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Liouvillian first integrals for a class of generalized Liénard polynomial differential systems

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We study the existence of Liouvillian first integrals for the generalized Liénard polynomial differential systems of the form $x' = y$, $y' = -g(x) - f(x)y$, where $f(x) = 3Q(x)Q'(x)P(x) + Q(x)^2P'(x)$ and $g(x) = Q(x)Q'(x)(Q(x)^2P(x)^2 - 1)$ with $P, Q \in \mathbb{C}[x]$. This class of generalized Liénard polynomial differential systems has the invariant algebraic curve $(y + Q(x)P(x))^2 - Q(x)^2 = 0$ of hyperelliptic type.

Keywords: Darboux polynomial; invariant algebraic curve; exponential factor; Liouvillian first integral; Liénard polynomial differential system

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1. Introduction and statement of the main result

One of the most classical and difficult problems in the qualitative theory of planar differential systems depending on parameters is to characterize the existence and non-existence of first integrals in functions of the parameters of the system.

We consider the polynomial differential system

$$x' = y, \qquad y' = -g(x) - f(x)y, \tag{1.1}$$

called the *generalized Liénard polynomial differential system*, where x and y are complex variables and the prime denotes the derivative with respect to the time t, which can be real or complex. Such differential systems appear in several branches of the sciences, such as biology, chemistry, mechanics and electronics (see, for example, [8,21] and the references therein). For $g(x) = x$ the Liénard differential system (1.1) is called the *classical Liénard polynomial differential system*.

Let

$$X = y\frac{\partial}{\partial x} - (g(x) + f(x)y)\frac{\partial}{\partial y}$$

be the polynomial vector field associated with system (1.1). Let U be an open and dense set in $\mathbb{C}^2$. We say that the non-locally constant function $H\colon U \to \mathbb{C}$ is a *first* *integral* of the polynomial vector field X on U if $H(x(t), y(t)) = \text{const.}$ for all values of t for which the solution $(x(t), y(t))$ of X is defined on U. Clearly, H is a first integral of X on U if and only if $XH = 0$ on U.

A *Liouvillian first integral* is a first integral H which is a Liouvillian function, that is, roughly speaking, one that can be obtained 'by quadratures' of elementary functions. For a precise definition see [19]. The study of Liouvillian first integrals is a classical problem of the integrability theory of differential equations, which goes back to Liouville.

As far as we know the Liouvillian first integrals of some multi-parameter family of planar polynomial differential systems have only been completely classified for the planar Lotka–Volterra systems (see [1, 9, 15–18]).

Note that when $g(x) = x$ system (1.1) is the well-known classical Liénard polynomial differential system whose Liouvillian first integrals were studied in [11]. Moreover, the Liouvillian first integrals of these systems when $2 \leqslant \deg g \leqslant \deg f$ were studied in [12], and the Liouvillian first integrals of these systems when $\deg g = \deg f + 1$ were studied in [13].

The case when f and g are general polynomials is still open. The study of Liouvillian first integrals is based, in particular, on the search for what is called an *invariant algebraic curve*. Let $h = h(x, y) \in \mathbb{C}[x, y] \setminus \mathbb{C}$. As usual $\mathbb{C}[x, y]$ denotes the ring of all complex polynomials in the variables x and y. We say that $h = 0$ is an *invariant algebraic curve* of the vector field X if it satisfies

$$y\frac{\partial h}{\partial x} - (g(x) + f(x)y)\frac{\partial h}{\partial y} = Kh$$

for some polynomial $K = K(x, y) \in \mathbb{C}[x, y]$, called the *cofactor* of $h = 0$. Clearly, h has degree at most $m = \max\{\deg f + 1, \deg g\} - 1$. We also say that h is a *Darboux polynomial* of system (1.1). Note that a polynomial first integral is a Darboux polynomial with zero cofactor.

The invariant algebraic curves are important because a sufficient number of them forces the existence of a first integral. This result is the basis of the Darboux theory of integrability (see, for example, [4–7, 10]).

An *exponential factor* E of system (1.3) is a function of the form $E = \exp(u/v) \notin \mathbb{C}$ with $u, v \in \mathbb{C}[x, y]$ satisfying

$$y\frac{\partial E}{\partial x} - (g(x) + f(x)y)\frac{\partial E}{\partial y} = LE \tag{1.2}$$

for some polynomial $L = L(x, y)$ of degree at most m, called the *cofactor* of E.

It is easy to check the following result for any generalized Liénard polynomial differential system (1.1).

Proposition 1.1. *System (1.1) has exponential factors* e^{x^j} *with cofactors* $x^{j-1}y$ *for* $j = 1, \dots, \max\{\deg f, \deg g - 1\}$ *and exponential factors of the form* $\exp(u(x))$ *with* $u(x)$ *a polynomial of degree at most* $\max\{\deg f, \deg g - 1\}$*. Moreover, if* $\deg g \leqslant \deg f$*, then system (1.1) has the exponential factors* $\exp(x + \int f(x)\,\mathrm{d}x)$ *with cofactor* $-g(x)$.

