

$$|t \circ L(c) - \gamma(c)(t)| < \eta \quad \text{and} \quad |t \circ h(c) - \gamma(c)(t)| < \eta$$

for all $c \in \mathcal{H}$ and all $t \in T(A)$. It follows from [25, Theorem 3.2] that there exists a unitary $W \in A$ such that

$$\|W^*h(u_\theta)W - L(u_\theta)\| < \epsilon/2 \quad \text{and} \quad \|W^*h(v_\theta)W - L(v_\theta)\| < \epsilon/2. \tag{4.9}$$

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Let

$$\widetilde{u} = W^* h(u_\theta) W \quad \text{and} \quad \widetilde{v} = W^* h(v_\theta) W.$$

Then, since h is a homomorphism, $\widetilde{u}\widetilde{v} = e^{2\pi i\theta}\widetilde{v}\widetilde{u}$. By (4.9) and (4.3),

$$\|\widetilde{u} - u\| < \epsilon \quad \text{and} \quad \|\widetilde{v} - v\| < \epsilon. \qquad \blacksquare$$

Remark 4.6 A version of Theorem 4.5 also holds in unital, amenable, purely infinite, simple C^*-algebras (see [23]).

5 Stability of Rational Rotation in Infinite Simple C^*-algebras

Now we consider the case where θ is a rational number.

Recall that the rational rotation C^*-algebra associated with the rational number θ is the universal C^*-algebra A_θ generated by a pair u_θ, v_θ of unitaries with $u_\theta v_\theta = e^{2\pi i\theta} v_\theta u_\theta$, where θ is a rational number. When $\theta = 0$, $A_0 \cong C(\mathbb{T}^2)$. If $\theta \neq 0$, write $\theta = \pm p/q$ with p, q coprime and $0 < 2p \leq q$. Let $\lambda = e^{2\pi i\theta}$, and define $q \times q$ matrices

$$S_1 = \begin{pmatrix} 1 & & & & \\ & \lambda^1 & & & \\ & & \lambda^2 & & \\ & & & \ddots & \\ & & & & \lambda^{q-1} \end{pmatrix} \quad \text{and} \quad S_2 = \begin{pmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix}. \tag{5.1}$$

Then $S_1 S_2 = e^{2\pi i\theta} S_2 S_1$. By the universal property, there is a unital homomorphism $\pi^{(0)} \colon A_\theta \to M_q$ such that $\pi^{(0)}(u_\theta) = S_1$ and $\pi^{(0)}(v_\theta) = S_2$. Since S_1 and S_2 generate M_q, this gives an irreducible representation of A_θ. Fix a pair of complex numbers $(t_1, t_2) \in \mathbb{T}^2$ and choose a pair of q-th roots $(r_1, r_2) \in \mathbb{T}^2$ such that $r_1^q = t_1$ and $r_2^q = t_2$. Define an automorphism $\alpha_{r_1,r_2} \colon A_\theta \to A_\theta$ such that $\alpha_{r_1,r_2}(u_\theta) = r_1 u_\theta$ and $\alpha_{r_1,r_2}(v_\theta) = r_2 v_\theta$. Then $\pi^{(0)} \circ \alpha_{r_1,r_2}$ also gives an irreducible representation. It is easy to verify that if $(r_1', r_2') \in \mathbb{T}^2$ and $(r_1')^q = t_1$ and $(r_2')^q = t_2$, then $\pi^{(0)} \circ \alpha_{r_1,r_2}$ and $\pi^{(0)} \circ \alpha_{r_1',r_2'}$ are unitarily equivalent (by considering permutations of the q-th roots). In particular, they have the same kernel I_{t_1,t_2}. Note that

$$\pi^{(0)} \circ \alpha_{r_1,r_2}(u_\theta^q) = t_1 \cdot 1_{M_q} = \pi^{(0)} \circ \alpha_{r_1',r_2'}(u_\theta^q)$$

and

$$\pi^{(0)} \circ \alpha_{r_1,r_2}(v_\theta^q) = t_2 \cdot 1_{M_q} = \pi^{(0)} \circ \alpha_{r_1',r_2'}(v_\theta^q).$$

Therefore, if $(t_1, t_2) \neq (t_1', t_2')$ in $\mathbb{T}^2$, then $I_{t_1,t_2} \neq I_{t_1',t_2'}$. In particular, they are not unitarily equivalent.

The following lemma will be used in the sequel. It is certainly known to many experts and should follow from [31] and the discussion in [2,4]. To clarify the matter, we include the proof here.

Lemma 5.1 *Let $\theta = p/q \in (-1/2, 1/2]$ be a non-zero rational number, where p and q are two integers, $p \neq 0$, $q > 0$, and $(p, q) = 1$.*

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Then there exist an integer N and a projection $P \in M_N(C(\mathbb{T}^2))$ (of rank q) and an isomorphism $H\colon A_\theta \to PM_N(C(\mathbb{T}^2))P$ such that $\pi_\xi \circ H(u_\theta^q) = t_1 P(t_1, t_2)$ and $\pi_\xi \circ H(v_\theta^q) = t_2 P(t_1, t_2)$ for any $\xi = (t_1, t_2) \in \mathbb{T}^2$, where π_ξ is the evaluation map at ξ.

Proof It follows from [31] and the proof of Corollary 3.11 that

$$A_\theta \cong P_1 M_N(C(\mathbb{T}^2))P_1,$$

where N is an integer and $P_1 \in M_N(C(\mathbb{T}^2))$ is a projection. Let ψ denote the isomorphism.

Let π be any irreducible representation of A_θ. Let $\lambda = e^{2\pi i\theta}$. Since $u_\theta^q v_\theta = \lambda^q v_\theta u_\theta^q = v_\theta u_\theta^q$, we have that u_θ^q lies in the center of A_θ and $\pi(u_\theta)^q$ lies in the center of $\pi(A_\theta)$, whence $\pi(u_\theta)^q$ is a scalar. Similarly, $\pi(v_\theta)^q$ is also a scalar. Let $t_1, t_2 \in \mathbb{T}$ such that $\pi(u_\theta^q) = t_1 I$ and $\pi(v_\theta^q) = t_2 I$. Thus $\pi(u_\theta)$ has possible eigenvalues $r_1\lambda^j$ for $0 \leq j < q$ and for some $r_1 \in \mathbb{T}$ such that $r_1^q = t_1$. Let $E_j = E_{\pi(u_\theta)}(r_1\lambda^j)$ be the corresponding spectral projections in $\pi(A_\theta)$ (which we do not know are non-zero at moment). We may write

$$\pi(u_\theta) = \sum_{j=0}^{q-1} r_1 \lambda^j E_j.$$

Since

$$\sum_{k=0}^{q-1} (\lambda^l)^k = 0$$

for all $l \in \{1, \dots, q-1\}$, we obtain that

$$E_i = \frac{1}{q} \sum_{k=0}^{q-1} (r_1\lambda^i)^{-k} \pi(u_\theta)^k.$$

Therefore, for $0 \leq i < q$,

$$\begin{aligned}\pi(v_\theta)E_i &= \frac{1}{q}\sum_{k=0}^{q-1}(r_1\lambda^i)^{-k}\pi(v_\theta)\pi(u_\theta)^k = \frac{1}{q}\sum_{k=0}^{q-1}(r_1\lambda^i)^{-k}\lambda^{-k}\pi(u_\theta)^k\pi(v_\theta)\\ &= \frac{1}{q}\sum_{k=0}^{q-1}(r_1\lambda^{i+1})^{-k}\pi(u_\theta)^k\pi(v_\theta) = E_{i+1}\pi(v_\theta).\end{aligned}$$

Let $r_2 \in \mathbb{T}$ such that $r_2^q = t_2$. We then verify that $E_{ij} = (\overline{r_2}\pi(v_\theta))^{i-j}E_j$ are partial isometries for $0 \leq i, j < q$. Since $(\overline{r_2}\pi(v_\theta))^q = I$, it is easy to verify that these form a set of matrix units for M_q. Moreover,

$$\pi(v_\theta) = \sum_{j=0}^{q-1} \pi(v_\theta)E_j = r_2 \sum_{j=0}^{q-1} E_{j+1,j},$$

where we interpret $E_{q,q-1}$ as $E_{0,q}$. Hence $C^*(\pi(u_\theta), \pi(v_\theta))$ is isomorphic to M_q. It follows that E_i are all one dimensional.