The main difficulty in studying the Liouvillian integrability of a polynomial differential system is the characterization of the invariant algebraic curves and of the exponential factors of the polynomial differential system. For that reason we restrict our study of the Liouvillian integrability of the generalized Liénard polynomial differential systems (1.1) to the following:

$$\left.\begin{aligned} x' &= y, \\ y' &= -g(x) - f(x)y \\ &= -Q(x)Q'(x)(Q(x)^2P(x)^2 - 1) \\ &\qquad - (3Q(x)Q'(x)P(x) + Q(x)^2P'(x))y. \end{aligned}\right\} \tag{1.3}$$

System (1.3) is motivated by the work of Żołądek [22]. More precisely, Żołądek studied the Liénard differential systems (1.1) having a hyperelliptic invariant algebraic curve of the form $(y+P(x))^2 - Q(x) = 0$, where $P(x)$ and $Q(x)$ are polynomials. We consider the subclass of Liénard differential systems (1.1) studied by Żołądek having a hyperelliptic invariant algebraic curve of the form $(y + Q(x)P(x))^2 - Q(x)^2 = 0$; this is because such a curve factorizes into the two invariant algebraic curves $y + Q(x)(P(x) - 1) = 0$ and $y + Q(x)(P(x) + 1) = 0$, which allows us to study the Liouvillian integrability of the Liénard differential systems (1.1) having such invariant algebraic curves.

Our main result on the Liouvillian integrability of the class of generalized Liénard polynomial differential system (1.3) is the following.

THEOREM 1.2. *The following statements hold for the generalized Liénard polynomial differential system (1.3).*

(a) *When* $\deg Q = 0$, *i.e.* $Q(x) = \kappa \in \mathbb{C}$, *system (1.3) is Liouvillian integrable, with the first integral* $H = y + \kappa^2 P(x)$.

(b) *When* $\deg P = 0$, *i.e.* $P(x) = \kappa \in \mathbb{C}$, *system (1.3) is Liouvillian integrable with the first integral*

$$H = \frac{\kappa^2 Q(x)^2 + \kappa y - 1}{\sqrt{y^2 + (2\kappa y - 1)Q(x)^2 + \kappa^2 Q(x)^4}}.$$

(c) *Assuming that* $\deg Q \geqslant 1$ *and* $\deg P \geqslant 1$,

   (i) *the unique irreducible Darboux polynomials are* $h_1 = y + Q(x)(P(x) - 1)$ *and* $h_2 = y + Q(x)(P(x) + 1)$ *with cofactors* $K_1 = -Q'(x)(Q(x)P(x) + 1)$ *and* $K_2 = -Q'(x)(Q(x)P(x) - 1)$, *respectively;*

   (ii) *system (1.3) is not Liouvillian integrable.*

Statements (a) and (b) can be checked directly from the definition of the first integral. We shall divide the proof of statement (c) into different parts: in § 3 we shall prove (i), while the proof of (ii) will be given in § 4.

Note that the main result in statement (i) is the uniqueness of h_1 and h_2 as irreducible Darboux polynomials, because their existence follows from [22]. We remark that $\exp(x^j)$ are exponential factors for any generalized Liénard polynomial differential system (1.1). The existence of rational first integrals of the form $H = y^2 + A(x)y + B(x)$ for the differential system (1.1) when $f(x)$ and $g(x)$ are rational functions was studied by Wilson in [20].

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2. Auxiliary notions and results

The following result is well known. For a proof see, for example, [7, proposition 8.4].

LEMMA 2.1. *Assume* $f \in \mathbb{C}[x,y]$ *and let* $f = f_1^{n_1} \cdots f_r^{n_r}$ *be its factorization into irreducible factors over* $\mathbb{C}[x,y]$. *Then, for a polynomial differential system (1.1),* $f = 0$ *is an invariant algebraic curve with cofactor* K_f *if and only if* $f_i = 0$ *is an invariant algebraic curve for each* $i = 1, \dots, r$ *with cofactor* K_{f_i}. *Moreover,* $K_f = n_1 K_{f_1} + \cdots + n_r K_{f_r}$.

PROPOSITION 2.2. *The following statements hold.*

(a) *If* $E = \exp(u/v)$ *is an exponential factor for the polynomial differential system (1.3) and* v *is not a constant polynomial, then* $v = 0$ *is an invariant algebraic curve.*

(b) *Eventually* $E = \exp(u)$ *can be exponential factors coming from the multiplicity of the invariant straight line at infinity.*

For a geometric meaning of exponential factors and a proof of proposition 2.2 see [3]. The existence of exponential factors $\exp(u/v)$ is due to the fact that the multiplicity of the invariant algebraic curve $v = 0$ is greater than 1 (again, for more details, see [3]).

The following result, given in [3], characterizes the algebraic multiplicity of an invariant algebraic curve using the number of exponential factors of system (1.3) associated with the invariant algebraic curve.

PROPOSITION 2.3. *Given an irreducible invariant algebraic curve* $v = 0$ *of degree* k *in system (1.3), it has algebraic multiplicity* ℓ *if and only if the vector field associated with system (1.3) has* $\ell - 1$ *exponential factors* $\exp(u_i/v^i)$, *where* u_i *is a polynomial of degree at most* ik *and* $(u_i, v) = 1$ *for* $i = 1, \dots, \ell - 1$.

In view of proposition 2.3 if we prove that $\mathrm{e}^{u/v}$ is not an exponential factor with $\deg u \leqslant \deg v$, there are no exponential factors associated with the invariant algebraic curve $v = 0$.

We say that a C^1 function $V = V(x,y)$ is an integrating factor if it satisfies

$$XV = -\operatorname{div} XV,$$

where div stands for the divergence of the vector field X.

In 1992 Singer [19] proved that a polynomial differential system has a Liouvillian first integral if and only if it has an integrating factor of the form

$$\exp\left(\int U_1(x,y)\,\mathrm{d}x + \int U_2(x,y)\,\mathrm{d}y\right),$$

where U_1 and U_2 are rational functions that verify $\partial U_1/\partial y = \partial U_2/\partial x$. In 1999 Christopher [2] improved the results of Singer, showing that there are integrating factors of the form

$$\exp\left(\frac{u}{v}\right) \prod_{i=1}^{k} f_i^{\lambda_i}, \tag{2.1}$$

where u, v and f_i are polynomials and $\lambda_i \in \mathbb{C}$. From the Darboux theory of integrability (see [7, 10, 19]) we have the following result.

THEOREM 2.4. *The polynomial differential system (1.3) has a Liouvillian first integral if and only if system (1.3) has an integrating factor of the form (2.1), or, equivalently, there exist p invariant algebraic curves $f_i = 0$ with cofactors K_i for $i = 1, \dots, p$, q exponential factors $E_j = \exp(u_j/v_j)$ with cofactors L_j for $j = 1, \dots, q$ and $\lambda_j, \mu_j \in \mathbb{C}$ not all zero such that*

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = -\textit{divergence of (1.3)} = f(x).$$

3. Proof of theorem 1.2(i)

The proof is a direct consequence of the following auxiliary results.

PROPOSITION 3.1. *Let $h = h(x, y)$ be a Darboux polynomial of system (1.3) with cofactor $K \neq 0$. Then $K = K(x)$.*

Proof. As system (1.3) has $\deg g = 2 \deg f + 1 = m + 1 \geqslant 3$, and K is a polynomial of degree at most m, we can write K as

$$K(x, y) = \sum_{j=0}^{m} K_j(x) y^j, \tag{3.1}$$

where $K_j(x)$ has degree at most $m - j$. By assumption, h satisfies

$$y \frac{\partial h}{\partial x} - (g(x) + f(x)y) \frac{\partial h}{\partial y} = h \sum_{j=0}^{m} K_j(x) y^j, \tag{3.2}$$

where f and g were given in (1.3). We write $h(x, y) = \sum_{j=0}^{l} h_j(x) y^j$. Without loss of generality we can assume that $h_l(x) \neq 0$. Computing the coefficient of y^{l+m} in (3.2), we get

$$0 = h_l(x) K_m(x), \quad \text{i.e. } K_m(x) = 0.$$

Therefore, repeating this argument for $y^{l+m-1}, \dots, y^{l+2}$, we get that $K_j(x) = 0$ for $j = 2, \dots, m-1$. Hence, $K(x) = K_0(x) + K_1(x)y$. Computing the coefficient of y^{l+1} in (3.2) we get $h_l'(x) = h_l(x) K_1(x)$, that is

$$h_l(x) = C \exp\left(\int K_1(x) \, \mathrm{d}x \right), \quad C \in \mathbb{C}.$$

Since $h_l(x)$ must be a polynomial in x, we have that $K_1(x) = 0$. This completes the proof of the proposition. □

PROPOSITION 3.2. *The unique irreducible Darboux polynomials of system (1.3) with non-zero cofactor are $h_1 = y + Q(x)(P(x) - 1)$ and $h_2 = y + Q(x)(P(x) + 1)$ with respective cofactors $K_1 = -Q'(x)(Q(x)P(x) + 1)$ and $K_2 = -Q'(x)(Q(x)P(x) - 1)$.*

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Proof. By direct computations we obtain that h_1 and h_2 are irreducible Darboux polynomials of system (1.3).

Now we shall prove that these are the only irreducible Darboux polynomials of system (1.3). Let $h = h(x, y)$ be another irreducible Darboux polynomial of system (1.3) with cofactor K. In view of proposition 3.1 we have that $K = K(x)$. Then,

$$y\frac{\partial h}{\partial x} - (g(x) + f(x)y)\frac{\partial h}{\partial y} = K(x)h,$$

with f and g as in (1.3).

Now we introduce the variables (X, Y) with

$$X = x \quad \text{and} \quad Y = h_1 = y + Q(x)(P(x) - 1). \tag{3.3}$$

Then in these variables system (1.3) becomes

$$X' = Y - Q(X)(P(X) - 1), \quad Y' = -Q'(X)(Q(X)P(X) + 1)Y. \tag{3.4}$$

Let $h = \bar{h}(X, Y)$. Then, if we denote by $\tilde{h} = \tilde{h}(X)$ the restriction of $\bar{h}$ to $Y = 0$ we get that $\tilde{h} \neq 0$ (otherwise h would not be irreducible). Note that $\tilde{h}$ is a Darboux polynomial of system (3.4) restricted to $Y = 0$, that is,

$$-Q(X)(P(X) - 1)\frac{\mathrm{d}\tilde{h}}{\mathrm{d}X} = K(X)\tilde{h}, \tag{3.5}$$

where $K(X)$ is the cofactor of $\tilde{h}$, equal to the cofactor of h.

Solving this linear differential equation, we deduce that

$$\tilde{h} = C\exp\left(-\int \frac{K(X)}{Q(X)(P(X) - 1)}\,\mathrm{d}X\right), \quad C \in \mathbb{C} \setminus \{0\}. \tag{3.6}$$

Let $r(X) = -Q(X)(P(X) - 1)$. Without loss of generality we can assume that K and r are coprime; otherwise, we divide by their common factor. We claim that

$$\deg K < \deg r. \tag{3.7}$$

We proceed by contradiction. Assume (3.7) and consider the Euclidean division of K and r. We have

$$K(X) = s(X)r(X) + \psi(X), \tag{3.8}$$

where $\psi(X)$ cannot be zero, taking into account that K and r are coprime and $\deg \psi < \deg r$. Hence, (3.8) becomes

$$\frac{K(X)}{r(X)} = s(X) + \frac{\psi(X)}{r(X)}. \tag{3.9}$$

Integrating this equation and taking into account (3.6), we have that

$$\tilde{h}(X) = C\exp(\tilde{s}(X))\exp\left(\int \frac{\psi(X)}{r(X)}\,\mathrm{d}X\right), \quad C \in \setminus\{0\}, \tag{3.10}$$

where $\tilde{s}'(X) = s(X)$. Therefore, the first factor in (3.10) cannot cancel the second factor of (3.10), and this contradicts the fact that $\tilde{h}(X)$ is a polynomial. Hence, we conclude that $\deg K < \deg r$, which proves (3.7).