We have just proved that $\pi = \operatorname{Ad} U \circ \pi^{(0)} \circ \alpha_{r_1,r_2}$ for some unitary $U \in M_q$ and with the primitive ideal space I_{t_1,t_2}.

Let $\xi \in \mathbb{T}^2$. Define $\pi_\xi(a) = a(\xi)$ for all $a \in P_1 M_N(C(\mathbb{T}^2))P_1$. Then $\pi_\xi \circ \psi$ gives an irreducible representation of A_θ. From what we have proved, there is $(t(\xi)_1, t(\xi)_2) \in \mathbb{T}^2$ such that π_ξ has the kernel $I_{t(\xi)_1, t(\xi)_2}$. We will show that the map $\sigma\colon \xi \to (t(\xi)_1, t(\xi)_2)$ is a homeomorphism. From the discussion preceding this

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lemma, we know that σ is injective and, from what we proved above, it is also surjective. Since $\mathbb{T}^2$ is a compact Hausdorff space, it suffices to show that σ is continuous.

For that, we assume that $\xi_n \to \xi_0$ in $\mathbb{T}^2$. Then

$$\|\pi_{\xi_n}(\psi(u_\theta^q)) - \pi_{\xi_0}(\psi(u_\theta^q))\| = \|t(\xi_n)_1 \cdot 1_{M_q} - t(\xi_0)_1 \cdot 1_{M_q}\| \to 0,$$
$$\|\pi_{\xi_n}(\psi(v_\theta^q)) - \pi_{\xi_0}(\psi(v_\theta^q))\| = \|t(\xi_n)_2 \cdot 1_{M_q} - t(\xi_0)_2 \cdot 1_{M_q}\| \to 0.$$

Therefore,

$$|t(\xi_n)_1 - t(\xi_0)_1| \to 0 \quad \text{and} \quad |t(\xi_n)_2 - t(\xi_0)_2| \to 0.$$

This proves that σ is continuous. Therefore, σ is a homeomorphism. Define

$$\psi_1\colon P_1M_N(C(\mathbb{T}^2)P_1 \longrightarrow P_1M_N(C(\mathbb{T}^2)P_1$$

by $\psi_1(f)(x) = f(\sigma^{-1}(x))$ for all $x \in \mathbb{T}^2$ and all $f \in P_1M_N(C(\mathbb{T}^2))P_1$, then ψ_1 is an isomorphism. Put $H = \psi_1 \circ \psi$. Let $(t_1, t_2) \in \mathbb{T}^2$ and $y = \sigma^{-1}((t_1, t_2))$, *i.e.*, $(t(y)_1, t(y)_2) = (t_1, t_2)$. Then

$$\pi_{(t_1,t_2)} \circ H(u_\theta^q) = \pi_{(t_1,t_2)} \circ \psi_1 \circ \psi(u_\theta^q) = \pi_y \circ \psi(u_\theta^q) = t_1 \cdot 1_{M_q}$$

and

$$\pi_{(t_1,t_2)} \circ H(v_\theta^q) = \pi_{(t_1,t_2)} \circ \psi_1 \circ \psi(v_\theta^q) = \pi_y \circ \psi(v_\theta^q) = t_2 \cdot 1_{M_q}.$$

Put $P = \psi_1(P_1)$, and the lemma follows. ■

The following is a consequence of [21, Theorem 2.7].