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We say that the polynomial $r(X)$ is square-free if $r(X) = \prod_{l=1}^{k}(X - \alpha_l)$ with $\alpha_l \neq \alpha_l$ for $l, j = 1, \dots, k$ and $l \neq j$. We claim that

$$\text{the polynomial } r \text{ must be square-free.} \tag{3.11}$$

We again proceed by contradiction. Using an affine transformation of the form $X \mapsto X + \alpha$ with $\alpha \in \mathbb{C}$ if necessary, we can assume that X is a factor of the polynomial r with multiplicity $\mu > 1$. Then we write it as $r(X) = X^{\mu}s(X)$ with $s(0) \neq 0$. We know that $K(0) \neq 0$, since K and r are coprime. Now we develop $K(X)/r(X)$ in simple fractions of X, that is

$$\frac{K(X)}{r(X)} = \frac{c_\mu}{X^\mu} + \frac{c_{\mu-1}}{X^{\mu-1}} + \cdots + \frac{c_1}{X} + \frac{\alpha_1(X)}{s(X)},$$

where $\alpha_1(X)$ is a polynomial with $\deg \alpha_1 < \deg s$ and $c_i \in \mathbb{C}$ for $i = 1, 2, \dots, \mu$. Equating both expressions, we get that $c_\mu = K(0)/s(0) \neq 0$. Therefore, (3.6) becomes

$$\tilde{h}(X) = C \exp\left(\frac{c_\mu}{1-\mu}\frac{1}{X^{\mu-1}}\right) \exp\left[\int \left(\frac{c_{\mu-1}}{X^{\mu-1}} + \cdots + \frac{c_1}{\mu} + \frac{\alpha_1(X)}{s(X)}\right) \mathrm{d}X\right],$$

where $C \in \mathbb{C} \setminus \{0\}$. The first exponential cannot be simplified with any part of the second exponential. Thus, we get a contradiction with the fact that $\tilde{h}$ must be a polynomial. Therefore, we conclude that r must be square-free, and (3.11) is proved.

Hence, we have

$$\frac{K(X)}{r(X)} = \frac{\gamma_1}{X - \alpha_1} + \cdots + \frac{\gamma_k}{X - \alpha_k}. \tag{3.12}$$

Integrating (3.6), we get

$$\tilde{h}(X) = C(X - \alpha_1)^{\gamma_1}(X - \alpha_2)^{\gamma_2} \cdots (X - \alpha_k)^{\gamma_k}, \quad C \in \mathbb{C} \setminus \{0\}.$$

Since $\tilde{h}$ must be a polynomial, we must have that $\gamma_i \in \mathbb{N} \cup \{0\}$ for $i = 1, \dots, k$.

Now we introduce the variables (X, Y) with

$$X = x \quad \text{and} \quad Y = h_2 = y + Q(x)(P(x) + 1). \tag{3.13}$$

Then, in these variables, system (1.3) becomes

$$X' = Y - Q(X)(P(X) + 1), \qquad Y' = -Q'(X)(Q(X)P(X) - 1)Y. \tag{3.14}$$

Let $h = \hat{h}(X, Y)$. Then, if we denote by $h^* = h^*(X)$ the restriction of $\hat{h}$ to $Y = 0$, we get that $h^* \neq 0$ (otherwise h would not be irreducible). Here h^* is a Darboux polynomial of system (3.14) restricted to $Y = 0$, that is

$$-Q(X)(P(X) + 1)\frac{\mathrm{d}h^*}{\mathrm{d}X} = K(X)h^*.$$

Solving this linear differential equation, we deduce that

$$h^* = C_1 \exp\left(-\int \frac{K(X)}{Q(X)(P(X) + 1)} \mathrm{d}X\right), \quad C_1 \in \mathbb{C} \setminus \{0\}. \tag{3.15}$$

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Proceeding as we did for $\tilde{h}$, if we define $r^*(X) = -Q(X)(P(X)+1)$, then we must have that r^* is square-free and that

$$\frac{K(X)}{r^*(X)} = \frac{\delta_1}{X-\beta_1} + \cdots + \frac{\delta_\ell}{X-\beta_\ell}. \tag{3.16}$$

Integrating (3.15), we get

$$h^*(X) = C_1(X-\beta_1)^{\delta_1}(X-\beta_2)^{\delta_2}\cdots(X-\beta_\ell)^{\delta_\ell}, \quad C_1 \in \mathbb{C}\setminus\{0\}.$$

Since h^* must be a polynomial, we must have that $\beta_i \in \mathbb{N}\cup\{0\}$ for $i = 1,\ldots,\ell$.

Note that if we denote by $h = h(x,y)$ a Darboux polynomial of system (1.3) with cofactor $K = K(x)$, then

$$h = \tilde{h} + (y + Q(x)(P(x)-1))h_1 = h^* + (y + Q(x)(P(x)+1))h_2$$

for some polynomials $h_1, h_2 \in \mathbb{C}[x,y]$. Moreover, from (3.12) we obtain

$$K(x) = -\frac{\tilde{h}'(x)}{\tilde{h}(x)}Q(x)(P(x)-1),$$

where the prime denotes the derivative with respect to x, and from (3.16) we get

$$K(x) = -\frac{h^{*'}(x)}{h^*(x)}Q(x)(P(x)+1).$$

Hence,

$$\frac{\tilde{h}'(x)}{\tilde{h}(x)}(P(x)-1) = \frac{h^{*'}(x)}{h^*(x)}(P(x)+1),$$

which yields

$$\left(\frac{\tilde{h}'(x)}{\tilde{h}(x)} - \frac{h^{*'}(x)}{h^*(x)}\right)P(x) = \frac{\tilde{h}'(x)}{\tilde{h}(x)} + \frac{h^{*'}(x)}{h^*(x)}.$$

That is,

$$P(x) = \frac{\tilde{h}'(x)h^*(x) + \tilde{h}(x)h^{*'}(x)}{\tilde{h}'(x)h^*(x) - \tilde{h}(x)h^{*'}(x)} = \frac{P_1(x)+P_2(x)}{P_1(x)-P_2(x)}, \tag{3.17}$$

where

$$P_1(x) = \tilde{h}'(x)h^*(x), \qquad P_2(x) = \tilde{h}(x)h^{*'}(x).$$

It follows from (3.17) that any zero of $P_1(x) - P_2(x)$ must be a zero of $P_1(x)$ and $P_2(x)$. This implies that $P_1(x) = aP_2(x)$ with $a \in \mathbb{C}$. However, since $P(x)$ is not constant, because $\deg P \geqslant 1$, this is not possible. Hence, we have a contradiction. This concludes the proof of the proposition. □

Proposition 3.3. *System (1.3) has no polynomial first integrals.*

Proof. We introduce the variables (X,Y) as in (3.3) and we get system (3.4). Let $h = \bar{h}(X,Y)$ be a polynomial first integral. Then, if we denote by $\tilde{h} = \tilde{h}(X)$ the restriction of $\bar{h}$ to $Y = 0$, $\tilde{h}$ satisfies (3.5) with $K(X) = 0$, i.e.

$$-Q(X)(P(X)-1)\frac{\mathrm{d}\tilde{h}}{\mathrm{d}X} = 0.$$

Then

$$\tilde{h}(X) = \tilde{c} \in \mathbb{C}.$$

Since we can assume without loss of generality that h has no constant terms, we have $\tilde{c} = 0$, and thus $\tilde{h} = 0$.

Now, introducing the variables (X, Y) as in (3.13), we get system (3.14). Then, if we denote by $h^* = h^*(X)$ the restriction of $\hat{h}$ to $Y = 0$, h^* satisfies (3.16) with $K(X) = 0$, i.e.

$$-Q(X)(P(X)+1)\frac{\mathrm{d}h^*}{\mathrm{d}X} = 0.$$

Then

$$h^*(X) = c^* \in \mathbb{C}.$$

In short, any polynomial first integral h can be written as

$$h = (y + Q(x)(P(x)-1))g_1 = c^* + (y + Q(x)(P(x)+1))g_2 \tag{3.18}$$

for some polynomials $g_1, g_2 \in \mathbb{C}[x, y]$. Restricting h to $y = -Q(x)(P(x)-1)$ and setting $\bar{g}_2 = \bar{g}_2(x) = g_2(x, -Q(x)(P(x)-1))$ (that is, $\bar{g}_2$ is the restriction of g_2 to $y = -Q(x)(P(x)-1)$) from (3.18) we get

$$0 = c^* + 2Q(x)\bar{g}_2(x),$$

but since $Q(x)$ is not constant because $\deg Q \geqslant 1$ this is not possible unless $c^* = 0$ and $\bar{g}_2(x) = 0$. Therefore, h can be written as

$$h = (y + Q(x)(P(x)-1))g_1 = (y + Q(x)(P(x)+1))g_2.$$

Hence,

$$h = [(y + Q(x)P(x))^2 - Q(x)^2]g_3$$

for some $g_3 \in \mathbb{C}[x, y]$ that satisfies

$$y\frac{\partial g_3}{\partial x} - (Q(x)Q'(x)(Q(x)^2P(x)^2-1) + (3Q(x)Q'(x)P(x) + Q(x)^2P'(x))y)\frac{\partial g_3}{\partial y} = Kg_3,$$

with $K = 2Q'(x)Q(x)P(x)$. In other words g_3 must be a Darboux polynomial of system (1.3) with cofactor $K = 2Q'(x)Q(x)P(x)$. In view of proposition 3.2 and lemma 2.1 we must have

$$m_1K_1(x) + m_2K_2(x) = 2Q'(x)Q(x)P(x), \quad m_1, m_2 \in \mathbb{N} \cup \{0\},$$

where $K_1(x) = -Q'(x)(Q(x)P(x)+1)$ and $K_2(x) = -Q'(x)(Q(x)P(x)-1)$. This is not possible because m_1 and m_2 must be positive integers, and this contradiction completes the proof of the proposition. □

Proof of theorem 1.2(i). The proof of theorem 1.2(i) follows directly from propositions 3.2 and 3.3. □

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4. Proof of theorem 1.2(ii)

We divide the proof of theorem 1.2 into different steps.