Lemma 5.2 *Let $P \in M_N(C(\mathbb{T}^2))$ be a projection of rank q (for some integer $N > q$) and let $C = PM_N(C(\mathbb{T}^2))P$. For any $\epsilon > 0$ and finite subset $\mathcal{F} \subset C$, there exists $\delta(\epsilon) > 0$ and finite subset $\mathcal{G}(\epsilon)$ satisfying the following. For any unital simple infinite dimensional C^*-algebra A with real rank zero and stable rank one, if $L\colon C \to A$ is a contractive, completely positive, linear map that is $\mathcal{G}$-δ-multiplicative and $[L]|_{\ker \rho_C} \subset \ker \rho_A$, then there exists a unital homomorphism $\phi\colon C \to A$ such that $\|L(f) - \phi(f)\| < \epsilon$ for all $f \in \mathcal{F}$.*

Proof First consider the case $C = C(\mathbb{T}^2)$. Note that $K_0(C) = \mathbb{Z} \oplus \mathbb{Z}$ with $\ker \rho_C = \mathbb{Z}$ (which may be identified with the second copy of $\mathbb{Z}$) and $K_1(C) = \mathbb{Z} \oplus \mathbb{Z}$. Thus $KL(C, A) \cong \mathrm{Hom}(K_0(C), K_0(A)) \oplus \mathrm{Hom}(K_1(C), K_1(A))$. Let $\alpha \in KL(C, A)$. Then, it follows from [19] that there is a unital homomorphism $h\colon C \to A$ such that $[h] = \alpha$ if and only if $\alpha([1_C]) = [1_A]$ and $\alpha(\ker \rho_C) \subset \ker \rho_A$. Thus this theorem follows immediately from [21, Theorem 2.7] when $C = C(\mathbb{T}^2)$. It is then clear that the theorem holds in the case where $C = M_n(C(\mathbb{T}^2))$ for any integer $n \geq 1$.

For the general case, we note that there exists an integer $N_1 \geq 1$ and a rank one projection $e \in M_{N_1}(C)$ such that $eM_{N_1}(C)e \cong C(\mathbb{T}^2)$. Therefore, there is a projection $Q \in M_{N_2}(C(\mathbb{T}^2)) \subset M_{N_1N_2}(C)$ for some integer N_2, and there is a unitary $W \in M_{N_2N_1}(C)$ such that $W^*QW = P$. Define $L_1 = L \otimes \mathrm{id}_{M_{N_1}}\colon M_{N_1}(C) = C \otimes M_{N_1} \to A \otimes M_{N_1}$. Let $\epsilon_1 > 0$ be given. If L is a $\mathcal{G}$-δ-multiplicative, contractive, completely

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positive, linear map with sufficiently small δ and sufficiently large $\mathcal{G}$, then there exists a unitary $V \in A \otimes M_{N_1N_2}$ such that

$$\|(L \otimes \mathrm{id}_{M_{N_1N_2}})(W) - V\| < \epsilon_1.$$

Then $L_2 = L_1|_{eM_{N_1}(C)e}$ is close to a unital completely positive linear map L_3 from $eM_{N_1}(C)e \cong C(\mathbb{T}^2)$ into $EM_{N_1}(A)E$ for some projection $E \in M_{N_1}(A)$ that is close to $L_2(e)$, whenever δ is sufficiently small and $\mathcal{G}$ is sufficiently large. Put $B = M_{N_2}(eM_{N_1}(C(\mathbb{T}^2))e)$. Then $B \cong M_{N_2}(C(\mathbb{T}^2))$. Define $L_4 = L_3 \otimes \mathrm{id}_{M_{N_2}} : B \to M_{N_1}(A) \otimes M_{N_2}$. If L_4 is close to a unital homomorphism, say $\psi : B \to M_{N_1}(A) \otimes M_{N_2}$, then $L_4|_{QBQ}$ is close to $\psi|_{QBQ}$. Note that there is a unitary $V_1 \in A \otimes M_{N_1N_2}$, which is close to $1_{M_{N_1N_2}}$ such that $V_1^*(V^*\psi(Q)V)V_1 = L(Q) = 1_A$. Therefore L is close to $\mathrm{Ad}(VV_1) \circ \psi|_{QBQ}$. Therefore, the general case can be reduced to the case $C = M_n(C(\mathbb{T}^2))$ for some integer n. ■