LEMMA 4.1. *System (1.3) has no exponential factors of the form* $\exp(u/h)$ *with* u *and* h *coprime and* $\deg u < \deg h$*,* h *being one of the two irreducible Darboux polynomials of proposition 3.2.*

Proof. Let $h_1 = y + Q(x)(P(x) - 1)$ and let $E = \exp(u/h_1)$, with u and h_1 being coprime. Clearly, after cancelling the $\exp(u/h_1)$, we get that u satisfies

$$y\frac{\partial u}{\partial x} - (Q(x)Q'(x)(Q(x)^2P(x)^2 - 1) + (3Q(x)Q'(x)P(x) + Q(x)^2P'(x))y)\frac{\partial u}{\partial y} + Q'(x)(Q(x)P(x) + 1)u = L(x, y)h_1, \tag{4.1}$$

where L is a polynomial of degree at most m. We introduce the change of variables of (3.3), and (4.1) becomes

$$(Y - Q(X)(P(X) - 1))\frac{\partial \bar{u}}{\partial X} - Q'(X)(Q(X)P(X) + 1)Y\frac{\partial \bar{u}}{\partial Y} + Q'(X)(Q(X)P(X) + 1)\bar{u} = \bar{L}Y, \tag{4.2}$$

where $\bar{u} = \bar{u}(X, Y) = u(x, y)$ and $\bar{L} = \bar{L}(X, Y) = L(x, y)$. If we denote by $\tilde{u}$ the restriction of $\bar{u}$ to $Y = 0$, we have that $\tilde{u} \neq 0$ (otherwise $\bar{u}$ would be divisible by Y). Evaluating (4.2) on $Y = 0$ we conclude that

$$-Q(X)(P(X) - 1)\frac{\mathrm{d}\tilde{u}}{\mathrm{d}X} + Q'(X)(Q(X)P(X) + 1)\tilde{u} = 0.$$

Therefore, $\tilde{u}$ must be a polynomial that satisfies (3.5) with

$$K(X) = -Q'(X)(Q(X)P(X) + 1).$$

Note that, proceeding as in the proof of proposition 3.2, we get that $\deg K(X)$ must be less than the degree of $Q(X)(P(X) - 1)$, which is not the case. Hence, system (1.3) has no exponential factors of the form $\exp(u/h_1)$ with u and h_1 being coprime.

Let $h_2 = y + Q(x)(P(x) + 1)$ and $E = \exp(u/h_2)$ with u and h_2 being coprime. After simplifying by u/h_2, we get that u satisfies

$$y\frac{\partial u}{\partial x} - (Q(x)Q'(x)(Q(x)^2P(x)^2 - 1) + (3Q(x)Q'(x)P(x) + Q(x)^2P'(x))y)\frac{\partial u}{\partial y} + Q'(x)(Q(x)P(x) - 1)u = L(x, y)h_2, \tag{4.3}$$

where L is a polynomial of degree at most m. We introduce the change of variables of (3.13), and (4.3) becomes

$$(Y - Q(X)(P(X)+1))\frac{\partial \bar{u}}{\partial X} - Q'(X)(Q(X)P(X)-1)Y\frac{\partial \bar{u}}{\partial Y} + Q'(X)(Q(X)P(X)-1)\bar{u} = \bar{L}Y, \quad (4.4)$$

where $\bar{u} = \bar{u}(X,Y) = u(x,y)$ and $\bar{L} = \bar{L}(X,Y) = L(x,y)$. If we denote by $\tilde{u}$ the restriction of $\bar{u}$ to $Y=0$, we have that $\tilde{u} \neq 0$ (otherwise $\bar{u}$ would be divisible by Y). Evaluating (4.4) on $Y=0$, we conclude that

$$-Q(X)(P(X)+1)\frac{\mathrm{d}\tilde{u}}{\mathrm{d}X} + Q'(X)(Q(X)P(X)-1)\tilde{u} = 0.$$

Note that by proceeding as in the proof of proposition 3.2 we get that the degree of $(Q'(X)(Q(X)P(X)-1))$ must be less than the degree of $Q(X)(P(X)+1)$, which is not the case. Hence, system (1.3) has no exponential factors of the form $\exp(u/h_2)$ with u and h_2 being coprime. □

LEMMA 4.2. *System (1.3) has no exponential factors of the form* $\exp(u/h_j^n)$ *with* $u \in \mathbb{C}[x,y]$ *coprime with* h_j *for* $j=1,2$ *and* $n \geqslant 1$, *and* h_1 *and* h_2 *being the two irreducible Darboux polynomials of proposition 3.2.*

Proof. The proof follows directly from proposition 2.3 and lemma 4.1. □

LEMMA 4.3. *System (1.3) has no exponential factors of the form* $\exp(u/(h_1^{n_1}h_2^{n_2}))$ *with* u, h_1 *and* h_2 *coprime,* $n_1 \geqslant 1$, $n_2 \geqslant 1$, *and* h_1 *and* h_2 *being the two irreducible Darboux polynomials of proposition 3.2.*

To prove lemma 4.3 we state the following result, whose proof was given in [14, lemma 3.2]. In fact, in [14] Llibre and Valls prove only one direction, but by working backwards in the proof we readily get the other direction.

LEMMA 4.4. *The functions* $\exp(g_1/h_1), \dots, \exp(g_r/h_r)$ *are exponential factors of some polynomial differential system with cofactors* L_j *for* $j=1,\dots,r$ *if and only if* $\exp(g_1/h_1 + \cdots + g_r/h_r)$ *is an exponential factor of the same differential system with cofactor* $L = \sum_{j=1}^{r} L_j$.

Proof of lemma 4.3. Assume that $\exp(u/(h_1^{n_1}h_2^{n_2}))$ is an exponential factor of system (1.3). Then, writing

$$\frac{u}{h_1^{n_1}h_2^{n_2}} = \frac{c_1}{h_1} + \cdots + \frac{c_{n_1}}{h_1^{n_1}} + \frac{d_1}{h_2} + \cdots + \frac{d_{n_2}}{h_2^{n_2}},$$

where c_k and d_l are polynomials of degree less than the degrees of h_1^k and h_2^l, respectively, for $k=1,\dots,n_1$ and $l=1,\dots,n_2$, and using lemma 4.4, we obtain that each c_k/h_1^k and d_l/h_1^l must be exponential factors, but this is impossible in view of lemma 4.2. This concludes the proof. □

Using lemmas 4.2 and 4.3 we get that the unique possible exponential factors of system (1.3) are of the form e^u with $u \in \mathbb{C}[x,y]$.

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Lemma 4.5. *If system (1.3) has a Liouvillian first integral, then it has an integrating factor of the form* $\exp(u(x,y))h_1^{\lambda_1}h_2^{\lambda_2}$, *where* $u \in \mathbb{C}[x,y]$, $\lambda_1, \lambda_2 \in \mathbb{C}$ *and* h_1 *and* h_2 *are the Darboux polynomials of theorem 1.2(i). Moreover, the cofactor of the exponential factor* $\exp(u(x,y))$ *is a polynomial* $L = L(x)$.

Proof. Let $L(x,y)$ be the cofactor of $\exp(u(x,y))$. In order that system (1.3) has a Liouvillian first integral, by theorems 2.4, 1.2(i) and lemma 4.1 we must have

$$\begin{aligned} -\lambda_1 Q'(x)(Q(x)P(x)+1) - \lambda_2 Q'(x)(Q(x)P(x)-1) + L(x,y) \\ = f(x) \\ = 3Q(x)Q'(x)P(x) + Q(x)^2P'(x). \end{aligned} \tag{4.5}$$

We expand L in power series in the variable y as $L(x,y) = \sum_{j=0}^m L_j(x)y^j$. Computing the coefficients of y^j with $j > 0$ in (4.5), we get that $L_j(x) = 0$ for $j = 1, \dots, n$ and thus $L = L_0(x)$. This concludes the proof. □

Since we are looking for Liouvillian first integrals of system (1.3), in view of lemma 4.5, we can restrict our study to the exponential factors with cofactor $L = L(x)$.

Proposition 4.6. *System (1.3) has no exponential factors of the form* $\exp(u)$, *where* $u \in \mathbb{C}[x,y]$ *with cofactor* $L = L(x)$.

Proof. Let $E = \exp(u)$ with $u \in \mathbb{C}[x,y] \setminus \mathbb{C}$ and let $L = L(x) = \sum_{k=0}^m \beta_k x^k$ be the cofactor associated with E with $\beta_k \in \mathbb{C}$. We write

$$u = \sum_{j=0}^r u_j(x)y^j.$$

Without loss of generality we can assume that $u_r(x) \neq 0$. By the definition of the exponential factor in (1.2) we have

$$y\frac{\partial u}{\partial x} - (g(x) + f(x)y)\frac{\partial u}{\partial y} = \sum_{k=0}^m \beta_k x^k, \tag{4.6}$$

with f and g as in (1.3). Then

$$\begin{aligned} \sum_{j=1}^r u_j'(x)y^{j+1} - Q(x)Q'(x)(Q(x)^2P(x)^2 - 1)\sum_{j=1}^r ju_j(x)y^{j-1} \\ - Q(x)(3Q'(x)P(x) + Q(x)P'(x))\sum_{j=1}^r ju_j(x)y^j = \sum_{k=0}^m \beta_k x^k. \end{aligned} \tag{4.7}$$

We write

$$Q(x) = a_q x^q + \text{l.o.t.} \quad \text{and} \quad P(x) = b_p x^p + \text{l.o.t.},$$

where ‘l.o.t.’ denotes the lower-order terms in x.

Now we consider two cases.

Case 1 ($r \geqslant 2$). Computing the coefficient of y^{r+1} in (4.7), we get that $u_r'(x) = 0$, i.e. without loss of generality we can take $u_r(x) = 1$. Now we claim that if we write

$$u = u(x, y) = y^r + \sum_{j=1}^{r} u_{r-j}(x) y^{r-j},$$

then, for $j = 1, \dots, r$,

$$u_{r-j}(x) = \frac{(a_q^2 b_p)^j A_j}{j!(2q+p)^j} x^{j(2q+p)} + \text{l.o.t.}, \tag{4.8}$$

where $A_1 = (3q+p)r$, $A_2 = q(2q+p)r + (3q+p)^2 r(r-1)$ and, for $\ell \geqslant 2$,

$$A_{\ell+1} = (3q+p)(r-\ell)A_\ell + q\ell(2q+p)(r-\ell-1)A_{\ell-1}. \tag{4.9}$$

Note that in view of (4.9) we have that $A_{\ell+1} > 0$ for any $\ell = 0, \dots, r-1$.

We start the proof of the claim. For $j = 1$, computing the coefficient of y^r in (4.7), we get that

$$u_{r-1}'(x) = rQ(x)(3Q'(x)P(x) + Q(x)P'(x)) = r(3q+p)a_q^2 b_p x^{2q+p-1} + \text{l.o.t.}$$

Integrating it, we obtain

$$u_{r-1}(x) = \frac{a_q^2 b_p (3q+p) r}{2q+p} x^{2q+p} + \text{l.o.t.},$$

which coincides with (4.8) for $j = 1$.