Theorem 5.3 *Let $\theta \in (-1/2, 1/2)$ be a rational number. Then for any $\epsilon > 0$ there exists $\delta > 0$ satisfying the following. For any unital simple C^*-algebra A with real rank zero and stable rank one and for any pair of unitaries u and v in A such that*

$$\|uv - e^{2\pi i\theta}vu\| < \delta \quad \textit{and} \quad \frac{1}{2\pi i}\tau\big(\log(uvu^*v^*)\big) = \theta$$

for all $\tau \in T(A)$, there exists a pair of unitaries $\widetilde{u}, \widetilde{v} \in A$ such that

$$\widetilde{u}\widetilde{v} = e^{2\pi i\theta}\widetilde{v}\widetilde{u}, \quad \|u - \widetilde{u}\| < \epsilon, \quad \textit{and} \quad \|v - \widetilde{v}\| < \epsilon.$$

Proof For the sub-class of simple finite dimensional C^*-algebras, the theorem follows from [6, Corollary 7.6]. In what follows we will assume that A is infinite dimensional.

The statement for $\theta = 0$ follows from [21, Corollary 2.11] immediately, or from Lemma 5.2. So, for the rest of the proof, we may assume that $\theta = \pm p/q$, where p and q are non-zero integers with $(p, q) = 1$, $0 < 2p < q$. By Lemma 5.1, we may write $A_\theta = PM_N(C(\mathbb{T}^2))P$, where N is an integer and $P \in M_N(C(\mathbb{T}^2))$ is a projection of rank q. Moreover, $\pi_\xi(u_\theta^q) = t_1 \circ P_\xi$ and $\pi_\xi(v_\theta^q) = t_2P_\xi$ for all $\xi = (t_1, t_2) \in \mathbb{T}^2$. Therefore $\ker \rho_{A_\theta}$ is generated by a single element $\mathrm{bott}(u_\theta^q, v_\theta^q)$. Let $\epsilon > 0$ and let $\mathcal{F} = \{u_\theta, v_\theta, 1_{A_\theta}\}$. Let $\delta_1 > 0$ (in place of $\delta(\epsilon)$) be a positive number and $\mathcal{G}_1 \subset A_\theta$ (in place of $\mathcal{G}(\epsilon)$) be finite subset required by Lemma 5.2 for $C = A_\theta = PM_N(C(\mathbb{T}^2))P$, $\epsilon/2$ and $\mathcal{F}$.

Let δ_{00} be as required by Lemma 4.4 for $\epsilon_0 = \min\{\delta_1, \epsilon/2\}$ and $\mathcal{G}_1$ (in place of $\mathcal{G}$). Let

$$\delta = \min\{\delta_{00}/2q^2, \delta_0/2q^2, 1/2q^2\},$$

where δ_0 is defined in Proposition 2.8. Suppose that A is a unital simple C^*-algebra of real rank zero and stable rank one and suppose that $u, v \in A$ are two unitaries such that

$$\|uv - e^{2\pi i\theta}vu\| < \delta \quad \text{and} \quad \frac{1}{2\pi i}\tau\big(\log(uvu^*v^*)\big) = \theta.$$

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It follows from Lemma 4.4 that there exists a unital δ_1-$\mathcal{G}_1$-multiplicative contractive, completely positive, linear map $L\colon A_\theta \to A$ such that

$$\|L(u_\theta)-u\|<\epsilon/2,\ \ \|L(v_\theta)-v\|<\epsilon/2 \quad \text{and} \quad [L]\big(\mathrm{bott}(u_\theta^q, v_\theta^q)\big) = \mathrm{bott}(u^q, v^q). \tag{5.2}$$

Let $S_1, S_2 \in M_q$ be as in (5.1). Put $U = u \otimes S_2$ and $V = v \otimes S_1$ in $A \otimes M_q$. We compute that

$$UV = uv \otimes S_2S_1 \approx_\delta e^{2\pi i\theta} vu \otimes (e^{-2\pi i\theta})S_1S_2 = VU.$$

Denote

$$Z = (u\otimes 1_{M_q})(v\otimes 1_{M_q})(u^*\otimes 1_{M_q})(v^*\otimes 1_{M_q}).$$

Then

$$\begin{aligned}
&\frac{1}{2\pi i}(\tau\otimes \mathrm{Tr})\big(\log(UVU^*V^*)\big)\\
&\quad= \frac{1}{2\pi i}(\tau\otimes \mathrm{Tr})\Big(\log\big((u\otimes S_2)(v\otimes S_1)(u^*\otimes S_2^*)(v^*\otimes S_1^*)\big)\Big)\\
&\quad= \frac{1}{2\pi i}(\tau\otimes \mathrm{Tr})\big(\log(Z(1_A\otimes S_2)(1_A\otimes S_1)(1_A\otimes S_2^*)(1_A\otimes S_1^*))\big)\\
&\quad= \frac{1}{2\pi i}(\tau\otimes \mathrm{Tr})(\log(Z\cdot e^{-2\pi i\theta}\cdot 1_{M_q(A)})),
\end{aligned} \tag{5.3}$$

for all $\tau \in T(A)$. Since $e^{-2\pi i\theta}\cdot 1_{M_q(A)}$ is in the center of $M_q(A)$, (5.3) equals

$$\frac{1}{2\pi i}(\tau\otimes \mathrm{Tr})(\log Z) - q\theta = \frac{1}{2\pi i}(\tau\otimes \mathrm{Tr})\big(\log(uvu^*v^*)\otimes 1_{M_q}\big) - q\theta = q\theta - q\theta = 0.$$

By the Exel trace formula, we conclude that

$$\mathrm{bott}(U, V) \in \ker \rho_A. \tag{5.4}$$

It follows that

$$\mathrm{bott}(U^q, V^q) = q^2\,\mathrm{bott}(U, V) \in \ker \rho_A.$$

Note that $U^q = u^q \otimes 1_{M_q}$ and $V^q = v^q \otimes 1_{M_q}$. It follows that

$$q\,\mathrm{bott}(u^q, v^q) = \mathrm{bott}(U^q, V^q) \in \ker \rho_A.$$

This implies that for all $\tau \in T(A)$,

$$q\tau\big(\mathrm{bott}(u^q, v^q)\big) = 0,$$

which implies that

$$\mathrm{bott}(u^q, v^q) \in \ker \rho_A. \tag{5.5}$$

It follows from (5.5) and (5.2) that

$$[L](\mathrm{bott}(u_\theta^q, v_\theta^q)) = \mathrm{bott}(u^q, v^q) \in \ker \rho_A.$$

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Consequently, $[L]|_{\ker \rho_{A_\theta}} \subset \ker \rho_A$. By applying Lemma 5.2, we obtain a unital homomorphism $\phi\colon A_\theta \to A$ such that

$$\|L(u_\theta) - \phi(u_\theta)\| < \epsilon/2 \quad \text{and} \quad \|L(v_\theta) - \phi(v_\theta)\| < \epsilon/2. \tag{5.6}$$

Put $\widetilde{u} = \phi(u_\theta)$ and $\widetilde{v} = \phi(v_\theta)$. Note that, since ϕ is a unital homomorphism,

$$\widetilde{u}\widetilde{v} = e^{2\pi i\theta}\widetilde{v}\widetilde{u}.$$

We also have, by (5.6) and (5.2),

$$\|\widetilde{u} - u\| < \epsilon \quad \text{and} \quad \|\widetilde{v} - v\| < \epsilon. \qquad \blacksquare$$

Next we consider that $\theta = \frac{1}{2}$.

Theorem 5.4 *For any $1 > \epsilon > 0$, there exists $\delta > 0$ satisfying the following. For any unital simple infinite dimensional C^*-algebra A with real rank zero and stable rank one and for any pair of unitaries u and v in A such that*

$$\|uv + vu\| < \delta \quad \textit{and} \quad \frac{1}{2\pi i}\tau(\log_0(uvu^*v^*)) = 1/2$$

for all $\tau \in T(A)$, where $\log_0$ is a continuous logarithm defined on a compact subset F of $\{e^{it} : t \in (0, 2\pi)\}$ with values in $\{ri : r \in (0, 2\pi)\}$, then there exists a pair of unitaries $\widetilde{u}, \widetilde{v} \in A$ such that

$$\widetilde{u}\widetilde{v} = -\widetilde{v}\widetilde{u}, \quad \|u - \widetilde{u}\| < \epsilon \quad \textit{and} \quad \|v - \widetilde{v}\| < \epsilon.$$