For $j = 2$, computing the coefficient of y^{r-1} in (4.7), we get that

$$\begin{aligned} u_{r-2}'(x) &= Q(x)Q'(x)(Q(x)^2 P(x)^2 - 1)r \\ &\quad + Q(x)(3Q'(x)P(x) + Q(x)P'(x))(r-1)u_{r-1}(x). \end{aligned}$$

Now, using that

$$u_{r-1}(x) = \frac{a_q^2 b_p (3q+p) r}{2q+p} x^{2q+p} + \text{l.o.t.},$$

we obtain

$$\begin{aligned} u_{r-2}'(x) &= q a_q^4 b_p^2 r x^{4q+2p-1} \\ &\qquad + (3q+p) a_q^2 b_p (r-1) x^{2q+p-1} \frac{a_q^2 b_p (3q+p) r}{2q+p} x^{2q+p} + \text{l.o.t.} \\ &= q a_q^4 b_p^2 r x^{4q+2p-1} + a_q^4 b_p^2 \frac{(3q+p)^2 r(r-1)}{2q+p} x^{4q+2p-1} + \text{l.o.t.} \\ &= \frac{a_q^4 b_p^2}{2q+p} (q(2q+p)r + (3q+p)^2 r(r-1)) x^{4q+2p-1} + \text{l.o.t.} \\ &= \frac{a_q^4 b_p^2 A_2}{2q+p} x^{4q+2p-1} + \text{l.o.t.} \end{aligned}$$

Integrating this, we get

$$u_{r-2}(x) = \frac{a_q^4 b_p^2 A_2}{2!(2q+p)^2} x^{4q+2p} + \text{l.o.t.},$$

which coincides with (4.8) for $j = 2$.

Now we assume that (4.8) holds for $j = 0, \dots, L$ with $L < r$, and we shall prove it for $j = L + 1$. Computing the terms in (4.7) with y^{r-L}, we get

$$\begin{aligned} u'_{r-L-1}(x) = Q'(x)Q(x)^3P(x)^2(r-L+1)u_{r-L+1}(x) \\ + Q(x)(3Q'(x)P(x) + Q(x)P'(x))(r-L)u_{r-L}(x) + \text{l.o.t.} \end{aligned}$$

Now, using the induction hypothesis and (4.9), we obtain that

$$\begin{aligned} u'_{r-L-1}(x) &= q a_q^4 b_p^2 x^{4q+2p-1}(r-L+1)\frac{(a_q^2 b_p)^{L-1} A_{L-1}}{(L-1)!(2q+p)^{L-1}} x^{(L-1)(2q+p)} \\ &\quad + (3q+p) a_q^2 b_p x^{2q+p-1}(r-L)\frac{(a_q^2 b_p)^L A_L}{L!(2q+p)^L} x^{L(2q+p)} + \text{l.o.t.} \\ &= \frac{(a_q^2 b_p)^{L+1}}{L!(2q+p)^L} x^{(L+1)(2q+p)-1}(qL(2q+p)(r-L+1)A_{L-1} \\ &\quad + (3q+p)(r-L)A_L) + \text{l.o.t.} \\ &= \frac{(a_q^2 b_p)^{L+1} A_{L+1}}{L!(2q+p)^L} x^{(L+1)(2q+p)-1} + \text{l.o.t.} \end{aligned}$$

Integrating the above equation yields

$$\begin{aligned} u_{r-L-1}(x) &= \frac{(a_q^2 b_p)^{L+1} A_{L+1}}{L!(2q+p)^L (L+1)(2q+p)} x^{(L+1)(2q+p)} + \text{l.o.t.} \\ &= \frac{(a_q^2 b_p)^{L+1} A_{L+1}}{(L+1)!(2q+p)^{L+1}} x^{(L+1)(2q+p)} + \text{l.o.t.}, \end{aligned}$$

which is (4.8) with $j = L + 1$. This completes the proof of the claim.

From (4.8) with $j = r - 1$ we obtain

$$u_1(x) = \frac{(a_q^2 b_p)^{r-1} A_{r-1}}{(r-1)!(2q+p)^{r-1}} x^{(r-1)(2q+p)} + \text{l.o.t.} \tag{4.10}$$

We recall that $A_{r-1} > 0$. Now, computing the coefficient of y^0 in (4.7), we get

$$-Q(x)Q'(x)(Q(x)^2P(x)^2 - 1)u_1(x) = \sum_{k=0}^{m} \beta_{0,k} x^k. \tag{4.11}$$

Using (4.10), the degree of the polynomial on the left-hand side of (4.11) is $(r-1)(2q+p) + 4q + 2p - 1 \geqslant 6q + 3p - 1$. Since the degree of the right-hand side is at most $m = 4q + 2p - 2$, we have a contradiction.

CASE 2 ($r \leqslant 1$). We write $u = u(x,y) = u_0(x) + u_1(x)y$. Computing the coefficient of y^2 in (4.7), we get

$$u_1'(x) = 0,$$

and without loss of generality we can take

$$u_1(x) = 1.$$

Furthermore, the coefficient of y in (4.7) gives

$$u_0'(x) - (3Q(x)Q'(x)P(x) + Q(x)^2P'(x)) = 0, \tag{4.12}$$

that is,

$$u_0(x) = \beta^0 + \int (3Q(x)Q'(x)P(x) + Q(x)^2P'(x))\,\mathrm{d}x,$$

β^0 being a constant.

Finally, the coefficient of y^0 in (4.7) gives

$$-Q(x)Q'(x)(Q(x)^2P(x)^2 - 1) = \sum_{k=0}^{m} \beta_k x^k. \tag{4.13}$$

Since $g(x) = Q(x)Q'(x)(Q(x)^2P(x)^2 - 1)$ has degree $m+1$, from (4.13) we get a contradiction. This concludes the proof of the proposition. □

Proof of theorem 1.2(ii). The proof of theorem 1.2 follows directly from lemma 4.5 and proposition 4.6. □

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