Proof The case that A is a unital simple finite dimensional C^*-algebra follows from [7, Theorem 8.3.4]. We will consider only infinite dimensional simple C^*-algebras of real rank zero and stable rank one. The proof is exactly the same as that of Theorem 5.3 except the part to verify (5.4), *i.e.*,

$$\text{bott}(U, V) \in \ker \rho_A.$$

In other words, using the Exel trace formula, we need to show that

$$(\tau \otimes \text{Tr})(\log(UVU^*V^*)) = 0 \quad \text{for all} \quad \tau \in T(A). \tag{5.7}$$

We compute

$$UV = uv \otimes S_2S_1 \approx_\delta e^{2\pi i\theta} vu \otimes (e^{-2\pi i\theta})S_1S_2 = VU.$$

We may assume that

$$\|uv + vu\| < 1/10 \tag{5.8}$$

and

$$\frac{1}{2\pi i}\tau(\log_0(uvu^*v^*) = 1/2. \tag{5.9}$$

for all $\tau \in T(A)$. Therefore (by (5.8))

$$uvu^*v^* = \exp(ia),$$

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for some $a \in A_{s.a.}$ with $\mathrm{spec}(a) \subset (\pi - \pi/10, \pi + \pi/10)$. Moreover, by (5.9)

$$\tau(a) = \pi,$$

for all $\tau \in T(A)$.

For any $\tau \in T(A)$, we have

$$\begin{aligned}
&\frac{1}{2\pi i}(\tau \otimes \mathrm{Tr})\big(\log(UVU^*V^*)\big)\\
&\quad = \frac{1}{2\pi i}(\tau \otimes \mathrm{Tr})\Big(\log\big((u \otimes S_2)(v \otimes S_1)(u^* \otimes S_2^*)(v^* \otimes S_1^*)\big)\Big)\\
&\quad = \frac{1}{2\pi i}(\tau \otimes \mathrm{Tr})\Big(\log\big((uvu^*v^* \otimes 1_{M_2}) \cdot (e^{-\pi i} \cdot 1_{M_2(A)})\big)\Big)\\
&\quad = \frac{1}{2\pi i}(\tau \otimes \mathrm{Tr})\Big(\log\big((e^{-\pi i/3} \cdot (e^{ia} \otimes 1_{M_2})\big) \cdot (e^{-\pi i + \pi i/3} \cdot 1_{M_2(A)})\big)\Big).
\end{aligned}$$

Note that $\mathrm{spec}\big((u \otimes S_2)(v \otimes S_1)(u^* \otimes S_2^*)(v^* \otimes S_1^*)\big)$, $\mathrm{spec}(e^{\frac{-\pi i}{3}} \cdot e^{ia} \otimes 1_{M_2})$, and $\mathrm{spec}(e^{-2\pi i/3} \cdot 1_{M_2(A)})$ are all in $\{e^{it} : t \in [-2\pi/3, \pi + \pi/10 - \pi/3]\}$.

Since $e^{-\pi i/3} \cdot e^{ia} \otimes 1_{M_2}$ commutes with $e^{-\pi i + \pi i/3} \cdot 1_{M_2(A)}$, we have

$$\begin{aligned}
&(\tau \otimes \mathrm{Tr})\Big(\log\big((e^{-\pi i/3} \cdot e^{ia} \otimes 1_{M_2}) \cdot (e^{-\pi i + \pi i/3} \cdot 1_{M_2(A)})\big)\Big)\\
&\quad = (\tau \otimes \mathrm{Tr})\big(\log(e^{-\pi i/3} \cdot e^{ia} \otimes 1_{M_2})\big) + (\tau \otimes \mathrm{Tr})\big(\log(e^{-\pi i + \pi i/3} \cdot 1_{M_2(A)})\big)\\
&\quad = (\tau \otimes \mathrm{Tr})\big(\log(e^{-\pi i/3 + ia} \otimes 1_{M_2})\big) - 2\frac{2\pi i}{3}\\
&\quad = 2\tau\Big(\frac{-\pi i}{3} + ai\Big) - 2\frac{2\pi i}{3} = 0.
\end{aligned}$$

It follows that (5.7) holds. ∎

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